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On invariants of certain symmetric algebras

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Abstract Let $\mathrm{Syz}_1(\mathfrak{m})$ be the first syzygy of the graded maximal ideal $\mathfrak{m}$ of a polynomial ring $K[x_1, \ldots, x_n]$ over a field K. The multiplicity and (Castelnuovo–Mumford) regularity of the symmetric algebra $\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))$ are estimated by using the theory of s-sequences. It is proved that the multiplicity of $\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))$ is 1 when $n \geq 5$, and $n-2$ is an upper bound for its regularity. In virtue of Gröbner bases, this bound is shown to be reached provided $n \leq 5$.

Keywords Multiplicity · Regularity · Symmetric algebra · s-Sequence

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1 Introduction

Let R be a Noetherian ring and $M = (f_1, \ldots, f_n)$ be a finitely generated R-module. The symmetric algebra $\mathrm{Sym}(M)$ of M is a quotient ring of the polynomial ring $R[y_1, \ldots, y_n]$ over R. Considering this presentation, s-sequences were introduced to study the properties of symmetric algebras in [5] (cf. [7,12]). If M is generated by an s-sequence, one obtains exact values for the dimension $\dim(\mathrm{Sym}(M))$ and the multiplicity $\mathrm{e}(\mathrm{Sym}(M))$, and bounds

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for the depth depth(Sym(M)) and the (Castelnuovo–Mumford) regularity reg(Sym(M)) by the same invariants of some special quotients of R by the annihilator ideals.

Let K be a field, $K[x_1, \dots, x_n]$ be a polynomial ring over K and $\mathfrak{m} = (x_1, \dots, x_n)$. The first syzygy of $\mathfrak{m}$ is denoted by $\mathrm{Syz}_1(\mathfrak{m})$. Our topic is the symmetric algebra $\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))$. In [10], the authors obtained the dimension and depth of $\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))$. In this paper, we will continue to study the multiplicity and regularity of $\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))$.

We calculate the multiplicity of $\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))$ in Sect. 3. In order to get the regularity of $\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))$, we need to estimate the regularity of the initial ideals of certain annihilator ideals in Sect. 4, where some new results in [3] and [8] are applied. In Sect. 5, an upper bound for the regularity of $\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))$ is given. When $n \leq 5$, using Buchberger's algorithm, we find a set of minimal generators for the second syzygy module of $\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))$. The degrees of these generators give a lower bound for the regularity. Then we obtain an equality for the regularity of $\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))$ provided $n \leq 5$.

2 Preliminaries

Let R be a Noetherian ring and $M = (f_1, \dots, f_n)$ be a finitely generated R-module. Then M has a presentation

$$R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0$$

with a relation matrix $A = (a_{ij})_{m \times n}$. The symmetric algebra $\mathrm{Sym}(M)$ has the presentation

$$R[y_1, \dots, y_n]/J,$$

where $J = (g_1, \dots, g_m)$ and $g_i = \sum_{j=1}^n a_{ij} y_j$, $i = 1, \dots, m$.

Let $P = R[y_1, \dots, y_n]$ which is a graded R-algebra. Then J is a graded ideal, and $\mathrm{Sym}(M)$ is a graded R-algebra. Assign degree one to each variable y_i and degree zero to the elements of R. Let $<$ be a monomial order induced by $y_1 < \cdots < y_n$. For any $f \in P$, $f = \sum_\alpha a_\alpha y^\alpha$, we put $\mathrm{in}(f) = a_\alpha y^\alpha$ where y^α is the largest monomial with respect to the given order such that $a_\alpha \neq 0$. We call $\mathrm{in}(f)$ the initial term of f and define the ideal

$$\mathrm{in}(J) = (\mathrm{in}(f) : f \in J),$$

which is generated by monomials in $y_1, \dots, y_n$ with coefficients in R and is finitely generated since P is Noetherian.

For $i = 1, \dots, n$, we set $M_i = \sum_{j=1}^i R f_j$ and let $I_i = M_{i-1} :_R f_i = \{a \in R : a f_i \in M_{i-1}\}$. We also set $I_0 = 0$. Then I_i is the annihilator ideal of the cyclic module M_i/M_{i-1}. It is clear that

$$(I_1 y_1, \dots, I_n y_n) \subseteq \mathrm{in}(J),$$

and the two ideals coincide in degree one.

Definition 2.1 The generators $f_1, \dots, f_n$ of M are called an s-sequence (with respect to $<$), if

$$(I_1 y_1, \dots, I_n y_n) = \mathrm{in}(J).$$

If, in addition, $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n$, then $f_1, \dots, f_n$ is called a strong s-sequence.

Let $S = K[x_1, \ldots, x_d]$ be a polynomial ring over a field K and M a finitely generated graded S-module. Let

$$\cdots \to F_j \xrightarrow{\phi_j} \cdots \to F_1 \xrightarrow{\phi_1} F_0 \to M \to 0$$

be a graded minimal free resolution of M, where $F_j = \oplus_i S(-a_{ji})$. $\operatorname{Im}(\phi_j)$ is called the j-th syzygy module of M. One says that M is m-regular if $a_{ji} - j \leq m$ for all i, j and defines the Castelnuovo–Mumford regularity (or regularity) of M by

$$\operatorname{reg}(M) = \min\{m : M \text{ is } m\text{-regular}\}.$$

Let J be a graded ideal of S. Notice that the i-th syzygy of J is just the $(i+1)$-th syzygy of S/J. It follows that $\operatorname{reg}(J) = \operatorname{reg}(S/J) + 1$. On the other hand, if g is a minimal generator of J, then $\operatorname{reg}(J) \geq \deg(g)$, and if h is a minimal generator of the first syzygy module of J, which is the second syzygy module of S/J, then $\operatorname{reg}(J) \geq \deg(h) - 1$, and so on. For the properties of the regularity, we refer to [1].

Lemma 2.2 ([5, Propositions 2.4 and 2.6]) *Suppose that $f_1, \ldots, f_n$ form a strong s-sequence and have the same degree. Let $d = \dim(\operatorname{Sym}(M))$. Then*

$$\operatorname{e}(\operatorname{Sym}(M)) = \sum_{r \geq 0, \dim(R/I_r) = d-r} \operatorname{e}(R/I_r),$$

and

$$\operatorname{reg}(\operatorname{Sym}(M)) \leq \max\{\operatorname{reg}(I_r) : r = 1, \ldots, n\}.$$

Assume from now on that $n \geq 3$. Let K be a field, $S = K[x_1, \ldots, x_n]$ be a polynomial ring and $\mathfrak{m} = (x_1, \ldots, x_n)$ be the graded maximal ideal of S. Denote the first syzygy of $\mathfrak{m}$ by $\operatorname{Syz}_1(\mathfrak{m})$. From the Koszul complex of S with respect to $x_1, \ldots, x_n$, one has a presentation of $\operatorname{Syz}_1(\mathfrak{m})$ as an S-module

$$S^{\binom{n}{3}} \longrightarrow S^{\binom{n}{2}} \longrightarrow \operatorname{Syz}_1(\mathfrak{m}) \longrightarrow 0.$$

It follows that the symmetric algebra of $\operatorname{Syz}_1(\mathfrak{m})$ has the presentation

$$\operatorname{Sym}_S(\operatorname{Syz}_1(\mathfrak{m})) = S[y_{ij} : 1 \leq i < j \leq n]/J,$$

where J is the ideal of $S[y_{ij} : 1 \leq i < j \leq n]$ generated by the set

$$\left\{x_i y_{jk} - x_j y_{ik} + x_k y_{ij} : 1 \leq i < j < k \leq n\right\}.$$

Since the generators of $\operatorname{Syz}_1(\mathfrak{m})$ do not form an s-sequence in general with respect to the term order $x_i < y_{12} < y_{13} < \cdots < y_{n-1,n}$, we cannot apply the theory of s-sequences in this form. However, the Jacobian dual of $\operatorname{Syz}_1(\mathfrak{m})$ can help us.

Let $Q = K[y_{ij} : 1 \leq i < j \leq n]$. Then the Jacobian dual $\operatorname{Syz}_1(\mathfrak{m})^\vee$ of $\operatorname{Syz}_1(\mathfrak{m})$ is a Q-module with a presentation

$$Q^{\binom{n}{3}} \longrightarrow Q^n \longrightarrow \operatorname{Syz}_1(\mathfrak{m})^\vee \longrightarrow 0,$$

and

$$\operatorname{Sym}(\operatorname{Syz}_1(\mathfrak{m})) \cong \operatorname{Sym}(\operatorname{Syz}_1(\mathfrak{m})^\vee) \cong Q[x_1, \ldots, x_n]/J,$$

cf. [11]. We will use this new presentation of $\operatorname{Sym}(\operatorname{Syz}_1(\mathfrak{m}))$ to estimate its multiplicity and regularity.

Lemma 2.3 ([7, Lemma 3.1 and Proposition 3.3]) *Let $Y = (y_{ij})_{n\times n}$ be the skew-symmetric matrix where $y_{ij} = -y_{ji}$ and $y_{ii} = 0$. Then the set*

$$\{x_i y_{jk} - x_j y_{ik} + x_k y_{ij} : 1 \le i < j < k \le n\} \cup \{x_r P_2(Y) : 1 \le r \le n\}$$

is a Gröbner basis of J with respect to the term order

$$x_n > x_{n-1} > \cdots > x_1 > y_{1n} > y_{1,n-1} > \cdots > y_{12} > y_{2n} > y_{2,n-1} > \cdots > y_{n-1,n},$$

where $P_2(Y)$ is the set of all 4-Pfaffians of Y.

Let x_i^* be the image of x_i in $\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))$, $i = 1, \ldots, n$. Then $x_1^*, \ldots, x_n^*$ is a strong s-sequence with the annihilator ideal

$$\begin{aligned} I_r &= (\{y_{ij} : 1 \le i < j < r\} \cup P_2(Y)) \\ &= (\{y_{ij} : 1 \le i < j < r\} \cup \{y_{il} y_{jk} - y_{ik} y_{jl} + y_{ij} y_{kl} : 1 \le i < j < k < \ell \le n\}), \\ &\quad r = 1, \ldots, n, \end{aligned}$$

which are ideals of Q.

Notice that $I_1 = I_2$, and when $n = 3$, $I_1 = I_2 = 0$.
Then, by Lemma 2.2,

$$\mathrm{e}(\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))) = \sum_{r \ge 0, \dim(Q/I_r) = d - r} \mathrm{e}(Q/I_r),$$

where $d = \dim(\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m})))$, and

$$\mathrm{reg}(\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))) \le \max\{\mathrm{reg}(I_r) : r = 1, \ldots, n\}.$$

3 Multiplicity of the symmetric algebra

For the multiplicity of the symmetric algebra $\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))$, we have the following equalities.

Theorem 3.1 *If $n \ne 4$, then*

$$\mathrm{e}(\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))) = 1,$$

and, if $n = 4$ then

$$\mathrm{e}(\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))) = 5.$$

Proof Let $d = \dim(\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m})))$. Then, by [10, Theorem 4.1], $d = \max\{\frac{n(n-1)}{2}, 2n-1\}$. Let us calculate the multiplicity $\mathrm{e}(Q/I_r)$ with $\dim(Q/I_r) = d - r$. Notice that $I_1 = I_2$ and, by [10, Proposition 3.4], $\dim(Q/I_r) = 2n - 1 - r$ for $r \ge 2$.

Firstly, suppose that $n = 3$. In this case, $d = 5$, $I_1 = I_2 = 0$, $I_3 = (y_{12})$ and $\dim(Q/I_3) = 2 = d - 3$. Then

$$\mathrm{e}(\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))) = \mathrm{e}(Q/I_3) = 1.$$

Now assume that $n = 4$. Then $d = 7$ and $\dim(Q/I_r) + r = d$ for $r = 2, 3, 4$. We have

$$\mathrm{e}(\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))) = \mathrm{e}(Q/I_2) + \mathrm{e}(Q/I_3) + \mathrm{e}(Q/I_4).$$

Notice that

$$\begin{aligned} Q/I_2 &= Q/(P_2(Y)), \\ Q/I_3 &= K[y_{13}, y_{14}, y_{23}, y_{24}, y_{34}]/(y_{14}y_{23} - y_{13}y_{24}), \\ Q/I_4 &= K[y_{14}, y_{24}, y_{34}]. \end{aligned}$$

For the multiplicity of a Pfaffian ideal, by [6, Theorem 5.6], we have the following result: If Y is a $2r \times 2r$ generic skew matrix of indeterminates and $R = K[Y]$, then

$$\mathrm{e}(R/(P_r(Y))) = \det\left[\binom{2}{-i+j+1} - \binom{2}{-i-j+1}\right]_{i,j=1,\ldots,r-1},$$

from which we have

$$\mathrm{e}(Q/I_2) = \binom{2}{1} - \binom{2}{-1} = 2.$$

For the multiplicity of a determinantal ideal, there is a well-known result (cf. [6]): If X is an $n \times n$ generic matrix of indeterminates and $R = K[X]$, then

$$\mathrm{e}(R/(\det(X))) = \det\left[\binom{2n-i-j}{n-i}\right]_{i,j=1,\ldots,n-1},$$

which implies that

$$\mathrm{e}(Q/I_3) = \mathrm{e}(K[y_{13}, y_{14}, y_{23}, y_{24}]/(y_{14}y_{23} - y_{13}y_{24})) = \binom{2}{1} = 2.$$

It is clear that $\mathrm{e}(Q/I_4) = 1$. Hence, in this case, $\mathrm{e}(\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))) = 5$.

Finally, suppose that $n \geq 5$. Then $d = \frac{n(n-1)}{2} \neq r + \dim(Q/I_r)$ for $r = 1, \ldots, n$. Hence

$$\mathrm{e}(\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))) = \mathrm{e}(Q) = 1.$$

The proof is complete. □

4 Regularity of annihilators

Let us estimate the regularity $\mathrm{reg}(I_r)$ of an annihilator ideal I_r. Notice that $Q/I_r = Q_r/I'_r$ where $Q_r = K[y_{ij} : 1 \leq i < j \leq n,\ j \geq r]$ and

$$\begin{aligned} I'_r = (&\{y_{il}y_{jk} - y_{ik}y_{jl} : 1 \leq i < j < r \leq k < l \leq n\} \\ &\cup \{y_{il}y_{jk} - y_{ik}y_{jl} + y_{ij}y_{kl} : 1 \leq i < j;\ r \leq j < k < l \leq n\}). \end{aligned}$$

Then $\mathrm{reg}(I_r) = \mathrm{reg}(I'_r)$.

In [10], in order to calculate the dimension and depth of $\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))$, the following result was proved.

Lemma 4.1 ([10, Lemma 3.3 and Proposition 3.4]) *Q_r/I'_r is Cohen–Macaulay of dimension $2n - 1 - r$ for $r \geq 2$ and*

$$\mathrm{in}(I'_r) = (\mathrm{in}(A)\ :\ A \text{ is a 2-minor of } Y'_r)$$

with the term order

$$y_{1n} > y_{1,n-1} > \cdots > y_{12} > y_{2n} > y_{2,n-1} > \cdots > y_{23} > \cdots\cdots > y_{n-1,n},$$

where

$$Y'_r = \begin{pmatrix} y_{1r} & y_{1,r+1} & \cdots & y_{1n} \\ & \cdots & \cdots & \\ y_{r-1,r} & y_{r-1,r+1} & \cdots & y_{r-1,n} \\ & y_{r,r+1} & \cdots & y_{rn} \\ & & \cdots & \\ & & & y_{n-1,n} \end{pmatrix}.$$

Let Z'_{nr} be the mirror symmetry of Y'_r:

$$Z'_{nr} = \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1,n-r} & z_{1,n-r+1} \\ & & \cdots & \cdots & \\ z_{r-1,1} & z_{r-1,2} & \cdots & z_{r-1,n-r} & z_{r-1,n-r+1} \\ z_{r1} & z_{r2} & \cdots & z_{r,n-r} & \\ & & \cdots & & \\ z_{n-1,1} & & & & \end{pmatrix},$$

i.e., $z_{ij} = y_{i,n-j+1}$, and let the term order $<'$ be as the following

$$z_{11} >' z_{12} >' \cdots >' z_{1,n-r+1} >' z_{21} >' z_{22} >' \cdots >' z_{2,n-r+1} >' \cdots\cdots >' z_{n-1,1}.$$

Notice that, by changing the variables from y_{ij} to z_{ij} and the term order from $<$ to $<'$, Q_r and $\mathrm{in}(I'_r)$ remain the same. Then

$$\mathrm{in}(I'_r) = (\mathrm{in}_{<'}(B) \ : \ B \text{ is a 2-minor of } Z'_{nr}).$$

Let $I_2(Z'_{nr})$ be the ideal of Q_r generated by all the 2-minors of Z'_{nr}. Since Z'_{nr} is a ladder, by [9, Corollary 3.4], the set of 2-minors of Z'_{nr} forms a Gröbner basis. It follows that $\mathrm{in}(I'_r) = \mathrm{in}_{<'}(I_2(Z'_{nr}))$.

Set $[n] = \{1, \ldots, n\}$. Let us identify z_{ij} with its index (i, j). Then Z'_{nr} is an ideal poset of $[n] \times [n]$ where $(i, j) \le (k, l)$ if and only if $i \le k$ and $j \le l$ (cf. [4, §9.1.2]). For any set S, denote the set of indeterminates x_s, $s \in S$, by x_S. Let $L(2, Z'_{nr})$ be the monomial ideal of $K[x_{[2]\times Z'_{nr}}]$ generated by the following monomials:

$$x_{1p}x_{2q}, \ \ p, q \in Z'_{nr}, \, p \le q.$$

By [2, Theorem 2.4], $L(2, Z'_{nr})$ is a Cohen–Macaulay ideal, hence, unmixed ($L(n, Z'_{nr})$ is just $I_{2,2}(Z'_{nr})$ with the notation of [2]). Then Theorem 4.4 of [8] claims that the regularity of $K[x_{[2]\times Z'_{nr}}]/L(2, Z'_{nr})$ is just the maximal cardinality of an antichain in Z'_{nr}, where an antichain in Z'_{nr} is a sequence of points in Z'_{nr} with the property that any two points are incomparable. From the shape of Z'_{nr}, it is easy to see that this maximal cardinality is $n - r + 1$. Then, we have proved the following

Lemma 4.2 $\mathrm{reg}(K[x_{[2]\times Z'_{nr}}]/L(2, Z'_{nr})) = n - r + 1.$

In [3], the authors developed one method by which one gets the same regularity by cutting down a regular sequence from $K[x_{[2]\times Z'_{nr}}]/L(2, Z'_{nr})$.

Let

$$\begin{aligned} \phi \ : \ [2] \times Z'_{nr} &\to [n+1] \times [n+1] \\ (1, i, j) &\mapsto (i, j) \\ (2, i, j) &\mapsto (i+1, j+1), \end{aligned}$$

and $L^{\phi}(2, Z'_{nr})$ be the monomial ideal of $K[x_{[n+1]\times[n+1]}]$ generated by the following monomials:

$$x_{ij}x_{i'+1,j'+1},\ \ (i,j),(i',j')\in Z'_{nr},(i,j)\le(i',j').$$

Then, by [3, Corollary 2.3], the following result holds.

Lemma 4.3 ([3, Corollary 2.3]) *The quotient rings* $K[x_{[2]\times Z'_{nr}}]/L(2, Z'_{nr})$ *and* $K[x_{[n+1]\times[n+1]}]/L^{\phi}(2, Z'_{nr})$ *have the same regularity.*

Notice that the generators of $L^{\phi}(2, Z'_{nr})$ are just the initial terms of the 2-minors of $Z'_{n+2,r+1}$ (identifying x_{ij} with z_{ij}) with respect to the term order $<'$, which form a Gröbner basis as we have noted before. It follows that $L^{\phi}(2, Z'_{nr}) = \mathrm{in}_{<'}(I_2(Z'_{n+2,r+1}))$.

Then we have the following crucial lemma.

Lemma 4.4 *For* $r = 3, \ldots, n$,

$$\mathrm{reg}(\mathrm{in}_{<'}(I_2(Z'_{nr}))) = n - r + 1.$$

Proof It is because

$$\begin{aligned}\mathrm{reg}(\mathrm{in}_{<'}(I_2(Z'_{nr}))) &= \mathrm{reg}(K[x_{[n+1]\times[n+1]}]/L^{\phi}(2, Z'_{n-2,r-1})) + 1\\ &= \mathrm{reg}(K[x_{[2]\times Z'_{n-2,r-1}}]/L(2, Z'_{n-2,r-1})) + 1\\ &= n - r + 1,\end{aligned}$$

where the last equality follows from Lemma 4.2. □

5 Regularity of the symmetric algebra

Now, we can estimate the regularity of the symmetric algebra $\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))$.

Theorem 5.1 *If* $n \ge 3$, *then*

$$\mathrm{reg}(\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))) \le n - 2.$$

Proof Notice that $I_1 = I_2$, $\mathrm{in}_{<'}(I_2(Z'_{n2})) = \mathrm{in}_{<'}(I_2(Z'_{n3}))$ and

$$\mathrm{reg}(I_r) = \mathrm{reg}(I'_r) \le \mathrm{reg}(\mathrm{in}(I'_r)) = \mathrm{reg}(\mathrm{in}_{<'}(I_2(Z'_{nr}))).$$

Then, by Lemmas 2.2 and 4.4, we have that

$$\begin{aligned}\mathrm{reg}(\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))) &\le \max\{\mathrm{reg}(I_r) : r = 2, \ldots, n\}\\ &\le \max\{\mathrm{reg}(\mathrm{in}_{<'}(I_2(Z'_{nr}))) : r = 2, \ldots, n\}\\ &= \max\{\mathrm{reg}(\mathrm{in}_{<'}(I_2(Z'_{nr}))) : r = 3, \ldots, n\}\\ &= \max\{n - r + 1 : r = 3, \ldots, n\}\\ &= n - 2.\end{aligned}$$

□

The above theorem obtains an inequality for the regularity of $\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))$. We wish that the other direction's inequality could also hold. For this purpose, let us check a graded minimal free resolution of $\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))$. For any $1 \le i < j < k \le n$, set $g_{ijk} =$

$y_{ij}x_k - y_{ik}x_j + y_{jk}x_i$. Then $\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m})) = Q[x_1, \ldots, x_n]/J$ and J is minimally generated by g_{ijk}, $1 \leq i < j < k \leq n$. We can construct the first two steps of a graded minimal free resolution of $\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))$:

$$\cdots \longrightarrow \bigoplus_{1 \leq i < j < k \leq n} Q[x_1, \ldots, x_n]e_{ijk}(-2) \xrightarrow{\phi_1} Q[x_1, \ldots, x_n] \longrightarrow \mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m})) \longrightarrow 0,$$

where $\phi_1(e_{ijk}) = g_{ijk}$. From the generators of the first syzygies of $\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))$, i.e., g_{ijk}, are all of degree 2, we see immediately from the definition of regularity that $\mathrm{reg}(\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))) \geq 1$. Hence $\mathrm{reg}(\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))) = n - 2$ when $n = 3$. However this bound is not big enough for $n \geq 4$. We have to consider the degrees of the minimal generators of the second syzygies of $\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))$.

Now assume that $n \geq 4$. In order to get that $\mathrm{reg}(\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))) \geq n - 2$, it is enough to find one minimal generator of $\mathrm{Ker}(\phi_1)$ which has degree n. Since we have found a Gröbner basis for J, it is possible, as pointed out in [1, page 335], to get a set of generators for the second syzygy module of $\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{m}))$ by Buchberger's algorithm:

Lemma 5.2 (cf. [1, Theorem 15.10]) *Let $S = K[X_1, \ldots, X_n]$ and $g_1, \ldots, g_s$ be a set of minimal generators of a graded ideal I of S. Suppose that $g_1, \ldots, g_s, g_{s+1}, \ldots, g_t$ form a Gröbner basis for I with respect to a term order $<$. Then, for any $1 \leq i < j \leq t$, the S-pair*

$$S(g_i, g_j) := m_{ji}g_i - m_{ij}g_j = \sum_u f_u^{(ij)} g_u, \ \mathrm{in}(f_u^{(ij)} g_u) < \mathrm{in}(m_{ji}g_i),$$

where $m_{ij} = \frac{\mathrm{in}(g_i)}{\gcd(\mathrm{in}(g_i), \mathrm{in}(g_j))}$. Substituting $g_{s+1}, \ldots, g_t$ in the above expressions in terms of $g_1, \ldots, g_s$, one has that

$$m_{ji}g_i - m_{ij}g_j = \sum_{u=1}^{s} h_u^{(ij)} g_u, \ \mathrm{in}(h_u^{(ij)} g_u) < \mathrm{in}(m_{ji}g_i), 1 \leq i < j \leq s.$$

Define an S-homomorphism

$$\phi : \bigoplus_{i=1}^{s} Se_i \to I$$
$$e_i \mapsto g_i.$$

Set $\tau_{ij} = m_{ji}e_i - m_{ij}e_j - \sum_{u=1}^{s} h_u^{(ij)} e_u$. Then the set $\{\tau_{ij} : 1 \leq i < j \leq s\}$ generates $\mathrm{Ker}(\phi)$*, i.e., the first syzygy module of I.*

Notice that, once a set of generators is given as above, some generator τ_{i_0, j_o} is minimal if and only if τ_{i_0, j_o} is not a linear combination of other generators in this set. We will use this idea to find a satisfied minimal generator.

It is clear that every element of $\mathrm{Ker}(\phi_1)$ is a linear combination:

$$\sum_{1 \leq i < j < k \leq n} f_{ijk}e_{ijk}, \ f_{ijk} \in Q[x_1, \ldots, x_n].$$

We call f_{ijk} the coefficient of e_{ijk}, which is a polynomial in variables $x_1, \ldots, x_n$, y_{ij}, $1 \leq i < j \leq n$. Sometimes, we write such a linear combination as $f_{123}e_{123} + \cdots$. Notice that the degree of $f_{ijk}e_{ijk}$ is equal to the degree of f_{ijk} plus two. Therefore, in order to get that

$\text{reg}(\text{Sym}(\text{Syz}_1(\mathfrak{m}))) \geq n-2$, it is enough to find one minimal homogeneous generator of $\text{Ker}(\phi_1)$

$$\sum_{1\leq i<j<k\leq n} f_{ijk}e_{ijk}, \quad f_{ijk} \in Q[x_1, \dots, x_n],$$

where all the nonzero coefficients f_{ijk} are of degree $n-2$. We will find such generators when $n = 4$ or 5 by using Lemma 5.2.

Theorem 5.3 *When* $n = 3$ *or* 4, *or* $n = 5$ *and* $\text{char}(K) \neq 2$,

$$\text{reg}(\text{Sym}(\text{Syz}_1(\mathfrak{m}))) = n-2.$$

Proof We may assume that $n \geq 4$. Define an order on $Q[x_1, \dots, x_n]$ as follows

$$x_n > x_{n-1} > \cdots > x_1 > y_{12} > y_{13} > \cdots > y_{23} > \cdots > y_{n-1,n}.$$

Set $P^{(r)}_{ijkl} = x_r(y_{ij}y_{kl} - y_{ik}y_{jl} + y_{il}y_{jk})$ for $1 \leq i < j < k < l \leq n$, $1 \leq r \leq n$. We will use the same notation for g_{ijk} and $P^{(r)}_{ijkl}$ when i, j, k or i, j, k, l are only different and not necessarily in the above order.

Let us follow the Buchberger's algorithm to get a Gröbner basis from g_{ijk}, $1 \leq i < j < k \leq n$ and then, try to find a minimal first syzygy of degree 4 or 5 when $n = 4$ or 5. The first step is to compute S-pairs $S(g_{ijk}, g_{stl})$ where we may assume that the initial terms of g_{ijk} and g_{stl} are not co-prime. There are two possibilities: $(i, j) = (s, t), k \neq l$ or $(i, j) \neq (s, t), k = l$. In the first case, we need to compute $S(g_{ijk}, g_{ijl})$ with $k < l$. One has

$$\begin{aligned} S(g_{ijk}, g_{ijl}) &= x_l g_{ijk} - x_k g_{ijl} \\ &= -y_{ik}x_jx_l + y_{jk}x_ix_l + y_{il}x_jx_k - y_{jl}x_ix_k \\ &= -x_j g_{ikl} + x_i g_{jkl}, \end{aligned}$$

which induces a generator of the first syzygy module of J:

$$x_l e_{ijk} - x_k e_{ijl} + x_j e_{ikl} - x_i e_{jkl}.$$

Notice that this generator is of degree 3 and with coefficients in variables x_u. We will see that this kind of generators would not appear in the second case. We discuss the second case according to $n = 4$ or 5.

Assume firstly that $n = 4$. Then $S(g_{ijk}, g_{stk})$ with $(i, j) \neq (s, t)$ are all the following

$$\begin{aligned} S(g_{124}, g_{134}) &= y_{13}g_{124} - y_{12}g_{134} = y_{14}g_{123} - P^{(1)}_{1234}, \\ S(g_{124}, g_{234}) &= y_{23}g_{124} - y_{12}g_{234} = y_{24}g_{123} - P^{(2)}_{1234}, \\ S(g_{134}, g_{234}) &= y_{23}g_{134} - y_{13}g_{234} = y_{34}g_{123} - P^{(3)}_{1234}. \end{aligned}$$

It follows that the following elements form a Gröbner basis (cf. Lemma 2.3)

$$P^{(1)}_{1234}, P^{(2)}_{1234}, P^{(3)}_{1234}, g_{ijk}, 1 \leq i < j < k \leq 4.$$

Substituting $P^{(1)}_{1234}$ and $P^{(2)}_{1234}$ in the S-pair $S(P^{(1)}_{1234}, P^{(2)}_{1234}) = x_2P^{(1)}_{1234} - x_1P^{(2)}_{1234}$, one gets that

$$(x_2y_{14} - x_1y_{24})g_{123} + (-x_2y_{13} + x_1y_{23})g_{124} + x_2y_{12}g_{134} - x_1y_{12}g_{234} = 0,$$

which induces to the following degree 4 syzygy:

$$(x_2y_{14} - x_1y_{24})e_{123} + (-x_2y_{13} + x_1y_{23})e_{124} + x_2y_{12}e_{134} - x_1y_{12}e_{234}.$$

It is clear that this syzygy is not a multiple of the unique degree 3 syzygy $x_4e_{123} - x_3e_{124} + x_2e_{134} - x_1e_{234}$. Therefore the above syzygy is minimal. It follows that $\operatorname{reg}(\operatorname{Sym}(\operatorname{Syz}_1(\mathfrak{m}))) = 2$ when $n = 4$.

Now assume that $n = 5$. The possible cases for $S(g_{ijk}, g_{stk})$ with $(i, j) \neq (s, t)$ are the following

$$\begin{aligned}
S(g_{125}, g_{345}) &= y_{34}g_{125} - y_{12}g_{345} = y_{35}g_{124} - y_{45}g_{123} - P^{(2)}_{1345} + P^{(1)}_{2345},\\
S(g_{135}, g_{245}) &= y_{24}g_{135} - y_{13}g_{245} = y_{25}g_{134} + y_{45}g_{123} - P^{(3)}_{1245} - P^{(1)}_{2345},\\
S(g_{145}, g_{235}) &= y_{23}g_{145} - y_{14}g_{235} = -y_{15}g_{234} + y_{45}g_{123} - P^{(3)}_{1245} + P^{(2)}_{1345},
\end{aligned}$$

and, for all $1 \leq i < j < k < l \leq 5$,

$$\begin{aligned}
S(g_{ijl}, g_{ikl}) &= y_{ik}g_{ijl} - y_{ij}g_{ikl} = y_{il}g_{ijk} - P^{(i)}_{ijkl},\\
S(g_{ijl}, g_{jkl}) &= y_{jk}g_{ijl} - y_{ij}g_{jkl} = y_{jl}g_{ijk} - P^{(j)}_{ijkl},\\
S(g_{ikl}, g_{jkl}) &= y_{jk}g_{ikl} - y_{ik}g_{jkl} = y_{kl}g_{ijk} - P^{(k)}_{ijkl}.
\end{aligned}$$

Assume that $\operatorname{char}(K) \neq 2$. Then from the above first three equations, we can solve $P^{(3)}_{1245}$, $P^{(2)}_{1345}$ and $P^{(1)}_{2345}$:

$$\begin{aligned}
P^{(3)}_{1245} &= \frac{1}{2}y_{45}g_{123} + \cdots,\\
P^{(2)}_{1345} &= -\frac{1}{2}y_{45}g_{123} + \cdots,\\
P^{(1)}_{2345} &= -\frac{1}{2}y_{45}g_{123} + \cdots.
\end{aligned}$$

It follows that the following elements form a Gröbner basis (cf. Lemma 2.3)

$$P^{(3)}_{1245}, P^{(2)}_{1345}, P^{(1)}_{2345}, P^{(i)}_{ijkl}, P^{(j)}_{ijkl}, P^{(k)}_{ijkl}, 1 \leq i < j < k < l \leq 5, g_{ijk}, 1 \leq i < j < k \leq 5.$$

Notice that, as a conclusion, there are no syzygies on $\{g_{ijk}\}$ of degree 3 with coefficients in variables y_{uv}.

From

$$\begin{aligned}
S\left(P^{(2)}_{1234}, P^{(2)}_{2345}\right) &= y_{23}y_{45}P^{(2)}_{1234} - y_{12}y_{34}P^{(2)}_{2345}\\
&= (y_{24}y_{35} - y_{25}y_{34})P^{(2)}_{1234} - (y_{13}y_{24} - y_{14}y_{23})P^{(2)}_{2345}
\end{aligned}$$

and substituting $P^{(2)}_{2345} = y_{23}g_{245} - y_{24}g_{235} + y_{25}g_{234}$ and $P^{(2)}_{1234} = y_{24}g_{123} - y_{23}g_{124} + y_{12}g_{234}$, we get a syzygy of degree 5 with coefficients in variables y_{uv}

$$(*) \qquad y_{24}(y_{23}y_{45} - y_{24}y_{35} + y_{25}y_{34})e_{123} + \cdots.$$

We will prove that the above syzygy $(*)$ is minimal. Then $\operatorname{reg}(\operatorname{Sym}(\operatorname{Syz}_1(\mathfrak{m}))) = 3$ follows.

Notice that if the syzygy $(*)$ is not minimal, then $(*)$ should be a linear combination of some degree 4 syzygies whose monomials of the coefficient of e_{123} divide the monomials of $y_{24}(y_{23}y_{45} - y_{24}y_{35} + y_{25}y_{34})$. Let us identify all such possible degree 4 syzygies. We will use the exclusive method.

The possible cases appear only in the S-pairs $S(P^{(r)}_{ijkl}, g_{uvr})$ with $(u, v) \neq (i, j)$ and (k, l) or $S(P^{(r)}_{ijkl}, P^{(r)}_{stuv})$ with $(i, j) = (s, t)$ or $(k, l) = (u, v)$.

For the first case $S(P_{ijkl}^{(r)}, g_{uvr})$, r must be 3 or 4. When $r=3$, there are only two subcases: $S(P_{1345}^{(3)}, g_{123})$ and $S(P_{2345}^{(3)}, g_{123})$. When $r=4$, it is just $S(P_{2345}^{(4)}, g_{234})$. From

$$\begin{aligned} S\left(P_{1345}^{(3)}, g_{123}\right) &= y_{12}P_{1345}^{(3)} - y_{13}y_{45}g_{123} \\ &= (-y_{14}y_{45} + y_{15}y_{34})g_{123} + y_{13}P_{1345}^{(2)} - y_{23}P_{1345}^{(1)}, \end{aligned}$$

we see that this case should be excluded because y_{1i} appears in the coefficients of e_{123}. However, one has that

$$\begin{aligned} S\left(P_{2345}^{(3)}, g_{123}\right) &= y_{12}P_{2345}^{(3)} - y_{23}y_{45}g_{123} \\ &= (-y_{24}y_{35} + y_{25}y_{34})g_{123} + y_{13}P_{2345}^{(2)} - y_{23}P_{2345}^{(1)}, \\ S\left(P_{2345}^{(4)}, g_{124}\right) &= y_{12}P_{2345}^{(4)} - y_{23}y_{45}g_{124} \\ &= (-y_{24}y_{35} + y_{25}y_{34})g_{124} + y_{14}P_{2345}^{(2)} - y_{24}P_{2345}^{(1)}, \end{aligned}$$

from which we get two syzygies:

$$\begin{aligned} &\left(\frac{1}{2}y_{23}y_{45} - y_{24}y_{35} + y_{25}y_{34}\right)e_{123} + \cdots, \\ &\frac{1}{2}y_{24}y_{45}e_{123} + \cdots. \end{aligned}$$

For the second case $S(P_{ijkl}^{(r)}, P_{stuv}^{(r)})$ with $(i,j)=(s,t)$ or $(k,l)=(u,v)$, there are six possibilities: $S(P_{1245}^{(r)}, P_{1345}^{(r)})$ with $r \le 4$, $S(P_{1235}^{(r)}, P_{1245}^{(r)})$ with $r \le 3$, $S(P_{1234}^{(r)}, P_{1245}^{(r)})$ with $r \le 3$, $S(P_{1234}^{(r)}, P_{1235}^{(r)})$ with $r \le 3$, $S(P_{1345}^{(r)}, P_{2345}^{(r)})$ with $r \le 4$ and $S(P_{1245}^{(r)}, P_{2345}^{(r)})$ with $r \le 4$.

Since

$$S\left(P_{1245}^{(r)}, P_{1345}^{(r)}\right) = y_{13}P_{1245}^{(r)} - y_{12}P_{1345}^{(r)} = y_{14}P_{1235}^{(r)} - y_{15}P_{1234}^{(r)},$$

and its coefficients are all with some y_{1s}, we see immediately that this possibility is excluded. Similarly for

$$S\left(P_{1235}^{(r)}, P_{1245}^{(r)}\right) = y_{45}P_{1235}^{(r)} - y_{35}P_{1245}^{(r)} = y_{15}P_{2345}^{(r)} - y_{25}P_{1345}^{(r)},$$

the coefficients of g_{123} in the explains of $y_{45}P_{1235}^{(r)}$, $y_{35}P_{1245}^{(r)}$ and $y_{25}P_{1345}^{(r)}$ do not divide any monomials of $y_{24}(y_{23}y_{45} - y_{24}y_{35} + y_{25}y_{34})$ in any cases; this possibility should also be excluded. Now consider $S(P_{1234}^{(r)}, P_{1245}^{(r)})$ with $r \le 3$. In the equality

$$S\left(P_{1234}^{(r)}, P_{1245}^{(r)}\right) = y_{45}P_{1234}^{(r)} - y_{34}P_{1245}^{(r)} = -y_{24}P_{1345}^{(r)} + y_{14}P_{2345}^{(r)},$$

only when $r=2$, $y_{45}P_{1234}^{(r)}$ contains $y_{24}y_{45}g_{123}$ and $y_{24}P_{1345}^{(r)}$ contains $-\frac{1}{2}y_{24}y_{45}g_{123}$. Thus, in this possibility, one gets only one required syzygy: $\frac{1}{2}y_{24}y_{45}e_{123} + \cdots$.

For the two possibilities that $S(P_{1234}^{(r)}, P_{1235}^{(r)})$ with $r \le 3$ and $S(P_{1345}^{(r)}, P_{2345}^{(r)})$ with $r \le 4$, in the equalities

$$\begin{aligned} S\left(P_{1234}^{(r)}, P_{1235}^{(r)}\right) &= y_{35}P_{1234}^{(r)} - y_{34}P_{1235}^{(r)} = -y_{23}P_{1345}^{(r)} + y_{13}P_{2345}^{(r)}, \\ S\left(P_{1345}^{(r)}, P_{2345}^{(r)}\right) &= y_{23}P_{1345}^{(r)} - y_{13}P_{2345}^{(r)} = -y_{35}P_{1234}^{(r)} + y_{34}P_{1235}^{(r)}, \end{aligned}$$

only when $r = 2$, $y_{23}P_{1345}^{(r)}$ has $-\frac{1}{2}y_{23}y_{45}g_{123}$, $y_{35}P_{1234}^{(r)}$ has $y_{24}y_{35}g_{123}$, and $y_{34}P_{1235}^{(r)}$ has $y_{25}y_{34}g_{123}$. It turns out, in these two possibilities, there is only one required syzygy: $(-\frac{1}{2}y_{23}y_{45} + y_{24}y_{35} - y_{25}y_{34})e_{123} + \cdots$.

Finally, for $S(P_{1245}^{(r)}, P_{2345}^{(r)})$ with $r \leq 4$, in the equality

$$S\left(P_{1245}^{(r)}, P_{2345}^{(r)}\right) = y_{23}P_{1245}^{(r)} - y_{12}P_{2345}^{(r)} = -y_{25}P_{1234}^{(r)} + y_{24}P_{1235}^{(r)},$$

only when $r = 3$, $y_{23}P_{1245}^{(r)}$ has $\frac{1}{2}y_{23}y_{45}g_{123}$; when $r = 2$, $y_{25}P_{1234}^{(r)}$ has $y_{24}y_{25}g_{123}$, and when $r = 3$, $y_{25}P_{1234}^{(r)}$ has $y_{25}y_{34}g_{123}$; when $r = 2$, $y_{24}P_{1235}^{(r)}$ has $y_{24}y_{25}g_{123}$, and when $r = 3$, $y_{24}P_{1235}^{(r)}$ has $y_{24}y_{35}g_{123}$. Therefore, only when $r = 3$, there is a syzygy $(\frac{1}{2}y_{23}y_{45} - y_{24}y_{35} + y_{25}y_{34})e_{123} + \cdots$.

To summarize the results obtained, there are only two degree 4 syzygies whose monomials of the coefficient of e_{123} divide the monomials of $y_{24}(y_{23}y_{45} - y_{24}y_{35} + y_{25}y_{34})$:

$$\left(\frac{1}{2}y_{23}y_{45} - y_{24}y_{35} + y_{25}y_{34}\right)e_{123} + \cdots$$

and

$$\frac{1}{2}y_{24}y_{45}e_{123} + \cdots .$$

It is clear that they cannot generate $y_{24}(y_{23}y_{45} - y_{24}y_{35} + y_{25}y_{34})e_{123} + \cdots$. Therefore the syzygy $(*)$ is minimal, as required. □

Remark 5.4 When $n = 3$, the equality $\text{reg}(\text{Sym}(\text{Syz}_1(\mathfrak{m}))) = 1$ can be easily seen because $\text{Sym}(\text{Syz}_1(\mathfrak{m})) = Q[x_1, x_2, x_3]/(x_1y_{23} - x_2y_{13} + x_3y_{12})$ and $x_1y_{23} - x_2y_{13} + x_3y_{12}$ is homogeneous of degree two.

On the other hand, when $n = 6$, one might hope naturally to find one degree 6 second syzygy by using the same method as above. Unfortunately it is impossible because, in this case, we can find a set of generators of degree at most 5 for the first syzygy of elements $\{g_{ijk}\}$ by using the computer algebra system CoCoA. Therefore, to get an equality for the regularity of $\text{Sym}(\text{Syz}_1(\mathfrak{m}))$ in the case $n \geq 6$, one might need to find a satisfied third syzygy of $\text{Sym}(\text{Syz}_1(\mathfrak{m}))$, which is challenging.

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