Annali di Matematica (2018) 197:1799–1819
https://doi.org/10.1007/s10231-018-0750-z

Global-in-time existence results for the two-dimensional Hasegawa–Wakatani equations

Shintaro Kondo[1]

Received: 10 November 2017 / Accepted: 18 April 2018 / Published online: 25 April 2018

Abstract In order to describe the resistive drift wave turbulence appearing in nuclear fusion plasma, the Hasegawa–Wakatani (HW) equations were proposed in 1983. We consider the two-dimensional HW equations, which have numerous structures (that is, they explain the branching phenomenon in turbulent and zonal flow in a two-dimensional plasma) and the generalized HW equations that include temperature fluctuation. We prove the global-in-time existence of a unique strong solution to both the HW equations and the generalized HW equations in a two-dimensional domain with double periodic boundary conditions.

Keywords Hasegawa–Wakatani equations · Drift wave turbulence · Zonal flow · Sobolev–Slobodetskiĭ spaces

Mathematics Subject Classification Primary 76W05; Secondary 35K45 · 35Q60 · 82D10

1 Introduction

We consider that a strong homogeneous magnetic field is added to a plasma in the x_3 direction, $\mathbf{e} = (0, 0, 1)$, in three-dimensional space $[X = (x_1, x_2, x_3) = (x', x_3)]$. Then, the electrostatic field $\mathbf{E}(X, t) = (E_1, E_2, E_3)(X, t)$ in the plasma can be described by the electrostatic potential $\phi(X, t)$, which satisfies $\mathbf{E}(X, t) = -\nabla\phi(X, t)$, where $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$. In this case, there is a large difference between the velocities of electrons and ions in the x_3 direction; hence, it is important to consider the current density $j = j(X, t)$ in the x_3 direction. Ohm's law for the x_3 direction is written as

$$\eta j = E_3 = -\frac{\partial\phi}{\partial x_3},$$

✉ Shintaro Kondo
skondo@gifu-u.ac.jp

1 Department of Electrical, Electronic and Computer Engineering, Gifu University, 1-1 Yanagido, Gifu City, Gifu 501-1193, Japan

where the resistivity η is positive constant. This relation can be generalized by adding a density fluctuation $n = n(X, t)$ as follows:

$$\eta j = -\frac{\partial}{\partial x_3}(\phi - n); \tag{1.1}$$

this equation is called the generalized Ohm's law.

For the following three-dimensional Hasegawa–Wakatani (HW) equations [16,17] for an inhomogeneous plasma equilibrium density $n^* = n^*(|x'|)$ (given function), a density fluctuation $n = n(X, t)$ (unknown function), and an electrostatic potential $\phi = \phi(X, t)$ (unknown function), we have several mathematical results [25–31]:

$$\begin{cases} \left(\frac{\partial}{\partial t} - (\nabla\phi \times \mathbf{e}) \cdot \nabla\right) \Delta\phi = \frac{\beta}{n^*}\frac{\partial j}{\partial x_3} + D_1\Delta^2\phi, \\ \left(\frac{\partial}{\partial t} - (\nabla\phi \times \mathbf{e}) \cdot \nabla\right) (n + \log n^*) = \frac{\beta}{n^*}\frac{\partial j}{\partial x_3}, \end{cases} \tag{1.2}$$

where j is given by (1.1), D_1 is a positive constant that is proportional to the kinematic ion viscosity coefficient, β is a positive constant, and $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2$. For two variables f and g, the following holds:

$$-(\nabla f \times \mathbf{e}) \cdot \nabla g = \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_1}.$$

Here, $v = -(\nabla\phi \times \mathbf{e})$ is called the electric field drift velocity (see, [18]). Setting $c_1 = \beta/\eta$ and inserting (1.1) into (1.2), we have

$$\begin{cases} \left(\frac{\partial}{\partial t} - (\nabla\phi \times \mathbf{e}) \cdot \nabla\right) \Delta\phi = -\frac{c_1}{n^*}\frac{\partial^2}{\partial x_3^2}(\phi - n) + D_1\Delta^2\phi, \\ \left(\frac{\partial}{\partial t} - (\nabla\phi \times \mathbf{e}) \cdot \nabla\right) (n + \log n^*) = -\frac{c_1}{n^*}\frac{\partial^2}{\partial x_3^2}(\phi - n). \end{cases} \tag{1.3}$$

In [25–27], we consider the initial boundary value problem for (1.3) in a cylindrical domain Ω with the following boundary conditions:

$$\begin{cases} \phi(X, t) = \Delta\phi(X, t) = n(X, t) = 0 \quad \text{for } x \in \Gamma, \\ \phi,\ n, \quad \text{periodic in the } x_3\text{-direction.} \end{cases} \tag{1.4}$$

Here $\Omega = \omega \times (-L, L)$, $\omega = \{x' = (x_1, x_2) \in \mathbf{R}^2 \mid |x'| < R\}$, $\partial\omega = \{x' = (x_1, x_2) \in \mathbf{R}^2 \mid |x'| = R\}$, and $\Gamma = \partial\omega \times [-L, L]$. It is assumed that the ions are singly ionized, and hence the densities of electrons n_e and ions n_i satisfy the Poisson equation, $-\Delta\phi = e(n_i - n_e)/\varepsilon_0$, where e is the elementary charge, and ε_0 is the permittivity of vacuum (see, [36]). Thus, the boundary condition $\Delta\phi(X, t) = 0$ states that the densities of electrons and ions are equal at the boundary. Here, we consider that the boundary of the plasma is away from the wall of the container. The existence and uniqueness of a strong solution to the initial-boundary-value problems for (1.3) and (1.4) were proven when the initial data are periodic in the x_3 direction [25]. When the temperature of the plasma is very high, the resistivity of the plasma approaches zero; therefore, it is important for nuclear fusion plasma research to consider the case of zero resistivity. In [26,27], we proved that as the resistivity tends to zero, the solution of (1.3) established in [25] converges strongly to that of the model equations of drift wave turbulence with zero resistivity. Note that (1.3) with zero resistivity under the additional condition that the mean value of n for x_3 is zero is similar to the Hasegawa–Mima equation with a higher-order correction term [see, (1.7)]. In nuclear fusion research, it is important to consider an irrational magnetic surface on which the line of force covers the surface ergodically without closing [46]. However, research into plasma phenomena in an irrational magnetic surface is difficult; therefore, we consider the following simple problem as the first step in researching plasma phenomena in a tokamak. In [28–31], we consider (1.3) in a cylindrical domain with almost-periodic initial data in the x_3 direction. The existence and uniqueness of a strong solution to the initial-boundary-value problems for (1.3) in a cylindrical domain were proven when the initial data are Stepanov almost-periodic [28]. In [29], we obtained two useful lemmas for Stepanov almost-periodic functions for the purpose of obtaining uniform *a priori* estimates for resistivity; additionally, we proved that the Stepanov almost-periodic solution of linearized HW equations converges strongly to that of linearized HW equations with zero resistivity as the resistivity tends to zero when the initial data are Stepanov almost-periodic. In [30], we used the lemmas presented in [29] to prove that the Stepanov almost-periodic solutions of (1.3) established in [28] converge strongly to that of (1.3) with zero resistivity under the additional condition that the mean value of n for x_3 is zero as the resistivity tends to zero. There is also a mathematical result [47] related to the HW equations.

If we formally replace the terms $\Delta\phi$, $\Delta^2\phi$, and $-c_1/n^*\partial^2(\phi-n)/\partial x_3^2$ in (1.3) with $\Delta_\perp\phi$, $\Delta_\perp^2\phi$, and $\alpha(\phi-n)$, respectively, we have the two-dimensional HW equations [16,17]:

$$\begin{cases} \dfrac{\partial\Delta_\perp\phi}{\partial t} + \{\phi, \Delta_\perp\phi\} = \alpha(\phi-n) + D_1\Delta_\perp^2\phi, \\ \dfrac{\partial n}{\partial t} + \left\{\phi, n + \log n^*\right\} = \alpha(\phi-n), \end{cases} \tag{1.5}$$

for $n = n(x,t)$ and $\phi = \phi(x,t)$. Here $x = (x_1, x_2)$, $\Delta_\perp = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$; α is a positive constant that can be written as $\alpha = \alpha'/\eta$, where η is the resistivity and α' is a positive constant independent of η (see, [25]). Here, $\{\cdot,\cdot\}$ denotes the Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_1}.$$

Let consider the two-dimensional domain $\mathbf{T}^2 = \{x = (x_1, x_2) |\, 0 < x_1 < L_1,\ 0 < x_2 < L_2\}$ with double periodic boundary conditions, where L_1 and L_2 are positive numbers. The periodic boundary condition for x_2 is natural, and that for x_1 is provided for simplicity. We assume that $n^* = n^*(x_1)$ is given by $\kappa = -\partial \log n^*/\partial x_1$, where κ is a positive constant. Then, (1.5) becomes

$$\begin{cases} \dfrac{\partial\Delta_\perp\phi}{\partial t} + \{\phi, \Delta_\perp\phi\} = \alpha(\phi-n) + D_1\Delta_\perp^2\phi, \\ \dfrac{\partial n}{\partial t} + \{\phi, n\} = \alpha(\phi-n) - \kappa\dfrac{\partial\phi}{\partial x_2}. \end{cases} \tag{1.6}$$

When we take the hydrodynamic limit $\alpha \to 0$, the first equation of (1.6) becomes the equation for the vorticity $\Delta_\perp\phi$ of the two-dimensional incompressible Navier–Stokes equation. When we take the adiabatic limit $\alpha \to \infty$, (1.6) become the Hasegawa–Mima (HM) equation [14,15]:

$$\frac{\partial(\Delta_\perp\phi - \phi)}{\partial t} + \{\phi, \Delta_\perp\phi\} = \kappa\frac{\partial\phi}{\partial x_2} + D_1\Delta_\perp^2\phi. \tag{1.7}$$

Here, taking the adiabatic limit $\alpha \to \infty$ means taking the limit $\eta \to 0$, that is, neglecting the effect of the resistivity. The HM equation has a dipolar vortex solution, which is called modon [19,33]. In a study of plasma turbulence, coherent vortex is an important research topic, since plasma turbulence may produce self-organized structures in the form of vortices, and indeed coherent vortices are observed in a variety of contents (see, for example, [24,39,44]). It is noteworthy that the same equation can be found in geophysics, Charney–Obukhov equation with respect to the quasi-geostrophic potential vorticity for Rossby wave [7,35,38,41]. For the HM equation, we have had some mathematical results. For the initial value problem, the temporally local existence and uniqueness of the strong solution and the temporally global existence of the weak solution were proved by Guo and Han [13] and Paumond [40] independently in 2004, and the global existence of a strong solution was proved by Gao and Zhu [10] in 2005. The global-in-time existence and uniqueness of the solution and the existence of a global attractor to the initial boundary value problem for the generalized HM equation with periodic boundary condition were proved by Zhang and Guo for the two-dimensional case [48] and the three-dimensional case [49]. In 2008, Hounkonnou and Kabir [20] investigated symmetry reductions and exact solutions for the HM equation. There are also mathematical results [4,5,12] related to the HM equation.

It is considered that drift wave turbulence is a natural cause of anomalous transport in plasma, and drift wave turbulence is suppressed through zonal flow generation [see (1.9)]. Therefore, zonal flow generation is an important phenomena in plasma physics. In 2007, Numata et al. [37] pointed out that the zonal components $\langle\phi\rangle$ and $\langle n\rangle$ are independent of x_3; hence, the relation $\partial^2(\phi-n)/\partial x_3^2 = \partial^2(\widetilde{\phi}-\widetilde{n})/\partial x_3^2$ holds. Here, the zonal and nonzonal components of the variable f are defined as

$$\text{zonal component}: \quad \langle f\rangle = \frac{1}{L_2}\int_0^{L_2} f(x)\,\mathrm{d}x_2,$$
$$\text{nonzonal component}: \quad \widetilde{f} = f - \langle f\rangle.$$

Therefore, they insisted that we should replace the term $-c_1/n^*\partial^2(\phi-n)/\partial x_3^2$ with $\alpha(\widetilde{\phi}-\widetilde{n})$ when we obtain two-dimensional HW equations. If we formally replace the terms $\Delta\phi$, $\Delta^2\phi$, and $-c_1/n^*\partial^2(\phi-n)/\partial x_3^2$ in (1.3) with $\Delta_\perp\phi$, $\Delta_\perp^2\phi$, and $\alpha(\widetilde{\phi}-\widetilde{n})$, respectively, we have the following two-dimensional HW equations:

$$\begin{cases} \dfrac{\partial\Delta_\perp\phi}{\partial t} + \{\phi,\Delta_\perp\phi\} = \alpha\left(\widetilde{\phi}-\widetilde{n}\right) + D_1\Delta_\perp^2\phi, \\ \dfrac{\partial n}{\partial t} + \{\phi, n\} = \alpha\left(\widetilde{\phi}-\widetilde{n}\right) - \kappa\dfrac{\partial\phi}{\partial x_2} \quad \text{for } x\in\mathbf{T}^2,\ t>0. \end{cases} \tag{1.8}$$

Although in [37], (1.6) and (1.8) are called the HW equations and the modified HW equations, respectively, in this paper, we will refer to both (1.6) and (1.8) as the two-dimensional HW equations. As far as the author knows, there are no mathematical results for (1.6) and (1.8). In this paper, we aim to prove the global-in-time existence of a unique strong solution to (1.6) and (1.8); however, before introducing the main theorems, we will further explain (1.8).

Numata et al. [37] pointed out that (1.8) with $D_1=0$ has a trivial solution $(\phi,n)=(0,0)$ and a zonal flow solution

$$(\phi,n)=(\phi_0,0),\quad \phi_0 = -\frac{V_0\cos(\lambda x_1)}{\lambda},$$

where V_0 and λ are positive constants. As the vorticity is $\Delta_\perp\phi_0$, the zonal flow velocity is given by

$$v = \left(-\frac{\partial\phi_0}{\partial x_2}, \frac{\partial\phi_0}{\partial x_1}\right) = (0, V_0\sin(\lambda x_1)). \tag{1.9}$$

They showed that (1.8) has numerous structures; that is, it explains the branching behavior of turbulent and zonal flow. Specifically, they presented a bifurcation diagram with respect

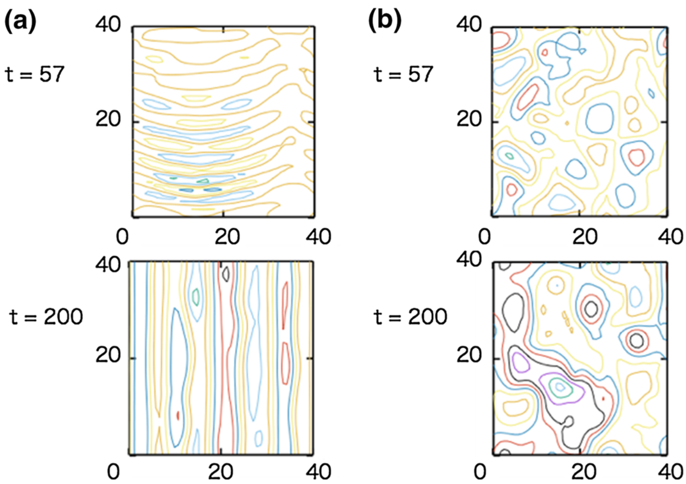

Fig. 1 Contour plots of ϕ for cases **a** ($D_1 = D_2 = 0.01,\ \kappa = 1.0$, and $\alpha = 4.0$), and **b** ($D_1 = D_2 = 0.01,\ \kappa = 1.0$, and $\alpha = 0.1$). Horizontal and vertical axes represent x_1 and x_2, respectively

to κ and α by combining linear stability analysis of the trivial and zonal flow solutions and numerical simulations for (1.8) with the hyperviscosity terms (see Figs. 3 and 11 in [37]). We show numerical simulations for the following equations:

$$\begin{cases} \dfrac{\partial \Delta_\perp \phi}{\partial t} + \{\phi, \Delta_\perp \phi\} = \alpha\left(\widetilde{\phi} - \widetilde{n}\right) + D_1 \Delta_\perp^2 \phi, \\ \dfrac{\partial n}{\partial t} + \{\phi, n\} = \alpha\left(\widetilde{\phi} - \widetilde{n}\right) - \kappa \dfrac{\partial \phi}{\partial x_2} + D_2 \Delta_\perp n \quad \text{for } x \in \mathbf{T}^2,\ t > 0, \end{cases} \tag{1.10}$$

where D_2 is a positive constant that is proportional to the collision frequency of electrons [42]. Details of the numerical simulation are shown in "Appendix" (see, [23,32,43]). In case (a) ($D_1 = D_2 = 0.01,\ \kappa = 1.0$, and $\alpha = 4.0$), drift waves appear owing to the linear instability of the trivial solution (Fig. 1, (a) $t = 57$). Here, a drift wave is a wave that propagates in the direction perpendicular to the equilibrium density gradient and the uniform magnetic field direction. After enough time has passed, zonal flow is generated (Fig. 1, (a) $t = 200$). This behavior is called self-organization of zonal flow. In case (a), the drift waves are well aligned; however, in case (b) ($D_1 = D_2 = 0.01,\ \kappa = 1.0$, and $\alpha = 0.1$), they do not persist, and a state of turbulence occurs immediately (Fig. 1, (b) $t = 57$). After enough time has passed, plasma turbulence appears (Fig. 1, (b) $t = 200$). It is called drift wave turbulence.

The aim of this paper is to establish the global-in-time existence of a unique strong solution in Sobolev spaces to the initial value problem for (1.8) under the conditions

$$\begin{cases} \phi(x,0) = \phi_0(x),\ n(x,0) = n_0(x) \quad \text{for } x \in \mathbf{T}^2, \\ \phi,\ n \quad \text{are periodic in the } x_i \text{ direction} \quad (i = 1, 2). \end{cases} \tag{1.11}$$

The definition of a Sobolev space is shown in Sect. 2. Our first result for the problem (1.8) and (1.11) is as follows.

Theorem 1 *Let D_1, α, and κ be positive constants. Assume that $(\phi_0, n_0) \in W_2^4(\mathbf{T}^2) \times W_2^2(\mathbf{T}^2)$ satisfies (1.11). Then, there exists a unique solution (ϕ, n) to the problem (1.8) and (1.11) in any time interval $[0, T]$ $(0 < T < \infty)$ such that $(\phi, n) \in \Big(L^2(0, T; W_2^4(\mathbf{T}^2)) \cap W_2^1(0, T; W_2^2(\mathbf{T}^2))\Big) \times W_2^{2,1}(Q_T)$. Here $Q_T \equiv \mathbf{T}^2 \times (0, T)$.*

Remark 1 (i) $\phi \in W_2^1(0, T; W_2^2(\mathbf{T}^2))$ means that

$$D_x^\alpha \phi \in L^2\left(\mathbf{T}^2; W_2^1(0, T)\right) \quad \text{where } |\alpha| = 0, 1, 2.$$

(ii) For the problem (1.6) and (1.11), the same result as Theorem 1 holds. For example, Lemma 1, which appears in the proof of Theorem 1, holds true if we replace $\alpha \int_0^t \|(\widetilde{\phi} - \widetilde{n})(\tau)\|^2 \,\mathrm{d}\tau$ with $\alpha \int_0^t \|(\phi - n)(\tau)\|^2 \,\mathrm{d}\tau$.

Next, we consider the generalized model of (1.8) with an inhomogeneous equilibrium temperature $T^* = T^*(x_1)$ (given function) and a temperature fluctuation $T = T(x, t)$ (unknown function) (see, [6,32,45]):

$$\begin{cases} \dfrac{\partial \Delta_\perp \phi}{\partial t} + \{\phi, \Delta_\perp \phi\} = \alpha\left(\widetilde{\phi} - \widetilde{n} - (1+\gamma)\widetilde{T}\right) + D_1 \Delta_\perp^2 \phi, \\ \dfrac{\partial n}{\partial t} + \{\phi, n\} = \alpha\left(\widetilde{\phi} - \widetilde{n} - (1+\gamma)\widetilde{T}\right) - \kappa \dfrac{\partial \phi}{\partial x_2}, \\ \dfrac{\partial T}{\partial t} + \{\phi, T\} = \chi\alpha\left(\widetilde{\phi} - \widetilde{n} - (1+\gamma)\widetilde{T}\right) - \kappa' \dfrac{\partial \phi}{\partial x_2} + D_3 \Delta_\perp T, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } x \in \mathbf{T}^2,\ t > 0, \end{cases} \tag{1.12}$$

where $\kappa' = -\partial \log T^*/\partial x_1$, γ, and χ are positive constants. The term $\alpha(\widetilde{\phi} - \widetilde{n} - (1+\gamma)\widetilde{T})$ represents a current term that arises from the generalized Ohm's law.

This paper also aims to establish the global-in-time existence of a unique strong solution in Sobolev spaces to the initial value problem for (1.12) under the conditions

$$\begin{cases} \phi(x, 0) = \phi_0(x),\ n(x, 0) = n_0(x),\ T(x, 0) = T_0(x) \quad \text{for } x \in \mathbf{T}^2, \\ \phi,\ n,\ T \quad \text{are periodic in the } x_i\text{-direction } (i = 1, 2). \end{cases} \tag{1.13}$$

Our second result for the problem (1.12) and (1.13) is as follows.

Theorem 2 *Let $D_1, D_3, \alpha, \kappa, \kappa', \gamma, \chi$ be positive constants. Assume that $(\phi_0, n_0, T_0) \in W_2^4(\mathbf{T}^2) \times W_2^2(\mathbf{T}^2) \times W_2^2(\mathbf{T}^2)$ satisfies (1.13). Then, there exists a unique solution (ϕ, n, T) to the problem (1.12) and (1.13) in any time interval $[0, T]$ $(0 < T < \infty)$ such that $(\phi, n, T) \in L^2(0, T; W_2^4(\mathbf{T}^2)) \times W_2^{2,1}(Q_T) \times W_2^{2,1}(Q_T)$, $\partial\phi/\partial t \in L^2(0, T; W_2^2(\mathbf{T}^2))$. Here $Q_T \equiv \mathbf{T}^2 \times (0, T)$.*

This paper is organized as follows. In Sect. 2, we present preliminary results. In Sect. 3, we prove Theorem 1. Since the second equation of (1.8) does not include the diffusion term of n, the existence theorem is proved according to the following procedure. First, we establish the global-in-time existence of a unique strong solution to the problem (1.10) and (1.11). Next, with the help of the uniform estimate for the solution with respect to D_2, by passing to the limit $D_2 \to 0$, we establish the global-in-time existence of a unique strong solution to the problem (1.8) and (1.11). In Sect. 4, we prove Theorem 2. First, we consider (4.1) which includes the diffusion term $D_2 \Delta_\perp n$, and establish the global-in-time existence of a unique strong solution to the problem (4.1) and (1.13). Next, by the similar way as the proof of Theorem 1, we establish the global-in-time existence of a unique strong solution to the problem (1.12) and (1.13). The procedures for proving Theorems 1 and 2 are similar to that in [25], in which we prove the existence of a unique strong solution in some time interval for the initial boundary value problems for (1.3) and (1.4).

2 Preliminaries

First, we recall function spaces and some notation. Let $\mathbf{T}^2 = \{x = (x_1, x_2) |\ 0 < x_1 < L_1,\ 0 < x_2 < L_2\}$. Here L_1 and L_2 are positive numbers.

- *L^p space* $L^p(\mathbf{T}^2)$ denotes the space of functions $u(x),\ x \in \mathbf{T}^2$, equipped with the finite norm

$$\|u\|_p = \|u\|_{L^p(\mathbf{T}^2)} = \left(\int_{\mathbf{T}^2} |u(x)|^p \,\mathrm{d}x\right)^{1/p}$$

for $1 \le p \le \infty$, and

$$\|u\|_\infty = \|u\|_{L^\infty(\mathbf{T}^2)} = \sup_{x \in \mathbf{T}^2} |u(x)|$$

for $p = \infty$. For simplicity, $\|\cdot\|$ is used instead of $\|\cdot\|_{L^2(\mathbf{T}^2)}$.

- *Sobolev space* $W_2^l(\mathbf{T}^2)$ $(l = 0, 1, 2\ldots)$ denotes the space of functions $u(x),\ x \in \mathbf{T}^2$, equipped with the finite norm

$$\|u\|^2_{W_2^l(\mathbf{T}^2)} = \sum_{|\alpha| \le l} \left\| D_x^\alpha \right\|^2_{L^2(\mathbf{T}^2)} .$$

Here $D_x^\alpha u = \partial^{|\alpha|} u / \partial x_1^{\alpha_1} \partial x_2^{\alpha_2}$ is the generalized derivative of order $|\alpha| = \alpha_1 + \alpha_2$, and $\alpha = (\alpha_1, \alpha_2)$ is a multi-index.

Similarly, the norm of the space $W_2^l(0, T)$ $(T \in \mathbf{R},\ T > 0)$ is defined as

$$\|u\|^2_{W_2^l(0,T)} = \sum_{j=0}^{l} \left\| D_t^j u \right\|^2_{L^2(0,T)} .$$

The anisotropic Sobolev space $W_2^{2l,l}(Q_T)$ $(Q_T \equiv \mathbf{T}^2 \times (0, T))$ is defined as $L^2(0, T; W_2^{2l}(\mathbf{T}^2)) \cap L^2(\mathbf{T}^2; W_2^l(0, T))$, equipped with the finite norm

$$\begin{aligned} \|u\|^2_{W_2^{2l,l}(Q_T)} &= \|u\|^2_{W_2^{2l,0}(Q_T)} + \|u\|^2_{W_2^{0,l}(Q_T)} \\ &\equiv \int_0^T \|u\|^2_{W_2^{2l}(\mathbf{T}^2)} \,\mathrm{d}t + \int_{\mathbf{T}^2} \|u\|^2_{W_2^l(0,T)} \,\mathrm{d}x. \end{aligned}$$

- *Sobolev–Slobodetskiĭ space* Sobolev–Slobodetskiĭ space is a generalization of Sobolev space $W_2^l(\mathbf{T}^2)$ in the case of the real number of the exponent of the derivatives l. $W_2^l(\mathbf{T}^2)$ $(l \in \mathbf{R},\ l \ge 0)$ denotes the space of functions $u(x),\ x \in \mathbf{T}^2$, equipped with the finite norm

$$\|u\|^2_{W_2^l(\mathbf{T}^2)} = \sum_{|\alpha| < l} \left\| D_x^\alpha \right\|^2_{L^2(\mathbf{T}^2)} + \|u\|^2_{\dot{W}_2^l(\mathbf{T}^2)},$$

where

$$\|u\|^2_{\dot{W}^l_2(\mathbf{T}^2)} = \begin{cases} \sum_{|\alpha|=l} \left\| D^\alpha_x u \right\|^2_{L^2(\mathbf{T}^2)} & \text{if } l \in \mathbf{Z}, \\ \sum_{|\alpha|=[l]} \int_{\mathbf{T}^2} \int_{\mathbf{T}^2} \frac{\left| D^\alpha_x u(x) - D^\alpha_y u(y) \right|^2}{|x-y|^{1+2(l-[l])}} \, \mathrm{d}x \, \mathrm{d}y & \text{if } l \notin \mathbf{Z}. \end{cases}$$

Here $[l]$ is the integral part of l.

Similarly, the norm of the space $W^l_2(0, T)$ $(T \in \mathbf{R},\ T > 0)$ is defined as

$$\|u\|^2_{W^l_2(0,T)} = \begin{cases} \sum_{j=0}^{l} \left\| D^j_t u \right\|^2_{L^2(0,T)} & \text{if } l \in \mathbf{Z}, \\ \sum_{j=0}^{l} \left\| D^j_t u \right\|^2_{L^2(0,T)} + \int_0^T \mathrm{d}t \int_0^t \frac{\left| D^{[l]}_t u(t) - D^{[l]}_\tau u(\tau) \right|^2}{|t-\tau|^{1+2(l-[l])}} \, \mathrm{d}\tau & \text{if } l \notin \mathbf{Z}. \end{cases}$$

The anisotropic Sobolev–Slobodetskiĭ space $W^{l,l/2}_2(Q_T)$ $(Q_T \equiv \mathbf{T}^2 \times (0, T))$ is defined as $L^2(0, T; W^l_2(\mathbf{T}^2)) \cap L^2(\mathbf{T}^2; W^{l/2}_2(0, T))$, equipped with the finite norm

$$\begin{aligned} \|u\|^2_{W^{l,l/2}_2(Q_T)} &= \|u\|^2_{W^{l,0}_2(Q_T)} + \|u\|^2_{W^{0,l/2}_2(Q_T)} \\ &\equiv \int_0^T \|u\|^2_{W^l_2(\mathbf{T}^2)} \, \mathrm{d}t + \int_{\mathbf{T}^2} \|u\|^2_{W^{l/2}_2(0,T)} \, \mathrm{d}x. \end{aligned}$$

Next, we recall a well-known lemma and several inequalities. In the following, we write $\nabla = \nabla_\perp$ and $\Delta = \Delta_\perp$, and the function $u(x)$, $x \in \mathbf{T}^2$ is periodic in the x_i direction $(i = 1, 2)$.

- *Gronwall's lemma* The functions $f(t)$ and $\psi(t)$, $t \in [0, \infty)$, are continuous; $\psi \geq 0$, c_0 is a constant, and

$$f(t) \leq c_0 + \int_0^t \psi(\tau) f(\tau) \, \mathrm{d}\tau, \quad t \geq 0$$

holds. Then

$$f(t) \leq c_0 \exp\left(\int_0^t \psi(\tau) \, \mathrm{d}\tau \right), \quad t \geq 0 \tag{2.1}$$

holds.

- *Young's inequality* [11] For any positive constants a, b, p, and q satisfying $1/p + 1/q = 1$,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{2.2}$$

holds.

- *Poincaré inequality* Let u satisfy $\int_{\mathbf{T}^2} u(x) \, \mathrm{d}x = 0$. For some positive constant c,

$$\|u\| \le c\|\nabla u\| \tag{2.3}$$

holds.

- *Schwarz's inequality*

$$\left|\int_{\mathbf{T}^2} f(x)g(x)\,\mathrm{d}x\right| \le \|f\|\|g\| \quad \text{for} \ \ f,\ g \in L^2\left(\mathbf{T}^2\right) \tag{2.4}$$

holds.

- *Gagliardo–Nirenberg inequalities* [9] For some positive constant c,

$$\|u\|_\infty \le c\|u\|^{1/2}_{W_2^2(\mathbf{T}^2)}\|u\|^{1/2} \tag{2.5}$$

holds. For some positive constant c,

$$\|\nabla u\|_4 \le c\|u\|^{3/4}_{W_2^2(\mathbf{T}^2)}\|u\|^{1/4} \tag{2.6}$$

holds.

- *Sobolev embedding theorem* [1] For some positive constant c,

$$\|u\|_\infty \le c\|u\|_{W_2^2(\mathbf{T}^2)} \tag{2.7}$$

holds.

- *Elliptic estimates* [2] Let u satisfy $\int_{\mathbf{T}^2} u(x)\,\mathrm{d}x = 0$. For some positive constant c,

$$\begin{aligned}
&\|u\|_{W_2^2(\mathbf{T}^2)} \le c\|\Delta u\|,\\
&\|u\|_{W_2^3(\mathbf{T}^2)} \le c\|\nabla\Delta u\|,\\
&\|u\|_{W_2^4(\mathbf{T}^2)} \le c\left(\|\nabla\Delta u\| + \|\nabla\Delta u\|\right),\\
&\|\Delta u\|_{W_2^2(\mathbf{T}^2)} \le c\left\|\Delta^2 u\right\|
\end{aligned} \tag{2.8}$$

hold.

3 Proof of theorem 1

First, we establish the global-in-time existence of a unique strong solution to the problem (1.10) and (1.11). Next, by passing to the limit $D_2 \to 0$, we establish the global-in-time existence of a unique strong solution to the problem (1.8) and (1.11).

3.1 Global-in-time existence for problem (1.10) and (1.11)

We can obtain the following proposition on the local-in-time existence of a unique strong solution to the problem (1.10) and (1.11) in Sobolev–Slobodetskiĭ space. The proof of Proposition 1 uses successive approximations; however, it is easier than that of [25], so we omit it.

Proposition 1 *Let D_1, D_2, α, and κ be positive constants. Assume that $(\phi_0, n_0) \in W_2^4(\mathbf{T}^2) \times W_2^2(\mathbf{T}^2)$ satisfies (1.11). Then, there exists a unique solution (ϕ, n) to the problem (1.10) and (1.11) in some time interval $[0, T]$ such that $(\phi, n) \in \Big(L^2(0, T; W_2^5(\mathbf{T}^2)) \cap W_2^{3/2}(0, T; W_2^2(\mathbf{T}^2))\Big) \times W_2^{3,3/2}(Q_T)$. Here $Q_T \equiv \mathbf{T}^2 \times (0, T)$.*

Remark 2 $\phi \in W_2^{3/2}(0, T; W_2^2(\mathbf{T}^2))$ means that

$$D_x^{\alpha}\phi \in L^2\left(\mathbf{T}^2; W_2^{3/2}(0, T)\right) \quad \text{where } |\alpha| = 0, 1, 2.$$

We prove the following theorem on the global-in-time existence of a unique strong solution.

Theorem 3 *Let D_1, D_2, α, and κ be positive constants. Assume that $(\phi_0, n_0) \in W_2^4(\mathbf{T}^2) \times W_2^2(\mathbf{T}^2)$ satisfies (1.11). Then, there exists a unique solution (ϕ, n) to the problem (1.10) and (1.11) in any time interval $[0, T]$ $(0 < T < \infty)$ such that $(\phi, n) \in \Big(L^2(0, T; W_2^5(\mathbf{T}^2)) \cap W_2^{3/2}(0, T; W_2^2(\mathbf{T}^2))\Big) \times W_2^{3,3/2}(Q_T)$. Here $Q_T \equiv \mathbf{T}^2 \times (0, T)$.*

To prove Theorem 3, we obtain *a priori* estimates of the solution (ϕ, n) established in Proposition 1 by using energy estimates. In this subsection, we write $\nabla = \nabla_\perp$ and $\Delta = \Delta_\perp$, and we denote by c a constant that may differ at each occurrence. First, we prove the following.

Lemma 1 *For any $t \geq 0$, the following holds.*

$$\frac{1}{2}\left(\|\nabla\phi(t)\|^2 + \|n(t)\|^2\right) + D_1\int_0^t \|\Delta\phi(\tau)\|^2\,\mathrm{d}\tau + D_2\int_0^t \|\nabla\phi(\tau)\|^2\,\mathrm{d}\tau$$
$$+\alpha\int_0^t \left\|\left(\widetilde{\phi} - \widetilde{n}\right)(\tau)\right\|^2\,\mathrm{d}\tau \leq \frac{\kappa}{2}\left(\|\nabla\phi_0\|^2 + \|n_0\|^2\right)\exp(\kappa t) \equiv C_1(t). \tag{3.1}$$

Proof Multiplying the first equation of (1.10) by ϕ and integrating over $\mathbf{T}^2$, we have, by virtue of integration by parts,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\phi(t)\|^2 + D_1\|\Delta\phi(t)\|^2 + \alpha\int_{\mathbf{T}^2}\left(\widetilde{\phi} - \widetilde{n}\right)\widetilde{\phi}\,\mathrm{d}x = 0. \tag{3.2}$$

Here, we use the following relation:

$$\int_{\mathbf{T}^2}\widetilde{f}g\,\mathrm{d}x = \int_{\mathbf{T}^2}\widetilde{f}\widetilde{g}\,\mathrm{d}x \quad \text{for } f,\ g \in L^2\left(\mathbf{T}^2\right). \tag{3.3}$$

Multiplying the second equation of (1.10) by n and integrating over $\mathbf{T}^2$, we have, by virtue of integration by parts, (2.2), and (3.3),

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|n(t)\|^2 + D_2\|\nabla n(t)\|^2 + \alpha\int_{\mathbf{T}^2}\left(\widetilde{\phi} - \widetilde{n}\right)(-\widetilde{n})\,\mathrm{d}x$$

$$= -\kappa \int_{\mathbf{T}^2} \frac{\partial \phi}{\partial x_2} n \, \mathrm{d}x \le \frac{\kappa}{2} \left(\|\nabla \phi(t)\|^2 + \|n(t)\|^2 \right). \tag{3.4}$$

Adding (3.2) and (3.4) yields

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left(\|\nabla \phi(t)\|^2 + \|n(t)\|^2 \right) + D_1 \|\Delta \phi(t)\|^2 + D_2 \|\nabla n(t)\|^2$$
$$+ \alpha \left\| \left(\widetilde{\phi} - \widetilde{n} \right)(t) \right\|^2 \le \frac{\kappa}{2} \left(\|\nabla \phi(t)\|^2 + \|n(t)\|^2 \right).$$

Integrating this inequality over $[0, t]$ and using (2.1), we obtain (3.1). □

Next, we prove the following.

Lemma 2 *For any $t > 0$, the following holds.*

$$\frac{1}{2} \|\Delta \phi(t)\|^2 + D_1 \int_0^t \|\nabla \Delta \phi(\tau)\|^2 \, \mathrm{d}\tau \le \frac{1}{2} \|\Delta \phi_0\|^2 + \frac{\sqrt{\alpha} C_1(t)}{2\sqrt{D_1}} \equiv C_2(t). \tag{3.5}$$

Proof Multiplying the first equation of (1.10) by $\Delta\phi$ and integrating over $\mathbf{T}^2$, we have, by virtue of integration by parts and (2.2),

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\Delta \phi(t)\|^2 + D_1 \|\nabla \Delta \phi(t)\|^2 = \alpha \int_{\mathbf{T}^2} \left(\widetilde{\phi} - \widetilde{n} \right) \Delta \phi \, \mathrm{d}x$$
$$\le \frac{\sqrt{\alpha}}{2\sqrt{D_1}} \left(\alpha \left\| \left(\widetilde{\phi} - \widetilde{n} \right)(t) \right\|^2 + D_1 \|\Delta \phi(t)\|^2 \right).$$

Integrating this inequality over $[0, t]$ and using (3.1), we obtain (3.5). □

Next, we prove the following.

Lemma 3 *For any $t > 0$, the following holds.*

$$\|\nabla \Delta \phi(t)\|^2 + D_1 \int_0^t \left\| \Delta^2 \phi(\tau) \right\|^2 \mathrm{d}\tau$$
$$\le \left(\|\nabla \Delta \phi_0\|^2 + \frac{2\alpha C_1(t)}{D_1} \right) \exp\left(\frac{c C_2(t)}{D_1^2} \right) \equiv C_3(t), \tag{3.6}$$

$$\int_0^t \left\| \frac{\partial \Delta \phi(\tau)}{\partial \tau} \right\|^2 \mathrm{d}\tau \le c \left(\int_0^t C_3(\tau)^2 \, \mathrm{d}\tau + C_1(t) + C_3(t) \right), \tag{3.7}$$

where c is a positive constant independent of D_2, and $C_3(t)$ is a positive function independent of D_2.

Proof Multiplying the first equation of (1.10) by $\Delta^2\phi$ and integrating over $\mathbf{T}^2$, we have, by virtue of integration by parts, (2.4), and (2.6),

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\nabla \Delta \phi(t)\|^2 + D_1 \left\| \Delta^2 \phi(t) \right\|^2 = \int_{\mathbf{T}^2} \left[\{\phi, \Delta\phi\} - \alpha \left(\widetilde{\phi} - \widetilde{n} \right) \right] \Delta^2 \phi \, \mathrm{d}x$$
$$\le \|\nabla \phi(t)\|_\infty \|\nabla \Delta \phi(t)\| \left\| \Delta^2 \phi(t) \right\| + \alpha \left\| \left(\widetilde{\phi} - \widetilde{n} \right)(t) \right\| \left\| \Delta^2 \phi(t) \right\|$$
$$\le \frac{D_1}{2} \left\| \Delta^2 \phi(t) \right\|^2 + \frac{1}{D_1} \left(c \|\nabla \Delta \phi(t)\|^4 + \alpha^2 \left\| \left(\widetilde{\phi} - \widetilde{n} \right)(t) \right\|^2 \right).$$

Here, we used

$$\|\nabla f\|_\infty \le c \|\nabla \Delta f\| \quad \text{for } f \in W_2^3\left(\mathbf{T}^2\right), \tag{3.8}$$

which was obtained from (2.7) and (2.8). Hence,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta\phi(t)\|^2+\frac{D_1}{2}\left\|\Delta^2\phi(t)\right\|^2 \leq \frac{1}{D_1}\left(c\|\nabla\Delta\phi(t)\|^4+\alpha^2\left\|\left(\widetilde{\phi}-\widetilde{n}\right)(t)\right\|^2\right).$$

If we integrate this inequality over $(0, t)$, then (2.1) and (3.1) yield

$$\|\nabla\Delta\phi(t)\|^2 \leq \left(\|\nabla\Delta\phi_0\|^2+\frac{2\alpha C_1(t)}{D_1}\right)\exp\left(\int_0^t \frac{c}{D_1}\|\nabla\Delta\phi(\tau)\|^2\,\mathrm{d}\tau\right).$$

From this inequality and (3.5), we have (3.6). From the first equation of (1.10), we have

$$\int_0^t\left\|\frac{\partial\Delta\phi(\tau)}{\partial\tau}\right\|^2\mathrm{d}\tau \leq 3\biggl(\int_0^t\|\nabla\phi(\tau)\|_\infty^2\,\|\nabla\Delta\phi(\tau)\|^2\,\mathrm{d}\tau + \alpha^2\int_0^t\left\|\left(\widetilde{\phi}-\widetilde{n}\right)(\tau)\right\|^2\mathrm{d}\tau + D_1^2\int_0^t\left\|\Delta^2\phi(\tau)\right\|^2\mathrm{d}\tau\biggr).$$

From this result, (3.1), (3.6), and (3.8), we have (3.7). □

Next, we prove the following.

Lemma 4 *For any $t > 0$, the following holds.*

$$\|\nabla n(t)\|^2 \leq \left(\|\Delta\phi_0\|^2+\|\nabla n_0\|^2\right)\exp\left(ct+c\sqrt{tC_2(t)}+c\sqrt{tC_3(t)}\right) \equiv C_4(t), \tag{3.9}$$

where c is a positive constant independent of D_2, and $C_4(t)$ is a positive function independent of D_2.

Proof Multiplying the first equation of (1.10) by $\Delta\phi$ and integrating over $\mathbf{T}^2$, we have, by virtue of integration by parts, (3.3), and $\widetilde{\Delta\phi}=\Delta\widetilde{\phi}$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\phi(t)\|^2+D_1\|\nabla\Delta\phi(t)\|^2+\alpha\int_{\mathbf{T}^2}\nabla\left(\widetilde{\phi}-\widetilde{n}\right)\cdot\nabla\widetilde{\phi}\,\mathrm{d}x=0. \tag{3.10}$$

Multiplying the second equation of (1.10) by Δn and integrating over $\mathbf{T}^2$, we have, by virtue of integration by parts, (2.2), (2.4), (2.8), and (3.3),

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla n(t)\|^2+D_2\|\Delta n(t)\|^2-\alpha\int_{\mathbf{T}^2}\nabla\left(\widetilde{\phi}-\widetilde{n}\right)\cdot\nabla\widetilde{n}\,\mathrm{d}x\\
&=\int_{\mathbf{T}^2}\left(\{\phi,n\}+\kappa\frac{\partial\phi}{\partial x_2}\right)\Delta n\,\mathrm{d}x\\
&\leq c\left\|D_x^2\phi(t)\right\|_\infty\|\nabla n(t)\|^2+\kappa\left\|D_x^2\phi(t)\right\|\|\nabla n(t)\|\\
&\leq c\left(\|\nabla\Delta\phi(t)\|+\left\|\Delta^2\phi(t)\right\|\right)\|\nabla n(t)\|^2+c\|\Delta\phi(t)\|\|\nabla n(t)\|\\
&\leq c\left(1+\|\nabla\Delta\phi(t)\|+\left\|\Delta^2\phi(t)\right\|\right)\left(\|\Delta\phi(t)\|^2+\|\nabla n(t)\|^2\right).
\end{aligned} \tag{3.11}$$

Here, we used

$$\begin{aligned}
\left\|D_x^2\phi(t)\right\|_\infty &\leq c\left(\|\nabla\Delta\phi(t)\|+\left\|\Delta^2\phi(t)\right\|\right)^{1/2}\|\Delta\phi(t)\|^{1/2}\\
&\leq c\left(\|\nabla\Delta\phi(t)\|+\left\|\Delta^2\phi(t)\right\|\right),
\end{aligned} \tag{3.12}$$

which was obtained from (2.2), (2.3), (2.5), and (2.8).

Adding (3.10) and (3.11) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\Delta\phi(t)\|^2+\|\nabla n(t)\|^2\right)+D_1\|\nabla\Delta\phi(t)\|^2+D_2\|\Delta n(t)\|^2+\alpha\left\|\nabla\left(\widetilde{\phi}-\widetilde{n}\right)(t)\right\|^2$$
$$\leq c\left(1+\|\nabla\Delta\phi(t)\|+\left\|\Delta^2\phi(t)\right\|\right)\left(\|\Delta\phi(t)\|^2+\|\nabla n(t)\|^2\right).$$

If we integrate this inequality over $(0,t)$, then (2.1), (3.5), and (3.6) yield

$$\|\Delta\phi(t)\|^2+\|\nabla n(t)\|^2$$
$$\leq\left(\|\Delta\phi_0\|^2+\|\nabla n_0\|^2\right)\exp\left(\int_0^t c\left(1+\|\nabla\Delta\phi(\tau)\|+\left\|\Delta^2\phi(\tau)\right\|\right)\mathrm{d}\tau\right)$$
$$\leq\left(\|\Delta\phi_0\|^2+\|\nabla n_0\|^2\right)\exp\left(ct+c\sqrt{\frac{tC_2(t)}{D_1}}+c\sqrt{\frac{tC_3(t)}{D_1}}\right).$$

From this result, we obtain (3.9). □

Next, we prove the following.

Lemma 5 *For any $t>0$, the following holds.*

$$\|\Delta n(t)\|^2\leq\left(\|\nabla\Delta\phi_0\|^2+\|\Delta n_0\|^2+2\int_0^t C_1(\tau)\,\mathrm{d}\tau\right)$$
$$\exp\left(ct+c\left(C_2(t)+C_3(t)\right)+c\sqrt{tC_3(t)}+c\sqrt{tC_4(t)}\right)\equiv C_5(t),\tag{3.13}$$

where c is a positive constant independent of D_2, and $C_5(t)$ is a positive function independent of D_2.

Proof Multiplying the first equation of (1.10) by $\Delta^2\phi$ and integrating over $\mathbf{T}^2$, we have, by virtue of integration by parts, (2.2), (3.3), (3.8), and $\widetilde{\Delta^2\phi}=\Delta^2\widetilde{\phi}$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta\phi(t)\|^2+D_1\left\|\Delta^2\phi(t)\right\|^2+\alpha\int_{\mathbf{T}^2}\Delta\left(\widetilde{\phi}-\widetilde{n}\right)\Delta\widetilde{\phi}\,\mathrm{d}x$$
$$\leq\|\nabla\phi(t)\|_\infty\|\nabla\Delta\phi(t)\|\left\|\Delta^2\phi(t)\right\|\leq c\|\nabla\Delta\phi(t)\|^2\left\|\Delta^2\phi(t)\right\|$$
$$\leq\frac{D_1}{2}\left\|\Delta^2\phi(t)\right\|^2+\frac{c}{D_1}\|\nabla\Delta\phi(t)\|^4.\tag{3.14}$$

Applying the Laplacian Δ to the second equation of (1.10), multiplying it by Δn, and integrating over $\mathbf{T}^2$, we have, by virtue of integration by parts, (2.2), (2.4), (2.6), (2.8), (3.3), (3.12), and $\widetilde{\Delta n}=\Delta\widetilde{n}$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta n(t)\|^2+D_2\|\nabla\Delta n(t)\|^2-\alpha\int_{\mathbf{T}^2}\Delta\left(\widetilde{\phi}-\widetilde{n}\right)\Delta\widetilde{n}\,\mathrm{d}x$$
$$=-\int_{\mathbf{T}^2}\Delta\left(\{\phi,n\}+\kappa\frac{\partial\phi}{\partial x_2}\right)\Delta n\,\mathrm{d}x$$
$$\leq\left(\left\|D_x^2\phi(t)\right\|_\infty\left\|D_x^2 n(t)\right\|+\|\nabla\Delta\phi(t)\|_4\|\nabla n(t)\|_4+\kappa\|\nabla\phi(t)\|\right)\|\Delta n(t)\|$$
$$\leq\Big(c\left(\|\nabla\Delta\phi(t)\|+\left\|\Delta^2\phi(t)\right\|\right)\|\Delta n(t)\|$$
$$+c\left\|\Delta^2\phi(t)\right\|^{3/4}\|\Delta\phi(t)\|^{1/4}\|\Delta n(t)\|^{3/4}\|n(t)\|^{1/4}+\kappa\|\nabla\phi(t)\|\Big)\|\Delta n(t)\|.$$

Hence, from this result, (2.2), and (2.3), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta n(t)\|^2 + D_2\|\nabla\Delta n(t)\|^2 - \alpha\int_{\mathbf{T}^2}\Delta\left(\widetilde{\phi}-\widetilde{n}\right)\Delta\widetilde{n}\,\mathrm{d}x\\
&\quad\le c\left(\|\nabla\Delta\phi(t)\| + \left\|\Delta^2\phi(t)\right\|\right)\|\Delta n(t)\|^2\\
&\qquad+\left\{c\left(\|\Delta\phi(t)\| + \left\|\Delta^2\phi(t)\right\|\right)(\|n(t)\| + \|\Delta n(t)\|) + \kappa\|\nabla\phi(t)\|\right\}\|\Delta n(t)\|\\
&\quad\le c\left(1 + \|\nabla\Delta\phi(t)\| + \left\|\Delta^2\phi(t)\right\| + \|\nabla\Delta\phi(t)\|^2 + \left\|\Delta^2\phi(t)\right\|^2\right)\|\Delta n(t)\|^2\\
&\qquad+\|\nabla\phi(t)\|^2 + \|n(t)\|^2.
\end{aligned}\tag{3.15}$$

Adding (3.14) and (3.15) yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla\Delta\phi(t)\|^2 + \|\Delta n(t)\|^2\right) + \frac{D_1}{2}\left\|\Delta^2\phi(t)\right\|^2 + D_2\|\nabla\Delta n(t)\|^2\\
&\qquad+\alpha\left\|\Delta\left(\widetilde{\phi}-\widetilde{n}\right)(t)\right\|^2\\
&\quad\le c\left(1 + \|\nabla\Delta\phi(t)\| + \left\|\Delta^2\phi(t)\right\| + \|\nabla\Delta\phi(t)\|^2 + \left\|\Delta^2\phi(t)\right\|^2\right)\\
&\qquad\left(\|\nabla\Delta\phi(t)\|^2 + \|\Delta n(t)\|^2\right) + \|\nabla\phi(t)\|^2 + \|n(t)\|^2.
\end{aligned}$$

If we integrate this inequality over $(0, t)$, then (2.1) and (3.1) yield

$$\begin{aligned}
&\|\nabla\Delta\phi(t)\|^2 + \|\Delta n(t)\|^2 \le \left(\|\nabla\Delta\phi_0\|^2 + \|\Delta n_0\|^2 + 2\int_0^t C_1(\tau)\,\mathrm{d}\tau\right)\\
&\quad\exp\left(\int_0^t c\left(1 + \|\nabla\Delta\phi(\tau)\| + \left\|\Delta^2\phi(\tau)\right\| + \|\nabla\Delta\phi(\tau)\|^2 + \left\|\Delta^2\phi(\tau)\right\|^2\right)\mathrm{d}\tau\right).
\end{aligned}$$

From this result, (3.5), and (3.6), we have (3.13). □

Next, we prove the following.

Lemma 6 *For any $t \ge 0$, the following holds.*

$$\begin{aligned}
\left\|\Delta^2\phi(t)\right\|^2 &\le \left(\left\|\Delta^2\phi_0\right\|^2 + cC_1(t) + \int_0^t cC_5(\tau)\,\mathrm{d}\tau\right)\\
&\quad\exp\left(ct + c\sqrt{tC_2(t)} + c\sqrt{tC_3(t)}\right),
\end{aligned}\tag{3.16}$$

where c is a constant independent of D_2.

Proof Applying the Laplacian Δ to the first equation of (1.10), multiplying it by $\Delta^2\phi$, and integrating over $\mathbf{T}^2$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Delta^2\phi(t)\right\|^2 + D_1\|\nabla\Delta^2\phi(t)\|^2\\
&\quad=\alpha\int_{\mathbf{T}^2}\Delta\left(\widetilde{\phi}-\widetilde{n}\right)\Delta^2\phi\,\mathrm{d}x - \int_{\mathbf{T}^2}\Delta\{\phi,\Delta\phi\}\Delta^2\phi\,\mathrm{d}x\\
&\quad\le c\alpha\left(\|\Delta\phi(t)\| + \|\Delta n(t)\|\right)\left\|\Delta^2\phi(t)\right\| + \left\|D_x^2\phi(t)\right\|_\infty\left\|D_x^2\Delta\phi(t)\right\|\left\|\Delta^2\phi(t)\right\|\\
&\quad\le c\alpha\left(\|\Delta\phi(t)\| + \|\Delta n(t)\|\right)\left\|\Delta^2\phi(t)\right\| + c\left(\|\nabla\Delta\phi(t)\| + \left\|\Delta^2\phi(t)\right\|\right)\left\|\Delta^2\phi(t)\right\|^2\\
&\quad\le c\left(1 + \|\nabla\Delta\phi(t)\| + \left\|\Delta^2\phi(t)\right\|\right)\left\|\Delta^2\phi(t)\right\|^2 + c\left(\|\Delta\phi(t)\|^2 + \|\Delta n(t)\|^2\right),
\end{aligned}$$

where we used (2.2), (2.8), (3.12), $\|\Delta\widetilde{\phi}\| \le c\|\Delta\phi\|$, and $\|\Delta\widetilde{n}\| \le c\|\Delta n\|$.

If we integrate this inequality over $(0, t)$, then (3.1) and (3.13) yield

$$\left\|\Delta^2\phi(t)\right\|^2 \leq \left\|\Delta^2\phi_0\right\|^2 + \frac{cC_1(t)}{D_1} + \int_0^t cC_5(\tau)\,\mathrm{d}\tau + \int_0^t c\left(1 + \|\nabla\Delta\phi(\tau)\| + \left\|\Delta^2\phi(\tau)\right\|\right)\left\|\Delta^2\phi(\tau)\right\|^2\,\mathrm{d}\tau.$$

From this result, (2.1), (3.5), and (3.6), we have (3.16). □

Remark 3 Because the regularity of the solution is not sufficient, the arguments of Lemmas 5 and 6 are formal. Indeed, the terms $\int_{\mathbf{T}^2}\Delta^2 n\Delta n\,\mathrm{d}x$ and $\int_{\mathbf{T}^2}\Delta^3\phi\Delta^2\phi\,\mathrm{d}x$ appear in the proofs of Lemmas 5 and 6. However, one can justify them by using mollifiers.

By the standard arguments based on the *a priori* estimates in Lemmas 1–6, the solution established in Proposition 1 can be extended to any time interval $[0, T]$ $(0 < T < \infty)$. Thus, the proof of Theorem 3 is complete.

3.2 Global-in-time existence for problem (1.8) and (1.11)

The *a priori* estimates in Lemmas 1–6 do not depend on D_2; that is, these estimates are uniform estimates with respect to D_2. With the help of the uniform estimate for the solution with respect to D_2, by passing to the limit $D_2 \to 0$, we can prove the following proposition on the local-in-time existence of a unique strong solution to the problem (1.8) and (1.11). The proof of Proposition 2 is similar to that of [25], so we omit the details.

Proposition 2 *Let D_1, α, and κ be positive constants. Assume that $(\phi_0, n_0) \in W_2^4(\mathbf{T}^2) \times W_2^2(\mathbf{T}^2)$ satisfies (1.11). Then, there exists a unique solution (ϕ, n) to the problem (1.8) and (1.11) in some time interval $[0, T]$ such that $(\phi, n) \in \left(L^2(0, T; W_2^4(\mathbf{T}^2)) \cap W_2^1(0, T; W_2^2(\mathbf{T}^2))\right) \times W_2^{2,1}(Q_T)$. Here $Q_T \equiv \mathbf{T}^2 \times (0, T)$.*

By using the *a priori* estimates in Lemmas 1–6 with $D_2 = 0$, the solution can be extended to any time interval $[0, T]$ $(0 < T < \infty)$. Thus, the proof of Theorem 1 is complete.

4 Proof of theorem 2

Let us add term $D_2\Delta_\perp n$ to the second equation of (1.12); then, we have

$$\begin{cases} \dfrac{\partial\Delta_\perp\phi}{\partial t} + \{\phi, \Delta_\perp\phi\} = \alpha\left(\widetilde{\phi} - \widetilde{n} - (1+\gamma)\widetilde{T}\right) + D_1\Delta_\perp^2\phi, \\ \dfrac{\partial n}{\partial t} + \{\phi, n\} = \alpha\left(\widetilde{\phi} - \widetilde{n} - (1+\gamma)\widetilde{T}\right) - \kappa\dfrac{\partial\phi}{\partial x_2} + D_2\Delta_\perp n, \\ \dfrac{\partial T}{\partial t} + \{\phi, T\} = \chi\alpha\left(\widetilde{\phi} - \widetilde{n} - (1+\gamma)\widetilde{T}\right) - \kappa'\dfrac{\partial\phi}{\partial x_2} + D_3\Delta_\perp T, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } x \in \mathbf{T}^2,\ t > 0, \end{cases} \tag{4.1}$$

First, we establish the global-in-time existence of a unique strong solution to the problem (4.1) and (1.13). Next, by passing to the limit $D_2 \to 0$, we establish the global-in-time existence of a unique strong solution to the problem (1.12) and (1.13).

4.1 Global-in-time existence for problem (4.1) and (1.13)

We can obtain the following proposition on the local-in-time existence of a unique strong solution to the problem (4.1) and (1.13) in Sobolev–Slobodetskiĭ space. The proof of Proposition 3 uses successive approximations; however, it is easier than that of [25], so we omit it.

Proposition 3 *Let* $D_1, D_2, D_3, \alpha, \kappa, \kappa', \gamma, \chi$ *be positive constants. Assume that* $(\phi_0, n_0, T_0) \in W_2^4(\mathbf{T}^2)\times W_2^2(\mathbf{T}^2)\times W_2^2(\mathbf{T}^2)$ *satisfies* (1.13). *Then, there exists a unique solution* (ϕ, n, T) *to the problem* (4.1) *and* (1.13) *in some time interval* $[0, T]$ *such that* $(\phi, n, T) \in \left(L^2(0, T; W_2^5(\mathbf{T}^2)) \cap W_2^{3/2}(0, T; W_2^2(\mathbf{T}^2))\right) \times W_2^{3,3/2}(Q_T) \times W_2^{3,3/2}(Q_T)$. *Here* $Q_T \equiv \mathbf{T}^2 \times (0, T)$.

In the following lemmas, we write $\nabla = \nabla_\perp$ and $\Delta = \Delta_\perp$, and we denote by c a constant that may differ at each occurrence. Let (ϕ, n, T) be the solution established in Proposition 2. The proofs of Lemmas 7–12 are similar to those of Lemmas 1–6, so we omit the proofs.

Lemma 7 *For any* $t \geq 0$, *the following holds.*

$$\begin{aligned}
&\frac{1}{2}\left(\|\nabla\phi(t)\|^2 + \|n(t)\|^2 + \frac{1+\gamma}{\chi}\|T(t)\|^2\right) + D_1\int_0^t \|\Delta\phi(\tau)\|^2\,\mathrm{d}\tau \\
&\quad + D_2\int_0^t \|\nabla n(\tau)\|^2\,\mathrm{d}\tau + \frac{(1+\gamma)D_3}{\chi}\int_0^t \|\nabla T(\tau)\|^2\,\mathrm{d}\tau \\
&\quad + \alpha\int_0^t \left\|\left(\widetilde{\phi} - \widetilde{n} - (1+\gamma)\widetilde{T}\right)(\tau)\right\|^2\,\mathrm{d}\tau \\
&\quad \leq \frac{\kappa}{2}\left(\|\nabla\phi_0\|^2 + \|n_0\|^2 + \frac{1+\gamma}{\chi}\|T_0\|^2\right)\exp(\kappa t) \equiv C_6(t),
\end{aligned}$$

where $C_6(t)$ *is a constant independent of* D_2.

Lemma 8 *For any* $t > 0$, *the following holds.*

$$\frac{1}{2}\|\Delta\phi(t)\|^2 + D_1\int_0^t \|\nabla\Delta\phi(\tau)\|^2\,\mathrm{d}\tau \leq \frac{1}{2}\|\Delta\phi_0\|^2 + \frac{\sqrt{\alpha}C_6(t)}{\sqrt{2D_1}} \equiv C_7(t),$$

where $C_7(t)$ *is a constant independent of* D_2.

Lemma 9 *For any* $t > 0$, *the following holds.*

$$\begin{aligned}
&\|\nabla\Delta\phi(t)\|^2 + D_1\int_0^t \left\|\Delta^2\phi(\tau)\right\|^2\,\mathrm{d}\tau \\
&\quad \leq \left(\|\nabla\Delta\phi_0\|^2 + \frac{2\alpha C_6(t)}{D_1}\right)\exp\left(\frac{cC_7(t)}{D_1^2}\right) \equiv C_8(t), \\
&\quad \int_0^t \left\|\frac{\partial\Delta\phi(\tau)}{\partial\tau}\right\|^2\,\mathrm{d}\tau \leq c\left(\int_0^t C_8(\tau)^2\,\mathrm{d}\tau + C_6(t) + C_8(t)\right),
\end{aligned}$$

where c *is a positive constant independent of* D_2, *and* $C_8(t)$ *is a positive function independent of* D_2.

Lemma 10 *For any* $t > 0$, *the following holds.*

$$\|\nabla n(t)\|^2 \le \left(\|\Delta\phi_0\|^2 + \|\nabla n_0\|^2 + \frac{1+\gamma}{\chi}\|\nabla T_0\|^2 \right)$$
$$\exp\left(ct + c\sqrt{tC_7(t)} + c\sqrt{tC_8(t)} \right) \equiv C_9(t),$$

where c is a positive constant independent of D_2, and $C_9(t)$ is a positive function independent of D_2.

Lemma 11 *For any $t > 0$, the following holds.*

$$\|\Delta n(t)\|^2 \le \left(\|\nabla\Delta\phi_0\|^2 + \|\Delta n_0\|^2 + \frac{1+\gamma}{\chi}\|\Delta T_0\|^2 + 2\int_0^t C_6(\tau)\,\mathrm{d}\tau \right)$$
$$\exp\left(ct + c\left(C_6(t) + C_7(t)\right) + c\sqrt{tC_7(t)} + c\sqrt{tC_8(t)} \right) \equiv C_{10}(t),$$

where c is a positive constant independent of D_2, and $C_{10}(t)$ is a positive function independent of D_2.

Lemma 12 *For any $t \ge 0$, the following holds.*

$$\left\|\Delta^2\phi(t)\right\|^2 \le \left(\left\|\Delta^2\phi_0\right\|^2 + cC_6(t) + \int_0^t cC_{10}(\tau)\,\mathrm{d}\tau \right)$$
$$\exp\left(ct + c\sqrt{tC_7(t)} + c\sqrt{tC_8(t)} \right),$$

where c is a constant independent of D_2.

By the standard arguments based on the *a priori* estimates in Lemmas 7–12, the solution established in Proposition 3 can be extended to any time interval $[0, T]$ $(0 < T < \infty)$. Thus, we have the following theorem.

Theorem 4 *Let $D_1,\ D_2,\ D_3,\ \alpha,\ \kappa,\ \kappa',\ \gamma,\ \chi$ be positive constants. Assume that $(\phi_0, n_0, T_0) \in W_2^4(\mathbf{T}^2)\times W_2^2(\mathbf{T}^2)\times W_2^2(\mathbf{T}^2)$ satisfies (1.13). Then, there exists a unique solution (ϕ, n, T) to the problem (4.1) and (1.13) for any time interval $[0, T]$ $(0 < T < \infty)$ such that $(\phi, n, T) \in \left(L^2(0, T; W_2^5(\mathbf{T}^2)) \cap W_2^{3/2}(0, T; W_2^2(\mathbf{T}^2))\right) \times W_2^{3,3/2}(Q_T) \times W_2^{3,3/2}(Q_T)$. Here $Q_T \equiv \mathbf{T}^2 \times (0, T)$.*

4.2 Global-in-time existence for problem (1.12) and (1.13)

The *a priori* estimates in Lemmas 7–12 do not depend on D_2; that is, these estimates are uniform estimates with respect to D_2. With the help of the uniform estimate for the solution with respect to D_2, by passing to the limit $D_2 \to 0$, we can prove the following proposition on the local-in-time existence of a unique strong solution to the problem (1.12) and (1.13). The proof of Proposition 4 is similar to that of [25], so we omit the details.

Proposition 4 *Let $D_1,\ D_3,\ \alpha,\ \kappa,\ \kappa',\ \gamma,\ \chi$ be positive constants. Assume that $(\phi_0, n_0, T_0) \in W_2^4(\mathbf{T}^2)\times W_2^2(\mathbf{T}^2)\times W_2^2(\mathbf{T}^2)$ satisfies (1.13). Then, there exists a unique solution (ϕ, n) to the problem (1.12) and (1.13) in some time interval $[0, T]$ such that $(\phi, n, T) \in \left(L^2(0, T; W_2^4(\mathbf{T}^2)) \cap W_2^1(0, T; W_2^2(\mathbf{T}^2))\right) \times W_2^{2,1}(Q_T) \times W_2^{2,1}(Q_T)$. Here $Q_T \equiv \mathbf{T}^2 \times (0, T)$.*

By using the *a priori* estimates in Lemmas 7–12 with $D_2 = 0$, the solution can be extended to any time interval $[0, T]$ $(0 < T < \infty)$. Thus, the proof of Theorem 2 is complete.

Acknowledgements The author would like to thank Professor Ryusuke Numata of Hyogo University in Japan for his valuable comments. The present study is partially supported by a Grant-in-Aid for Young Scientists (B) (No. 16K17632) from the Japan Society for the Promotion of Science.

Appendix

When we obtain Fig. 1, we use the same numerical scheme as in [8,37], so we recall it. Let $D_1 = D_2$, and rewrite the equations of (1.10) as

$$\begin{cases} \dfrac{\partial \psi}{\partial t} = \alpha\left(\widetilde{\phi} - \widetilde{n}\right) + D_1 \Delta_\perp \psi - \{\phi, \psi\}, \\ \dfrac{\partial n}{\partial t} = \alpha\left(\widetilde{\phi} - \widetilde{n}\right) + D_1 \Delta_\perp n - \{\phi, n\} - \kappa \dfrac{\partial \phi}{\partial x_2}, \\ \Delta_\perp \phi = \psi. \end{cases} \tag{4.2}$$

Then, we solve it with periodic boundary conditions by the finite differential method. We apply the third-order Karniadakis time integration scheme to the first and second equations of (4.2), and we apply the successive over-relaxation (SOR) method to the third equation of (4.2). The SOR method is well known, so we recall only the third-order Karniadakis time integration scheme [22].

When we use the third-order Karniadakis time integration scheme, we use the central difference for the spatial difference $\partial\phi/\partial x_2$ and $\Delta_\perp$, and Arakawa's method for the Poisson bracket $\{\cdot, \cdot\}$. We set $L_1 = L_2 = 40$; we use a grid width of $dx_1 = dx_2 = h$, and we use 256×256 grid points. Let us write the grid number as $N = 256$ and the time step width of the numerical calculation as dt; further, we write $t_k = k\mathrm{d}t$ for $k = 0, 1, 2, \ldots$. At the grid point (i, j) and time step k, we write the unknown function f as $f^k_{i,j}$. As the initial value, we choose the following function containing low-frequency waves:

$$\phi_0 = n_0 = 0.1 \sin\left(\frac{4\pi x_1}{L_1}\right) \sin\left(\frac{4\pi x_2}{L_2}\right) + 0.1 \exp\left(-(x_1 - 15)^2 - (x_2 - 15)^2\right).$$

Arakawa [3] introduced the following discretization of $\{f, g\}^k_{i,j}$ at the grid point (i, j) and time step k:

$$\begin{aligned} \{f, g\}^k_{i,j} = -\frac{1}{12h^2} \Big[&\left(f^k_{i,j-1} + f^k_{i+1,j-1} - f^k_{i,j+1} - f^k_{i+1,j+1}\right)\left(g^k_{i+1,j} + g^k_{i,j}\right) \\ &- \left(f^k_{i-1,j-1} + f^k_{i,j-1} - f^k_{i-1,j+1} - f^k_{i,j+1}\right)\left(g^k_{i,j} + g^k_{i-1,j}\right) \\ &+ \left(f^k_{i+1,j} + f^k_{i+1,j+1} - f^k_{i-1,j} - f^k_{i-1,j+1}\right)\left(g^k_{i,j+1} + g^k_{i,j}\right) \\ &- \left(f^k_{i+1,j-1} + f^k_{i+1,j} - f^k_{i-1,j-1} - f^k_{i-1,j}\right)\left(g^k_{i,j} + g^k_{i,j-1}\right) \\ &+ \left(f^k_{i+1,j} - f^k_{i,j+1}\right)\left(g^k_{i+1,j+1} + g^k_{i,j}\right) \\ &- \left(f^k_{i,j-1} - f^k_{i-1,j}\right)\left(g^k_{i,j} + g^k_{i-1,j-1}\right) \\ &+ \left(f^k_{i,j+1} - f^k_{i-1,j}\right)\left(g^k_{i-1,j+1} + g^k_{i,j}\right) \\ &- \left(f^k_{i+1,j} - f^k_{i,j-1}\right)\left(g^k_{i,j} + g^k_{i+1,j-1}\right)\Big]. \end{aligned}$$

This discretization of the convection term exactly conserves energy, enstrophy, and circulation. Arakawa's method is compared with the spectral method in [34].

Let us denote the discretization of the right-hand sides of the first and second equations of (4.2) at the grid point (i, j) and time step k as $F^k_{i,j}$ and $G^k_{i,j}$, as follows:

$$
\begin{aligned}
F^k_{i,j} &= \alpha\left(\phi^k_{i,j} - \frac{1}{N}\sum_{j=0}^{N}\phi^k_{i,j} - n^k_{i,j} + \frac{1}{N}\sum_{j=0}^{N} n^k_{i,j}\right) - \{\phi, \psi\}^k_{i,j} \\
&\quad + D_1\left(\frac{\psi^k_{i-1,j} - 2\psi^k_{i,j} + \psi^k_{i+1,j}}{h^2} + \frac{\psi^k_{i,j-1} - 2\psi^k_{i,j} + \psi^k_{i,j+1}}{h^2}\right),\\
G^k_{i,j} &= \alpha\left(\phi^k_{i,j} - \frac{1}{N}\sum_{j=0}^{N}\phi^k_{i,j} - n^k_{i,j} + \frac{1}{N}\sum_{j=0}^{N} n^k_{i,j}\right) - \{\phi, n\}^k_{i,j} \\
&\quad + D_1\left(\frac{n^k_{i-1,j} - 2n^k_{i,j} + n^k_{i+1,j}}{h^2} + \frac{n^k_{i,j-1} - 2n^k_{i,j} + n^k_{i,j+1}}{h^2}\right) \\
&\quad - \kappa\frac{\phi^k_{i,j+1} - \phi^k_{i,j-1}}{2h}.
\end{aligned}
$$

We apply the third-order Karniadakis time integration scheme [22] to the first and second equations of (4.2):

$$
\begin{aligned}
\psi^{k+1}_{i,j} &= \frac{6}{11}\left(3\psi^k_{i,j} - \frac{3}{2}\psi^{k-1}_{i,j} + \frac{1}{3}\psi^{k-2}_{i,j} + \mathrm{d}t\left(3F^k_{i,j} - 3F^{k-1}_{i,j} + F^{k-2}_{i,j}\right)\right),\\
n^{k+1}_{i,j} &= \frac{6}{11}\left(3n^k_{i,j} - \frac{3}{2}n^{k-1}_{i,j} + \frac{1}{3}n^{k-2}_{i,j} + \mathrm{d}t\left(3G^k_{i,j} - 3G^{k-1}_{i,j} + G^{k-2}_{i,j}\right)\right).
\end{aligned}
$$

When we solve the first and second equations of (4.2) using the third-order Karniadakis time integration scheme, we take $\mathrm{d}t$ as follows:

$$
\mathrm{d}t = C^* \min\left\{\frac{h^2}{D_1}, \frac{h}{\kappa}, \frac{h}{M^*}\right\},
$$

where C^* is a positive constant, and

$$
M^* = \max_{i,j}\left\{\frac{|\phi_{i+1,j} - \phi_{i-1,j}|}{2h}, \frac{|\phi_{i,j+1} - \phi_{i,j-1}|}{2h}\right\}.
$$

Because the CFL constant C^* must satisfy $C^* \le 1/4$ (see, [21]), we take $C^* = 0.1$.

References

1. Adams, R.A.: Sobolev Spaces. Academic Press, New York (1975)
2. Agmon, S.: Lectures on Elliptic Boundary Value Problems. Van Nostrand, Princeton (1965)
3. Arakawa, A.: Computational design for long-term numerical integration of the equations of fluid motion: two-dimensional incompressible flow. Part I. J. Comput. Phys. **1**, 119–143 (1966)
4. Bronski, J.C., Fetecau, R.C.: An alternative energy bound derivation for a generalized Hasegawa–Mima equation. Nonlinear Anal. Real World Appl. **13**, 1362–1368 (2012)
5. Cao, C., Farhat, A., Titi, E.S.: Global well-posedness of an inviscid three-dimensional pseudo-Hasegawa–Mima model. Commun. Math. Phys. **319**, 195–229 (2013)
6. Chang, Z., Callen, J.D.: Unified fluid/kinetic description of plasma microinstabilities. Phys. Fluids B **4**, 1182–1192 (1992)
7. Charney, J.G.: On the scale of atmospheric motions. Geofys. Publ. Norske Vid.-Akad. Oslo **17**, 1–17 (1948)

8. Dewhurst, J.M., Hnat, B., Dendy, R.O.: Finite Larmor radius effects on test particle transport in drift wave-zonal flow turbulence. Plasma Phys. Control. Fusion **52**, 025004 (2010)
9. Friedman, A.: Partial Differential Equations. Holt, Rinehart and Winston, Inc., New York (1969)
10. Gao, H., Zhu, A.: The global strong solutions of Hasegawa–Mima–Charney–Obukhov equation. J. Math. Phys. **46**, 083517 (2005)
11. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Springer, Berlin (2001)
12. Grauer, R.: An energy estimate for a perturbed Hasegawa–Mima equation. Nonlinearity **11**, 659–666 (1998)
13. Guo, B., Han, Y.: Existence and uniqueness of global solution of the Hasegawa–Mima equation. J. Math. Phys. **45**, 1638–1647 (2004)
14. Hasegawa, A., Mima, K.: Stationary spectrum of strong turbulence in magnetized plasma. Phys. Rev. Lett. **39**, 205–208 (1977)
15. Hasegawa, A., Mima, K.: Pseudo-three-dimensional turbulence in magnetized nonuniform plasma. Phys. Fluids **21**, 87–92 (1978)
16. Hasegawa, A., Wakatani, M.: Plasma edge turbulence. Phys. Rev. Lett. **50**, 682–686 (1983)
17. Hasegawa, A., Wakatani, M.: A collisional drift wave description of plasma edge turbulence. Phys. Fluids **27**, 611–618 (1984)
18. Horton, W.: Drift waves and transport. Rev. Mod. Phys. **71**, 735–778 (1999)
19. Horton, W., Hasegawa, A.: Quasi-two-dimensional dynamics of plasmas and fluids. Chaos **4**, 227–251 (1994)
20. Hounkonnou, M.N., Kabir, M.M.: Hasegawa–Mima–Charney–Obukhov equation: symmetry reductions and solutions. Int. J. Contemp. Math. Sci. **3**, 145–157 (2008)
21. Johnston, H., Liu, J.G.: Accurate, stable and efficient Navier–Stokes solvers based on explicit treatment of the pressure term. J. Comput. Phys. **199**, 221–259 (2004)
22. Karniadakis, G.E., Israeli, M., Orszag, S.A.: High-order splitting methods for the incompressible Navier–Stokes equations. J. Comput. Phys. **97**, 414–443 (1991)
23. Kim, J., Terry, P.W.: Numerical investigation of frequency spectrum in the Hasegawa–Wakatani model. Phys. Plasmas **20**, 102303 (2013)
24. Kiwamoto, Y., Ito, K., Sanpei, A., Mohri, A.: Dynamics of electron-plasma vortex in background vorticity distribution. Phys. Rev. Lett. **85**, 3173–3176 (2000)
25. Kondo, S., Tani, A.: Initial boundary value problem for model equations of resistive drift wave turbulence. SIAM J. Math. Anal. **43**, 925–943 (2011)
26. Kondo, S., Tani, A.: Initial boundary value problem of Hasegawa–Wakatani equations with vanishing resistivity. Adv. Math. Sci. Appl. **21**, 223–253 (2011)
27. Kondo, S., Tani, A.: On the Hasegawa–Wakatani equations with vanishing resistivity. Proc. Jpn. Acad. **87**, 156–161 (2011)
28. Kondo, S., Tani, A.: Almost-periodic solutions to initial boundary value problem for model equations of resistive drift wave turbulence. Ann. Scuola Norm. Sup. Pisa **XVI**, 291–333 (2016)
29. Kondo, S.: Almost-periodic solution of linearized Hasegawa–Wakatani equations with vanishing resistivity. Rend. Sem. Mat. Univ. Padova **133**, 215–239 (2015)
30. Kondo, S.: On the almost-periodic solution of Hasegawa–Wakatani equations. J. Evol. Equ. **16**, 155–172 (2016)
31. Kondo, S.: An almost-periodic solution of Hasegawa–Wakatani equations with vanishing resistivity. Proc. R. Soc. Edinb. Sect. A **146**, 983–1003 (2016)
32. Korsholm, S.B.: Coherent structures and transport in drift wave plasma turbulence. Risø-R-Report 1337 (2011)
33. Makino, M., Kamimura, T., Taniuti, T.: Dynamics of two-dimensional solitary vortices in a low-plasma with convective motion. J. Phys. Soc. Jpn. **50**, 980–989 (1981)
34. Naulin, V., Nielsen, A.H.: Accuracy of spectral and finite difference schemes in 2D advection problems. SIAM J. Sci. Comput. **25**, 104–126 (2003)
35. Nezlin, M.V., Chernikov, G.P., Rylov, A.Y., Titishov, K.B.: Self-organization of the large-scale planetary and plasma drift vortices. Chaos **6**, 309–327 (1996)
36. Nishikawa, K., Wakatani, M.: Plasma Physics: Basic Theory with Fusion Applications. Springer, New York (1999)
37. Numata, R., Ball, R., Dewar, R.L.: Bifurcation in electrostatic resistive drift wave turbulence. Phys. Plasmas **14**, 102312 (2007)
38. Obukhov, A.M.: On the question of geostrophic wind. Izv. Akad. Nauk SSSR Geogr. Geophiz. **13**, 281–306 (1949). (in Russian)

39. Okamoto, A., Hara, K., Nagaoka, K., Yoshimura, S., Vranješ, J., Kono, M., Tanaka, M.Y.: Experimental observation of a tripolar vortex in a plasma. Phys. Plasmas **10**, 2211–2216 (2003)
40. Paumond, L.: Some remarks on a Hasegawa–Mima—Charney—Obukhov equation. Physica D **195**, 379–390 (2004)
41. Pedlosky, J.: Geophysical Fluids Dynamics. Springer, New York (1987)
42. Sandberg, I., Isliker, H., Pavlenko, V.P., Hizanidis, K., Vlahos, L.: Generation and saturation of large-scale flows in flute turbulence. Phys. Plasmas **12**, 032503 (2005)
43. Stals, L.: A study of the Hasegawa–Wakatani equations using an implicit explicit backward differentiation formula. ANZIAM J. **50**, C519–C533 (2008)
44. Sundkvist, D., Krasnoselskikh, V., Shukla, P.K., Vaivads, A., André, M., Buchert, S., Réme, H.: In situ multi-satellite detection of coherent vortices as a manifestation of Alfvénic turbulence. Nature **436**, 825–828 (2005)
45. Tassi, E., Garcia, O.E., Paulsen, J.V., Rypdal, K., Riccardi, C.: Three-field model for drift waves in a simple magnetized torus. Phys. Scr. **T113**, 121–129 (2004)
46. Wakatani, M.: Stellarator and Heliotron Devices. Oxford University Press, Oxford (1998)
47. Wu, X.: On the global well-posedness of the magnetic-curvature-driven plasma equations with random effects in $\mathbf{R}^3$. Commun. Math. Sci. **13**, 1665–1681 (2015)
48. Zhang, R., Guo, B.: Global attractor for Hasegawa–Mima equation. Appl. Math. Mech. **27**, 567–574 (2006)
49. Zhang, R., Guo, B.: Dynamical behavior for the three-dimensional generalized Hasegawa–Mima equations. J. Math. Phys. **48**, 012703 (2007)