Ann Glob Anal Geom (2009) 35:227–230
DOI 10.1007/s10455-008-9131-y

ORIGINAL PAPER

Correction to "All regular Landsberg metrics are Berwald"

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Received: 8 July 2008 / Accepted: 17 September 2008 / Published online: 5 October 2008

Abstract Just recently, incompleteness in the proof of the theorem appearing in the title [published in Szabó (Ann Glob Anal Geom, to appear, 2008)] has been discovered. Without this problematic part, the theorem is established only in the following restricted form: "A regular Finsler metric is Berwald if and only if it satisfies the dual Landsberg condition." The incompleteness appears in proving that the original Landsberg condition implies the dual one.

Keywords Finsler metrics · Landsberg metrics · Berwald metrics

Landsberg metrics are such Finsler metrics where the Berwald parallel transports are isometries regarding the metrics $g_{ij} = (1/2)\partial_{y^i}\partial_{y^j}L^2$ defined on the tangent spaces. All Berwald metrics are Landsberg; however, the existence of non-Berwald Landsberg metrics is one of the oldest problems in Finsler geometry. In [1], the proof of non-existence of such metrics is based on the following idea: For a given Landsberg metric one constructs, first, a Riemannian metric tensor $\mathbf{g}_{ij}(p)$ by averaging the Landsberg metric tensor $g_{ij}(p, y)$ on the unit balls $B_p \subset T_p(M^n)$ by the measure $\mu_p = \sqrt{det(g_{ij})(p, y)}dy^1 \wedge \cdots \wedge dy^n$, that is, $\mathbf{g}_{ij}(p) = \int_{B_p} g_{ij}(p, y)\mu_p / \int_{B_p} \mu_p$. Furthermore, since the unit balls are always invariant under the action of Berwald parallel transports, the Berwald covariant derivative ∇_k of a Landsberg space satisfies the relations $\nabla_k g_{ij} = 0$ and $\nabla_k \mu = 0$, therefore also the relation $\nabla_k \mathbf{g}_{ij} = 0$ must hold. This latter identity implies, in two different ways, that the torsion free Berwald connection must be the Levi Civita connection of the Riemannian metric $\mathbf{g}_{ij}(p)$. That is, it is linear indeed.

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Let it be mentioned, first, that this argument applies without any difficulty to linear Berwald connections, i.e., when the metric is Berwald. In this case, the statement says that, on Berwald manifolds, the linear Berwald connection is nothing but the Levi Civita connection of the averaged metric $\mathbf{g}_{ij}(p)$. In [2,3], such Riemannian metrics are corresponded to Berwald metrics in a completely different way, by using the holonomy groups of the linear Berwald connections.

In a general situation, the Berwald connection is non-linear and linear coordinates $(y^1, \ldots, y^n)$ defined on a fixed tangent space $T_p(M^n)$ are transported along curves $c(t)$ to proper non-linear coordinates $(\tilde{y}^1(t), \ldots, \tilde{y}^n(t))$ of the other tangent spaces $T_{c(t)}(M^n)$. The proof of the theorem must assume this general situation and the desired linearity should be established by the Landsberg property. However, this property a priory implies only that the functions $g(\partial_{\tilde{y}^i(t)}, \partial_{\tilde{y}^j(t)})$ and their averages do not change during parallel transports. On the other hands, the averaged Riemann metric $\mathbf{g}_{ij}$ is defined in linear coordinates and the question arises if the same metric is defined by these two averages?

Actually, this problem appears in a simplified situation in the paper. In fact, by Theorem 2.1, it is enough to prove that the Berwald parallel transports $\tau_{pq} : T_p(M^n) \to T_q(M^n)$ keep the $\mathbf{g}$-length of the radially directed vectors. That is, $\mathbf{L}_{|k} = 0$ holds, where $\mathbf{L}$ is the Finsler function belonging to $\mathbf{g}$ and $|k$ means derivative with respect to the Berwald connection. Note that the rigidity condition $\mathbf{L}_{|k} = 0$ imposed on metric $\mathbf{g}_{ij}$ is very similar to Landsberg's original condition imposed on g_{ij}. By this reason, it is called *dual Landsberg condition regarding radial directions*.

After this simplification, the above question arises in the following form. Let $c(t)$, $p = c(0)$, be a curve and $X(0) \in T_p(M^n)$ a vector whose parallel extension onto $c(t)$ is denoted by $X(t)$. Let $X^v(0)$ be the vertical vector field defined by lifting $X(0)$ in the tangent spaces $T_{\mathbf{y}}(T_p(M^n))$. Its parallel extension along curves $(c(t), Y(t))$, where $Y(t)$ is parallel along $c(t)$, is denoted by $\tilde{X}(t)$. In general, $\tilde{X}(c(t), Y(t))$ is the vertical lift of $X(t)$ only for $Y(t)$'s, which are proportional to $X(t)$. In this situation, the question is if the average of $g(\tilde{X}(t), \tilde{X}(t))$ is equal to $\mathbf{g}(X(t), X(t))$? The latter quantity is defined by averaging $g(X^v(t), X^v(t))$, while, on Landsberg manifolds, the first quantity is a priory equal only to $\mathbf{g}(X(0), X(0))$. For linear connections, $\tilde{X}(t) = X^v(t)$ holds, yielding an obvious positive answer to this question. This identity also explains why the Berwald connection of a Berwald space is equal to the Levi Civita connection of $\mathbf{g}$. Since the identity of averages does not require the identity of functions, this answer is also possible without assuming linearity. In fact, based on the following arguments, it is rather suggestive that the two averaging must be equal on Landsberg manifolds.

The main source of confusion is that Finsler tensors are not tensors in the regular sense on the tangent spaces. Generically speaking, they behave as tensors there just regarding linear transformations. From this respect, even the characteristic equation $l_i G^i_{jkl} = 0$ (which is equivalent to $g_{jk|l} = 0$) of Landsberg metrics is not a regular tensor equation. On a general Finsler manifold, the metric tensor $g_{ij} = (1/2)\partial_{y^i y^j} L^2$ is a regular tensor on each $T_p(M^n)$, and, on Landsberg manifolds, this tensor is invariant under the actions of Berwald parallel transports. The partial derivatives present in g_{ij} impose strong conditions on these transports and also applications of simple techniques such as Fubini's theorem combined with integration by parts are available for explicit investigation of the above integral equation. Note, for instance, that the averaged metric $\mathbf{g}$ does not depend on the radius R of the balls, where the averaging is computed. Limiting $R \to \infty$ reveals then that $\mathbf{g}(X, X)$ depends on $L(X)$ and averages of functions involving certain derivatives of the density function regarding the parallel vector field X^v. These arguments have been very suggestive in concluding that this integral equation must obviously be true. Actually, this flaw is hidden under the cover of a short formal computation and these arguments suggest an easy direct settling of the question. Unfortunately, it is not so! The above technique also produces non-regular tensors that confuse the transformation of these explicit integral formulas from linear charts to the non-linear ones. At the present time, I cannot completely eliminate this flaw from the proof. Thus, the Landsberg space problem remains an open question. I am convinced that this problem can be settled by the averaging technique only on those Landsberg manifolds where also the Riemann volume from attached to the averaged metric is invariant under the actions of the Berwald transports. The proof of this additional rigidity condition on a general Landsberg manifold certainly needs new ideas. By omitting this incomplete part, the statement is correctly established only in restricted forms that concern the dual conditions. Yet, these statements are interesting Landsberg-type characterizations of Berwald metrics. The complete solution of the problem requires to prove that the original Landsberg condition implies the dual conditions.

Theorem 1 *A regular Finsler metric is Berwald if and only if it satisfies the dual Landsberg condition* $\mathbf{L}_{|k} = 0$ *concerning radial directions. In this case, the Berwald connection is nothing but the Levi Civita connection of* $\mathbf{g}$.

If the metric is dual Landsberg, then $\mathbf{L}_{|k} = 0$ and Theorem 2.1 of [1] imply that the Berwald connection must be the Levi Civita connection of $\mathbf{g}$. Theorem 2.1 claims that the Berwald connection is uniquely determined by the following properties: (1) The connection symbols G^i_k are 1-homogeneous and torsion free, meaning $G^i_{jk} = \partial_{y^j} G^i_k = G^i_{kj}$. (2) The metric condition $L_{|k} = 0$ holds. Also, the Levi Civita connections of Riemann manifolds among non-linear connections are uniquely determined by these conditions. For them, $G^i_{kj} = \Gamma^i_{jk}$ are the Christoffel symbols. Conversely, a Berwald metric is Landsberg having linear Berwald connection. Thus, the above arguments applied to linear connections yield: $\mathbf{L}_{|k} = 0$.

A Finsler metric satisfies the *dual Landsberg condition regarding the tangent-to-indicatrix directions* if the Berwald parallel transports keep the $\mathbf{g}$-length of vectors tangent to the indicatrices. That is, $\overline{\mathbf{g}}_{|k} = 0$, where $\overline{\mathbf{g}}$ is the metric induced by $\mathbf{g}$ on the indicatrix. Then, we have:

Theorem 2 *A regular Finsler manifold of dimension* ≥ 3 *is Berwald if and only if the Berwald parallel transports keep the* $\mathbf{g}$*-norm of vectors tangent to the indicatrices.*

By the above argument, Berwald manifolds are also dual Landsberg in this sense. Interestingly enough, the proof in the opposite direction can be completed in a completely different way, which is based on global rigidity theorems concerning smooth strictly convex bodies. In this case, the Berwald transports, $\tau : T_p(M^n) \to T_q(M^n)$, leave the Riemann metric $\overline{\mathbf{g}}$ induced by the Euclidean metric $\mathbf{g}$ on the indicatrix field invariant. That is, they are isometries between two indicatrices $(I_p, \overline{\mathbf{g}})$ and $(I_q, \overline{\mathbf{g}})$ endowed with these induced metrics. By the classical rigidity theorem, yielded for $n \geq 3$, the τ must be an orthogonal transformation between the Euclidean spaces defined by $\mathbf{g}(p)$ and $\mathbf{g}(q)$. The linearity of the Berwald connection follows by formula (5) of [1].

Acknowledgements This problem about my paper was recognized by Vladimir Matveev who asked the question if the complications due to the non-linear Berwald parallel transports are overlooked there, and, if they are not overlooked, then how can the paper cope with this problem? Detailed explanations are available also in his most recent arxiv-paper: 0809.1581. The actual flaw appearing in the paper is explained above. In my opinion, one can cope with this difficulty after developing an adequate invariance theory regarding

non-linear transformations on the tangent spaces. This is a completely missing item in Finsler geometry. This paper is partially supported by NSF grant DMS-0604861.

References

1. Szabó, Z.I.: All regular Landsberg metrics are Berwald. Ann. Glob. Anal. Geom. (2008) (to appear). http://dx.doi.org/10.1007/s10455-008-9115-y
2. Szabó, Z.I.: Positive definite Berwald spaces (Structure theorems). Tensor N. S. **35**, 25–39 (1981)
3. Szabó, Z.I.: Berwald metrics constructed by Chevalley's polynomials. arXiv:math.DG/0601522 (2006)