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Vertical–horizontal decomposition of Laplacians and cohomologies of manifolds with trivial tangent bundles

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Abstract

In this paper, we obtain a vertical–horizontal decomposition formula of Laplacians on manifolds with a special foliation structure. Two Nomizu-type theorems for cohomologies of nilmanifolds follow as applications.

Keywords Künneth formula · Horizontal lift · Vertical form · Laplacian · Nilpotent foliation · Torus fibration · Nomizu-type theorem · Nilpotent group

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Contents

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1 Introduction

Let X be an n-dimensional compact smooth manifold. Assume that the tangent bundle T_X of X is trivial (thus the cotangent bundle T_X^* is also trivial). Let

$$\Phi := \{\sigma^1, \ldots, \sigma^n\}$$

be a global smooth frame of T_X^*. Inspired by [9,20], we introduce the following

Definition 1.1 We call Φ a *nilpotent frame* if

$$d\sigma^j = \sum_{k,l>j} A^j_{kl}\, \sigma^k \wedge \sigma^l, \qquad \forall\, 1 \le j \le n, \tag{1.1}$$

where A^j_{kl} are "real constants."

In the complex case, assume that the holomorphic tangent bundle $\wedge^{1,0}T_X$ of a compact complex manifold X is *smoothly* trivial (may not be trivial as a holomorphic vector bundle). Now assume that the complex dimension of X is n. Let

$$\Psi := \{\xi^1, \ldots, \xi^n\}$$

be a global smooth frame of $\wedge^{1,0}T_X^*$. We shall use the following

Definition 1.2 We call Ψ a *complex nilpotent frame* if

$$d\xi^j = \sum_{k,l>j} B^j_{kl}\, \xi^k \wedge \xi^l + \sum_{k,l>j} B^j_{k\bar{l}}\, \xi^k \wedge \overline{\xi^l}, \qquad \forall\, 1 \le j \le n, \tag{1.2}$$

where B^j_{kl} and $B^j_{k\bar{l}}$ are "complex constants."

We have the following generalization of the main results in [9,13,20].

Main Theorem *Let X be a compact smooth (resp. complex) manifold. Assume that T_X^* (resp. $\wedge^{1,0}T_X^*$) possesses a nilpotent (resp. complex nilpotent) frame Φ (resp. Ψ). Then, every de Rham (resp. Dolbeault) cohomology class of X can be represented by $\mathbb{R}$ (resp. $\mathbb{C}$) linear combination of finite wedge products of forms in Φ (resp. $\Psi \cup \overline{\Psi}$).*

Remark 1 Denote by $A^\star$ (resp. $A^{\star,\star}$) the finite dimension $\mathbb{R}$ (resp. $\mathbb{C}$) linear space spanned by wedge products of Φ (resp. $\Psi \cup \overline{\Psi}$). Then we know that the d-cohomology is well defined on $A^\star$, the $\overline{\partial}$-cohomology is well defined on $A^{\star,\star}$ and they are also called *the Lie algebra cohomologies*. Let us denote them by $H^\star_{d,\Phi}$ and $H^{\star,\star}_{\overline{\partial},\Psi}$, respectively. Then, our main theorem is equivalent to say that

$$H_d^\star \simeq H^\star_{d,\Phi}, \quad H^{\star,\star}_{\overline{\partial}} \simeq H^{\star,\star}_{\overline{\partial},\Psi},$$

where $H_d^\star$ (resp. $H^{\star,\star}_{\overline{\partial}}$) denotes the usual *de Rham (resp. Dolbeault) cohomology group*. See Sect. 8 for a more explicit description of $H^\star_{d,\Phi}$ and $H^{\star,\star}_{\overline{\partial},\Psi}$ in certain cases and Sect. 9 for related results.

Remark 2 The main ingredient in the proof of our main theorem is the following vertical–horizontal decomposition of Laplacians (see Theorem 4.1)

$$\Box_d = \Box_{d^v} + \Box_{d^h+R_d},$$

associated to the following decomposition

$$d = d^h + d^v + R_d,$$

of d on a smooth manifold with a special foliation structure, where d^h only increases the horizontal degree, d^v only increases the vertical degree and the remaining term R_d is a tensor (see [1–3,18,19,21,26] for the background and related results).

Remark 3 The proof of our main theorem in Sect. 7 also gives the following result: Let X be a compact smooth manifold. Assume that $\mathbb{C} \otimes T_X^*$ possesses a nilpotent frame Φ. Then every complex de Rham cohomology class of X can be represented by $\mathbb{C}$ linear combination of finite wedge products of forms in Φ.

Our main theorem suggests to study the following problem:

($\star$): *Let G be a Lie group, let Γ be a discrete subgroup of G. Assume that with respect to the left action of Γ, $X := \Gamma\backslash G$ is a compact manifold. When does T_X^* possess a nilpotent frame ?*

If G is nilpotent then of course T_X^* possesses a nilpotent frame. But it is also interesting to study the general case, e.g., $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})$ (non-compact!). In Sect. 9, we shall give an example where T_X^* possesses a nilpotent frame but G is not nilpotent. For related results, see [6,7].

2 Motivations

2.1 First motivation: Künneth formula

Our first motivation comes from the following well-known Künneth formula:

Theorem 2.1 (Künneth formula) *Let (X, g_X) and (Y, g_Y) be two compact Riemannian manifolds. Let (E, h_E), (F, h_F) be Hermitian complex vector bundles over X and Y, respectively.*

- *If E and F are flat, then we have the following formula for de Rham cohomologies:*
$$H_d^k(X \times Y, E \otimes F) = \oplus_{p+q=k} H_d^p(X, E) \otimes H_d^q(Y, F);$$
- *If X, Y are complex manifolds and E, F are holomorphic vector bundles, then*
$$H_{\overline{\partial}}^{p,q}(X \times Y, E \otimes F) = \oplus_{j+k=p,l+m=q} H_{\overline{\partial}}^{j,l}(X, E) \otimes H_{\overline{\partial}}^{k,m}(Y, F).$$

One way to prove the above formulas is to use the Leray spectral sequence for fibrations, see [9]. Our motivation comes from the proof of using the following decomposition formulas of Laplacians:
$$\Box_d^{X \times Y} = \Box_d^X + \Box_d^Y, \tag{2.1}$$
and
$$\Box_{\overline{\partial}}^{X \times Y} = \Box_{\overline{\partial}}^X + \Box_{\overline{\partial}}^Y. \tag{2.2}$$
More precisely, we will study the following problem:

Problem: *How to generalize* (2.1) *and* (2.2) *to non-product fibrations?*

Remark One way to study the above problem is to develop the L^2-theory of the Leray spectral sequence for fibrations (see [5,11] for related results). We know that for the spectral sequence of the double complex $(\partial, \overline{\partial})$, $d = \partial + \overline{\partial}$, the associated L^2-theory is based on the classical Bochner–Kodaira–Nakano formula.

2.2 Second motivation: Nomizu-type theorems

Our second motivation is based on the following celebrated Nomizu's theorem [20] proved in 1954:

Nomizu's theorem (weak version) *Let G be a simply connected nilpotent Lie group with a discrete subgroup Γ. Assume that $X := \Gamma \backslash G$ is compact and the ascending central series of the Lie algebra of G (see Sect. 6.2 for the definition) defines a torus fibration resolution of X. Then the de Rham cohomology of X can be represented by G-invariant forms.*

In 1976, Sakane [24] proved that the Nomizu theorem is also true for compact complex parallelisable solvmanifolds. The following theorem of Cordero–Fernández–Gray–Ugarte [9] is a generalization of Sakane's theorem:

Cordero–Fernández–Gray–Ugarte's theorem (weak version) *Let G be a simply connected nilpotent Lie group with a discrete subgroup Γ. Assume that $X := \Gamma \backslash G$ is a compact manifold with a left invariant integrable almost complex structure J. Assume that the J-compatible ascending series of the Lie algebra of G (see Sect. 6.3 for the definition) defines a holomorphic torus fibration resolution of X. Then, the Dolbeault cohomology of X can be represented by G-invariant forms.*

Remark The assumption that the J-compatible ascending series of the Lie algebra of G defines a holomorphic torus fibration resolution is contained in the proof of the main theorem in [9].

In real case the ascending central series will always give a torus fibration resolution (see page 208 in [10]). Thus the following result is still true:

Nomizu's theorem (original version) *Let G be a simply connected nilpotent Lie group with a discrete subgroup Γ. Assume that $X := \Gamma \backslash G$ is compact. Then, the de Rham cohomology of X can be represented by G-invariant forms.*

In complex case, the J-compatible ascending series may not give a torus fibration resolution (see Example 3.6 in [23] or [13]). But our main theorem implies the following result.

Cordero–Fernández–Gray–Ugarte's theorem (strong version) *Let G be a simply connected nilpotent Lie group with a discrete subgroup Γ. Assume that $X := \Gamma\backslash G$ is a compact manifold with a nilpotent complex structure (see* [9]*, page 2, for the definition). Then, the Dolbeault cohomology of X can be represented by G-invariant forms. In particular, it is independent of Γ.*

The above result applies in a number of important cases.

Corollary 2.2 *Let G be a simply connected nilpotent Lie group with a discrete cocompact subgroup Γ and left-invariant complex structure J. If G is 2-step nilpotent, then the Dolbeault cohomology of $X = (\Gamma\backslash G, J)$ can be computed by left-invariant forms.*

Proof We only have to observe that if G is 2-step nilpotent, then every left-invariant complex structure on G is nilpotent in the above sense by [22, Prop. 3.3] so the strong version of Cordero–Fernández–Gray–Ugarte's theorem applies. □

A different way to generalize the weak version of Cordero–Fernández–Gray–Ugarte's theorem was considered in [13] and like in loc. cit. we are able to settle all cases of low dimension.

Corollary 2.3 *Let X be a nilmanifold of real dimension at most 6 with left-invariant complex structure. Then, the Dolbeault cohomology of X is computed by left-invariant forms.*

Proof In dimension 2 and 4, there are only tori and the Kodaira–Thurston manifold to consider, for which the result is well known.

In real dimension 6, there are only finitely many nilpotent Lie algebras and the ones admitting complex structures are classified by Salamon [25]. In [22, Proof of Thm. B] the statement was shown to hold for all complex structures on all such nilmanifolds except possibly for those with Lie algebra $\mathfrak{h}_7$, in the notation of Salamon (see also [8] for the original definition). Since $\mathfrak{h}_7$ is 2-step nilpotent, indeed the free 2-step nilpotent Lie algebra on 3 generators, the previous corollary applies to this remaining case. □

3 Foliations of nilpotent type

3.1 Nilpotent foliation

Let us recall the definition of distribution first.

Definition 3.1 (*Distribution*) Let X be a smooth manifold. We call

$$\mathcal{V} := \{\mathcal{V}_x\}_{x\in X},$$

a *rank-r distribution* on X if for every $x \in X$, $\mathcal{V}_x$ is an r-dimensional real linear subspace of T_xX (space of vectors at x) and there exist smooth vector fields $V_1, \ldots, V_r$ on an open neighborhood, say U_x, of x such that

$$\mathcal{V}_y = \mathrm{Span}_{\mathbb{R}}\{V_1(y), \ldots, V_r(y)\},$$

for every $y \in U_x$. We call $\{V_1, \ldots, V_r\}$ a local basis of $\mathcal{V}$.

Remark If V is a smooth vector field on X such that $V(x) \in \mathcal{V}_x$ for every $x \in X$, then we say that V lies in $\mathcal{V}$ and write $V \in \mathcal{V}$. Denote by $C^\infty(T_X)$ the space of smooth vector fields on X. Then, one may look at a rank-r distribution as a subspace of $C^\infty(T_X)$ that is locally generated by r linearly independent smooth vector fields.

Definition 3.2 (*Integrable distribution*) A distribution $\mathcal{V}$ is said to be *integrable* if $[V, W] \in \mathcal{V}$, for every $V, W \in \mathcal{V}$ (see the remark above). We call an integrable distribution a *foliation* on X.

Remark It is enough to check integrability for local basis of $\mathcal{V}$. The classical Frobenius theorem tells us that a rank-r distribution $\mathcal{V}$ is integrable if and only if for every $x \in X$ there exists a smooth local coordinate system, say $\{x^1, \ldots, x^n\}$, near x such that $\mathcal{V}$ is generated by $\{\partial/\partial x^1, \ldots, \partial/\partial x^r\}$ near x (i.e., $\mathcal{V}$ is tangent to the fibers of the map $(x^1, \ldots, x^n) \mapsto (x^{r+1}, \ldots, x^n)$). Thus, a rank-$r$ integrable distribution is equivalent to a foliation of r-dimensional local smooth manifolds.

We shall use the following lemma to define the notion of nilpotent foliation.

Lemma 3.3 *Let $\mathcal{V}$ be a distribution on X. Let g_X be a smooth Riemannian metric on X. Then*

$$\mathcal{V}^\perp := \{\mathcal{V}_x^\perp\}_{x \in X},$$

is also a distribution on X, where each $\mathcal{V}_x^\perp$ denote the orthogonal complement of $\mathcal{V}_x$ in $T_x X$ with respect to g_X.

Proof Let $\{V_1, \ldots, V_r\}$ be a local basis of $\mathcal{V}$. Then we can extend it to a local frame, say $\{V_1, \ldots, V_n\}$, of T_X. Denote by $V_j^\perp$, $j > r$, the orthogonal projection of V_j to $\mathcal{V}^\perp$. Then, we know that $\{V_j^\perp\}_{j>r}$ generates $\mathcal{V}^\perp$ locally. □

Definition 3.4 (*Nilpotent foliation*) Let $\mathcal{V}$ be a distribution on a Riemannian manifold (X, g_X). We call $(\mathcal{V}, g_X)$ a *nilpotent foliation structure* on X if locally there exists an orthonormal frame $\{V_1, \ldots, V_n\}$ of (T_X, g_X) such that

(1) $\{V_j\}_{j \le r}$ is a local basis of $\mathcal{V}$ and $\{V_j\}_{j>r}$ is a local basis of $\mathcal{V}^\perp$;
(2) $[V_j, V_k] = 0$ for every $1 \le j \le r$, $1 \le k \le n$.

Remark Notice that condition (2) in the above definition implies that a nilpotent foliation is always integrable.

We shall also study nilpotent foliations on complex manifold.

Definition 3.5 (*Complex nilpotent foliation*) Let $\mathcal{V}$ be a distribution on a complex manifold (X, J). Let g_X be a J-Hermitian metric on X. We call $(\mathcal{V}, J, g_X)$ a *complex nilpotent foliation structure* on X if locally there exists an orthonormal frame $\{V_1, \ldots, V_n\}$ of $(T_X^{1,0}, g_X)$ such that

(1) $\{V_j, \bar{V}_j\}_{j \le r}$ is a local basis of $\mathcal{V}$ and $\{V_j, \bar{V}_j\}_{j>r}$ is a local basis of $\mathcal{V}^\perp$;
(2) $[V_j, V_k] = [V_j, \bar{V}_k] = 0$ for every $1 \le j \le r$, $1 \le k \le n$.

Remark Since g_X is J-Hermitian, a complex nilpotent foliation also satisfies $J(\mathcal{V}^\perp) = \mathcal{V}^\perp$.

3.2 Vertical–horizontal decomposition of *d*

Definition 3.6 (*Vertical–horizontal vector field*) Let $\mathcal{V}$ be a distribution on a Riemannian manifold (X, g_X). We call $V \in \mathcal{V}$ a *vertical vector field* and $W \in \mathcal{V}^{\perp}$ a *horizontal vector field*.

We also need the dual of the notion of vertical–horizontal vector field (motivated by [4], see also formula (1.3) in [1]).

Definition 3.7 (*Vertical–horizontal one-form*) A differential one-form u on X is said to be *horizontal* (resp. *vertical*) if $V \rfloor u = 0$ for every vertical (resp. horizontal) vector field V on X.

Definition 3.8 (*Vertical–horizontal degree*) Denote by T_h^* and T_v^* the subbundles of T^*X generated by horizontal one-forms and vertical one-forms, respectively. Then, we have

$$\wedge^p T^*X = \oplus_{k+l=p}(\wedge^k T_h^*) \wedge (\wedge^l T_v^*).$$

We call a section of $(\wedge^k T_h^*) \wedge (\wedge^l T_v^*)$ a *degree* $(k|l)$*-form* and say that it has *horizontal degree* k and *vertical degree* l.

The following lemma suggests to study vertical–horizontal decomposition of the exterior derivative.

Lemma 3.9 *Let $\mathcal{V}$ be a distribution on a Riemannian manifold (X, g_X). Let u be a smooth degree $(k|l)$-form on X. Assume that $\mathcal{V}$ is integrable. Then, we can write*

$$du = d^v u + d^h u + R_d u,$$

where $d^v u$ is degree $(k|l+1)$, $d^h u$ is degree $(k+1|l)$ and $R_d u$ is degree $(k+2|l-1)$.

Proof Let us locally write

$$u = \sum u_h^j \wedge u_v^j,$$

where u_v^j are $(0|l)$-forms and u_h^j are $(k|0)$-forms. Since $\mathcal{V}$ is integrable, we know that $d(u_h^j)$ has no degree $(k-1|2)$ components. Thus, du has no degree $(k-1|l+2)$ components. □

Definition 3.10 (*Atiyah tensor*) Let $\mathcal{V}$ be an integrable distribution on a Riemannian manifold (X, g_X). Then, we define d^h as the $(1|0)$-part of d and d^v as the $(0|1)$-part of d. We call the following degree $(2|-1)$ tensor

$$R_d := d - d^h - d^v,$$

the *Atiyah tensor*.

Remark 1 From the proof of the above Lemma, we know that the Atiyah tensor is zero if and only if $\mathcal{V}^{\perp}$ is integrable. In case $\mathcal{V}$ is associated to the Lie algebra $\mathfrak{g}$ of a G–bundle, then cohomology class of each Lie–algebra component of the Atiyah tensor is also called the *Atiyah class*.

Remark 2 d^h, d^v are also well defined on the space of all smooth forms on X. The reason is we can always write a smooth form u as

$$u = \sum u^{(k|l)},$$

where each $u^{(k|l)}$ denotes the degree $(k|l)$-component of u. Then we can define

$$d^h u = \sum d^h u^{(k|l)}, \quad d^v u = \sum d^v u^{(k|l)}.$$

3.3 Vertical–horizontal decomposition of $\overline{\partial}$

Now let us consider the case that $\mathcal{V}$ is a distribution on a complex manifold (X, J) with a J-Hermitian Riemannian metric g_X (we call (X, J, g_X) a *Hermitian complex manifold*) such that $J(\mathcal{V}) = \mathcal{V}$. Then we have

$$\wedge^{p,q} T^*X = \oplus_{k+j=p,l+s=q} (\wedge^{k,l} T_h^*) \wedge (\wedge^{j,s} T_v^*).$$

We call smooth section of $(\wedge^{k,l} T_h^*) \wedge (\wedge^{j,s} T_v^*)$ a *degree* $(k, l|j, s)$*-form* and say that it has *horizontal degree* (k, l) and *vertical degree* (j, s). Similar as the real case, we have

Lemma 3.11 *Let $\mathcal{V}$ be an integrable distribution on a hermitian complex manifold (X, J, g_X). Let u be a smooth degree $(k, l|j, s)$-form on X. Assume that $J(\mathcal{V}) = \mathcal{V}$. Then we can write*

$$\overline{\partial} u = \overline{\partial}^v u + \overline{\partial}^h u + R_{A_1} u + R_{A_2} u + R_{KS} u,$$

where $\overline{\partial}^v u$ is degree $(k, l|j, s+1)$, $\overline{\partial}^h u$ is degree $(k, l+1|j, s)$ and $R_{A_1} u$ is degree $(k+1, l+1|j-1, s)$, $R_{A_2} u$ is degree $(k, l+2|j, s-1)$ and $R_{KS} u$ is degree $(k+1, l|j-1, s+1)$.

Definition 3.12 (*Complex Atiyah Tensor and Kodaira–Spencer Tensor*) Let $\mathcal{V}$ be a J-invariant integrable distribution on a Hermitian complex manifold (X, J, g_X). We define $\overline{\partial}^h$ as the $(0, 1|0, 0)$-part of $\overline{\partial}$ and $\overline{\partial}^v$ as the $(0, 0|0, 1)$-part of $\overline{\partial}$. We call

$$R_A := R_{A_1} + R_{A_2},$$

the *complex Atiyah tensor*and R_{KS} the *Kodaira–Spencer tensor*.

Remark In case $\mathcal{V}$ is given by the fiber-tangent distribution of a proper holomorphic submersion, then cohomology class of each component of R_{KS} is just the well known Kodaira–Spencer class. In general, put

$$R_{\overline{\partial}} = R_A + R_{KS},$$

If $R_{KS} \neq 0$ then $R_d \neq R_{\overline{\partial}} + R_{\partial}$. In fact, we have

$$R_d = R_A + \overline{R_A}, \;\; d^v = \overline{\partial}^v + \partial^v,$$

and

$$d^h = R_{\mathrm{KS}} + \overline{R_{\mathrm{KS}}} + \partial^h + \overline{\partial}^h.$$

In case $\mathcal{V}$ is a complex nilpotent foliation, we can prove that

Lemma 3.13 *Assume that $(\mathcal{V}, J, g_X)$ is a complex nilpotent foliation structure. Then $R_{KS} \equiv 0$.*

Proof It suffices to show that if u is an vertical $(1, 0)$-form then $\overline{\partial} u$ has no degree $(1, 0|0, 1)$-component. Since

$$du(W, V) = L_W(u(V)) - L_V(u(W)) - u([W, V]), \tag{3.1}$$

it is enough to show that for every vertical $(0, 1)$-vector field V and horizontal $(1, 0)$-vector field W, the vertical $(1, 0)$-component of $[V, W]$ is zero, which follows from 2) in Definition 3.5. ◻

4 Vertical–horizontal decomposition of Laplacians

4.1 Fundamental theorem

The fundamental theorem in this paper is the following:

Theorem 4.1 (Real case)*: Let (X, g_X) be an oriented Riemannian manifold with a nilpotent foliation structure (see Definition 3.4). Then on the space of smooth forms on X, we have*

$$\Box_d = \Box_{d^v} + \Box_{d^h+R_d}. \tag{4.1}$$

(Complex case)*: Let (X, J, g_X) be a hermitian complex manifold with a complex nilpotent foliation structure (see Definition 3.5). Then, on the space of smooth forms on X, we have*

$$\Box_{\bar\partial} = \Box_{\bar\partial^v} + \Box_{\bar\partial^h+R_{\bar\partial}}, \ \Box_{\bar\partial^v} = \Box_{\partial^v}. \tag{4.2}$$

Remark 1 In our proof, we shall use the following notation: if P is a differential operator on the space of smooth forms on X, then we shall write P^* as the adjoint of P and write

$$\Box_P := PP^* + P^*P.$$

Recall that P^* satisfies

$$(Pu, v) = (u, P^*v),$$

if u is a smooth form and v is a smooth form with compact support. Thus, P^* and $\Box_P$ are well defined on the space of smooth forms. If P maps a degree k form to a degree $(k+p)$ form, then we say that P has degree p. If P is a degree p operator and Q is a degree q operator, then we write

$$[P, Q] := PQ - (-1)^{pq}QP.$$

Since $d, d^v, d^h + R_d$ are degree one operators, we have

$$\Box_d = [d, d^*], \ \Box_{d^v} = [d^v, (d^v)^*],$$

and

$$\Box_{d^h+R_d} = [d^h + R_d, (d^h + R_d)^*].$$

Thus (4.1) is equivalent to

$$[d^h, (d^v)^*] = 0, \ \ [d^v, R_d^*] = 0. \tag{4.3}$$

Remark 2 Notice that if X is compact, then

$$((\Box_P + \Box_Q)u, u) = ||Pu||^2 + ||P^*u||^2 + ||Qu||^2 + ||Q^*u||^2,$$

for every smooth form u on X. Thus $(\Box_P + \Box_Q)u = 0$ is equivalent to

$$Pu = P^*u = Qu = Q^*u = 0,$$

which gives the following corollary:

Corollary 4.2 (Real case)*: Let* (X, g_X) *be an oriented compact Riemannian manifold with a nilpotent foliation structure (see Definition 3.4). Then, a smooth form* u *lies in* $\ker \square_d$ *if and only if*

$$d^v u = (d^v)^* u = (d^h + R_d) u = (d^h + R_d)^* u = 0 \ \ on\ X. \tag{4.4}$$

(Complex case)*: Let* (X, J, g_X) *be a hermitian compact complex manifold with a complex nilpotent foliation structure (see Definition 3.5). Then, a smooth form* u *lies in* $\ker \square_{\overline{\partial}}$ *if and only if*

$$\overline{\partial}^v u = (\overline{\partial}^v)^* u = (\overline{\partial}^h + R_{\overline{\partial}}) u = (\overline{\partial}^h + R_{\overline{\partial}})^* u = \partial^v u = (\partial^v)^* u = 0 \ \ on\ X. \tag{4.5}$$

4.2 Proof of the real case

Let $(\mathcal{V}, g_X)$ be a nilpotent foliation structure on X (see Definition 3.4). Let $\{V_j\}$ be the local frame of (T_X, g_X) in Definition 3.4. Let us write

$$X_j^h = V_{r+j}, \quad X_k^v = V_k, \qquad 1 \le k \le r, \quad 1 \le j \le n-r.$$

We know each X_k^v is vertical and each X_j^h is horizontal. Denote by

$$\{\varphi_h^j, \varphi_v^k\}_{1\le k\le r, 1\le j\le n-r},$$

the dual frame of $\{X_j^h, X_k^v\}$. By 2) in Definition 3.4 and (3.1), we have

$$d\varphi_h^j = \sum_{k,l=1}^{n-r} C_{kl}^j\, \varphi_h^k \wedge \varphi_h^l, \quad 1 \le j \le n-r,$$

and

$$d\varphi_v^k = \sum_{j,l=1}^{n-r} D_{jl}^k\, \varphi_h^j \wedge \varphi_h^l, \quad 1 \le k \le r,$$

where C_{kl}^j and D_{jl}^k are smooth functions. Thus we have

Lemma 4.3 *The components* R_d, d^v, d^h *of* d *can be written as*

$$\begin{aligned}
R_d &= \sum_{k=1}^{r} \sum_{j,l=1}^{n-r} \left(D_{jl}^k\, \varphi_h^j \wedge \varphi_h^l \right) \wedge (X_k^v \rfloor), \\
d^v &= \sum_{p=1}^{r} \varphi_v^p \wedge (X_p^v), \\
d^h &= \sum_{j=1}^{n-r} \varphi_h^j \wedge (X_j^h) + \sum_{j,k,l=1}^{n-r} C_{kl}^j \left(\varphi_h^k \wedge \varphi_h^l \right) \wedge (X_j^h \rfloor).
\end{aligned}$$

Now let us finish the proof of the first identity in (4.3).

Lemma 4.4 $[d^v, R_d^*] = 0$.

Proof Since R_d is a tensor, we have

$$R_d^* = \sum_{k=1}^{r} \sum_{j,l=1}^{n-r} D_{jl}^k \, \varphi_v^k \wedge (X_l^h \rfloor)(X_j^h \rfloor). \tag{4.6}$$

Thus R_d^* commutes with $\varphi_v^p \wedge$. Moreover, $d^2 \varphi_v^k = 0$ gives that $d^v D_{jl}^k \equiv 0$. Thus $[d^v, R_d^*] = 0$. □

We need the following proposition to prove $[d^h, (d^v)^*] = 0$.

Proposition 4.5 *Denote by $*$ the Hodge star operator on our oriented manifold (X, g_X). Assume that the orientation of X is given by $\Omega_h \wedge \Omega_v$, where*

$$\Omega_h := \varphi_h^1 \wedge \cdots \wedge \varphi_h^{n-r}, \quad \Omega_v := \varphi_v^1 \wedge \cdots \wedge \varphi_v^r.$$

Denote by $*_h$ (resp. $*_v$) the Hodge star operator with respect to Ω_h (resp. Ω_v) on the space of horizontal (resp. vertical) forms, respectively. Then

$$* \, (u_h \wedge u_v) = (-1)^{(n-r-p)q} *_h u_h \wedge *_v u_v, \tag{4.7}$$

where u_h is a degree p horizontal form, u_v is a degree q vertical form.

Proof Notice that

$$(u_h \wedge u_v) \wedge (-1)^{(n-r-p)q} (*_h u_h \wedge *_v u_v) = (u_h \wedge *_h u_h) \wedge (u_v \wedge *_v u_v) = (u_h \wedge u_v) \wedge *(u_h \wedge u_v).$$

Thus (4.7) follows. □

Lemma 4.6 *Let $u = f \, u_h \wedge u_v$ be a smooth degree $(a|b)$ form, where u_h (resp. u_v) is a finite wedge product of φ_h^k (resp. φ_v^l) and f is a smooth function. Then*

$$(d^v)^* u = (-1)^a u_h \wedge \left((-1)^{r(b+1)+1} *_v d^v *_v (f u_v) \right), \tag{4.8}$$

and

$$(d^h)^* u = \left((-1)^{(n-r)(a+1)+1} *_h d^h *_h (f u_h) \right) \wedge u_v. \tag{4.9}$$

Proof The main idea is to use the fact that $(d^v)^* u$ (resp. $(d^h)^* u$) is the degree $(a|b-1)$ (resp. $(a-1|b)$) part of $d^* u$ and $d^* u = (-1)^{n(a+b+1)+1} * d * u$. Thus, the above proposition applies. We shall only prove the first formula. By (4.7), we have

$$*u = (-1)^{(n-r-a)b} f \; *_h u_h \wedge *_v u_v.$$

Thus

$$d * u = (-1)^{(n-r-a)b} \left(df \wedge (*_h u_h \wedge *_v u_v) + f \, d(*_h u_h \wedge *_v u_v) \right).$$

Using (4.7) again, we know that the degree $(a|b-1)$-part of $(-1)^{n(a+b+1)+1} * d * u$ is equal to the right hand side of (4.8). □

Remark Since

$$(-1)^{r(b+1)} *_v (\varphi_v^j \wedge) *_v u_v = X_j^v \rfloor u_v,$$

(4.8) gives the following formula:

Lemma 4.7 $(d^v)^* = -\sum_{j=1}^r (X_j^v \rfloor) \wedge (X_j^v)$.

Now we can prove the second identity in (4.3).

Lemma 4.8 $[d^h, (d^v)^*] = 0$.

Proof Notice that $d^2\varphi_h^j = 0$ gives $d^v C_{kl}^j \equiv 0$. Thus Lemma 4.3 and the above lemma give

$$[d^h, (d^v)^*] = \sum([X_j^h, X_k^v])(X_k^v \rfloor) \wedge (\varphi_h^j \wedge). \tag{4.10}$$

But by (2) in Definition 3.4, we have

$$[X_j^h, X_k^v] = 0. \tag{4.11}$$

Thus the lemma follows. □

The proof of the real case is complete.

4.3 Proof of the complex case

Let $(\mathcal{V}, J, g_X)$ be a complex nilpotent foliation structure on X. Let $\{V_j\}_{1\le j\le n}$ be the local frame of $(T_X^{1,0}, g_X)$ in Definition 3.5). Put

$$X_j^h := V_{r+j}, \quad X_k^v = V_k, \qquad 1 \le k \le r, \;\; 1 \le j \le n-s.$$

We know each X_k^v is vertical and each X_j^h is horizontal. Denote by

$$\{\varphi_h^j, \varphi_v^k\}_{1\le k\le r, 1\le j\le n-r},$$

the dual frame of $\{X_j^h, X_k^v\}$. By (2) in Definition 3.5 and (3.1), we have

$$d\varphi_h^j = \sum_{k,l=1}^{n-r} C_{kl}^j \varphi_h^k \wedge \varphi_h^l + \sum_{k,l=1}^{n-r} C_{k\bar{l}}^j \varphi_h^k \wedge \overline{\varphi_h^l}, \;\; 1 \le j \le n-r,$$

and

$$d\varphi_v^k = \sum_{j,l=1}^{n-r} D_{jl}^k \varphi_h^j \wedge \varphi_h^l + \sum_{j,l=1}^{n-r} D_{j\bar{l}}^k \varphi_h^j \wedge \overline{\varphi_h^l}, \;\; 1 \le k \le r,$$

where C_{kl}^j, $C_{k\bar{l}}^j$, D_{jl}^k and $D_{j\bar{l}}^k$ are smooth functions, which gives

Lemma 4.9 *The components $\overline{\partial}^h$, $\overline{\partial}^v$, $R_{\overline{\partial}}$ can be written as*

$$\overline{\partial}^h = \sum_{j=1}^{n-r} \overline{\varphi_h^j} \wedge (\overline{X_j^h}) + \sum_{j,k,l=1}^{n-r} \overline{C_{kl}^j \left(\varphi_h^k \wedge \varphi_h^l\right) \wedge (X_j^h \rfloor)} + \sum_{j,k,l=1}^{n-r} C_{k\bar{l}}^j \left(\varphi_h^k \wedge \overline{\varphi_h^l}\right) \wedge (X_j^h \rfloor),$$

$$\overline{\partial}^v = \sum_{k=1}^{r} \overline{\varphi_v^k} \wedge (\overline{X_k^v}),$$

and $R_{\overline{\partial}} = R_{KS} + R_{A_1} + R_{A_2}$ satisfies

$$R_{KS} = 0, \;\; R_{A_1} = \sum_{k=1}^{r}\sum_{j,l=1}^{n-r} \left(D_{j\bar{l}}^k \varphi_h^j \wedge \overline{\varphi_h^l}\right) \wedge (X_k^v \rfloor), \;\; R_{A_2} = \sum_{k=1}^{r}\sum_{j,l=1}^{n-r} \overline{\left(D_{jl}^k \varphi_h^j \wedge \varphi_h^l\right) \wedge (X_k^v \rfloor)}.$$

By a similar proof as the real case, we have

$$[\overline{\partial}^v, R^*_{\overline{\partial}}] = 0, \tag{4.12}$$

and the following analogy of Lemma 4.7.

Lemma 4.10 $(\overline{\partial}^v)^* = -\sum_{j=1}^r (\overline{X_j^v} \rfloor) \wedge (X_j^v)$.

Similar as the real case, the above lemma gives

$$[\overline{\partial}^h, (\overline{\partial}^v)^*] = 0 \tag{4.13}$$

We know that (4.12) and (4.13) together give

$$\Box_{\overline{\partial}} = \Box_{\overline{\partial}^v} + \Box_{\overline{\partial}^h + R_{\overline{\partial}}}.$$

Now it suffices to prove

$$\Box_{\overline{\partial}^v} = \Box_{\partial^v}.$$

By Lemmas 4.10 and 4.9, we have

$$\Box_{\overline{\partial}^v} = -\sum_{j=1}^r (X_j^v)(\overline{X_j^v}) = -\sum_{j=1}^r (\overline{X_j^v})(X_j^v), \tag{4.14}$$

which gives

$$\Box_{\overline{\partial}^v} = \overline{\Box_{\overline{\partial}^v}} = \Box_{\partial^v}.$$

The proof of Theorem 4.1 is complete. □

Remark One may also prove the complex case of Theorem 4.1 by using vertical–horizontal decomposition of the following Demailly–Griffiths–Kähler identity (see page 306 in [12,15] or [27] for a pure algebraic proof)

$$\overline{\partial}^* = i[\partial, \Lambda] + [L, \theta^*], \quad Lu := \omega \wedge u, \quad \theta := [\overline{\partial}, L], \quad \Lambda := L^*, \tag{4.15}$$

where ω denotes the real Hermitian $(1, 1)$-form associated to (g_X, J).

5 An example: the Kodaira–Thurston manifold

The Kodaira–Thurston surface was first found by Kodaira [17]. It is the first example [28] of complex symplectic manifold without Kähler structure. Let us recall its definition in [27]. Consider the following group structure

$$a * b := (a^1 + b^1, a^2 + b^2, a^3 + a^1 b^2 + b^3, a^4 + b^4),$$

on $\mathbb{R}^4$. The Kodaira–Thurston surface X is defined as the quotient manifold with respect to the *left* action of $\mathbb{Z}^4$ on (notice that $\mathbb{Z}^4$ is a discrete subgroup of G)

$$G := (\mathbb{R}^4, *).$$

It is easy to see that X is a compact manifold with respect to the quotient topology. Let (x^1, x^2, x^3, x^4) be the canonical coordinate system on $\mathbb{R}^4$. We know

$$\begin{cases} \varphi_h = \mathrm{d}x^1 + i\mathrm{d}x^2, \\ \varphi_v = \mathrm{d}x^3 - x^1\mathrm{d}x^2 + i\mathrm{d}x^4, \\ \overline{\varphi_h} = \mathrm{d}x^1 - i\mathrm{d}x^2, \\ \overline{\varphi_v} = \mathrm{d}x^3 - x^1\mathrm{d}x^2 - i\mathrm{d}x^4, \end{cases} \tag{5.1}$$

is a frame of the space of G-invariant (with respect to the left action of G) 1-forms on X. Let J be the almost complex structure on X such that the associated $\wedge^{1,0}T^*X$ is spanned by $\{\varphi_h, \varphi_v\}$. Notice that

$$\begin{cases} \mathrm{d}\varphi_h = 0, \\ \mathrm{d}\varphi_v = -\frac{i}{2}\varphi_h \wedge \overline{\varphi_h}. \end{cases} \tag{5.2}$$

implies that J is integrable on X. One may check that

$$\begin{cases} \overline{\partial}(x^1 + ix^2) = 0, \\ \overline{\partial}\left(x^3 + ix^4 - \frac{i}{2}(x^1)^2\right) = 0. \end{cases}$$

Thus

$$\begin{cases} z := x^1 + ix^2, \\ w := x^3 + ix^4 - \frac{i}{2}(x^1)^2. \end{cases}$$

are local holomorphic coordinates on X. Now we know that the following holomorphic map from $(\mathbb{R}^4, J)$ to $\mathbb{C}$

$$(z, w) \mapsto z,$$

defines a holomorphic submersion, say π, from X to the torus $\mathbb{T} := \mathbb{C}/\mathbb{Z}^2$. Let $\{X^h, X^v\}$ be the global frame of $T^{1,0}(X)$ that is dual to $\{\varphi_h, \varphi_v\}$. Then we know that

$$\begin{cases} \partial/\partial z := X^h + ix^1X^v, \\ \partial/\partial w := X^v. \end{cases}$$

is a holomorphic π-local (i.e., well defined on the π-inverse of a sufficiently small open set in $\mathbb{T}$) frame for $T^{1,0}X$. Thus we have

Proposition 5.1 $\pi : X \to \mathbb{T}$ *is locally trivial.*

Remark Notice that (5.2) implies that the fibers of π defines a complex nilpotent foliation structure on (X, J, g_X), where g_X is J-hermitian such that the fundamental form of (g_X, J) is

$$\omega = i\varphi_h \wedge \overline{\varphi_h} + i\varphi_v \wedge \overline{\varphi_v}.$$

We know that $\varphi_h, \overline{\varphi_h}$ are horizontal forms and $\varphi_v, \overline{\varphi_v}$ are vertical forms. By (5.2), we know that

$$R_{\overline{\partial}} = -\frac{i}{2}\varphi_h \wedge \overline{\varphi_h} \wedge (X^v \lrcorner \ldots),$$

is of degree $(1, 1|-1, 0)$. We shall use Theorem 4.1 and Corollary 4.2 to give another proof of the following well known result (see Sect. 5 in [9]).

Theorem 5.2 *Denote by $\mathcal{H}^{p,q}(\bar\partial)$ the space of $\bar\partial$-harmonic (p,q)-forms on the Kodaira–Thurston surface (X, J, g_X), we have*

$$\begin{cases} \mathcal{H}^{0,0}(\bar\partial) = \mathrm{Span}_{\mathbb{C}}\langle 1\rangle, \\ \mathcal{H}^{1,0}(\bar\partial) = \mathrm{Span}_{\mathbb{C}}\langle \varphi_h\rangle, \\ \mathcal{H}^{0,1}(\bar\partial) = \mathrm{Span}_{\mathbb{C}}\langle \overline{\varphi_h},\ \overline{\varphi_v}\rangle, \\ \mathcal{H}^{2,0}(\bar\partial) = \mathrm{Span}_{\mathbb{C}}\langle \varphi_h\wedge\varphi_v\rangle, \\ \mathcal{H}^{1,1}(\bar\partial) = \mathrm{Span}_{\mathbb{C}}\langle \overline{\varphi_h}\wedge\varphi_v,\ \varphi_h\wedge\overline{\varphi_v}\rangle, \\ \mathcal{H}^{0,2}(\bar\partial) = \mathrm{Span}_{\mathbb{C}}\langle \overline{\varphi_h}\wedge\overline{\varphi_v}\rangle, \\ \mathcal{H}^{2,1}(\bar\partial) = \mathrm{Span}_{\mathbb{C}}\langle \varphi_h\wedge\varphi_v\wedge\overline{\varphi_h},\ \varphi_h\wedge\varphi_v\wedge\overline{\varphi_v}\rangle, \\ \mathcal{H}^{1,2}(\bar\partial) = \mathrm{Span}_{\mathbb{C}}\langle \varphi_v\wedge\overline{\varphi_h}\wedge\overline{\varphi_v}\rangle, \\ \mathcal{H}^{2,2}(\bar\partial) = \mathrm{Span}_{\mathbb{C}}\langle \varphi_h\wedge\varphi_v\wedge\overline{\varphi_h}\wedge\overline{\varphi_v}\rangle. \end{cases}$$

Proof $\mathcal{H}^{0,0}(\bar\partial) = \mathrm{Span}_{\mathbb{C}}\langle 1\rangle$ is trivial. By Corollary 4.2, we know that all harmonic forms in $\mathcal{H}^{p,q}(\bar\partial)$ lie in the kernel of $\bar\partial^v$, $(\bar\partial^v)^*$, $\bar\partial^h + R_{\bar\partial}$ and $(\bar\partial^h + R_{\bar\partial})^*$.

Degree $(1,0)$ *case*: Let

$$u = a\varphi_h + b\varphi_v,$$

be in $\mathcal{H}^{1,0}(\bar\partial)$. Notice that $\bar\partial^v u = 0$ is equivalent to

$$\bar\partial^v a = \bar\partial^v b = 0,$$

and $(\bar\partial^h + R_{\bar\partial})u = 0$ is equivalent to

$$\bar\partial^h b = 0,\quad \bar\partial^h a + \frac{i}{2} b\,\overline{\varphi_h} = 0.$$

Thus $\bar\partial b = 0$ and b is a constant. Notice that $\bar\partial^v a = 0$ and $\bar\partial^h a + \frac{i}{2} b\,\overline{\varphi_h} = 0$ together imply $\frac{i}{2} b\,\overline{\varphi_h} = -\bar\partial a$ is $\bar\partial$-exact. Since $\overline{\varphi_h}$ is not $\bar\partial$-exact, we know that $b = 0$. Thus

$$\bar\partial^h a = \bar\partial^v a = 0,$$

which gives $\bar\partial a = 0$ and a is a constant. Thus $\mathcal{H}^{1,0}(\bar\partial) = \mathrm{Span}_{\mathbb{C}}\langle \varphi_h\rangle$.

Degree $(0,1)$ *case*: Let

$$u = a\overline{\varphi_h} + b\overline{\varphi_v},$$

be in $\mathcal{H}^{0,1}(\bar\partial)$. $\bar\partial^v u = (\bar\partial^h + R_{\bar\partial})u = 0$ is equivalent to

$$\bar\partial^v a = \bar\partial^h b = 0.$$

$(\bar\partial^v)^* u = 0$ is equivalent to

$$\partial^v b = 0.$$

Since $\Box_{\bar\partial^v} = \Box_{\partial^v}$, we know that $\partial^v b = 0$ implies $\bar\partial^v b = 0$. Thus $\bar\partial b = 0$ and b is a constant. $(\bar\partial^h + R_{\bar\partial})^* u = 0$ is equivalent to $\partial^h a = 0$, thus $\partial a = 0$ and a is a constant.

Degree $(2,0)$ *case*: Let

$$u = a\varphi_h\wedge\varphi_v,$$

be in $\mathcal{H}^{2,0}(\bar\partial)$. $\bar\partial u = 0$ is equivalent to $\bar\partial a = 0$, which is equivalent to that a is a constant.

Degree (1, 1) *case*: Let

$$u = a\varphi_h \wedge \overline{\varphi_h} + b\varphi_h \wedge \overline{\varphi_v} + c\varphi_v \wedge \overline{\varphi_h} + f\varphi_v \wedge \overline{\varphi_v},$$

be a harmonic (1, 1)-form. We have

$$\Box_{\overline{\partial}^v} u = (\Box_{\overline{\partial}^v} a)\, \varphi_h \wedge \overline{\varphi_h} + (\Box_{\overline{\partial}^v} b)\, \varphi_h \wedge \overline{\varphi_v} + (\Box_{\overline{\partial}^v} c)\, \varphi_v \wedge \overline{\varphi_h} + (\Box_{\overline{\partial}^v} f)\, \varphi_v \wedge \overline{\varphi_v},$$

Then $\Box_{\overline{\partial}^v} u = 0$ is equivalent to

$$\Box_{\overline{\partial}^v} a = \Box_{\overline{\partial}^v} b = \Box_{\overline{\partial}^v} c = \Box_{\overline{\partial}^v} f = 0.$$

Together with $\Box_{\partial^v} = \Box_{\overline{\partial}^v}$, the above identities give

$$d^v a = d^v b = d^v c = d^v f = 0.$$

$(\overline{\partial}^h + R_{\overline{\partial}})u = 0$ is equivalent to

$$\overline{\partial}^h f = 0, \qquad \overline{\partial}^h b + \frac{i}{2} f \overline{\varphi_h} = 0,$$

thus $\overline{\partial} f = 0$ and f is a constant. Since $\overline{\partial}^v b = 0$, we have

$$\overline{\partial} b + \frac{i}{2} f \overline{\varphi_h} = 0.$$

By the above computation of $\mathcal{H}^{0,1}(\overline{\partial})$, we know that $\overline{\varphi_h}$ is not $\overline{\partial}$-exact. Thus $f = 0$ and $\overline{\partial} b = 0$. Now we know that b is a constant. $(\overline{\partial}^h + R_{\overline{\partial}})^* u = 0$ is equivalent to

$$\partial^h a = 0, \qquad \frac{i}{2} a\varphi_h + \partial^h c = 0.$$

thus $\partial a = 0$ and a is a constant. Again

$$-\frac{i}{2} \bar{a} \overline{\varphi_h} + \overline{\partial} \bar{c} = 0,$$

gives $a = 0$ and c is a constant.

For the remaining cases, by the following well-known formula

$$\dim \mathcal{H}^{p,q}(\overline{\partial}) = \dim \mathcal{H}^{n-p,n-q}(\overline{\partial}),$$

it is enough to check that the listed forms lie in the $\overline{\partial}$*-harmonic spaces, which follows by a direct computation.* □

5.1 Nilpotent fibrations

For the Kodaira–Thurston manifold, the complex nilpotent foliation structure comes from a holomorphic fibration. The general definition is as follows:

Definition 5.3 (*Nilpotent fibration*) We call a proper smooth submersion $\pi : (X, g_X) \to (B, g_B)$ between two Riemannian manifolds a *nilpotent fibration* if the associated foliation $\mathcal{V}$ of the fibers defines a nilpotent foliation structure on (X, g_X) and

$$g_X(V, W) = g_B(\pi_* V, \pi_* W), \tag{5.3}$$

for every horizontal vector fields V, W on X.

Remark Let V_B be a vector field on B, we call a vector field V_X on X a lift of V_B if

$$\pi_* V_X = V_B.$$

It is clear that V_B has a unique lift V_X such that V_X is horizontal. (5.3) says that the norm of a vector field on B is equal to the norm of its horizontal lift.

Definition 5.4 (*Complex nilpotent fibration*) We call a proper holomorphic submersion $\pi : (X, \omega_X) \to (B, \omega_B)$ between two Hermitian complex manifolds a *complex nilpotent fibration* if the associated foliation $\mathcal{V}$ of the fibers defines a complex nilpotent foliation structure on (X, ω_X) and

$$\omega_X(V, W) = \omega_B(\pi_* V, \pi_* W), \tag{5.4}$$

for every horizontal $(1, 0)$-vector fields V, W on X.

Theorem 4.1 gives

Theorem 5.5 *On the total space of a nilpotent fibration, we have*

$$\Box_d = \Box_{d^v} + \Box_{d^h + R_d}.$$

On the total space of a complex nilpotent fibration, we have

$$\Box_{\overline{\partial}} = \Box_{\overline{\partial}^v} + \Box_{\overline{\partial}^h + R_{\overline{\partial}}}, \quad \Box_{\overline{\partial}^v} = \Box_{\partial^v}.$$

Remark Associated to a fibration there is a natural Leray-Serre spectral sequence, which plays a crucial role in the proof of Nomizu-type theorems. But in general a foliation does not give a good fibration structure, thus one has to use other methods. Our main idea is: with the help of Theorem 4.1, one may use the spectral sequence for double complex to continue the reduction process (as in the fibration case), in which the natural setup is a manifold with a nilpotent frame.

6 Nilpotent frame

6.1 Real case

Let X be a compact smooth manifold with trivial T_X^*. Let

$$\Phi := \{\sigma^1, \ldots, \sigma^n\}$$

be a global smooth frame of T_X^*. Then we have the following Maurer–Cartan equations

$$d\sigma^j = \sum_{k,l=1}^{n} A_{kl}^j \sigma^k \wedge \sigma^l, \tag{6.1}$$

where A_{kl}^j are globally defined smooth functions on X. Recall that Φ is a *nilpotent frame* if A_{kl}^j are real *constants* and the above equations reduce to

$$d\sigma^j = \sum_{k,l>j} A_{kl}^j \sigma^k \wedge \sigma^l. \tag{6.2}$$

Definition 6.1 Let Φ be a nilpotent frame. Put $r_0 = 0$ and define r_j $(j \geq 1)$ inductively by

$$r_j + 1 = \min U_j,\ U_j := \cup_{j \geq r_{j-1}+1}\{k, l : A^j_{kl} \neq 0\},$$

where $r_j := n$ if U_j is empty. Fix k such that

$$0 = r_0 < r_1 < \cdots < r_{k-1} < r_k = n,$$

we call Φ a *k-nilpotent frame*. Let S^j be the subbundle of T^*_X generated by $\{\sigma^1, \ldots, \sigma^{r_j}\}$. We call

$$0 = S^0 \hookrightarrow S^1 \hookrightarrow \cdots \hookrightarrow S^k = T^*_X,$$

the *Φ-filtration* of T^*_X.

Remark We always have $1 \leq k \leq n$. For the Kodaira–Thurston manifold (see (5.1)), put

$$\sigma^1 = \mathrm{d}x^4,\ \sigma^2 = \mathrm{d}x^3 - x^1\mathrm{d}x^2,\ \sigma^3 = \mathrm{d}x^2,\ \sigma^4 = \mathrm{d}x^1,$$

we have

$$\begin{cases} \mathrm{d}\sigma^1 = 0, \\ \mathrm{d}\sigma^2 = \sigma^3 \wedge \sigma^4, \\ \mathrm{d}\sigma^3 = 0, \\ \mathrm{d}\sigma^4 = 0. \end{cases}$$

Thus $k = 2$ and the Φ-filtration is

$$0 \hookrightarrow \mathrm{Span}\{\sigma^1, \sigma^2\} \hookrightarrow \mathrm{Span}\{\sigma^1, \sigma^2, \sigma^3, \sigma^4\} = T^*_X.$$

Put

$$T^*_v = \mathrm{Span}\{\sigma^1, \sigma^2\},\ T^*_h = \mathrm{Span}\{\sigma^3, \sigma^4\}.$$

We get the following vertical–horizontal decomposition of the T^*_X

$$T^*_X = T^*_v \oplus T^*_h.$$

with respect the Riemannian metric $\sum \sigma^j \otimes \sigma^j$. In general, we shall introduce the following definition

Definition 6.2 Denote by $T^*_{v_j}$ $(1 \leq j \leq k)$ the subbundle of T^*_X generated by $\{\sigma^{r_{j-1}+1}, \ldots, \sigma^{r_j}\}$. Put

$$T^*_{h_j} = \oplus^k_{l=j+1} T^*_{v_l},\ 0 \leq j \leq k-1,\ \ T^*_{h_k} = 0.$$

We call

$$T^*_{h_{j-1}} := T^*_{v_j} \oplus T^*_{h_j},\ \ 1 \leq j \leq k,$$

the jth*vertical horizontal decomposition* with respect to the following Riemannian metric

$$g_X := \sum_{j=1}^n \sigma^j \otimes \sigma^j, \tag{6.3}$$

on X associated to the nilpotent frame Φ.

Remark The first vertical horizontal decomposition of T^*X gives

$$d := d^{h_0} = d^{v_1} + d^{h_1} + R_d^1,$$

where d^{v_1} (resp. d^{h_1}) increases the first vertical (resp. horizontal) degree by one and R_d^1 is the remaining term. In general, the jth vertical horizontal decomposition gives

$$d^{h_{j-1}} = d^{v_j} + d^{h_j} + R_d^j, \qquad 1 \leq j \leq k.$$

Notice that the kth vertical horizontal decomposition reduces to

$$d^{h_{k-1}} = d^{v_k}, \quad R_d^k = 0, \; d^{h_k} = 0.$$

We shall use our fundamental theorem (see Theorem 4.1) to prove the following result.

Theorem 6.3 *With respect to the notation above, we have*

$$\Box_d = \Box_{d^{v_1}} + \Box_{d^{h_1}+R_d^1}, \tag{6.4}$$

on the space of smooth forms on X. *Moreover, if* $k \geq 2$ *then for every* $2 \leq j \leq k$, *we have*

$$\Box_{d^{h_{j-1}}} = \Box_{d^{v_j}} + \Box_{d^{h_j}+R_d^j}, \tag{6.5}$$

on the space of smooth forms in $\ker \Box_{d^{v_1}} \cap \cdots \cap \ker \Box_{d^{v_{j-1}}}$.

Proof Denote by $\{T^{h_j}, T^{v_k}\}$ the subbundles of T_X that are dual to $\{T^*_{h_j}, T^*_{v_k}\}$. From Definition 6.1, we know that the distribution $\mathcal{V}^1$ associated to T^{v_1} is integrable and $\mathcal{V}^1$ defines a nilpotent foliation with respect to the Riemannian metric g_X in (6.3). Thus Theorem 4.1 gives (6.4).

Now let us prove (6.5) for $j = 2$. Let us write a smooth form u on X as

$$u = \sum f_{p,q}\, u_{h_1}^p \wedge u_{v_1}^q, \tag{6.6}$$

where $\{u_{h_1}^p\}$ (resp. $\{u_{v_1}^q\}$) denotes a basis of the exterior algebra generated by $\{\sigma^{r_1+1}, \ldots, \sigma^n\}$ (resp. $\{\sigma^1, \ldots, \sigma^{r_1}\}$), respectively. Denote by $\{V_l\}$ the frame of T_X that is dual to $\{\sigma^l\}$. Put

$$\varphi_{v_1}^l = \sigma^l, \quad \varphi_{h_1}^m = \sigma^{r_1+m}, \qquad X_l^{v_1} = V_l, \quad X_m^{h_1} = X_{r_1+m}.$$

By Lemmas 4.3 and 4.10, we have

$$\Box_{d^{v_1}} = -\sum (X_l^{v_1})(X_l^{v_1}),$$

which gives

$$\Box_{d^{v_1}} u = \sum (\Box_{d^{v_1}} f_{p,q})\, u_{h_1}^p \wedge u_{v_1}^q.$$

Thus $\Box_{d^{v_1}} u = 0$ is equivalent to

$$\Box_{d^{v_1}} f_{p,q} \equiv 0,$$

for all p, q. Since $f_{p,q}$ are globally defined smooth functions on X, we know that $\Box_{d^{v_1}} f_{p,q} \equiv 0$ is equivalent to that $d^{v_1} f_{p,q} \equiv 0$. Thus we get the following lemma.

Lemma 6.4 $\Box_{d^{v_1}} u = 0$ *is equivalent to that all* $f_{p,q}$ *are constants on leaves of* $\mathcal{V}^1$.

By the above lemma and Definition 6.1, we know that the distribution $\mathcal{V}^2$ associated to T^{v_2} is integrable on the space of smooth forms in $\ker \Box_{d^{v_1}}$. Moreover, on the space of smooth forms in $\ker \Box_{d^{v_1}}$, $\mathcal{V}^2$ defines a nilpotent foliation with respect to the following decomposition

$$T^*_{h_1} = T^*_{v_2} \oplus T^*_{h_2}.$$

Thus the proof of Theorem 4.1 gives (6.5) for $j = 2$. The general case follows by induction on j. $\square$

6.2 Complex case

Let X be a compact complex manifold with smoothly trivial holomorphic cotangent bundle $\wedge^{1,0}T^*_X$. Let

$$\Psi := \{\xi^1, \ldots, \xi^n\}$$

be a global smooth frame of $\wedge^{1,0}T^*_X$. Recall that Ψ is a *complex nilpotent frame* if

$$d\xi^j = \sum_{k,l>j} B^j_{kl}\, \xi^k \wedge \xi^l + \sum_{k,l>j} B^j_{k\bar{l}}\, \xi^k \wedge \overline{\xi^l}, \tag{6.7}$$

where B^j_{kl} and $B^j_{k\bar{l}}$ are *complex constants*.

Definition 6.5 Let Ψ be a complex nilpotent frame. Put $r_0 = 0$ and define r_j $(j \geq 1)$ inductively by

$$r_j + 1 = \min U_j, \;\; U_j := \cup_{j \geq r_{j-1}+1} \{k, l : B^j_{kl} \text{ or } B^j_{k\bar{l}} \neq 0\},$$

where $r_j := n$ if U_j is empty. Fix k such that

$$0 = r_0 < r_1 < \cdots < r_{k-1} < r_k = n,$$

we call Ψ a complex *k-nilpotent frame*. Let S^j be the subbundle of $\wedge^{1,0}T^*_X$ generated by $\{\xi^1, \ldots, \xi^{r_j}\}$. We call

$$0 = S^0 \hookrightarrow S^1 \hookrightarrow \cdots \hookrightarrow S^k = \wedge^{1,0}T^*_X,$$

the Ψ-*filtration* of $\wedge^{1,0}T^*_X$.

Similar as the real case, we have

Definition 6.6 Denote by $T^*_{v_j}$ $(1 \leq j \leq k)$ the (smooth, may not be holomorphic) subbundle of $\wedge^{1,0}T^*_X$ generated by $\{\xi^{r_{j-1}+1}, \ldots, \xi^{r_j}\}$. Put

$$T^*_{h_j} = \oplus_{l=j+1}^k T^*_{v_l}, \;\; 0 \leq j \leq k-1, \;\; T^*_{h_k} = 0.$$

We call

$$T^*_{h_{j-1}} := T^*_{v_j} \oplus T^*_{h_j}, \;\;\; 1 \leq j \leq k,$$

the *jth vertical horizontal decomposition* with respect to the following Hermitian form

$$\omega_X := i \sum_{j=1}^n \xi^j \wedge \overline{\xi^j}, \tag{6.8}$$

on X associated to the complex nilpotent frame Ψ.

Remark The first vertical horizontal decomposition of T^*X gives

$$\overline{\partial} := \overline{\partial}^{h_0} = \overline{\partial}^{v_1} + \overline{\partial}^{h_1} + R^1_{\overline{\partial}},$$

where $\overline{\partial}^{v_1}$ (resp. $\overline{\partial}^{h_1}$) increases the first vertical (resp. horizontal) degree by (0, 1) and $R^1_{\overline{\partial}}$ is the remaining term. In general, the jth vertical horizontal decomposition gives

$$\overline{\partial}^{h_{j-1}} = \overline{\partial}^{v_j} + \overline{\partial}^{h_j} + R^j_{\overline{\partial}}, \qquad 1 \leq j \leq k.$$

Notice that the kth vertical horizontal decomposition reduces to

$$\overline{\partial}^{h_{k-1}} = \overline{\partial}^{v_k}, \quad R^k_{\overline{\partial}} = 0, \ \overline{\partial}^{h_k} = 0.$$

Similar as the real case, Theorem 4.1 implies the following result.

Theorem 6.7 *With respect to the notation above, we have*

$$\Box_{\overline{\partial}} = \Box_{\overline{\partial}^{v_1}} + \Box_{\overline{\partial}^{h_1} + R^1_{\overline{\partial}}}, \quad \Box_{\overline{\partial}^{v_1}} = \Box_{\partial^{v_1}}, \tag{6.9}$$

on the space of smooth forms on X. Moreover, if $k \geq 2$ then for every $2 \leq j \leq k$, we have

$$\Box_{\overline{\partial}^{h_{j-1}}} = \Box_{\overline{\partial}^{v_j}} + \Box_{\overline{\partial}^{h_j} + R^j_{\overline{\partial}}}, \quad \Box_{\overline{\partial}^{v_j}} = \Box_{\partial^{v_j}}, \tag{6.10}$$

on the space of smooth forms in $\ker \Box_{\overline{\partial}^{v_1}} \cap \cdots \cap \ker \Box_{\overline{\partial}^{v_{j-1}}}$.

7 Proof of the main theorem

7.1 Cohomology description of Theorem 4.1

Let X be a compact smooth manifold. By the de Rham theorem, we have the following isomorphism

$$H^\star_d \simeq \mathcal{H}^\star_d := \{u \in \mathcal{A}^* : \Box_d u = 0\},$$

where $H^\star_d$ denotes the de Rham cohomology group of X, $\mathcal{A}^*$ denotes the space of smooth forms on X. With the assumption in Theorem 4.1 (real case), every $u \in \mathcal{A}^*$ must satisfies

$$\Box_{d^v} u = 0,$$

which gives, by Lemma 6.4, that all coefficients of u are constants on leaves of $\mathcal{V}$. Denote by $\mathcal{A}^*_1$ the space of smooth forms on X whose coefficients are constants on leaves of $\mathcal{V}$, the usual d operator reduces to $d^h + R_d$ on $\mathcal{A}^*_1$, which suggests to look at the following cohomology (notice that $(d^h + R_d)^2 = 0$ on $\mathcal{A}^*_1$, since the coefficients of R_d are constants on leaves of $\mathcal{V}$)

$$H^*_{d^h+R_d} := \frac{\{u \in \mathcal{A}^*_1 : (d^h + R_d)u = 0\}}{(d^h + R_d)(\mathcal{A}^*_1)}.$$

A representative of a class in $H^*_{d^h+R_d}$ is minimal if and only if it lies in $\ker \Box_{d^h+R_d}$. Thus Theorem 4.1 implies that

$$\mathcal{H}^\star_d \simeq H^*_{d^h+R_d}.$$

A similar argument also works in the complex case, to summarize, we get

Theorem 7.1 *Theorem 4.1 (real case) implies*

$$H_d^* \simeq H_{d^h+R_d}^*.$$

Theorem 4.1 (real case) implies

$$H_{\bar\partial}^{*,*} \simeq H_{\bar\partial^h+R_{\bar\partial}}^{*,*}.$$

Remark In the setting of Theorems 6.7 and 6.3, one may continue to get a sequence of isomorphisms. In fact, with the notation in Theorem 6.3, denote by $\mathcal{V}^j$ $(1 \le j \le k)$ the distribution associated to $T^{v_1} \oplus \cdots \oplus T^{v_j}$, let $\mathcal{A}_j^*$ be the space of smooth forms on X whose coefficients are constants on leaves of $\mathcal{V}^j$, then one may verify that

$$(d^{h_j} + R_d^j)^2 = 0, \qquad \text{on } \mathcal{A}_j^*.$$

Put

$$H_{d^{h_j}+R_d^j}^* := \frac{\{u \in \mathcal{A}_j^* : (d^{h_j} + R_d^j)u = 0\}}{(d^{h_j} + R_d^j)(\mathcal{A}_j^*)}, \quad H_{d^{h_j}}^* := \frac{\{u \in \mathcal{A}_j^* : d^{h_j}u = 0\}}{d^{h_j}(\mathcal{A}_j^*)}$$

we have

Theorem 7.2 *Theorem 6.3 implies*

$$H_d^* \simeq H_{d^{h_1}+R_d^1}^*, \quad H_{d^{h_{j-1}}}^* \simeq H_{d^{h_j}+R_d^j}^*, \quad \forall\, 2 \le j \le k.$$

Theorem 6.7 implies

$$H_{\bar\partial}^{*,*} \simeq H_{\bar\partial^{h_1}+R_{\bar\partial}^1}^{*,*}, \quad H_{\bar\partial^{h_{j-1}}}^{*,*} \simeq H_{\bar\partial^{h_j}+R_{\bar\partial}^j}^{*,*}, \quad \forall\, 2 \le j \le k.$$

Remark In the proper fibration case, the above theorem is essentially equivalent to the use of the E_2 term in the Leray–Serre spectral sequence.

In order to apply the above theorem, one has to study the relation between $H_{d^{h_j}+R_d^j}^*$ and $H_{d^{h_j}}^*$; in the next section, we shall prove that, with respect to the spectral sequence of the double complex $d^{h_j} + R_d^j$, the E_2 term is equal to $H_{d^{h_j}}^*$ and the E_∞ term is $H_{d^{h_j}+R_d^j}^*$.

7.2 Spectral sequence for double complex

We shall use the notation in [14]. We look at the complex

$$d^{h_j} + R_d^j : \mathcal{A}_j^* \to \mathcal{A}_j^*,$$

denote its kernel by Z, image by B and put

$$H := \frac{Z}{B} = H_{d^{h_j}+R_d^j}^*.$$

With the notation in [14], we denote by ${}^pT$ as the space of forms in $\mathcal{A}_j^*$ whose jth horizontal degree is no less than p. By Definition 6.1, the maximal jth horizontal degree is $n - r_j$, thus we have

$$\mathcal{A}_j^* = {}^0T \supset {}^1T \supset \cdots \supset {}^{n-r_j}T \supset {}^{n-r_j+1}T = \{0\}.$$

Denote by T^m the space of degree-m forms in $\mathcal{A}_j^*$. We have

$$\mathcal{A}_j^* = \oplus_{m=0}^n T^m.$$

Put

$${}^pT^m := {}^pT \cap T^m,$$

we know that ${}^pT^m$ is the space of degree-m forms in $\mathcal{A}_j^*$ whose horizontal degree is no less that p. One may easily verify the following compatibility conditions in [14]

$$(d^{h_j} + R_d^j)\, T^m \subset T^{m+1}, \quad (d^{h_j} + R_d^j)\, {}^pT \subset {}^pT.$$

In fact, notice that $d^{h_j}\, {}^pT \subset {}^{p+1}T$ and $R_d^j\, {}^pT \subset {}^{p+2}T$, the following stronger compatibility condition holds

$$(d^{h_j} + R_d^j)\, {}^pT \subset {}^{p+1}T.$$

Moreover, we shall introduce

$${}^pZ^m := Z \cap {}^pT^n, \quad {}^pB^m := B \cap {}^pT^n, \quad {}^pH^m := \frac{{}^pZ^m}{{}^pB^m}.$$

The fundamental definition is the following

Definition 7.3 (*Spectral sequence of a filtered complex* [14]) The *modules of spectral sequence* are

$$E_r^{p,q} := \frac{{}^pT_r^{p+q}}{{}^pT_{r-1}^{p+q} + {}^pB_{r-1}^{p+q}}, \quad E_\infty^{p,q} := \frac{{}^pZ^{p+q}}{{}^pZ^{p+q} + {}^pB^{p+q}}, \quad r = 1, 2, \ldots,$$

where

$${}^pT_r^m := \{u \in {}^pT^m : (d^{h_j} + R_d^j)u \in {}^{p+r}T\}, \quad {}^pB_r^m := \{(d^{h_j} + R_d^j)u \in {}^pT^m : u \in {}^{p-r}T\}.$$

One may verify that $d^{h_j} + R_d^j$ induces the following d_r complex

$$E_r^{p-r,q+r-1} \xrightarrow{d_r} E_r^{p,q} \xrightarrow{d_r} E_r^{p+r,q-r+1}.$$

As general properties of the spectral sequence, we have

Proposition 7.4 *The associated d_r cohomology gives the E_{r+1} term, more precisely*

$$E_{r+1}^{p,q} \simeq \frac{\ker d_r \cap E_r^{p,q}}{\operatorname{Im} d_r \cap E_r^{p,q}}.$$

Moreover, we have

$$E_\infty^{p,q} \simeq \frac{{}^pH^{p+q}}{{}^{p+1}H^{p+q}}, \quad E_r^{p,q} = E_\infty^{p,q}, \quad \forall\, r \geq n - r_j + 1.$$

Remark For the double complex $d = \partial + \overline{\partial}$ (also called Frölicher spectral sequence), it is well known that [14]

$$E_1^{p,q} \simeq H_{\overline{\partial}}^{p,q}.$$

In our case, we obtain the following

Lemma 7.5 $\oplus_{p+q=m} E_2^{p,q} \simeq H^m_{d^{h_j}}$.

Proof Recall that

$$E_2^{p,q} := \frac{{}^pT_2^{p+q}}{{}^pT_1^{p+q} + {}^pB_1^{p+q}}.$$

Let us write $u \in {}^pT^m$ as

$$u = u^{p|m-p} + u^{p+1|m-p-1} + \cdots,$$

where $u^{\alpha|\beta}$ denotes the component of u with horizontal degree α and vertical degree β. Notice that $(d^{h_j} + R_d^j)u$ has no horizontal degree $p+1$ component if and only if

$$d^{h_j} u^{p|m-p} = 0.$$

Thus u lies in ${}^pT_2^{p+q}$ if and only if $d^{h_j} u^{p|m-p} = 0$. A similar discussion gives that u lies in ${}^pT_1^{p+q} + {}^pB_1^{p+q}$ if and only if $u^{p|m-p} \in \operatorname{Im} d^{h_j}$. Thus the $(p|q)$-component of $H^m_{d^{h_j}}$ is isomorphic to $E_2^{p,q}$. Hence the lemma follows. □

7.3 The final proof

Let us introduce the following notation: let D be a complex on $\mathcal{A}_j^*$, denote its associated cohomology by H_D^*. Recall that $\mathcal{A}_j^*$ means the space of smooth forms on X whose coefficients are constants on leaves of $\mathcal{V}^j$, in particular

$$A^* := \mathcal{A}_k^*$$

is the finite dimensional real vector space spanned by wedge products of the frame Φ. Assume that D maps A^* to itself. Put

$$H^*_{D,\Phi} := \frac{\ker D|_{A^*}}{\operatorname{Im} D|_{A^*}},$$

we say that H_D^* is *simple* if it reduces to $H^*_{D,\Phi}$, i.e.,

$$H_D^* \simeq H^*_{D,\Phi}.$$

The real case of our main theorem is of course equivalent to that H_d^* is simple. By Theorem 7.2, H_d^* is simple if and only if $H^*_{d^{h_1}+R_d^1}$ is simple. Now let us look at the double complex (d^{h_1}, R_d^1), by Lemma 7.5, its E_2 term is $H^*_{d^{h_1}}$. Thus in order to prove that $H^*_{d^{h_1}+R_d^1}$ is simple, it suffices to show that $H^*_{d^{h_1}}$ is simple. Apply Theorem 7.2 again, we know that $H^*_{d^{h_1}}$ is simple if and only if $H^*_{d^{h_2}+R_d^2}$ is simple. Now again Lemma 7.5 ensures that it is enough to prove simplicity of $H^*_{d^{h_2}}$. Repeat the above argument, we know it suffices to show that $H^*_{d^{h_k}}$ is simple, which holds trivially since $\mathcal{A}_k^* = A^*$. The complex case follows by a similar argument.

Remark Notice that the Lie algebra cohomology $H^*_{d,\Phi}$ has the following "harmonic" representative

$$H^*_{d,\Phi} \simeq \mathcal{H}^*_{d,\Phi} := \{u \in A^* : \Box_{R_d^1+\cdots+R_d^{k-1}} u = 0\}.$$

Since

$$\mathcal{H}^*_{d,\Phi} \in \ker \Box_d,$$

our main theorem actually also gives a formula for the following d-harmonic space

$$\mathcal{H}_d^* := \ker \Box_d = \mathcal{H}_{d,\Phi}^*.$$

A more careful study of $\mathcal{H}_{d,\Phi}^*$ will be given in Sect. 8.

7.4 Proof of the strong version of Cordero–Fernández–Gray–Ugarte's theorem

Let us choose an appropriate basis of left-invariant vector fields for the nilpotent Lie group and a left-invariant metric making this a global orthonormal frame. Then we know that our main theorem applies.

Remark The above argument also applies to the original version of Nomizu's theorem.

8 An explicit version of our main theorem

In case $k = 2$, the remark at the end of Sect. 7.3 gives

$$\ker \Box_d = \{u \in A^* : \Box_{R_d^1} u = 0\}. \tag{8.1}$$

The above formula can also be proved using the E_3 term of the spectral sequence of the double complex (d^{h_1}, R_d^1). In fact, by Lemma 7.5, we know that the E_3 term is defined by the following complex

$$R_d^1 : A^* \to A^*.$$

The key point here is that both R_d^1 and its adjoint send A^* to itself, from which we know that the above spectral sequence degenerates at E_3.

Example 1 By the example in Sect. 5, if X is the Kodaira–Thurston manifold, then $k = 2$ and

$$R_{\bar\partial}^1 = -\frac{i}{2}\varphi_h \wedge \overline{\varphi_h} \wedge (X^v \rfloor), \quad (R_{\bar\partial}^1)^* = \frac{i}{2}\varphi_v \wedge (\overline{X^h} \rfloor)(X^h \rfloor).$$

The complex version of (8.1) gives

$$\mathcal{H}_{\bar\partial}^{*,*} = \{u \in A^{*,*} : R_{\bar\partial}^1 u = (R_{\bar\partial}^1)^* u = 0\}.$$

Example 2 Let $\mathfrak{g}$ be the 3-dimensional 2-step complex nilpotent Lie algebra admitting a complex $(1, 0)$-coframe $\{\xi^1, \xi^2, \xi^3\}$ satisfying

$$\mathrm{d}\xi^1 = \xi^2 \wedge \overline{\xi^2} + D\xi^3 \wedge \overline{\xi^3}, \qquad \mathrm{d}\xi^2 = 0, \qquad \mathrm{d}\xi^3 = 0,$$

where D is a complex parameter having non-negative imaginary part. Let G be the simply connected Lie group having $\mathfrak{g}$ as Lie algebra. Let X be any compact quotient of G. Then, $k = 2$, the decomposition of $\wedge^{1,0}T_X^*$ into vertical and horizontal subbundles is given by

$$\wedge^{1,0}T_X^* = \langle \xi^1 \rangle \oplus \langle \xi^2, \xi^3 \rangle$$

and

$$R_{\bar\partial}^1 = (\xi^2 \wedge \overline{\xi^2} + D\xi^3 \wedge \overline{\xi^3}) \wedge (\xi_1 \rfloor), \quad (R_{\bar\partial}^1)^* = \xi^1 \wedge [(\overline{\xi_2} \rfloor)(\xi_2 \rfloor) + \bar{D}(\overline{\xi_3} \rfloor)(\xi_3 \rfloor)],$$

where $\{\xi_1, \xi_2, \xi_3\}$ satisfies $\xi^r(\xi_s) = \delta^r_s$. Also in this case, the complex version of (8.1) gives

$$\mathcal{H}^{*,*}_{\overline{\partial}} = \{u \in A^{*,*} : R^1_{\overline{\partial}} u = (R^1_{\overline{\partial}})^* u = 0\}.$$

For general k, we shall introduce the following:

Definition 8.1 With the notation in our main theorem. In the real case, the nilpotent frame Φ is said to be *admissible* if $k \geq 2$ or

$$R_d^{k-j-1}\left(\Box_{R_d^{k-j}} + \cdots + \Box_{R_d^{k-1}}\right) = \left(\Box_{R_d^{k-j}} + \cdots + \Box_{R_d^{k-1}}\right) R_d^{k-j-1}, \quad \forall\, 1 \leq j \leq k-2,$$

when $k \geq 3$. In the complex case, the nilpotent frame Ψ is said to be *admissible* if $k \geq 2$ or

$$R_{\overline{\partial}}^{k-j-1}\left(\Box_{R_{\overline{\partial}}^{k-j}} + \cdots + \Box_{R_{\overline{\partial}}^{k-1}}\right) = \left(\Box_{R_{\overline{\partial}}^{k-j}} + \cdots + \Box_{R_{\overline{\partial}}^{k-1}}\right) R_{\overline{\partial}}^{k-j-1}, \quad \forall\, 1 \leq j \leq k-2,$$

when $k \geq 3$.

We obtain the following result:

Theorem 8.2 *With the assumptions in our main theorem. Assume further that the associated frame is admissible. Then:*

(1) *In the real case, we have*

$$\mathcal{H}^{\star}_d = \{u \in A^{\star} : R^j_d u = (R^j_d)^* u = 0,\ \forall\, 1 \leq j \leq k-1\}.$$

(2) *In the complex case, we have*

$$\mathcal{H}^{\star,\star}_{\overline{\partial}} = \{u \in A^{\star,\star} : R^j_{\overline{\partial}} u = (R^j_{\overline{\partial}})^* u = 0,\ \forall\, 1 \leq j \leq k-1\}.$$

Proof Notice that

$$R_d^{k-2}\left(\Box_{R_d^{k-1}}\right) = \left(\Box_{R_d^{k-1}}\right) R_d^{k-2}$$

implies (notice that $\mathcal{H}^*_{d^{h_{k-2}}} = \ker \Box_{R_d^{k-1}}$)

$$R_d^{k-2}(\mathcal{H}^*_{d^{h_{k-2}}}) \subset \mathcal{H}^*_{d^{h_{k-2}}},$$

from which we know that the following complex

$$R_d^{k-2} : H^*_{d^{h_{k-2}}} \to H^*_{d^{h_{k-2}}},$$

is also well defined on the corresponding harmonic space $\mathcal{H}^*_{d^{h_{k-2}}}$. Thus the spectral sequence $(d^{h_{k-2}}, R_d^{k-2})$ degenerates at E_3 and

$$\mathcal{H}^*_{d^{h_{k-2}} + R_d^{k-2}} = \mathcal{H}^*_{d^{h_{k-2}}} \cap \ker \Box_{R_d^{k-2}},$$

which is equivalent to

$$\mathcal{H}^*_{d^{h_{k-3}}} = \ker \Box_{R_d^{k-2} + R_d^{k-1}} = \ker \Box_{R_d^{k-2}} \cap \Box_{R_d^{k-1}}.$$

Repeating the above argument gives the proof of the real case, the complex case follows by a similar argument. □

Remark Not all nilpotent frames are admissible, we shall give a counterexample in the next section.

9 Further examples

9.1 A nilmanifold not satisfying Theorem 8.2

Let $\mathfrak{h}_7$ be the real 6-dimensional nilpotent Lie algebra with basis $\{e_1, \dots, e_6\}$ such that

$$[e_1, e_2] = -e_4, \quad [e_1, e_3] = -e_5, \quad [e_2, e_3] = -e_6,$$

the other brackets vanishing. Then, denoting by $\{e^1, \dots, e^6\}$ the dual basis of $\{e_1, \dots, e_6\}$, we obtain the following structure equations

$$\mathrm{d}e^1 = 0, \quad \mathrm{d}e^2 = 0, \quad \mathrm{d}e^3 = 0, \quad \mathrm{d}e^4 = e^1 \wedge e^2, \quad \mathrm{d}e^5 = e^1 \wedge e^3, \quad \mathrm{d}e^6 = e^2 \wedge e^3.$$

Then, the simply connected Lie group H_7 whose Lie algebra is $\mathfrak{h}_7$ admits compact quotients $M = \Gamma\backslash H_7$. Define an almost complex structure J_0 on M by the following complex $(1,0)$-coframe

$$\omega^1 = e^1 + ie^2, \quad \omega^2 = -2(e^3 + ie^4), \quad \omega^3 = -4(e^5 + ie^6).$$

Then

$$\mathrm{d}\omega^1 = 0, \quad d\omega^2 = \omega^1 \wedge \overline{\omega^1}, \quad \mathrm{d}\omega^3 = \omega^1 \wedge \omega^2 + \omega^1 \wedge \overline{\omega^2},$$

that is J_0 is in fact integrable and by Ugarte [29] (see also [23]) any other complex structure on M is equivalent to J_0. Set

$$\omega^1 = \xi^3, \quad \omega^2 = \xi^2, \quad \omega^3 = \xi^1.$$

Then,

$$\mathrm{d}\xi^1 = -\xi^2 \wedge \xi^3 + \xi^3 \wedge \overline{\xi^2}, \quad \mathrm{d}\xi^2 = \xi^3 \wedge \overline{\xi^3}, \quad \mathrm{d}\xi^3 = 0 \tag{9.1}$$

and $k = 3$. The decomposition of $\wedge^{1,0}T^*_M$ into vertical and horizontal subbundles reads as

$$\wedge^{1,0}T^*_M = \langle \xi^1 \rangle \oplus \langle \xi^2, \xi^3 \rangle.$$

A direct computation taking into account (9.1) gives

$$R^1_{\bar\partial} = -\overline{\xi^2} \wedge \overline{\xi^3} \wedge (\overline{\xi_1} \lrcorner\,) + \xi^3 \wedge \overline{\xi^2} \wedge (\xi_1 \lrcorner\,),\; R^2_{\bar\partial} = \xi^3 \wedge \overline{\xi^3} \wedge (\xi_2 \lrcorner\,),$$
$$(R^1_{\bar\partial})^* = -\overline{\xi^1} \wedge [(\overline{\xi_3} \lrcorner\,)(\overline{\xi_2} \lrcorner\,)] + \xi^1 \wedge [(\overline{\xi_2} \lrcorner\,)(\xi_3 \lrcorner\,)],\; (R^2_{\bar\partial})^* = \xi^2 \wedge [(\overline{\xi_3} \lrcorner\,)(\xi_3 \lrcorner\,)],$$

where $\{\xi_1, \xi_2, \xi_3\}$ satisfies $\xi^r(\xi_s) = \delta^r_s$.

Theorem 9.1 *The Dolbeault cohomology group of* $\Gamma\backslash H_7$ *satisfies*

$$\mathcal{H}^{1,1}_{\bar\partial} \simeq \mathrm{Span}_{\mathbb{C}} \langle \xi^{1\bar2}, \xi^{2\bar3}, \xi^{2\bar2} + \xi^{3\bar1},\; \xi^{2\bar2} + \xi^{1\bar3}, \rangle. \tag{9.2}$$

where $\xi^{i\bar j} := \xi^i \wedge \overline{\xi^j}$. *In particular,* $\mathcal{H}^{1,1}_{\bar\partial}$ *does not satisfy Theorem 8.2.*

Proof Let

$$u = a_1\xi^{1\bar1} + b_1\xi^{2\bar1} + c_1\xi^{3\bar1} + a_2\xi^{1\bar2} + b_2\xi^{2\bar2} + c_2\xi^{3\bar2} + a_3\xi^{1\bar3} + b_3\xi^{2\bar3} + c_3\xi^{3\bar3}$$

be a harmonic $(1,1)$-form. By our main theorem, one may assume that all the above coefficients are constants. Then, u is harmonic if and only if

$$\bar\partial u = \left(R^1_{\bar\partial} + R^2_{\bar\partial}\right) u = \bar\partial^* u = \left((R^1_{\bar\partial})^* + (R^2_{\bar\partial})^*\right) u = 0,$$

which gives (9.2). Notice that $\mathcal{H}^{1,1}_{\bar\partial}$ satisfies Theorem 8.2 if and only if

$$\mathcal{H}^{1,1}_{\bar\partial} \subset \ker R^1_{\bar\partial} \cap \ker(R^1_{\bar\partial})^* \cap \ker R^2_{\bar\partial} \cap \ker(R^2_{\bar\partial})^*$$

But obviously

$$R^2_{\bar\partial}(\xi^{2\bar1} + \xi^{3\bar1}) \neq 0,$$

thus $\mathcal{H}^{1,1}_{\bar\partial}$ does not satisfy Theorem 8.2. □

9.2 Nilpotent frame in a non-nilpotent Lie group

Let us look at the following example from [16].

Example Consider the following group structure

$$(a, \lambda) * (t, z) = (a + t, \lambda + e^{2\pi i a} z),$$

on $\mathbb{R} \times \mathbb{C}$. Then we know that

$$G := (\mathbb{R} \times \mathbb{C}, *),$$

is a Lie group. It is clear that

$$\Gamma := \mathbb{Z} \times (\mathbb{Z} + i\mathbb{Z})$$

is a discrete subgroup of G such that the quotient space (with respect to the left action)

$$X := \Gamma\backslash G,$$

is a compact smooth manifold. It is clear that

$$\mathrm{d}t, \quad e^{-2\pi i t}\mathrm{d}z,$$

are G-invariant with respect to the left action of G. Put

$$e^1 = r\mathrm{d}t, \ e^2 = \mathrm{Re}(e^{-2\pi i t}\mathrm{d}z), \ e^3 = \mathrm{Im}(e^{-2\pi i t}\mathrm{d}z).$$

we know that $\{e^1, e^2, e^3\}$ is a basis of the dual Lie algebra $\mathfrak{g}^*$ of G. Notice that if $z = x + iy$ then

$$e^2 = (\cos 2\pi t)\, \mathrm{d}x + (\sin 2\pi t)\, \mathrm{d}y,$$

and

$$e^3 = -\,(\sin 2\pi t)\, \mathrm{d}x + (\cos 2\pi t)\, \mathrm{d}y.$$

Thus

$$\mathrm{d}e^1 = 0, \ \mathrm{d}e^2 = 2\pi\, e^1 \wedge e^3, \ \mathrm{d}e^3 = -\,2\pi\, e^1 \wedge e^2,$$

which gives

$$[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = [\mathfrak{g}, \mathfrak{g}], \quad [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] = 0.$$

Thus $\mathfrak{g}$ is solvable but not nilpotent. But notice that

$$\sigma^1 := \mathrm{d}t, \quad \sigma^2 := \mathrm{d}x, \quad \sigma^3 := \mathrm{d}y,$$

give a smooth frame of T_X^* such that

$$d\sigma^j = 0, \ \ j = 1, 2, 3.$$

Thus

$$\Phi := \{\sigma^1, \sigma^2, \sigma^3\},$$

is a nilpotent frame of X and our main theorem implies that the de Rham cohomology of X is isomorphic to the exterior algebra generated by Φ (in fact it is easy to see that X is diffeomorphic to a real torus).

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