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On a Hecke-type functional equation with conductor $q = 5$

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Abstract We give a complete characterization of the solutions $F(s)$ of the analog in the Selberg class of Hecke's functional equation of conductor 5, namely

$$\left(\frac{\sqrt{5}}{2\pi}\right)^s \Gamma(s+\mu)F(s) = \omega \left(\frac{\sqrt{5}}{2\pi}\right)^{1-s} \Gamma(1-s+\overline{\mu})\overline{F(1-\overline{s})}$$

with $\Re\mu \geq 0$ and $|\omega| = 1$. The proof is based on several results from our theory of nonlinear twists of L-functions, applied to obtain a full description of the Euler factor of $F(s)$ at $p = 2$, and then on some ideas from a 1995 paper by J. B. Conrey and D. W. Farmer on converse theorems for Euler products.

Keywords L-functions · Hecke theory · Selberg class · Converse theorems

Mathematics Subject Classification 11M41 · 11F66

1 Introduction

Hecke's functional equation of signature (λ, k, ω), where $\lambda, k > 0$ and $|\omega| = 1$, is in normalized form

$$\Phi(s) = \omega\Phi(1-s), \tag{1.1}$$

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where

$$\Phi(s) = \left(\frac{\lambda}{2\pi}\right)^s \Gamma(s+\mu)F(s) \tag{1.2}$$

with $\mu = (k-1)/2$. The solutions $F(s)$ of (1.1) are required to be absolutely convergent Dirichlet series for $\sigma > 1$ such that $(s-1)F(s)$ is entire of finite order. In this situation, the conductor of $F(s)$, or of the functional equation (1.1), is $q = \lambda^2$. A well-known theorem of Hecke asserts that if $\lambda > 2$, then (1.1) has an uncountable number of linearly independent solutions $F(s)$; see Chapter 2 of Hecke [4], Chapter 1 of Ogg [16] and Chapter 4 of Berndt-Knopp [1].

In this paper, we deal with the case $q = 5$, the first integral conductor with an infinite-dimensional space of solutions, with a twofold aim. On the one hand, we show that as soon as standard arithmetical requirements, i.e., Euler product and Ramanujan conjecture, are added to $F(s)$, then the solutions of (1.1) drastically reduce to a finite number. Thus, in particular, all but finitely many solutions of (1.1) do not have at least one of these arithmetical properties. In addition, we give a full description of the solutions. Actually, we deal with a slightly different, but essentially equivalent, functional equation, namely the general case of degree $d = 2$ and conductor $q = 5$ with a single Γ-factor in the Selberg class; see (1.3) and (1.4), and Sect. 2 for definitions. Thus, we get a sharp converse theorem in this framework. Note that, from the point of view of modular forms, the Selberg class functional equation (1.3) is more appropriate than Hecke's (1.1), since the conjugation is required when dealing with newforms of level > 1.

The results in this paper are obtained by developing certain ideas in our previous papers on nonlinear twists of L-functions and their applications to converse theorems, see in particular [12] and [15], and then using some ideas in Conrey–Farmer [3]. More precisely, we first exploit the properties of nonlinear twists to detect the exact form of μ in (1.4) and of the Euler factor of $F(s)$ at $p = 2$. This is done in Sect. 2 and forms the bulk of the paper; actually, the results in Sect. 2 hold in a more general framework, see Propositions 1 and 2. Then, in Sect. 3 we suitably modify certain arguments in [3] to deduce the final result.

In order to state our results, we first recall the definition of the Selberg class in the special case under investigation in the paper; the general definition of the Selberg class $\mathcal{S}$, and of degree, conductor and other invariants, is given in the next section. Let $F(s)$ be an absolutely convergent Dirichlet series for $\sigma > 1$ such that $(s-1)^m F(s)$ is an entire function of finite order for some integer m, and satisfying the functional equation

$$\Phi(s) = \omega\overline{\Phi(1-\overline{s})}, \tag{1.3}$$

with $|\omega| = 1$ and

$$\Phi(s) = \left(\frac{\sqrt{q}}{2\pi}\right)^s \Gamma(s+\mu)F(s), \qquad q > 0,\ \Re\mu \geq 0. \tag{1.4}$$

Moreover, the Dirichlet coefficients $a(n)$ satisfy the Ramanujan conjecture $a(n) \ll n^\varepsilon$ for every $\varepsilon > 0$, and $F(s)$ has a general Euler product representation expressed in the form

$$\log F(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}, \tag{1.5}$$

with $b(n) = 0$ unless $n = p^m$, $m \geq 1$, and $b(n) \ll n^\vartheta$ for some $\vartheta < 1/2$. We denote by $\mathcal{S}_2(q,\mu)$ the class of such functions. Actually, $\mathcal{S}_2(q,\mu)$ is the subclass of $\mathcal{S}$ of the functions with degree 2, conductor q and a single Γ-factor in the functional equation. We refer to our surveys [6], [9], [17], [18], [19] and [20] for definitions and the basic theory of the Selberg class. Moreover, we recall that $S_k(\Gamma_0(N), \chi)$ denotes the space of holomorphic cusp forms $f(z)$ for $\Gamma_0(N)$ of weight k and character χ, and $L(s, f)$ is the Hecke L-function associated with $f(z)$. We also recall that a newform in $S_k(\Gamma_0(N), \chi)$ is a Hecke eigenform for all Hecke operators T_n that belongs to the subspace $S_k^\sharp(\Gamma_0(N), \chi)$; see Section 6.6 of Iwaniec [5].

Theorem *Let $F \in \mathcal{S}_2(5, \mu)$. Then there exist an integer $k \geq 1$ and a Dirichlet character χ* (mod 5) *such that*

$$\Re\mu = \frac{k-1}{2}, \qquad \chi(-1) = (-1)^k$$

and either

$$F(s) = \zeta(s)L(s, \chi)$$

or

$$F(s) = L(s + \mu, f)$$

for some newform $f \in S_k(\Gamma_0(5), \chi)$. In the first case, we have $\mu = 0$, and hence, in particular $k = 1$ and χ is odd and primitive.

We conclude with some remarks. First, arguing similarly as in Hecke's theory, one can show that the Dirichlet series solutions of (1.3) and (1.4) with $q > 4$, satisfying the above regularity conditions, form a real vector space with uncountable basis; see Carletti et al. [2]. We remark that this can also be derived by a direct argument from Hecke's theory. As already pointed out, our result shows, when $q = 5$, that adding the Euler product and Ramanujan conjecture requirements, the number of such solutions becomes finite. In this case, the solutions are automatically linearly independent thanks to the results by Kaczorowski–Molteni–Perelli [7] and [8]. Moreover, we note that the Ramanujan conjecture is used only in the proof of the density estimate in (ii) of Lemma 3, while the Euler product is exploited at several stages in the proof. Further, the above theorem gives some support to the conjectures that if $F \in \mathcal{S}$ has a pole at $s = 1$ then $\zeta(s)$ divides $F(s)$ and that the degree 2 functions in the Selberg class coincide with the GL_2 L-functions. We finally remark that, although here we deal explicitly only with the case of conductor $q = 5$, the involved ideas have a more general flavor, and we believe that other conductors can be dealt with along these lines. However, as for Conrey–Farmer [3], getting results of this type for every integer conductor $q \geq 5$ remains an open problem.

2 The Euler factor at $p = 2$

We first recall some further notation; again, we refer to our surveys listed in Introduction for a comprehensive account of the various definitions and results we need in the paper. The extended Selberg class $\mathcal{S}^\sharp$ contains the Dirichlet series $F(s)$, absolutely convergent for $\sigma > 1$ and such that $(s-1)^m F(s)$ extends to an entire function of finite order for some integer m, satisfying the functional equation (1.3) with

$$\Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s),$$

where $Q, \lambda_j > 0$ and $\Re\mu_j \geq 0$. The Selberg class $\mathcal{S}$ is the subclass of the functions $F \in \mathcal{S}^\sharp$ with an Euler product as in (1.5) and satisfying the Ramanujan conjecture, see Sect. 1. Given

$F \in \mathcal{S}$, for $\sigma > 1$ and p prime we write

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \quad \text{and} \quad F_p(s) = \sum_{m=0}^{\infty} \frac{a(p^m)}{p^{ms}}, \tag{2.1}$$

hence the Euler product (1.5) can be written in the more familiar form

$$F(s) = \prod_p F_p(s). \tag{2.2}$$

Degree, conductor, ξ-invariant and root number of $F \in \mathcal{S}^\sharp$ are, respectively, defined as

$$d_F = 2\sum_{j=1}^{r} \lambda_j, \qquad q_F = (2\pi)^{d_F} Q^2 \prod_{j=1}^{r} \lambda_j^{2\lambda_j},$$
$$\xi_F = 2\sum_{j=1}^{r} (\mu_j - 1/2) = \eta_F + i d_F \theta_F, \qquad \omega_F^* = \omega \prod_{j=1}^{r} \lambda_j^{2i\Im\mu_j};$$

for simplicity, *in what follows we drop the suffix F when referring to such invariants*. As already pointed out, if $\Phi(s)$ is as in (1.4), then $F(s)$ has degree 2. For $F \in \mathcal{S}^\sharp$, $\sigma > 1$ and $\alpha, \beta \in \mathbb{R}$, we consider the nonlinear twist

$$F(s, \alpha, \beta) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e(-\alpha n - \beta\sqrt{n}), \tag{2.3}$$

where $e(x) = \mathrm{e}^{2\pi i x}$; we write $\overline{F}(s, \alpha, \beta)$ for the twist in (2.3) with $\overline{a(n)}$ in place of $a(n)$. Note that $F(s, \alpha, \beta)$ reduces to the *linear twist* when $\beta = 0$ and, if $F(s)$ has degree 2, to the *standard twist* when $\alpha = 0$; the properties of such twists, reported in Lemma 2, are crucial in this paper. Moreover, if $F(s)$ has degree 2 and conductor q, we also write

$$n_\beta = \frac{q\beta^2}{4}, \qquad \mathrm{Spec}(F) = \{\beta > 0 : a(n_\beta) \neq 0\} = \left\{2\sqrt{\frac{m}{q}} : m \geq 1 \text{ integer}, a(m) \neq 0\right\};$$

here, and later, we write $a(x) = 0$ if $x \notin \mathbb{N}$. Further, we recall from [12] that the class $M(d, q)$, where $d, q > 0$, consists of the functions $f(s)$, meromorphic over $\mathbb{C}$ and holomorphic for $\sigma < 1$, with the property that for every $A < B$ there exists a constant $C = C(A, B)$ such that

$$f(\sigma + it) \ll |\sigma|^{d|\sigma|} \left(\frac{q}{(2\pi e)^d}\right)^{|\sigma|} |\sigma|^C$$

as $\sigma \to -\infty$, uniformly for $A \leq t \leq B$. Finally, for $F \in \mathcal{S}$ we write

$$N_F(\sigma, T) = |\{\rho : F(\rho) = 0, \Re\rho > \sigma, |\Im\rho| \leq T\}|.$$

The first lemma allows the normalization $\Im\mu = 0$ when dealing with $F \in \mathcal{S}^\sharp$ with $\Phi(s)$ as in (1.4).

Lemma 1 *If $F \in \mathcal{S}^\sharp$ has a pole at $s = 1$ and $\Phi(s)$ as in (1.4), then $\mu = 0$ and the pole is simple.*

Proof If $F(s)$ has a pole at $s = 1$, then $\overline{F}(1 - s)$ has a pole at $s = 0$. Moreover, if $\mu \neq 0$, then both $\Gamma(s + \mu)$ and $\Gamma(1 - s + \overline{\mu})$ are holomorphic and nonvanishing at $s = 0$. Therefore, in this case from the functional equation (1.3) and (1.4) we immediately see that $F(s)$ has a pole at $s = 0$, a contradiction; thus, $\mu = 0$. Further, since $\Gamma(s)$ has a simple pole at $s = 0$, again from the functional equation we see that the pole of $F(s)$ at $s = 1$ must be simple. □

Remark 1 As already mentioned, thanks to Lemma 1 we may assume that $\Im\mu = 0$. Indeed, if $F \in \mathcal{S}_2(q, \mu)$ with $\Re\mu \geq 0$ is entire, we can make the shift $s \mapsto s - i\Im\mu$ thus getting that $G(s) = F(s - i\Im\mu)$ belongs to $\mathcal{S}_2(q, \Re\mu)$. Hence, we may work with $G(s)$, and only in the end we shift back to $F(s)$.

Since $\theta = \Im\mu$ if $\Phi(s)$ is as in (1.4), in view of Remark 1 we may assume that $\theta = 0$ in what follows, which simplifies a bit the notation. Next we recall some results from our theory of nonlinear twists, specialized to the case of degree 2 functions (*recall that we drop the suffix F in the invariants of $F(s)$*).

Lemma 2 *Let $F \in \mathcal{S}^\sharp$ with degree 2 and $\theta = 0$.*

(i) *Let $\alpha > 0$. For every integer $K \geq 0$, there exist polynomials $Q_0(s), \ldots, Q_K(s)$, independent of α and with $Q_0(s) = 1$ identically, such that the linear twist $F(s, \alpha, 0)$ satisfies the transformation formula*

$$F(s, \alpha, 0) = -\omega^* e^{-i\frac{\pi}{2}\eta}(\alpha\sqrt{q})^{2s-1} \sum_{\nu=0}^{K} \alpha^\nu Q_\nu(s) \overline{F}\left(s + \nu, -\frac{1}{q\alpha}, 0\right) + H_K(s, \alpha), \tag{2.4}$$

where $H_K(s, \alpha)$ is holomorphic for $-K + 1/2 < \sigma < 2$ and $|s| \leq 2(K + 1)$ and satisfies $H_K(s, \alpha) \ll (A(K + 1))^K$ for some constant $A > 1$. Moreover, the polynomials $Q_\nu(s)$ satisfy

$$Q_\nu(s) \ll \frac{(A(|s| + 1))^{2\nu}}{\nu!} \quad \text{for} \quad \nu \leq \min(|s|, K)$$

and

$$Q_\nu(s) \ll (A(K + 1))^K \quad \text{for} \quad |s| \leq 2(K + 1) \text{ and } \nu \leq K.$$

(ii) *Let $\beta > 0$. If $\beta \notin Spec(F)$ then the standard twist $F(s, 0, \beta)$ is entire, while if $\beta \in$ Spec(F) then $F(s, 0, \beta)$ is meromorphic over $\mathbb{C}$ with at most simple poles at the points*

$$s_k = 3/4 - k/2, \qquad k = 0, 1, \ldots$$

and

$$\text{res}_{s=s_0} F(s, 0, \beta) = c_0(F) \frac{\overline{a(n_\beta)}}{q^{3/4} n_\beta^{1/4}}, \qquad c_0(F) \neq 0.$$

(iii) *If the first nonvanishing coefficient of $F(s)$ equals 1, $\alpha > 0$ and $\beta \in \mathbb{R}$, we have*

$$F(s, \alpha, \beta) = -\omega^* e^{-i\frac{\pi}{2}\eta}(\alpha\sqrt{q})^{2s-1} e(\beta^2/4\alpha) \overline{F}\left(s, -\frac{1}{q\alpha}, \frac{\beta}{\alpha\sqrt{q}}\right) + H(s, \alpha, \beta), \tag{2.5}$$

where $H(s, \alpha, \beta)$ is holomorphic for $\sigma > 1/2$.

Proof Equations (2.4) and (2.5) are special explicit cases of a general transformation formula for nonlinear twists of functions in $\mathcal{S}^\sharp$, see [11] and [14], and their meaning is that the difference between the LHS and the explicit terms on the RHS is holomorphic in the stated region. Moreover, (ii) contains some of the properties of the standard twist, see [10] and [13]. More precisely, (i) follows from Theorem 1.2 of [12], (ii) is a very special case of Theorems 1 and 2 of [10] (or of Theorem 2 of [13]), and (iii) is Lemma in [15]. □

Remark 2 If $F \in \mathcal{S}^\sharp$, then the conjugate function $\overline{F}(s) = \overline{F(\overline{s})}$ belongs to $\mathcal{S}^\sharp$, has conjugate Dirichlet coefficients and $d_{\overline{F}} = d_F$, $q_{\overline{F}} = q_F$, $\eta_{\overline{F}} = \eta_F$, $\omega^*_{\overline{F}} = \overline{\omega^*_F}$. Hence we may apply transformation formula (2.4) with $\overline{F}(s)$ in place of $F(s)$, with suitable modifications. In particular, we denote by $\overline{Q}_\nu(s)$ the Q_ν-polynomials in (2.4) referred to $\overline{F}(s)$. Note that what really matters in this paper about such polynomials for $\nu \geq 1$ are only the bounds they satisfy. Also, we take this opportunity to correct a slip occurred in [14], namely the function $\overline{F}(s^* + \eta_j; f^*)$, appearing in (1.10) of [14] and in later occasions, should be replaced by $\overline{F}(s^* + \eta_j; -f^*)$.

Further results from our previous papers, needed later on, are summarized in the following lemma.

Lemma 3 *(i) If $F \in \mathcal{S}^\sharp$ has degree d and conductor q, then $F(s)$ belongs to $M(d, q)$. Moreover, uniformly for $A \leq |t| \leq B$ with A sufficiently large, as $\sigma \to -\infty$ we have*

$$F(s) \gg |\sigma|^{d|\sigma|} \left(\frac{q}{(2\pi e)^d} \right)^{|\sigma|} |\sigma|^C \tag{2.6}$$

for some $C = C(A, B)$.

(ii) If $F \in \mathcal{S}$ has degree 2, then $N_F(\sigma, T) = o(T)$ for every fixed $\sigma > 1/2$.

Proof (i) is Lemma 2.1 in [12], while (ii) follows from the last displayed formula on p. 474 of [12]. □

The next lemma contains some arithmetical relations satisfied by the Euler products (2.2) and their linear twists. For simplicity, we assume that $F \in \mathcal{S}$, but actually the lemma holds in general for any Dirichlet series, absolutely convergent for $\sigma > 1$, with Euler factors at $p = 2$ and $p = 3$.

Lemma 4 *For $F \in \mathcal{S}$ and $\sigma > 1$, we have*

$$F(s, 1/4, 0) + F(s, 3/4, 0) = 2\big(1 - a(2)2^{1-s} F_2(s)^{-1} - F_2(s)^{-1}\big) F(s), \tag{2.7}$$

$$F(s, 1/3, 0) + F(s, 2/3, 0) = 2F(s) - 3F_3(s)^{-1} F(s) \tag{2.8}$$

and

$$F(s, 1/2, 0) = F(s) - 2F_2(s)^{-1} F(s). \tag{2.9}$$

Proof We have

$$\begin{aligned} F(s, 1/4, 0) &= \sum_{4|n} \frac{a(n)}{n^s} - \sum_{2\|n} \frac{a(n)}{n^s} + \sum_{2\nmid n} \frac{a(n)e(-n/4)}{n^s} \\ &= F(s) - 2\sum_{2\|n} \frac{a(n)}{n^s} - \sum_{2\nmid n} \frac{a(n)}{n^s} + \sum_{2\nmid n} \frac{a(n)e(-n/4)}{n^s} \\ &= F(s) - (a(2)2^{1-s} + 1) \sum_{2\nmid n} \frac{a(n)}{n^s} + \sum_{2\nmid n} \frac{a(n)e(-n/4)}{n^s} \\ &= \big(1 - a(2)2^{1-s} F_2(s)^{-1} - F_2(s)^{-1}\big) F(s) + \sum_{2\nmid n} \frac{a(n)e(-n/4)}{n^s}. \end{aligned}$$

Similarly, we have

$$F(s, 3/4, 0) = \big(1 - a(2)2^{1-s} F_2(s)^{-1} - F_2(s)^{-1}\big) F(s) + \sum_{2\nmid n} \frac{a(n)e(-3n/4)}{n^s}.$$

Adding the two identities, we obtain (2.7), since $2 \nmid n$ if and only if $n = 4k + a$ with $a \in \{1, 3\}$, and $e(-n/4) + e(-3n/4) = i^{-a} + i^{-3a} = i + i^{-1} = 0$.

To prove (2.8), we proceed in a similar way. Indeed

$$F(s, 1/3, 0) + F(s, 2/3, 0) = 2\sum_{3|n} \frac{a(n)}{n^s} + \sum_{3\nmid n} \frac{a(n)}{n^s}\big(e(-n/3) + e(-2n/3)\big);$$

but for $3 \nmid n$ we have $e(-n/3) + e(-2n/3) = e(-1/3) + e(-2/3) = -1$, hence

$$F(s, 1/3, 0) + F(s, 2/3, 0) = 2\left(F(s) - \sum_{3\nmid n} \frac{a(n)}{n^s}\right) - \sum_{3\nmid n} \frac{a(n)}{n^s} = 2F(s) - 3F_3(s)^{-1}F(s).$$

Finally,

$$F(s, 1/2, 0) = \sum_{2|n} \frac{a(n)}{n^s} - \sum_{2\nmid n} \frac{a(n)}{n^s} = F(s) - 2\sum_{2\nmid n} \frac{a(n)}{n^s},$$

and (2.9) follows. □

Now we deduce further results from the above lemmas. We explicitly remark that these results hold for any function in $\mathcal{S}$ with degree 2, conductor $q = 5$ and $\theta = 0$, not necessarily satisfying the special functional equation of form (1.4).

Lemma 5 *If $F \in \mathcal{S}$ has degree 2, conductor $q = 5$ and $\theta = 0$, then*

$$\big(1 + a(2)2^{1-s}\big)F_2(s)^{-1}$$

is a Dirichlet polynomial in 2^{-s} of degree ≤ 4.

Proof For $s = \sigma + it$ with $\sigma \geq 3/2$, we write

$$E(2^{-s}) = \big(1 + a(2)2^{1-s}\big)F_2(s)^{-1}.$$

We follow the basic strategy in the proof of Theorem 1.1 in [12], namely we exploit link (2.7) between $E(2^{-s})$ and linear twists to prove that $E(z)$ is an entire function satisfying a suitable bound, which then implies the final result. Of course, here we have first to prove that the involved linear twists have good analytic properties; this is done using (i) of Lemma 2.

Thanks to the assumption $\vartheta < 1/2$ in the definition of the Selberg class, see Sect. 1, we already know that $F_2(s)^{-1}$, and hence $E(2^{-s})$, is holomorphic, bounded and t-periodic of period $2\pi/\log 2$ for $\sigma > 1/2 - 2\delta$ for some $\delta > 0$; see, e.g., p. 956 of [9]. Hence, we may assume that $\sigma < 1/2 - \delta$. Let $c_1 > 0$ be sufficiently large, $K > 0$ be an arbitrarily large integer and let

$$D_K = \{s \in \mathbb{C} : 1 - K < \sigma < 1/2 - \delta,\ c_1 < t < K/2\}.$$

In view of the above properties, if $E(2^{-s})$ is holomorphic on D_K for arbitrarily large K, then $E(2^{-s})$ is an entire function. In what follows we use expressions of type $O(H)$ to denote holomorphic functions $f(s)$ on D_K satisfying $f(s) = O(H)$ in a specified range of s. Moreover, $A > 1$ denotes a constant whose value is not necessarily the same at each occurrence.

We start with (i) of Lemma 2, with $\alpha = 1/4$, $q = 5$ and a large integer K. Thanks to the 1-periodicity in α of (2.3), from (2.4) we have

$$F(s, 1/4, 0) = -\omega^* \mathrm{e}^{-i\frac{\pi}{2}\eta}(\sqrt{5}/4)^{2s-1} \sum_{\nu=0}^{K} 4^{-\nu} Q_\nu(s) \overline{F}(s+\nu, 1/5, 0) + O\left((A|\sigma|)^{|\sigma|}\right)$$

for $s \in D_K$ with $-K+1 < \sigma < -K+3$. In view of Remark 2, we may apply again the transformation formula (2.4) to $\overline{F}(s+\nu, 1/5, 0)$, with $\alpha = 1/5$, $q = 5$ and the sum extended to $\ell = 0, \dots, K-\nu$, thus getting that

$$\begin{aligned} F(s, 1/4, 0) &= \mathrm{e}^{-i\pi\eta}(\sqrt{5}/4)^{2s-1} \sum_{\nu=0}^{K} 4^{-\nu} Q_\nu(s) (\sqrt{5}/5)^{2(s+\nu)-1} \\ &\quad \times \sum_{\ell=0}^{K-\nu} 5^{-\ell} \overline{Q}_\ell(s+\nu) F(s+\nu+\ell) + O\big((A|\sigma|)^{|\sigma|}\big) \\ &\quad + O\left(4^{-2\sigma} \sum_{\nu=0}^{K} 4^{-\nu} |Q_\nu(s)| (A|\sigma|)^{|\sigma|-\nu+3}\right) \\ &= \Sigma(s) + O\big((A|\sigma|)^{|\sigma|}\big) + R, \end{aligned} \tag{2.10}$$

say, again for $s \in D_K$ with $-K+1 < \sigma < -K+3$. In particular, (2.10) shows that $F(s, 1/4, 0)$ is holomorphic on D_K, since the possible poles of $F(s+\nu+\ell)$ are on the real line. Thanks to the bounds in (i) of Lemma 2, we have

$$R \ll A^{|\sigma|} \sum_{\nu=0}^{K} \frac{(|\sigma|+2)^{2\nu}}{\nu!} (|\sigma+2)^{|\sigma|-\nu+3} + (A(|\sigma|+2))^{|\sigma|} \ll (A|\sigma|)^{|\sigma|}.$$

Moreover, in view of (i) of Lemmas 2 and 3 and the definition of the class $M(2, 5)$ we also have

$$\begin{aligned} \Sigma(s) &\ll (\sqrt{5}/4)^{2\sigma} \sum_{\nu=0}^{K} \sum_{\ell=0}^{K-\nu} 4^{-\nu} 5^{-\ell} (\sqrt{5}/5)^{2(\sigma+\nu)} \\ &\quad \times |Q_\nu(s) \overline{Q}_\ell(s+\nu)| |\sigma|^{2(|\sigma|-\nu-\ell)} \left(\frac{5}{(2\pi e)^2}\right)^{|\sigma|-\nu-\ell} |\sigma|^{c_2} \\ &\ll \left(\frac{16 \times 5}{(2\pi e)^2}\right)^{|\sigma|} |\sigma|^{2\sigma|} \sum_{\nu=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{A^{\nu+\ell}}{\nu! \ell!} |\sigma|^{c_2} \ll \left(\frac{16 \times 5}{(2\pi e)^2}\right)^{|\sigma|} |\sigma|^{2\sigma|+c_2} \end{aligned}$$

for some $c_2 > 0$. Therefore, from (2.10) we obtain that $F(s, 1/4, 0)$ is holomorphic on D_K, and for $s \in D_K$ with $-K+1 < \sigma < -K+3$ satisfies

$$F(s, 1/4, 0) \ll \left(\frac{16 \times 5}{(2\pi e)^2}\right)^{|\sigma|} |\sigma|^{2\sigma|+c}. \tag{2.11}$$

Since $F(s, 3/4, 0) = \overline{\overline{F}(\overline{s}, 1/4, 0)}$, the same properties hold for $F(s, 3/4, 0)$ as well.

By (2.7) in Lemma 4, we have

$$E(2^{-s}) = 1 - \frac{F(s, 1/4, 0) + F(s, 3/4, 0)}{2F(s)}, \tag{2.12}$$

hence $E(2^{-s})$ is holomorphic on D_K apart, possibly, at the zeros of $F(s)$. To show that $E(2^{-s})$ is actually holomorphic on D_K, we follow an argument already used in the proof of Theorem 1.1 of [12], which we outline for completeness. Indeed, since c_1 is sufficiently large, thanks to the density estimate (ii) of Lemma 3 and the functional equation, $F(s)$ has $o(K)$ zeros in D_K. But, by the uniqueness of meromorphic continuation, $E(2^{-s})$ is t-periodic on D_K, hence if not holomorphic, it must have $\gg K$ poles in D_K, a contradiction if K is sufficiently large. Therefore, $E(2^{-s})$ is holomorphic on D_K with K is arbitrarily large and hence, as already pointed out at the beginning of the proof, it is an entire function. Moreover, $E(2^{-s})$ is bounded for $\sigma > 1/2 - \delta$ and, thanks to (2.11), (2.12), (2.6) in (i) of Lemma 3 and the t-periodicity, satisfies

$$E(2^{-s}) \ll 2^{4|\sigma|+\varepsilon}$$

uniformly in t as $\sigma \to -\infty$, for every $\varepsilon > 0$. Thus, $E(z)$ is an entire function satisfying $E(z) \ll z^{4+\varepsilon}$ as $|z| \to \infty$; hence, it is a polynomial of degree ≤ 4. □

Lemma 6 *If $F \in \mathcal{S}$ has degree 2, conductor $q = 5$ and $\theta = 0$, then $F_2(s)^{-1}$ is a Dirichlet polynomial of degree ≤ 4, and has degree ≤ 3 if $a(2) \neq 0$.*

Proof The argument is partly similar as in Lemma 5, but simpler since no estimates are needed here. Hence, we proceed more sketchy, and we use the convenient notation

$$f(s) \sim \sum_{\nu=0}^{\infty} g_\nu(s)$$

to mean that if we cut the summation at $\nu = K$, then the remainder is holomorphic on D_K, where D_K is as in Lemma 5. With this notation, applying (i) of Lemma 2 with $\alpha = 1/3$, $q = 5$ and an arbitrarily large integer K, thanks to the 1-periodicity in α of (2.3) we get

$$F(s, 1/3, 0) \sim (\sqrt{5}/3)^{2s} \sum_{\nu=0}^{\infty} P_\nu(s, F)\overline{F}(s+\nu, 2/5, 0)$$

with certain polynomials $P_\nu(s, F)$ depending on $F(s)$. Applying again (2.4) to $\overline{F}(s + \nu, 2/5, 0)$, we obtain

$$\begin{aligned} F(s, 1/3, 0) &\sim (\sqrt{5}/3)^{2s} \sum_{\nu=0}^{\infty} P_\nu(s, F)(2/\sqrt{5})^{2(s+\nu)} \sum_{\ell=0}^{\infty} P_\ell(s+\nu, \overline{F})F(s+\nu+\ell, 1/2, 0) \\ &\sim (2/3)^{2s} \sum_{m=0}^{\infty} R_m(s, F)F(s+m, 1/2, 0) \end{aligned} \tag{2.13}$$

with certain polynomials $R_m(s, F)$. Similarly, since $F(s, 2/3, 0) = \overline{\overline{F}(\overline{s}, 1/3, 0)}$ we have

$$F(s, 2/3, 0) \sim (2/3)^{2s} \sum_{m=0}^{\infty} R_m(s, \overline{F})F(s+m, 1/2, 0) \tag{2.14}$$

with certain polynomials $R_m(s, \overline{F})$.

Combining (2.8) and (2.9) in Lemma 4 with (2.13) and (2.14), and writing $V_m(s) = R_m(s, F) + R_m(s, \overline{F})$, we get

$$2F(s) - 3F_3(s)^{-1}F(s) \sim (2/3)^{2s} \sum_{m=0}^{\infty} V_m(s)F(s+m)\big(1 - 2F_2(s+m)^{-1}\big),$$

hence

$$F_3(s)^{-1} \sim \frac{2}{3} - \frac{1}{3F(s)}(2/3)^{2s} \sum_{m=0}^{\infty} V_m(s) F(s+m)\bigl(1 - 2F_2(s+m)^{-1}\bigr). \tag{2.15}$$

Note that by (i) of Lemma 2 we have $Q_0(s) = 1$ identically; thus, $V_0(s) = V_0$ identically, and a simple computation based on (2.4) shows that $V_0 \neq 0$. Thanks to Lemma 5, (2.15) gives the meromorphic continuation of $F_3(s)^{-1}$ to D_K. Moreover, $F_3(s)^{-1}$ is holomorphic and t-periodic of period $2\pi/\log 3$ for $\sigma > 1/2 - 2\delta$, exactly by the same reason explained for $F_2(s)^{-1}$ in the proof of Lemma 5. But from Lemma 5 we also see that either $F_2(s)^{-1}$ is entire or its poles are at the points

$$\rho + \frac{2\pi k i}{\log 2}, \qquad k \in \mathbb{Z} \tag{2.16}$$

with a certain $\rho \in \mathbb{C}$ with $\Re\rho < 1/2 - \delta$. These facts imply that

$$F_2(s)^{-1} \text{ and } F_3(s)^{-1} \text{ are entire.} \tag{2.17}$$

Indeed, picking the term with $m = 0$, we rewrite (2.15) as

$$F_3(s)^{-1} = (2/3)^{2s+1} V_0 F_2(s)^{-1} + \frac{\Sigma_1(s)}{F(s)} + \frac{2}{3} - (2/3)^{2s} V_0/3 \tag{2.18}$$

and suppose that $F_2(s)^{-1}$ is not entire. Then, we note that, due to the shift by $m \geq 1$, the poles of $\Sigma_1(s)$ in D_K are disjoint from those of $F_2(s)^{-1}$. Moreover, as we already observed in the proof of Lemma 5, $1/F(s)$ has $o(K)$ poles in D_K. Hence, the poles of $\Sigma_1(s)/F(s)$ may cancel at most $o(K)$ poles of $(2/3)^{2s+1} V_0 F_2(s)^{-1}$ in D_K. Therefore, the first term on the RHS of (2.18) induces poles of $F_3(s)^{-1}$ in D_K of form (2.16) for all but $o(K)$ values $\frac{c_1 \log 2}{2\pi} < k < \frac{K \log 2}{4\pi}$. But $\log 2$ and $\log 3$ are linearly independent over $\mathbb{Q}$, thus leading to a contradiction in view of the above t-periodicity of $F_3(s)^{-1}$. Hence, $F_2(s)^{-1}$ is entire. Moreover, the same argument used in Lemma 5 shows that the possible poles of $F_3(s)^{-1}$ and $\Sigma_1(s)/F(s)$ cannot match; thus, $F_3(s)^{-1}$ is holomorphic in D_K. Therefore, as in Lemma 5, $F_3(s)^{-1}$ is also entire, and (2.17) is proved.

Finally, since $\bigl(1 + a(2)2^{1-s}\bigr)F_2(s)^{-1}$ is a Dirichlet polynomial $P_4(s)$, say, of degree ≤ 4 by Lemma 5, and $F_2(s)^{-1}$ is entire, we have that $1 + a(2)2^{1-s}$ divides $P_4(s)$; the lemma is therefore proved. □

In view of Lemma 6, we write

$$F_2(s)^{-1} = \sum_{\ell=0}^{4} \frac{A_\ell}{2^{\ell s}}, \tag{2.19}$$

where $A_0 = 1$ and $A_1 = -a(2)$, and $A_4 = 0$ if $a(2) \neq 0$. Up to this point, only the linear twists have been used; now we also exploit the standard twists and the nonlinear twists (2.3).

Lemma 7 *If $F \in \mathcal{S}$ has degree 2, conductor $q = 5$ and $\theta = 0$, then for $\beta \in \mathbb{R}$ we have*

$$\left(e(5\beta^2 - \eta/2) + e(-5\beta^2 + \eta/2) + a(2)A_3 + \frac{1}{2}A_4\right) F(s, 0, 4\beta)$$
$$+ 2^s \left(a(2)A_2 + \frac{1}{2}A_3\right) F(s, 0, 2^{3/2}\beta)$$
$$+ 4^s \left(a(2)A_1 + \frac{1}{2}A_2\right) F(s, 0, 2\beta)$$
$$+ 8^s \left(a(2) + \frac{1}{2}A_1\right) F(s, 0, 2^{1/2}\beta) = H(s),$$

where $H(s)$ is holomorphic for $\sigma > 1/2$.

Proof We first show that, thanks to (2.19), for $\sigma > 1$ we have

$$\sum_{2\nmid n} \frac{a(n)e(-\beta\sqrt{n})}{n^s} = \sum_{\ell=0}^{4} \frac{A_\ell}{2^{\ell s}} F(s, 0, 2^{\ell/2}\beta). \tag{2.20}$$

Indeed, the Euler product implies that

$$\sum_{2\nmid n} \frac{a(n)}{n^s} = F_2(s)^{-1} F(s),$$

hence in view of (2.19), for every $n \geq 1$ we get

$$\frac{1}{2}(1 - (-1)^n)a(n) = \sum_{\substack{2^\ell | n \\ 0 \leq \ell \leq 4}} A_\ell a(n/2^\ell).$$

Therefore

$$\sum_{2\nmid n} \frac{a(n)e(-\beta\sqrt{n})}{n^s} = \sum_{n=1}^{\infty} \sum_{\substack{2^\ell | n \\ 0 \leq \ell \leq 4}} A_\ell a(n/2^\ell) \frac{e(-\beta\sqrt{n})}{n^s}$$

and (2.20) follows by inverting the order of summation. Next we use (2.20) to show, similarly as in (2.7) in Lemma 4, that for $\sigma > 1$

$$F(s, 1/4, \beta) + F(s, 3/4, \beta) = 2F(s, 0, \beta) - 2^{2-s}a(2) \sum_{\ell=0}^{3} \frac{A_\ell}{2^{\ell s}} F(s, 0, 2^{(\ell+1)/2}\beta)$$
$$- 2\sum_{\ell=0}^{4} \frac{A_\ell}{2^{\ell s}} F(s, 0, 2^{\ell/2}\beta). \tag{2.21}$$

Indeed, as in the first steps of the proof of (2.7) we obtain for $\sigma > 1$ that

$$F(s, 1/4, \beta) + F(s, 3/4, \beta) = 2F(s, 0, \beta) - 4\frac{a(2)}{2^s} \sum_{2\nmid n} \frac{a(n)e(-2^{1/2}\beta\sqrt{n})}{n^s}$$
$$- 2\sum_{2\nmid n} \frac{a(n)e(-\beta\sqrt{n})}{n^s},$$

and (2.21) follows at once from (2.20), since $A_4 = 0$ if $a(2) \neq 0$ by (2.19).

Now we apply (iii) of Lemma 2 to the LHS of (2.21); this is possible since $\theta = 0$ and $a(1) = 1$. By two applications of (2.5), the fact that $e(4n/5) = e(-n/5)$ and Remark 2, a simple computation shows that

$$\begin{aligned} F(s, 1/4, \beta) &= -\omega^* \mathrm{e}^{-i\frac{\pi}{2}\eta}(\sqrt{5}/4)^{2s-1} e(\beta^2)\overline{F}(s, -4/5, 4\beta/\sqrt{5}) + h_1(s) \\ &= \mathrm{e}^{-i\pi\eta} 4^{1-2s} e(5\beta^2) F(s, 0, 4\beta) + h_2(s), \end{aligned} \tag{2.22}$$

where the $h_j(s)$'s denote, here and later, holomorphic functions for $\sigma > 1/2$. Moreover, since

$$F(s, 3/4, \beta) = \overline{\overline{F}(\overline{s}, 1/4, -\beta)},$$

from (2.22) we deduce that

$$F(s, 3/4, \beta) = \mathrm{e}^{i\pi\eta} 4^{1-2s} e(-5\beta^2) F(s, 0, 4\beta) + h_3(s). \tag{2.23}$$

Finally, adding (2.22) and (2.23) and substituting the result in the LHS of (2.21) we obtain

$$\begin{aligned} 2^{1-4s}\big(e(5\beta^2 - \eta/2) + e(-5\beta^2 + \eta/2)\big) F(s, 0, 4\beta) &= F(s, 0, \beta) \\ - 2\frac{a(2)}{2^s} \sum_{\ell=0}^{3} \frac{A_\ell}{2^{\ell s}} F(s, 0, 2^{(\ell+1)/2}\beta) - \sum_{\ell=0}^{4} \frac{A_\ell}{2^{\ell s}} F(s, 0, 2^{\ell/2}\beta) + h_4(s), \end{aligned} \tag{2.24}$$

and the lemma follows by a rearrangement of (2.24). □

Now we use the above lemmas to detect the admissible values of η (see the definition at the beginning of the section) and the finer structure of $F_2(s)$.

Proposition 1 *If $F \in \mathcal{S}$ has degree 2, conductor $q = 5$ and $\theta = 0$, then*

$$\eta \in \mathbb{Z} \tag{2.25}$$

and the degree of $F_2(s)^{-1}$ is ≤ 2.

Proof Recalling the notation at the beginning of this section, for integer $\nu \geq 0$ we choose

$$\beta_\nu = 2^{(\nu-2)/2}/\sqrt{5} \ \text{ in order to have } \ n_{2^{j/2}\beta_\nu} = 2^{\nu+j-4} \ \text{ for } \ j = 1, \ldots, 4. \tag{2.26}$$

Next we choose $\beta = \beta_\nu$ and compute the residue at $s = 3/4$ of both sides of the identity in Lemma 7. Thanks to (2.26) and (ii) of Lemma 2, a simple computation shows that the following recurrence relation holds

$$\begin{aligned} &\left(2\pi_\nu(\eta) + a(2)A_3 + \frac{1}{2}A_4\right)\frac{\overline{a(2^\nu)}}{2^{\nu/4}} + 2^{3/4}\left(a(2)A_2 + \frac{1}{2}A_3\right)\frac{\overline{a(2^{\nu-1})}}{2^{(\nu-1)/4}} \\ &+ 4^{3/4}\left(a(2)A_1 + \frac{1}{2}A_2\right)\frac{\overline{a(2^{\nu-2})}}{2^{(\nu-2)/4}} + 8^{3/4}\left(a(2) + \frac{1}{2}A_1\right)\frac{\overline{a(2^{\nu-3})}}{2^{(\nu-3)/4}} = 0, \end{aligned} \tag{2.27}$$

where $a(x) = 0$ if $x \notin \mathbb{N}$ and

$$\pi_\nu(\eta) = \begin{cases} \sin(\pi\eta) & \text{if } \nu = 0 \\ -\cos(\pi\eta) & \text{if } \nu = 1 \\ \cos(\pi\eta) & \text{if } \nu \geq 2. \end{cases}$$

Now we consider the generating function

$$G(z) = \sum_{\nu=0}^{\infty} a(2^\nu) z^\nu,$$

hence clearly $G(2^{-s}) = F_2(s)$. From the properties of $F_2(s)$ seen at the beginning of the proof of Lemma 5, this series is absolutely convergent for $|z| < 2^{-\vartheta}$, and thanks to (2.19) we have

$$G(z) = \frac{1}{1 + A_1 z + A_2 z^2 + A_3 z^3 + A_4 z^4}. \tag{2.28}$$

But, in view of the recurrence relation (2.27), we can get another explicit expression for $G(z)$ by a fairly standard computation with formal power series; Proposition 1 will then follow by a comparison of such expressions. Indeed, (2.27) implies that for $\nu \geq 0$

$$B_{0,\nu} a(2^\nu) + B_1 a(2^{\nu-1}) + B_2 a(2^{\nu-2}) + B_3 a(2^{\nu-3}) = 0 \tag{2.29}$$

where

$$\begin{aligned} B_{0,\nu} &= 2\pi_\nu(\eta) + \overline{a(2)A_3} + \overline{A_4}/2 \\ B_0 &= 2\cos(\pi\eta) + \overline{a(2)A_3} + \overline{A_4}/2 \\ B_1 &= 2\overline{a(2)A_2} + \overline{A_3} \\ B_2 &= 4\overline{a(2)A_1} + 2\overline{A_2} \\ B_3 &= 8\overline{a(2)} + 4\overline{A_1}; \end{aligned} \tag{2.30}$$

note that B_0 is the value of $B_{0,\nu}$ for $\nu \geq 2$. Then, multiplying (2.29) by z^ν, summing over $\nu \geq 0$ and observing that

$$B_{0,0}a(1) + B_{0,1}a(2)z - B_0 a(1) - B_0 a(2)z = -2(\cos(\pi\eta) - \sin(\pi\eta)) - 4\cos(\pi\eta)a(2)z,$$

after a rearrangement we obtain

$$G(z) = \frac{2(\cos(\pi\eta) - \sin(\pi\eta)) + 4\cos(\pi\eta)a(2)z}{B_0 + B_1 z + B_2 z^2 + B_3 z^3}. \tag{2.31}$$

Suppose first that $a(2) \neq 0$. Then by (2.19) we have $A_4 = 0$. If $A_3 \neq 0$, then $\cos(\pi\eta) = 0$ and

$$\frac{-2\sin(\pi\eta)}{B_0 + B_1 z + B_2 z^2 + B_3 z^3} = \frac{1}{1 + A_1 z + A_2 z^2 + A_3 z^3}$$

by a comparison of (2.28) and (2.31); moreover, $\sin(\pi\eta) = \pm 1$. Thus, for $j = 0, \ldots, 4$ we have

$$B_j = -2A_j \sin(\pi\eta). \tag{2.32}$$

Now from (2.29) with $\nu = 0$ we obtain that $B_{0,0} = 0$, and hence, by (2.30) we have

$$a(2)A_3 = -2\sin(\pi\eta). \tag{2.33}$$

Therefore, taking $j = 3$ in (2.32) and multiplying by $a(2)$ we get $a(2)B_3 = 4\sin^2(\pi\eta) = 4$. But from (2.19) and (2.30) we deduce that $B_3 = 4\overline{a(2)}$, hence

$$|a(2)| = 1 \text{ and therefore } \overline{a(2)} = a(2)^{-1}. \tag{2.34}$$

Next we choose $j = 1$ in (2.32), thus getting from (2.19) that $B_1 = 2a(2)\sin\pi\eta$. Hence multiplying by $a(2)$, comparing with (2.30) and using (2.34) gives $2\overline{A_2} + a(2)^2\overline{a(2)A_3} = 2a(2)\sin\pi\eta$. In view of (2.33), taking conjugates we therefore obtain

$$A_2 = 2\overline{a(2)}^2\sin(\pi\eta). \tag{2.35}$$

Take now $j = 2$ in (2.32) and compare with (2.35), thus getting $B_2 = -4\overline{a(2)}^2$. Hence comparing with (2.30) and inserting (2.35), in view of the value of A_1 in (2.19), we finally obtain

$$a(2)^2\sin(\pi\eta) = 0,$$

a contradiction since we already know that $\cos(\pi\eta) = 0$. Therefore, recalling also (2.33), we have that $a(2) \neq 0$ implies

$$A_3 = A_4 = \sin(\pi\eta) = 0. \tag{2.36}$$

Suppose now that $a(2) = 0$. Then from (2.30) and the value of A_1 in (2.19), we have that $B_3 = 0$, hence comparing (2.28) and (2.31) we immediately get $A_3 = A_4 = 0$, and hence $\sin(\pi\eta) = 0$ as well since we already observed that $B_{0,0}$ in (2.30) vanishes. Therefore, (2.36) holds also if $a(2) = 0$. In particular, we deduce that $\eta \in \mathbb{Z}$, and Proposition 1 follows. □

Remark 3 Actually, we can prove that under the hypotheses of Proposition 1

$$F_2(s) = \left(1 - \frac{\alpha}{2^s}\right)^{-1}\left(1 - \frac{\beta}{2^s}\right)^{-1}$$

with certain $\alpha, \beta \in \mathbb{C}$ satisfying $|\alpha|, |\beta| \le 1$. Indeed, $F_2(s)$ can be written in this form with some $\alpha, \beta \in \mathbb{C}$ since $F_2(s)^{-1}$ has degree ≤ 2. But then the Ramanujan conjecture implies that $|\alpha|, |\beta| \le 1$, see the argument at the end of the proof of Theorem 1.1 in [12]. Moreover, it is interesting to note that (2.29) agrees with the action of the Hecke operator T_2. Indeed, inserting the value of the coefficients A_j given by Proposition 2, as well as the relations $\cos(\pi\eta) = \chi(-1)$ and $a(2) = \chi(2)\overline{a(2)}$, recurrence (2.29) becomes (after dividing by $2\chi(-1)$ all terms)

$$a(2^\nu) - a(2)a(2^{\nu-1}) + \chi(2)a(2^{\nu-2}) + 2a(2)(\text{the same with } \nu \text{ replaced by } \nu - 1) = 0.$$

We need one more lemma.

Lemma 8 *If $F \in \mathcal{S}$ has degree 2, conductor $q = 5$, $\theta = 0$ and $a(2) = 0$, then $|a(4)| = 1$.*

Proof According to Proposition 1, $F_2(s)^{-1}$ is a Dirichlet polynomial of degree ≤ 2; hence, comparing with the shape of $F_2(s)$ in (2.1), we deduce that in this case

$$F_2(s)^{-1} = 1 - a(4)4^{-s}.$$

Therefore, comparing with (2.1) we get

$$a(n) = 0 \text{ if } 2\|n \text{ and } a(4m) = a(4)a(m) \text{ for every } m \ge 1. \tag{2.37}$$

Since for $n \equiv 1, 3, 5 \pmod 6$ we have $e(-2n/3) + e(-n/6) = 0$, and $e(-8m/3) = e(-4m/6)$ for every $m \ge 1$, in view of (2.37) for $\sigma > 1$ and $\beta \in \mathbb{R}$ we have

$$\begin{aligned} F(s, 2/3, \beta) + F(s, 1/6, \beta) &= \sum_{4|n} \frac{a(n)}{n^s}\big(e(-2n/3) + e(-n/6)\big)e(-\beta\sqrt{n}) \\ &= \frac{2a(4)}{4^s} F(s, 2/3, 2\beta). \end{aligned} \tag{2.38}$$

The lemma will follow applying (iii) of Lemma 2 to both sides of (2.38) and then comparing the outputs in a suitable way.

In the following computations, for ease of notation, we denote by λ (or $\lambda_1, \dots$) complex numbers with $|\lambda| = 1$ and by $h(s)$ holomorphic functions for $\sigma > 1/2$, not necessarily the same at each occurrence. Moreover, we apply without explicit mention the conjugation and periodicity tricks, in the same fashion already used in Lemma 7, when several applications of the transformation formula in (iii) of Lemma 2 are performed. By repeated applications of (2.5), we have

$$\begin{aligned} F(s, 2/3, \beta) &= \overline{\overline{F}(\overline{s}, 1/3, -\beta)} = \lambda(\sqrt{5}/3)^{2s-1}\overline{F}(s, 3/5, 3\beta/\sqrt{5}) + h(s) \\ &= \lambda(\sqrt{5}/3)^{2s-1}\overline{F}(s, -2/5, 3\beta/\sqrt{5}) + h(s) \\ &= \lambda(\sqrt{5}/3)^{2s-1}\overline{F(\overline{s}, 2/5, -3\beta/\sqrt{5})} + h(s) \\ &= \lambda(2/3)^{2s-1}F(s, 1/2, 3\beta/2) + h(s). \end{aligned} \tag{2.39}$$

But, thanks to (2.37),

$$\begin{aligned} F(s, 1/2, 3\beta/2) &= \sum_{4|n} \frac{a(n)}{n^s} e(-3\beta\sqrt{n}/2) - \sum_{2\nmid n} \frac{a(n)}{n^s} e(-3\beta\sqrt{n}/2) \\ &= \frac{2a(4)}{4^s} F(s, 0, 3\beta) - F(s, 0, 3\beta/2) \end{aligned}$$

hence (2.39) becomes

$$F(s, 2/3, \beta) = \lambda(2/3)^{2s-1}\left(\frac{2a(4)}{4^s}F(s, 0, 3\beta) - F(s, 0, 3\beta/2)\right) + h(s). \tag{2.40}$$

Similarly

$$\begin{aligned} F(s, 1/6, \beta) &= \lambda(\sqrt{5}/6)^{2s-1}\overline{F}(s, -6/5, 6\beta/\sqrt{5}) + h(s) \\ &= \lambda(\sqrt{5}/6)^{2s-1}\overline{F}(s, 1/5, 6\beta/\sqrt{5}) + h(s) \\ &= \lambda(1/6)^{2s-1}F(s, 0, 6\beta) + h(s), \end{aligned} \tag{2.41}$$

and from (2.40), we also have

$$F(s, 2/3, 2\beta) = \lambda(2/3)^{2s-1}\left(\frac{2a(4)}{4^s}F(s, 0, 6\beta) - F(s, 0, 3\beta)\right) + h(s). \tag{2.42}$$

Inserting (2.40), (2.41) and (2.42) in (2.38) and rearranging terms, and then choosing $\beta = \beta_0 > 0$ such that $n_{6\beta_0} = 1$ and recalling (ii) of Lemma 2, we obtain

$$(\lambda_1 - \lambda_2 a(4)^2)(1/6)^{2s-1}F(s, 0, 6\beta_0) = h(s). \tag{2.43}$$

Indeed, both $n_{3\beta_0/2}$ and $n_{3\beta_0}$ are smaller than 1; hence, $3\beta_0/2$ and $3\beta_0$ do not belong to $\mathrm{Spec}(F)$, and therefore, $F(s, 0, 3\beta_0/2)$ and $F(s, 0, 3\beta_0)$ are entire functions. Computing the residue of both sides of (2.43) at $s = 3/4$, thanks to the formula in (ii) of Lemma 2 we get

$$c(F)((\lambda_1 - \lambda_2 a(4)^2) = 0$$

with some $c(F) \neq 0$, and the lemma follows since $a(4)^2 = \lambda_1/\lambda_2$. □

Further information on the finer structure of $F_2(s)$ is given by the next result.

Proposition 2 *If $F \in \mathcal{S}$ has degree 2, conductor $q = 5$, $\theta = 0$ and $\eta \in \mathbb{Z}$, then*

$$F_2(s)^{-1} = 1 - \frac{a(2)}{2^s} + \frac{\chi(2)}{2^{2s}} \tag{2.44}$$

with a certain Dirichlet character $\chi \pmod 5$ such that $\chi(-1) = (-1)^\eta$ and $a(2) = \chi(2)\overline{a(2)}$.

Proof Assume first that $a(2) \neq 0$. Taking $\nu = 1$ in (2.29) and recalling from (2.36) that $A_3 = A_4 = 0$, we get

$$A_2 = \cos(\pi\eta)\frac{\overline{a(2)}}{a(2)}, \tag{2.45}$$

hence in particular

$$|A_2| = 1 \text{ and hence } \overline{A_2} = A_2^{-1}. \tag{2.46}$$

Now taking $\nu = 2$ in (2.29) and recalling that $A_1 = -a(2)$, see (2.19), we obtain

$$\cos(\pi\eta)a(4) = 2\overline{a(2)}^2 - \left(1 + |a(2)|^2\right)\overline{A_2}. \tag{2.47}$$

By (2.28) and (2.46), we have $a(4) = a(2)^2 - A_2 = a(2)^2 - \overline{A_2}^{-1}$; hence, substituting in (2.47) we have

$$\cos(\pi\eta)a(2)^2 - \cos(\pi\eta)\overline{A_2}^{-1} = 2\overline{a(2)}^2 - (1 + |a(2)|^2)\overline{A_2}.$$

Multiplying by $\overline{A_2}$ and observing that

$$\overline{a(2)}^2\overline{A_2} = |a(2)|^2\cos(\pi\eta) \text{ and } \overline{A_2}a(2)^2\cos(\pi\eta) = |a(2)|^2\overline{A_2}^2$$

thanks to (2.45), we therefore obtain

$$|a(2)|^2\overline{A_2}^2 - \cos(\pi\eta) = 2\cos(\pi\eta)|a(2)|^2 - (1 + |a(2)|^2)\overline{A_2}^2.$$

Taking conjugates this gives

$$(1 + 2|a(2)|^2)A_2^2 = \cos(\pi\eta)(1 + 2|a(2)|^2)$$

and hence

$$A_2^2 = \cos(\pi\eta) = (-1)^\eta. \tag{2.48}$$

Therefore, the admissible values for A_2 are $\pm 1, \pm i$; thus, there exists a Dirichlet character $\chi \pmod 5$ such that

$$A_2 = \chi(2), \tag{2.49}$$

which also satisfies $\chi(3) = \overline{\chi(2)}$ and $\chi(4) = \chi(2)^2 = (-1)^\eta$. Hence, in view of (2.19) and (2.36), (2.44) follows in this case since $\chi(-1) = \chi(4)$. Moreover, multiplying (2.45) by A_2 and using (2.48) and (2.49) we have that $a(2) = \chi(2)\overline{a(2)}$.

Assume now that $a(2) = 0$. In this case, we take $\nu = 2$ in (2.29) and, recalling again that $A_3 = A_4 = 0$, we obtain

$$\overline{A_2} = -\cos(\pi\eta)a(4). \tag{2.50}$$

But comparing (2.1) with (2.19), we see that

$$A_2 = a(2)^2 - a(4) = -a(4), \tag{2.51}$$

and $|a(4)| = 1$ by Lemma 8. Thus comparing (2.50) and (2.51), we have $|A_2| = 1$ and $\overline{A_2} = \cos(\pi\eta)A_2$, and hence, exactly as before, we obtain that the admissible values for A_2 are ± 1, $\pm i$ and $A_2 = \chi(2)$ for some $\chi \pmod 5$. Moreover, still comparing (2.50) and (2.51) we get that $\overline{A_2} = \cos(\pi\eta)A_2$, hence $A_2^2 = \cos(\pi\eta) = (-1)^\eta$ and therefore $\chi(-1) = (-1)^\eta$ as before. □

3 Proof of the Theorem

3.1 Notation

First we recall some notation from the theory of modular forms and from Conrey–Farmer [3]. As usual we denote by $\mathbb{H}$ the upper half-plane, for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ we write $\chi(\gamma) = \chi(d)$, and for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$, $k \in \mathbb{N}$ and $f : \mathbb{H} \to \mathbb{C}$ the k-slash operator is

$$f_{|\gamma}(z) = (\det\gamma)^{k/2}(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right),$$

and is well defined. If $\gamma \in \mathbb{C}[\mathrm{GL}_2^+(\mathbb{R})]$ has the shape

$$\gamma = \sum_{j=1}^{J} c_j\gamma_j \tag{3.1}$$

we write

$$f_{|\gamma} = \sum_{j=1}^{J} c_j f_{|\gamma_j}. \tag{3.2}$$

Moreover, we write

$$\gamma^* = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \;\text{ if }\; \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\text{ and }\; \gamma^* = \sum_{j=1}^{J} \overline{c_j}\gamma_j^* \;\text{ if } \gamma \text{ is as in (3.1)},$$

and $\overline{f}(z) = \overline{f(-\overline{z})}$ for any function $f : \mathbb{H} \to \mathbb{C}$. Clearly, $\gamma^{**} = \gamma$, $\overline{\overline{f}} = f$ and $f = 0$ if and only if $\overline{f} = 0$. We further write

$$\Omega_f = \left\{\gamma \in \mathbb{C}[\mathrm{GL}_2^+(\mathbb{R})] : f_{|\gamma} = 0\right\}$$

and $\gamma \equiv \gamma' \pmod{\Omega_f}$ to mean that $\gamma - \gamma' \in \Omega_f$. Since $\det\gamma^* = \det\gamma$, a simple computation shows that

$$\overline{f_{|\gamma}} = \overline{f}_{|\gamma^*}, \quad \text{and} \quad \overline{f}_{|\gamma} = \overline{f_{|\gamma^*}}. \tag{3.3}$$

We recall that the Hecke operators of weight k and character χ are defined for $n \geq 1$ as

$$T_n = \frac{1}{n}\sum_{ad=n} \chi(a)a^k \sum_{1\leq b<d} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix},$$

and their action on $f : \mathbb{H} \to \mathbb{C}$ is given by

$$T_n f(z) = \frac{1}{n}\sum_{ad=n} \chi(a)a^k \sum_{0\leq b<d} f\left(\frac{az+b}{d}\right); \tag{3.4}$$

see Section 6.2 of Iwaniec [5]. We also consider the modified Hecke operator $\widetilde{T}_2$, which essentially mimics the action of T_2 via the slash operator (see (3.7)), defined as

$$\widetilde{T}_2 = \chi(2)\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}. \tag{3.5}$$

Finally, we write

$$H_5 = \begin{pmatrix} 0 & -1 \\ 5 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad W_5 = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

3.2 The holomorphic case

We start with the proof of the theorem, assuming that $F(s)$ is entire. By Remark 1, we may assume that $\mu \geq 0$, i.e., $\theta = 0$, since otherwise we consider $G(s) = F(s - \Im\mu)$ in place of $F(s)$. Note that in the special case where $\Phi(s)$ is as in (1.4), we have $\eta = 2\mu - 1$ and, by Proposition 1, $\eta \in \mathbb{Z}$. But $\mu \geq 0$, thus writing $\eta + 2 = k$ we have that $k \geq 1$ is an integer and

$$\mu = \frac{k-1}{2}.$$

Moreover, in view of Proposition 2 we have

$$\chi(-1) = (-1)^k;$$

we shall use these properties without further mention. Let

$$f(z) = \sum_{n=1}^{\infty} a_n e(nz), \qquad a_n = a(n)n^{(k-1)/2}; \tag{3.6}$$

clearly, $f(z)$ is absolutely convergent on $\mathbb{H}$. Having obtained full information on the Euler factor $F_2(s)$ of $F(s)$ in Proposition 2, now we modify some ideas in Conrey–Farmer [3] to characterize $F(s)$.

In view of (3.1), (3.2), (3.4) and (3.5), a simple computation shows that, for any $f : \mathbb{H} \to \mathbb{C}$,

$$f_{|\widetilde{T}_2}(z) = \sqrt{2}\, 2^{-(k-1)/2}\, T_2 f(z). \tag{3.7}$$

Choose now $f(z)$ as in (3.6). By Proposition 2 and standard computations from the theory of Hecke operators, see, e.g., Chapters 2 and 4 of Ogg [16], in this case we have

$$T_2 f(z) = a_2 f(z) = 2^{(k-1)/2} a(2) f(z), \tag{3.8}$$

hence comparing (3.7) and (3.8) we get

$$f_{|\widetilde{T}_2}(z) = \alpha_2 f(z), \qquad \alpha_2 = \sqrt{2}\, a(2). \tag{3.9}$$

The next steps in the proof are as follows. First we combine (3.9) with the information coming from the functional equation of $F(s)$ to show that $f(z)$ satisfies the modularity relation

$$f_{|\gamma}(z) = \chi(\gamma) f(z) \quad \text{for} \quad \gamma \in \{P, W_5, M_2\}. \tag{3.10}$$

Since $\{P, W_5, M_2\}$ is a set of generators of $\Gamma_0(5)$, see p. 450 of Conrey–Farmer [3], from (3.10) we obtain that $f(z)$ satisfies the same modularity relation for every $\gamma \in \Gamma_0(5)$. Then we show that $f(z)$ is a newform in $S_k(\Gamma_0(5), \chi)$. Hence, by Remark 1 and (3.6), we obtain that

$$F(s) = L\left(s + \frac{k-1}{2} + \Im\mu, f\right) = L(s + \mu, f), \tag{3.11}$$

as required.

Now we prove (3.10). Clearly, by (3.6) we have

$$f_{|P}(z) = \chi(P) f(z). \tag{3.12}$$

Moreover, it is well known that the functional equation of $F(s)$ implies, via the Mellin transform, that

$$f_{|H_5} = \omega \overline{f} \quad \text{and} \quad \overline{f}_{|H_5} = \omega^{-1} f, \tag{3.13}$$

hence

$$f_{|_{H_5^{-1}}} = \omega \overline{f} \tag{3.14}$$

since $\overline{f} = \overline{f}_{|_{H_5 H_5^{-1}}} = \omega^{-1} f_{|_{H_5^{-1}}}$ by the chain rule satisfied by the k-slash operator. In order to deal with M_2 we note that by (3.13), (3.9) and (3.14) we have

$$\overline{f}_{|_{H_5 \widetilde{T}_2 H_5^{-1}}} = \omega^{-1} f_{|_{\widetilde{T} H_5^{-1}}} = \omega^{-1} \alpha_2 f_{|_{H_5^{-1}}} = \alpha_2 \overline{f},$$

hence $H_5 \widetilde{T}_2 H_5^{-1} - \alpha_2 I \in \Omega_{\overline{f}}$ and therefore by (3.3)

$$\left(H_5 \widetilde{T}_2 H_5^{-1}\right)^* - \overline{\alpha_2} I \in \Omega_f. \tag{3.15}$$

But a direct computation shows that

$$H_5 \widetilde{T}_2 H_5^{-1} = \chi(2) \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ -5 & 1 \end{pmatrix},$$

hence

$$\left(H_5 \widetilde{T}_2 H_5^{-1}\right)^* = \overline{\chi(2)} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 5 & 1 \end{pmatrix}.$$

Multiplying this identity by $\chi(2)$, from (3.15), we have

$$\chi(2) \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \chi(2) \begin{pmatrix} 2 & 0 \\ 5 & 1 \end{pmatrix} - \chi(2) \overline{\alpha_2} I \in \Omega_f,$$

and subtracting this relation from (3.9), we obtain

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} - \chi(2) \begin{pmatrix} 2 & 0 \\ 5 & 1 \end{pmatrix} - (\alpha_2 - \chi(2)\overline{\alpha_2}) I \in \Omega_f.$$

Thanks to Proposition 2 and in view of (3.9), we have $\alpha_2 - \chi(2)\overline{\alpha_2} = 0$, and hence, we get

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} - \chi(2) \begin{pmatrix} 2 & 0 \\ 5 & 1 \end{pmatrix} \in \Omega_f. \tag{3.16}$$

Multiplying (3.16) on the right by $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$ and dividing by 2, we obtain

$$P - \chi(2) M_2 \in \Omega_f,$$

and since $P \equiv I \pmod{\Omega_f}$ by (3.12), and $\chi(2)^{-1} = \chi(3)$, the above relation shows that

$$f_{|M_2}(z) = \chi(M_2) f(z). \tag{3.17}$$

Finally, to deal with W_5 we note that by (3.13), (3.12) and (3.14) we have

$$\overline{f}_{|_{H_5 P H_5^{-1}}} = \omega^{-1} f_{|_{P H_5^{-1}}} = \omega^{-1} f_{|_{H_5^{-1}}} = \overline{f},$$

hence $H_5 P H_5^{-1} - I \in \Omega_{\overline{f}}$ and therefore by (3.3)

$$\left(H_5 P H_5^{-1}\right)^* - I \in \Omega_f. \tag{3.18}$$

Again, a direct computation shows that

$$H_5 P H_5^{-1} = \begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix},$$

hence $(H_5 P H_5^{-1})^* = W_5$ and (3.18) becomes

$$f_{|W_5}(z) = \chi(W_5) f(z). \tag{3.19}$$

The modularity relations (3.10) follow now from (3.12), (3.17) and (3.19).

To show that $f(z)$ is a newform in $S_k(\Gamma_0(5), \chi)$, we argue as follows. Since this is probably known in the theory of modular forms thanks to the multiplicity one property, we give a sketch of proof based on the linear independence property of L-functions, which in fact is a kind of analog of the multiplicity one in the framework of L-functions. As in Section 5 of [3], the above convergence and modularity properties imply that $f(z)$ is holomorphic at the cusps of $\Gamma_0(5)$; hence, $f \in M_k(\Gamma_0(5), \chi)$ and (3.11) holds. By Proposition 2.6 of Iwaniec [5], $\Gamma_0(5)$ has only two inequivalent cusps, 0 and ∞; moreover, (3.6) is the Fourier expansion of $f(z)$ at ∞ and $F(s-\mu)$ is its associated L-function. By the functional equation of $F(s)$, we deduce (see Section 7.2 of [5]) that

$$\overline{f}(z) = \sum_{n=1}^{\infty} \overline{a_n} e(nz)$$

is the Fourier expansion of $f(z)$ at 0, and hence, $f(z)$ is a cusp form since both 0-coefficients vanish. Now we recall that

$$S_k(\Gamma_0(5), \chi) = S_k^{\sharp}(\Gamma_0(5), \chi) \oplus S_k^{\flat}(\Gamma_0(5), \chi),$$

$S_k^{\sharp}(\Gamma_0(5), \chi)$ has a basis of newforms $f_j(z)$ of level 5 and $S_k^{\flat}(\Gamma_0(5), \chi)$ has a basis of type $g_j(d_j z)$ with $g_j(z)$ Hecke eigenform of level 1 and $d_j|5$; see Section 6 of [5]). Hence, we have

$$f(z) = \sum_{j=1}^{h} \alpha_j f_j(z) + \sum_{j=1}^{r} \beta_j g_j(d_j z), \quad \alpha_j, \beta_j \in \mathbb{C}. \tag{3.20}$$

Passing to the associated L-functions, we obtain that

$$L(s, f) = \sum_{j=1}^{h} \alpha_j L(s, f_j) + \sum_{j=1}^{r} \beta_j d_j^{-s} L(s, g_j),$$

hence $L(s, f)$ is a linear combination over Dirichlet polynomials of L-functions in the Selberg class $\mathcal{S}$. Thanks to the linear independence property of L-functions in $\mathcal{S}$, see Kaczorowski–Molteni–Perelli [7] and [8], we see that either $L(s, f) = L(s, f_j)$ or $L(s, f) = L(s, g_j)$ for some j. But $F(s)$ has conductor $q = 5$; hence, in view of (3.11) we deduce that $f(z) = f_j(z)$ for some j, as required.

3.3 The polar case

Suppose now that $F(s)$ has a pole at $s = 1$; the argument is similar to the holomorphic case, so we only give a brief sketch; but first we need a further lemma.

Lemma 9 *If $F \in \mathcal{S}_2(5, \mu)$ is polar, then $a(2) = 1 + \chi(2)$, where χ is as in Proposition 2.*

Proof By Lemma 1, we have $\mu = 0$ and the pole of $F(s)$ at $s = 1$ is simple. Hence, in this case we have $k = 1$ and $\cos(\pi\eta) = -1$; moreover, by Proposition 2 we also have $A_0 = 1$, $A_1 = -a(2)$ and $A_2 = \chi(2)$. Now we choose $\beta = 0$ in Lemma 7; recalling that $A_3 = A_4 = 0$ and inserting the above values of A_0, A_1, A_2 and $\cos(\pi\eta)$, we obtain

$$\big(-2 + 2^s a(2)\chi(2) + 4^s(-a(2)^2 + \chi(2)/2) + 8^s a(2)/2\big)F(s) = H(s),$$

where $H(s)$ is holomorphic for $\sigma > 1/2$. Computing residues at $s = 1$ on both sides, dividing by 4 and rearranging terms we get

$$a(2)^2 - (1 + \chi(2)/2)a(2) + (1 - \chi(2))/2 = 0.$$

Now observe that $\chi(2)^2 = -1$, since $\chi(2)^2 = \chi(4) = \chi(-1) = (-1)^k = -1$. Hence, a simple computation shows that the two values

$$a(2) = 1 + \chi(2) \quad \text{and} \quad a(2) = -\chi(2)/2$$

satisfy the above equation, which therefore are the only possibilities for $a(2)$. But inserting $a(2) = -\chi(2)/2$ in (2.45) and recalling that $A_2 = \chi(2)$ and $\cos(\pi\eta) = -1$, we get

$$\chi(2) = -\frac{\overline{\chi(2)}}{\chi(2)},$$

hence $-1 = \chi(2)^2 = -\overline{\chi(2)}$, thus $\overline{\chi(2)} = 1$ and hence $\chi(4) = 1$, a contradiction. Therefore, $a(2) = 1 + \chi(2)$, and the lemma follows. □

As already observed in the proof of Lemma 9, the pole is simple, $\mu = 0$ and $k = 1$; in this case, we replace (3.6) by

$$f(z) = \sum_{n=0}^{\infty} a(n)e(nz), \qquad a(0) = F(0). \tag{3.21}$$

Then, (3.9) and (3.12)–(3.14) hold also for $f(z)$ defined by (3.21). Indeed, (3.13) and (3.14) hold in this case since $F(s)$ has a simple pole at $s = 1$, while (3.12) is trivial. Moreover, by (3.9) and a direct computation we have

$$\begin{aligned} f_{|\widetilde{T}_2}(z) &= a(0)_{|\widetilde{T}_2} + \alpha_2\big(f(z) - a(0)\big) \\ &= \sqrt{2}(1 + \chi(2))a(0) + \alpha_2\big(f(z) - a(0)\big) = \alpha_2 f(z) \end{aligned}$$

in view of Lemma 9. As a consequence, (3.10) holds as well, exactly by the same arguments as in the holomorphic case. Further, exactly as before we have that $f \in M_1(\Gamma_0(5), \chi)$. But

$$\dim M_1(\Gamma_0(5), \chi) = \dim S_1(\Gamma_0(5), \chi) + 1$$

and $M_1(\Gamma_0(5), \chi)$ has a basis consisting of cusp forms plus the normalized Eisenstein series $f_0(z)$, whose associated L-function is $\zeta(s)L(s,\chi)$; see Section 6.3 of Stein [21]. Therefore, with notation similar as in Sect. 3.2 with $k = 1$, in analogy with (3.20) we have

$$f(z) = \sum_{j=1}^{\tilde{h}} \tilde{\alpha}_j \tilde{f}_j(z) + \sum_{j=1}^{\tilde{r}} \tilde{\beta}_j \tilde{g}_j(\tilde{d}_j z) + \gamma_0 f_0(z), \quad \tilde{\alpha}_j, \tilde{\beta}_j, \gamma_0 \in \mathbb{C}.$$

Passing to the associated L-functions, and using again the linear independence property of $\mathcal{S}$, we finally obtain that $F(s) = \zeta(s)L(s,\chi)$ since all the $L(s, \tilde{f}_j)$ and $L(s, \tilde{g}_j)$ are entire. The proof of the theorem is now complete.

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