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Uniform boundedness of the attractor in H^2 of a non-autonomous epidemiological system

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Abstract In this paper, we prove the uniform boundedness of the pullback attractor of a non-autonomous SIR (susceptible, infected, recovered) model from epidemiology considered in Anguiano and Kloeden (Commun Pure Appl Anal 13(1):157–173, 2014). We prove two uniform bounds of this pullback attractor, firstly in the norm H_0^1 and later, under appropriate additional assumptions, in the norm H^2.

Keywords SIR epidemic model with diffusion · Invariant sets · Uniform boundedness in H^2

Mathematics Subject Classification 35B41 · 37B55

1 Introduction and setting of the problem

Epidemiology is the study of the spread of diseases with the objective of tracing factors that are responsible for or contribute to their occurrence. Mathematical models are used extensively in the study of epidemiological phenomena. Most models for the transmission of infectious diseases (see, for instance, Anderson and May [1], Brauer et al. [4]) descend from the classical SIR model of Kermack and McKendrick [8] established in 1927. Its classical form involves a system of autonomous ordinary differential equations for three classes, the susceptible S, infective I and recovered R, of a constant total population.

There is a strong biological motivation to include time-dependent terms into epidemiological models; for instance, temporally varying forcing is typical of seasonal variation of a disease (see Keeling et al. [7], Stone et al. [10]).

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We consider the following model (1)–(3) below, a classical and well-known model from mathematical epidemiology in the form of the SIR equations, with diffusion, in which a temporal forcing term is considered.

Several approaches have been used for this model, like the theory of non-autonomous dynamical systems. Some questions addressed concerning this model are the existence of solution or the existence of a pullback attractor, i.e., a family of time-dependent compact sets which is invariant and pullback attracts autonomous bounded sets. An important matter is why the attractor has to be unique. It is obvious that the attractor is minimal with respect to set inclusion, and that is the only way to talk about uniqueness when dealing with a universe of autonomous bounded sets, since the attractor is not an object of the universe and cannot be attracted by itself.

In this sense, in Anguiano and Kloeden [2] we prove the existence and uniqueness of positive solutions of (1)–(3) for initial data in L^2, and we establish that if the non-autonomous term takes positive bounded values, the process associated with (1)–(3) has a unique pullback attractor $\mathcal{A}$.

Recently, Tan and Ji [11] have proved the existence of pullback attractors in higher integrable spaces. In particular, the authors show, for $\delta \geq 0$, the existence of a $(L^2, L^{2+\delta})$ pullback attractor for (1)–(3) establishing *a priori* estimates for the difference of solutions of (1)–(3) by a bootstrap argument.

Another question is the study of regularity for this model. For instance, in Anguiano [3] we establish a regularity result for the unique positive solution to problem (1)–(3), and we prove some regularity results for the pullback attractor $\mathcal{A}$ obtained in [2]. This study motivated the investigation of the problem considered in this paper. Moreover, as far as we know, there are no results in the literature concerning the uniform boundedness of the pullback attractor $\mathcal{A}$ as we will consider in the present paper.

Let us introduce the model we will be involved with in this paper. Let $\Omega \subset \mathbb{R}^d$, where $d \geq 1$, be a bounded domain with a smooth boundary $\partial\Omega$. We consider the following problem for a temporally forced SIR (susceptible, infected, recovered) model with diffusion

$$\left.\begin{aligned} &\frac{\partial S}{\partial t} - \Delta S = aq(t) - aS + bI - \gamma \frac{SI}{N} && \text{in } \Omega \times (t_0, +\infty), \\ &\frac{\partial I}{\partial t} - \Delta I = -(a+b+c)I + \gamma \frac{SI}{N} && \text{in } \Omega \times (t_0, +\infty), \\ &\frac{\partial R}{\partial t} - \Delta R = cI - aR && \text{in } \Omega \times (t_0, +\infty), \end{aligned}\right\} \tag{1}$$

where $S(x,t)$, $I(x,t)$ and $R(x,t)$ denote the number of individuals at time t in susceptible class, infective class and recovered class, respectively, $N = S + I + R$, and $t_0 \in \mathbb{R}$. The parameter a is the per capita disease-induced death rate, b is the excess per capita death rate of the infective class, c is the per capita recovery rate of the infected individuals, and γ is the contact transmission rate.

We deal the problem with Dirichlet boundary condition

$$S(x,t) = I(x,t) = R(x,t) = 0 \text{ on } \quad \partial\Omega \times (t_0, +\infty), \tag{2}$$

and initial condition

$$S(x,t_0) = S_0(x), \quad I(x,t_0) = I_0(x), \quad R(x,t_0) = R_0(x) \text{ for } x \in \Omega. \tag{3}$$

We assume that the parameters a, b, c and γ are positive constants such that $\gamma + \frac{b}{2} + \frac{c}{2} < \lambda_1$, where $\lambda_1 > 0$ is the first eigenvalue of the negative Laplacian with zero Dirichlet boundary condition in Ω. The temporal forcing term is given by a continuous function $q : \mathbb{R} \to \mathbb{R}$

taking positive bounded values, i.e., $q(t) \in [q^-, q^+]$ for all $t \in \mathbb{R}$ where $0 < q^- \leq q^+$, such that $q' \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ and satisfies

$$\sup_{t_0 \in \mathbb{R}} \int_{t_0}^{t_0+1} |q'(s)|^2_{L^2(\Omega)} \, ds < \infty. \tag{4}$$

The choice of Dirichlet boundary conditions and the space $L^2(\Omega)$ here are to facilitate the derivation of the required estimates. Solutions in the space $L^1(\Omega)$ are more typical in many biological situations, but due to the special structure of the system (and its possible variants) we note that the solutions have stronger regularity, in particular they are also in the space $L^\infty(\Omega)$, and $L^1(\Omega) \cap L^\infty(\Omega)$ is a subspace of $L^2(\Omega)$.

The structure of the paper is as follows. In Sect. 2, we prove the uniform boundedness of the attractor $\mathcal{A}$ in $H_0^1(\Omega)^3$. Then, under appropriate additional assumptions, the uniform boundedness in $H^2(\Omega)^3$ of $\mathcal{A}$ is proved in Sect. 3. A conclusion section is established in Sect. 4.

2 Uniform boundedness of the pullback attractor in $H_0^1(\Omega)^3$

Let us introduce the functions spaces we will be used with in this paper. $L^2(\Omega)$ denotes the space of square integrable real-valued functions defined on Ω with the norm $|\cdot|_{L^2(\Omega)}$ corresponding to the scalar product defined by

$$(u, v) = \int_\Omega u \cdot v \, dx \quad \forall u, v \in L^2(\Omega),$$

while $H_0^1(\Omega)$ denotes the space of such functions satisfying the Dirichlet boundary condition that have square integrable generalized derivatives with the scalar product

$$((u, v)) := (\nabla u, \nabla v) = \int_\Omega \nabla u \cdot \nabla u \, dx \quad \forall u, v \in H_0^1(\Omega),$$

and the norm

$$||u|| := |\nabla u|_{L^2(\Omega)} \quad \forall u \in H_0^1(\Omega).$$

We will denote by $\langle \cdot, \cdot \rangle$ the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

In addition, X_3 denotes the space of functions $(u_1, u_2, u_3) \in L^2(\Omega)^3$ with the scalar product

$$((u_1, u_2, u_3), (v_1, v_2, v_3)) = (u_1, v_1) + (u_2, v_2) + (u_3, v_3),$$

and norm

$$|(u_1, u_2, u_3)|_{L^2(\Omega)} = |u_1|_{L^2(\Omega)} + |u_2|_{L^2(\Omega)} + |u_3|_{L^2(\Omega)},$$

for all (u_1, u_2, u_3),$(v_1, v_2, v_3) \in X_3$, while Y_3 denotes the space of functions $(u_1, u_2, u_3) \in H_0^1(\Omega)^3$ with the scalar product

$$(((u_1, u_2, u_3), (v_1, v_2, v_3))) = ((u_1, v_1)) + ((u_2, v_2)) + ((u_3, v_3)),$$

and norm

$$||(u_1, u_2, u_3)|| = ||u_1|| + ||u_2|| + ||u_3||,$$

for all $(u_1, u_2, u_3),(v_1, v_2, v_3) \in Y_3$. Finally, let X_3^+ be the subspace of nonnegative functions in X_3 and Y_3^+ be the subspace of nonnegative functions in Y_3.

The globally defined nonnegative solutions of (1)–(3) generate a process in the Banach space X_3^+ (see Anguiano and Kloeden [2] for more details), i.e., a family of mappings $U_{t,t_0} : X_3^+ \to X_3^+$ with $t \geq t_0$ in $\mathbb{R}$ satisfying

$$U_{t_0,t_0}x = x, \quad U_{t,t_0}x = U_{t,r} \circ U_{r,t_0}x,$$

for all $t_0 \leq r \leq t$ and $x \in X_3^+$. In [2, Proposition 1] we established that the 2-parameter family of mappings $U_{t,t_0} : X_3^+ \to X_3^+$, $t_0 \leq t$, given by

$$U_{t,t_0}(S_0, I_0, R_0) = (S(t), I(t), R(t)), \tag{5}$$

where $(S(t), I(t), R(t))$ is the unique positive solution of (1)–(3) with the initial value (S_0, I_0, R_0), defines a continuous process on X_3^+.

Recall that a pullback attractor for the process U_{t,t_0} (e.g., cf. Crauel et al. [5]) in the space X_3^+ is a family $\mathcal{A} = \{\mathcal{A}(t), t \in \mathbb{R}\}$ of non-empty compact subsets of X_3^+, which is invariant in the sense that

$$U_{t,t_0}\mathcal{A}(t_0) = \mathcal{A}(t), \qquad \text{for all } t \geq t_0,$$

and pullback attracts bounded subsets D of X_3^+, i.e.,

$$\operatorname{dist}_{X_3^+}\left(U_{t,t_0}D, \mathcal{A}(t)\right) \to 0 \quad \text{as} \quad t_0 \to -\infty,$$

where we denote by $\operatorname{dist}_{X_3^+}(\cdot, \cdot)$ the Hausdorff semi-distance in X_3^+.

In [2, Theorem 6.2, Remark 6] we establish that the process associated with (1)–(3) has a unique pullback attractor $\mathcal{A}$, which satisfies

$$\mathcal{A}(t) \subset \Sigma_3^+, \quad \text{for each } t \in \mathbb{R}, \tag{6}$$

where Σ_3^+ is a closed and bounded subset of X_3^+.

We recall a lemma (see Robinson [9] for more details) which is necessary for the proof of our results.

Lemma 1 *Let X, Y be Banach spaces such that X is reflexive, and the inclusion $X \subset Y$ is continuous. Assume that $\{u_n\}$ is a bounded sequence in $L^\infty(t_0, T; X)$ such that $u_n \rightharpoonup u$ weakly in $L^q(t_0, T; X)$ for some $q \in [1, +\infty)$ and $u \in C^0([t_0, T]; Y)$. Then, $u(t) \in X$ for all $t \in [t_0, T]$ and*

$$\|u(t)\|_X \leq \sup_{n \geq 1} \|u_n\|_{L^\infty(t_0,T;X)} \quad \forall t \in [t_0, T].$$

Let $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ be the linear operator associated with the negative Laplacian. The operator A is symmetric, coercive and continuous.

Since the space $H_0^1(\Omega)$ is included in $L^2(\Omega)$ with compact injection, as a consequence of the Hilbert–Schmidt Theorem there exists a non-decreasing sequence $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ of eigenvalues of A with zero Dirichlet boundary condition in Ω, with $\lim_{j\to\infty} \lambda_j = +\infty$ and there exists an orthonormal basis of Hilbert $\{w_j : \ j \geq 1\}$ of $L^2(\Omega)$ and orthogonal in $H_0^1(\Omega)$ with $V_n := \operatorname{span}\{w_j : \ 1 \leq j \leq n\}$ and $\{V_n : n \in \mathbb{N}\}$ densely embedded in $H_0^1(\Omega)$, such that

$$Aw_j = \lambda_j w_j \text{ for all } j \geq 1.$$

For each integer $n \geq 1$, we denote by

$$(S_n(t), I_n(t), R_n(t)) = (S_n(t; t_0, S_0), I_n(t; t_0, I_0), R_n(t; t_0, R_0))$$

the Galerkin approximation of the solution $(S(t; t_0, S_0), I(t; t_0, I_0), R(t; t_0, R_0))$ of (1)–(3), which is given by

$$S_n(t) = \sum_{j=1}^{n} \gamma_{nj}^1(t) w_j, \; I_n(t) = \sum_{j=1}^{n} \gamma_{nj}^2(t) w_j, \; R_n(t) = \sum_{j=1}^{n} \gamma_{nj}^3(t) w_j,$$

and is the solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(S_n(t), w_j\big) = \big\langle \Delta S_n(t), w_j\big\rangle + \big(f_1(S_n(t), I_n(t), R_n(t), t), w_j\big),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\big(I_n(t), w_j\big) = \big\langle \Delta I_n(t), w_j\big\rangle + \big(f_2(S_n(t), I_n(t), R_n(t)), w_j\big),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\big(R_n(t), w_j\big) = \big\langle \Delta R_n(t), w_j\big\rangle + \big(f_3(S_n(t), I_n(t), R_n(t)), w_j\big),$$

with initial data

$$\big(S_n(t_0), w_j\big) = \big(S_0, w_j\big), \big(I_n(t_0), w_j\big) = \big(I_0, w_j\big), \big(R_n(t_0), w_j\big) = \big(R_0, w_j\big),$$

for all $w_j \in V_n$, where

$$\gamma_{nj}^1(t) = (S_n(t), w_j), \;\; \gamma_{nj}^2(t) = (I_n(t), w_j), \;\; \gamma_{nj}^3(t) = (R_n(t), w_j).$$

We denote

$$f_1(S_n(t), I_n(t), R_n(t), t) := aq(t) - aS_n(t) + bI_n(t) - \gamma \frac{S_n(t)I_n(t)}{N_n(t)},$$
$$f_2(S_n(t), I_n(t), R_n(t)) := -(a+b+c)I_n(t) + \gamma \frac{S_n(t)I_n(t)}{N_n(t)},$$
$$f_3(S_n(t), I_n(t), R_n(t)) := cI_n(t) - aR_n(t),$$

where

$$N_n(t) = S_n(t) + I_n(t) + R_n(t).$$

On the other hand, if we denote

$$D(A) = \big\{v \in H_0^1(\Omega) : Av \in L^2(\Omega)\big\},$$

with the scalar product

$$(v, w)_{D(A)} = (Av, Aw) \;\; \forall v, w \in D(A),$$

then $D(A)$ is a Hilbert space, and $D(A)$ is included in $H_0^1(\Omega)$ with continuous and dense injection. Let $D(A)^+$ be the subspace of nonnegative functions in $D(A)$.

Remark 1 We note that if $\Omega \subset \mathbb{R}^d$ is a bounded C^2 domain, then we have that $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, and moreover, the norm induced by $(\cdot, \cdot)_{D(A)}$ in $D(A)$ and the norm of $H^2(\Omega)$ are equivalent.

Now, in our first main result, we prove the uniform boundedness of the attractor $\mathcal{A}(t)$ in $H_0^1(\Omega)^3$.

Theorem 1 *Suppose that $\Omega \subset \mathbb{R}^d$ is a bounded C^2 domain and assume that $\gamma + \frac{b}{2} + \frac{c}{2} < \lambda_1$ where λ_1 is the first eigenvalue of the operator A on the domain Ω with Dirichlet boundary condition. Then $\mathcal{A}(t)$ is uniformly bounded in t in $H_0^1(\Omega)^3$.*

Proof From the inequality (27) of [3], for any $t \geq t_0$ we have

$$\begin{aligned} &|S_n(r)|^2_{L^2(\Omega)} + |I_n(r)|^2_{L^2(\Omega)} + |R_n(r)|^2_{L^2(\Omega)} \\ &\quad + \int_{t_0}^r \left(\|S_n(s)\|^2 + \|I_n(s)\|^2 + \|R_n(s)\|^2\right) \mathrm{d}s \\ &\leq C_1 \left(|S_0|^2_{L^2(\Omega)} + |I_0|^2_{L^2(\Omega)} + |R_0|^2_{L^2(\Omega)} + (t - t_0)\right), \end{aligned} \tag{7}$$

for all $r \in [t_0, t]$, and all $n \geq 1$, where $C_1 := \dfrac{\max\left\{1, \frac{a}{2}(q^+)^2\,|\Omega|\right\}}{\min\left\{1, 2 - \lambda_1^{-1}(b + c + 2\gamma)\right\}}$.

From (7) and (26) in [3], we now obtain that

$$\begin{aligned} &(r - t_0)\left(\|S_n(r)\|^2 + \|I_n(r)\|^2 + \|R_n(r)\|^2\right) \\ &\leq C_1 \left(|S_0|^2_{L^2(\Omega)} + |I_0|^2_{L^2(\Omega)} + |R_0|^2_{L^2(\Omega)} + (t - t_0)\right) \\ &\quad + (q^+)^2\,|\Omega|\,(t - t_0)^2 (2a^2 + \frac{a}{2} k_1 C) \\ &\quad + k_1 C \left(|S_0|^2_{L^2(\Omega)} + |I_0|^2_{L^2(\Omega)} + |R_0|^2_{L^2(\Omega)}\right)(t - t_0), \end{aligned} \tag{8}$$

for any $t \geq t_0$, all $r \in [t_0, t]$, and all $n \geq 1$, where $C := (2\lambda_1 - b - c - 2\gamma)^{-1}$ and k_1 is a positive constant.

In particular, from (8) we deduce

$$\|S_n(r)\|^2 + \|I_n(r)\|^2 + \|R_n(r)\|^2 \leq C_2 \left(|S_0|^2_{L^2(\Omega)} + |I_0|^2_{L^2(\Omega)} + |R_0|^2_{L^2(\Omega)} + 1\right), \tag{9}$$

for all $r \in [t_0 + 1, t_0 + 2]$, and any $n \geq 1$, where

$$C_2 := \max\left\{C_1 + 2k_1 C, 2C_1 + 4(q^+)^2 |\Omega| \left(2a^2 + \frac{a}{2} k_1 C\right)\right\}.$$

Using Lemma 3 in [3], we have that $(S_n(\cdot), I_n(\cdot), R_n(\cdot)) = (S_n(\cdot; t_0, S_0), I_n(\cdot; t_0, I_0), R_n(\cdot; t_0, R_0))$ converges weakly to the unique solution to the problem (1)–(3) $(S(\cdot), I(\cdot), R(\cdot)) = (S(\cdot; t_0, S_0), I(\cdot; t_0, I_0), R(\cdot; t_0, R_0))$ in $L^2(t_0, t; (Y^+)^3)$, for all $t > t_0$. Thus, from (9) and Lemma 1, we in particular obtain

$$\|S(t_0 + 1)\|^2 + \|I(t_0 + 1)\|^2 + \|R(t_0 + 1)\|^2 \leq C_2 \left(|S_0|^2_{L^2(\Omega)} + |I_0|^2_{L^2(\Omega)} + |R_0|^2_{L^2(\Omega)} + 1\right),$$

which together with (6) imply that $\mathcal{A}(t)$ is uniformly bounded in t in $H_0^1(\Omega)^3$. □

3 Uniform boundedness of the pullback attractor in $H^2(\Omega)^3$

The aim of this section is to continue with the analysis of the model in the sense of proving that the attractor $\mathcal{A}(t)$ is uniformly bounded in the space $H^2(\Omega)^3$, provided some additional assumptions are fulfilled. Our second main result is the following.

Theorem 2 *In addition to the assumptions in Theorem 1, assume moreover that* $q' \in L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega))$ *and satisfies (4). Then* $\mathcal{A}(t)$ *is uniformly bounded in* t *in* $H^2(\Omega)^3$.

Proof From inequality (35) in [3], taking $t = t_0 + 3$ and $\varepsilon = 2$, we have

$$\begin{aligned} &\left|S_n'(r)\right|^2_{L^2(\Omega)} + \left|I_n'(r)\right|^2_{L^2(\Omega)} + \left|R_n'(r)\right|^2_{L^2(\Omega)} \\ &\quad \le (4k_3+1)\int_{t_0+1}^{t_0+3} \left(\left|S_n'(\theta)\right|^2_{L^2(\Omega)} + \left|I_n'(\theta)\right|^2_{L^2(\Omega)} + \left|R_n'(\theta)\right|^2_{L^2(\Omega)}\right) \mathrm{d}\theta \\ &\qquad + a\int_{t_0+1}^{t_0+3} \left|q'(\theta)\right|^2_{L^2(\Omega)} \mathrm{d}\theta, \end{aligned} \tag{10}$$

for all $r \in [t_0+2, t_0+3]$, and any $n \ge 1$, where k_3 is a positive constant.

Analogously, and if we take $s = t_0 + 1$ and $r = t = t_0 + 3$ in inequality (25) of [3], we, in particular, have

$$\begin{aligned} &\int_{t_0+1}^{t_0+3} \left(\left|S_n'(\theta)\right|^2_{L^2(\Omega)} + \left|I_n'(\theta)\right|^2_{L^2(\Omega)} + \left|R_n'(\theta)\right|^2_{L^2(\Omega)}\right) \mathrm{d}\theta \\ &\quad \le \|S_n(t_0+1)\|^2 + \|I_n(t_0+1)\|^2 + \|R_n(t_0+1)\|^2 \\ &\qquad + 3(q^+)^2\,|\Omega|\,(2a^2 + \frac{a}{2}k_1 C) \\ &\qquad + k_1 C\left(|S_0|^2_{L^2(\Omega)} + |I_0|^2_{L^2(\Omega)} + |R_0|^2_{L^2(\Omega)}\right), \end{aligned} \tag{11}$$

for all $n \ge 1$, where k_1 is a positive constant and $C := (2\lambda_1 - b - c - 2\gamma)^{-1}$.

From (10) and (11), we obtain

$$\begin{aligned} &\left|S_n'(r)\right|^2_{L^2(\Omega)} + \left|I_n'(r)\right|^2_{L^2(\Omega)} + \left|R_n'(r)\right|^2_{L^2(\Omega)} \\ &\quad \le (4k_3+1)\left(\|S_n(t_0+1)\|^2 + \|I_n(t_0+1)\|^2 + \|R_n(t_0+1)\|^2\right) \\ &\qquad + (4k_3+1)\,3(q^+)^2\,|\Omega|\,(2a^2 + \frac{a}{2}k_1 C) + a\int_{t_0+1}^{t_0+3} \left|q'(\theta)\right|^2_{L^2(\Omega)} \mathrm{d}\theta \\ &\qquad + (4k_3+1)\,k_1 C\left(|S_0|^2_{L^2(\Omega)} + |I_0|^2_{L^2(\Omega)} + |R_0|^2_{L^2(\Omega)}\right), \end{aligned}$$

for all $r \in [t_0+2, t_0+3]$, and any $n \ge 1$.

Owing to this inequality and (9), there exists a constant $\widetilde{C}_1 > 0$ such that

$$\begin{aligned} &\left|S_n'(r)\right|^2_{L^2(\Omega)} + \left|I_n'(r)\right|^2_{L^2(\Omega)} + \left|R_n'(r)\right|^2_{L^2(\Omega)} \\ &\quad \le \widetilde{C}_1\left(|S_0|^2_{L^2(\Omega)} + |I_0|^2_{L^2(\Omega)} + |R_0|^2_{L^2(\Omega)} + \int_{t_0+1}^{t_0+3} \left|q'(\theta)\right|^2_{L^2(\Omega)} \mathrm{d}\theta + 1\right), \end{aligned} \tag{12}$$

for all $r \in [t_0+2, t_0+3]$, and any $n \ge 1$.

From inequality (36) of [3], and thanks to (12), we have

$$\begin{aligned} &|\Delta S_n(r)|^2_{L^2(\Omega)} + |\Delta I_n(r)|^2_{L^2(\Omega)} + |\Delta R_n(r)|^2_{L^2(\Omega)} \\ &\quad \le 4\widetilde{C}_1\left(|S_0|^2_{L^2(\Omega)} + |I_0|^2_{L^2(\Omega)} + |R_0|^2_{L^2(\Omega)} + \int_{t_0+1}^{t_0+3} \left|q'(\theta)\right|^2_{L^2(\Omega)} \mathrm{d}\theta + 1\right) \\ &\qquad + 8a^2(q^+)^2\,|\Omega| + 4k_2\left(|S_n(r)|^2_{L^2(\Omega)} + |I_n(r)|^2_{L^2(\Omega)} + |R_n(r)|^2_{L^2(\Omega)}\right), \end{aligned}$$

for all $r \in [t_0+2, t_0+3]$, and any $n \ge 1$, where k_2 is a positive constant.

Therefore, by (7) we obtain that there exists a constant $\widetilde{C}_2 > 0$ such that

$$\begin{aligned} &|\Delta S_n(r)|^2_{L^2(\Omega)} + |\Delta I_n(r)|^2_{L^2(\Omega)} + |\Delta R_n(r)|^2_{L^2(\Omega)} \\ &\quad \leq \widetilde{C}_2 \left(|S_0|^2_{L^2(\Omega)} + |I_0|^2_{L^2(\Omega)} + |R_0|^2_{L^2(\Omega)} + \int_{t_0+1}^{t_0+3} \left| q'(\theta) \right|^2_{L^2(\Omega)} \mathrm{d}\theta + 1 \right), \end{aligned} \tag{13}$$

for all $r \in [t_0 + 2, t_0 + 3]$, and any $n \geq 1$.

By Theorem 6 in [3], we have that $(S(\cdot; t_0, S_0), I(\cdot; t_0, I_0), R(\cdot; t_0, R_0)) \in C\left([t_0 + 2, t_0 + 3]; Y_3^+\right)$. On the other hand, in the proof of Theorem 4 in [3], we proved that $\{(S_n(\cdot; t_0, S_0), I_n(\cdot; t_0, I_0), R_n(\cdot; t_0, R_0))\}$ is bounded in $L^2(t_0, t; (D(A)^+)^3)$ for all $t > t_0$. Then, we have that $(S_n(\cdot), I_n(\cdot), R_n(\cdot)) = (S_n(\cdot; t_0, S_0), I_n(\cdot; t_0, I_0), R_n(\cdot; t_0, R_0))$ converges weakly to the unique solution, $(S(\cdot), I(\cdot), R(\cdot)) = (S(\cdot; t_0, S_0), I(\cdot; t_0, I_0), R(\cdot; t_0, R_0))$, to (1)–(3) in $L^2(t_0 + 2, t_0 + 3; (D(A)^+)^3)$.

Then, by Lemma 1, inequality (13) and the equivalence of the norms $|\Delta v|_{L^2(\Omega)}$ and $\|v\|_{H^2(\Omega)}$, we have that there exists a constant $\widetilde{C}_3 > 0$ such that

$$\begin{aligned} &\|(S(r; t_0, S_0), I(r; t_0, I_0), R(r; t_0, R_0))\|^2_{H^2(\Omega)^3} \\ &\quad \leq \widetilde{C}_3 \left(|S_0|^2_{L^2(\Omega)} + |I_0|^2_{L^2(\Omega)} + |R_0|^2_{L^2(\Omega)} + \int_{t_0+1}^{t_0+3} \left| q'(\theta) \right|^2_{L^2(\Omega)} \mathrm{d}\theta + 1 \right), \end{aligned} \tag{14}$$

for all $r \in [t_0 + 2, t_0 + 3]$, any $t_0 \in \mathbb{R}$, and $(S_0, I_0, R_0) \in X_3^+$.

Thus, from (14), and using (5), we deduce that there exists a constant $\widetilde{C}_4 > 0$ such that

$$\begin{aligned} &\|U_{t_0+2,t_0}(S_0, I_0, R_0)\|^2_{H^2(\Omega)^3} \\ &\quad \leq \widetilde{C}_4 \left(|(S_0, I_0, R_0)|^2_{L^2(\Omega)} + \int_{t_0+1}^{t_0+3} \left| q'(\theta) \right|^2_{L^2(\Omega)} \mathrm{d}\theta + 1 \right), \end{aligned}$$

for all $t_0 \in \mathbb{R}$, $(S_0, I_0, R_0) \in X_3^+$.

From this inequality, and the fact that $\mathcal{A}(t_0) = U_{t_0,t_0-2}\mathcal{A}(t_0 - 2)$, we obtain

$$\begin{aligned} &\|(v_1, v_2, v_3)\|^2_{H^2(\Omega)^3} \\ &\quad \leq \widetilde{C}_4 \left(\sup_{(w_1,w_2,w_3)\in\mathcal{A}(t_0-2)} |(w_1, w_2, w_3)|^2_{L^2(\Omega)} + \int_{t_0-1}^{t_0+1} \left| q'(\theta) \right|^2_{L^2(\Omega)} \mathrm{d}\theta + 1 \right), \end{aligned} \tag{15}$$

for all $(v_1, v_2, v_3) \in \mathcal{A}(t_0)$, and any $t_0 \in \mathbb{R}$.

Now, from (6) and (15), we have that there exists $M > 0$ such that

$$\left(\sup_{(v_1,v_2,v_3)\in\mathcal{A}(t_0)} \|(v_1, v_2, v_3)\|_{H^2(\Omega)^3} \right)^2 \leq M + \int_{t_0-1}^{t_0+1} \left| q'(\theta) \right|^2_{L^2(\Omega)} \mathrm{d}\theta,$$

for any $t_0 \in \mathbb{R}$. Finally, assumption (4) implies the uniform boundedness of $\mathcal{A}(t)$ in $H^2(\Omega)^3$. □

4 Conclusions

An infectious disease is considered where all classes, susceptible, infective and recovered, diffuse in space with the same diffusion constant. The model considered in this paper is more general than the typical SIR model as it allows some infective individual to move directly back into the susceptible class rather than into the recovered class. Moreover, the model considered is non-autonomous because there is seasonal recruitment into the susceptible class.

In Anguiano and Kloeden [2], we prove that the process associated with this model has a unique pullback attractor $\mathcal{A} = \{A(t) : t \in \mathbb{R}\}$ in L^2, which is obtained by pullback convergence that makes use of information about the past of the non-autonomous dynamical system. It includes and is perhaps most realistic, when the non-autonomity arises from asymptotic autonomity or some sort of temporal recurrence such as periodicity or almost periodicity.

In the present paper, we have proved that $\mathcal{A} = \{A(t) : t \in \mathbb{R}\}$ is uniformly bounded in H^2, i.e., $\cup_{t\in\mathbb{R}} A(t)$ is bounded in H^2, which means the component subsets of pullback attractor are uniformly bounded; then the pullback attractor is characterized by the bounded entire solutions of the process. In particular, Proposition 7.1 in Kloeden et al. [6] guarantees us that a uniformly bounded pullback attractor $\mathcal{A}$ admits the dynamical characterization: for each $t_0 \in \mathbb{R}$

$$x_0 \in A(t_0) \Leftrightarrow \text{ there exists a bounded entire solution } (S, I, R) \\ \text{with } (S(t_0), I(t_0), R(t_0)) = x_0.$$

Such a pullback attractor is therefore uniquely determinated in H^2. Therefore, the pullback attractor gives us information about the state of the disease at a particular time, provided the disease has started long enough ago.

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References

1. Anderson, R.M., May, R.M.: Infectious Diseases of Humans, Dynamics and Control. Oxford University Press, Oxford (1992)
2. Anguiano, M., Kloeden, P.E.: Asymptotic behavior of the nonautonomous SIR equations with diffusion. Commun. Pure Appl. Anal. **13**(1), 157–173 (2014)
3. Anguiano, M.: H^2-boundedness of the pullback attractor for the non-autonomous SIR equations with diffusion. Nonlinear Ana. Theory Methods Appl. **113**, 180–189 (2015)
4. Brauer, F., van den Driessche, P., Wu, J. (eds.): Mathematical Epidemiology, Springer Lecture Notes in Mathematics, 1945. Springer, Heidelberg (2008)
5. Crauel, H., Debussche, A., Flandoli, F.: Random attractors. J. Dyn. Differ. Equ. **9**, 307–341 (1997)
6. Kloeden, P.E., Pötzsche, C., Rasmussen, M.: Discrete-time nonautonomous dynamical systems. In: Stability and Bifurcation Theory for Non-autonomous Differential Equations, Lectures Notes in Mathematics, 2065. Springer, Berlin, Heidelberg (2013)
7. Keeling, M.J., Rohani, P., Grenfell, B.T.: Seasonally forced disease dynamics explored as switching between attractors. Physica D **148**, 317–335 (2001)
8. Kermack, W.O., McKendrick, A.G.: Contributions to the mathematical theory of epidemics (part I). Proc. R. Soc. Lond. Ser. A **115**, 700–721 (1927)
9. Robinson, J.C.: Infinite-Dimensional Dynamical Systems. Cambridge University Press, Cambridge (2001)
10. Stone, L., Olinky, R., Huppert, A.: Seasonal dynamics of recurrent epidemics. Nature **446**, 533–536 (2007)
11. Tan, W., Ji, Y.: On the pullback attractor for the non-autonomous SIR equations with diffusion. J. Math. Anal. Appl. **449**(2), 1850–1862 (2017)