Annali di Matematica (2018) 197:1631–1635
https://doi.org/10.1007/s10231-018-0740-1

Iitaka dimensions of vector bundles

Shin-Yao Jow[1]

Received: 5 December 2017 / Accepted: 21 March 2018 / Published online: 27 March 2018

Abstract Let X be a projective variety. If L is a line bundle on X, for each positive integer m in $\mathbf{N}(L) = \{m \in \mathbb{N} \mid H^0(X, L^{\otimes m}) \neq 0\}$, the global sections of $L^{\otimes m}$ define a rational map

$$\phi_m : X \dashrightarrow Y_m \subseteq \mathbb{P}\big(H^0(X, L^{\otimes m})\big),$$

where Y_m is the closure of $\phi_m(X)$. It is well-known that for all sufficiently large $m \in \mathbf{N}(L)$, the rational maps $\phi_m : X \dashrightarrow Y_m$ are birationally equivalent to a fixed fibration (the Iitaka fibration), and $\kappa(L) := \dim Y_m$ is called the Iitaka dimension of L. In a recent paper titled "Iitaka fibrations for vector bundles", Mistretta and Urbinati generalized this to a vector bundle E on X. Let $\mathbf{N}(E)$ be the set of positive integers m such that the evaluation map $H^0(X, S^m E) \to S^m E_x$ is surjective for all points x in some nonempty open subset of X. For each $m \in \mathbf{N}(E)$, the global sections of $S^m E$ define a rational map

$$\varphi_m : X \dashrightarrow Y_m \subseteq \mathbb{G}(H^0(X, S^m E), \operatorname{rank} S^m E),$$

where $\mathbb{G}(H^0(X, S^m E), \operatorname{rank} S^m E)$ is the Grassmannian of rank $S^m E$-dimensional quotients of $H^0(X, S^m E)$. Mistretta and Urbinati showed that for every $m \in \mathbf{N}(E)$, the rational maps φ_{km} are birationally equivalent for sufficiently large k, and called $\kappa(E) := \dim Y_{km}$ the Iitaka dimension of E. Here we first slightly improve Mistretta and Urbinati's result to show that the rational maps φ_m are birationally equivalent for all sufficiently large $m \in \mathbf{N}(E)$. Then we show that

$$\kappa(E) \geq \kappa\big(\mathcal{O}_{\mathbb{P}(E)}(1)\big) - \operatorname{rank} E + 1.$$

An immediate corollary of this inequality is that if E is big then $\kappa(E) = \dim X$, which answers a question of Mistretta and Urbinati. Another corollary is that if E is big then $\det E$ is big, provided that $\mathbf{N}(E) \neq \emptyset$.

Keywords Asymptotically generically generated · Big vector bundle · Iitaka dimension

✉ Shin-Yao Jow
syjow@math.nthu.edu.tw

1 Department of Mathematics, National Tsing Hua University, Hsinchu City, Taiwan

Mathematics Subject Classification 14C20 · 14J60 · 32L10

1 Introduction

Notations and conventions We will work over an algebraically closed field. Varieties are assumed to be *irreducible*. Points of a variety mean *closed* points. If V is a vector space, $\mathbb{P}(V)$ denotes the projective space of one-dimensional *quotients* of V, and $\mathbb{G}(V, r)$ denotes the Grassmannian of r-dimensional *quotients* of V. If E is a vector bundle on a variety X and $x \in X$ is a point, we denote by $\mathbb{P}(E)$ the projective bundle of one-dimensional *quotients* of E, and by E_x the *fiber* of E over x (**not** the stalk of the germs of sections of E at x).

Let X be a projective variety. It is a basic construction in algebraic geometry that a line bundle L on X such that $H^0(X, L^{\otimes m}) \neq 0$ for some $m > 0$ naturally induces a rational map from X to the projective space $\mathbb{P}\big(H^0(X, L^{\otimes m})\big)$. In [2], Mistretta and Urbinati studied a generalization of this construction to vector bundles as follows.

Definition 1 Let E be a vector bundle on a projective variety X, and let U be an open subset of X. We say that

(1) E is *globally generated on* U if the evaluation map $H^0(X, E) \to E_x$ is surjective for every point $x \in U$.
(2) E is *generically generated* if it is globally generated on some nonempty open subset of X.
(3) E is *asymptotically generically generated* (AGG) if for some positive integer m, the mth symmetric power $S^m E$ of E is generically generated.

Definition 2 Let E be an AGG vector bundle of rank r on a projective variety X. Let m be a positive integer such that $S^m E$ is globally generated on a nonempty open subset $U \subseteq X$. Denote

$$\sigma_m(r) = \operatorname{rank} S^m E = \binom{m + r - 1}{m}.$$

Then one can define a rational map

$$\varphi_m : X \dashrightarrow \mathbb{G}\big(H^0(X, S^m E), \sigma_m(r)\big)$$

by sending a point $x \in U$ to the $\sigma_m(r)$-dimensional quotient $[H^0(X, S^m E) \twoheadrightarrow S^m E_x]$ of $H^0(X, S^m E)$ under the evaluation map. We call φ_m the mth *Kodaira map* of E.

Our first result is a slight improvement on [2, Theorem 4.4].

Theorem 3 *Let E be an AGG vector bundle on a complex projective variety X, and denote*

$$\mathbf{N}(E) = \{m \in \mathbb{N} \mid S^m E \textit{ is generically generated}\}.$$

For each $m \in \mathbf{N}(E)$, let φ_m be the mth Kodaira map of E, and let Y_m be the closure of $\varphi_m(X)$. Then for all sufficiently large $m \in \mathbf{N}(E)$, the rational maps $\varphi_m : X \dashrightarrow Y_m$ are birationally equivalent to a fixed surjective morphism of projective varieties

$$\varphi_{\mathbb{G}} : X_{\mathbb{G}} \to Y_{\mathbb{G}}.$$

That is, there exists a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{u_{\mathbb{G}}} & X_{\mathbb{G}} \\ {\scriptstyle \varphi_m} \downarrow & & \downarrow {\scriptstyle \varphi_{\mathbb{G}}} \\ Y_m & \underset{\nu_m}{\dashleftarrow} & Y_{\mathbb{G}} \end{array}$$

where the horizontal maps are birational and $u_{\mathbb{G}}$ *is a morphism.*

The statement in [2, Theorem 4.4] is the same except that instead of "for all sufficiently large $m \in \mathbf{N}(E)$", they have "for every $m \in \mathbf{N}(E)$ and for $k \gg 0$" the rational maps $\varphi_{km}: X \dashrightarrow Y_{km}$ are birationally equivalent to a fixed surjective morphism $\varphi_{\mathbb{G}}: X_{\mathbb{G}} \to Y_{\mathbb{G}}$. Our version is more in line with the original Iitaka fibrations for line bundles [1, Theorem 2.1.33]. We also remark that compared to the case of line bundles, the difficulty in the proof comes from the fact that for a vector bundle E, $S^p E \otimes S^q E$ is not isomorphic to $S^{p+q} E$, and also that the morphism $\varphi_{\mathbb{G}}: X_{\mathbb{G}} \to Y_{\mathbb{G}}$ is in general not a fibration (i.e., may not have connected fibers: see [2, Example 3.7]).

Following [2], we call the dimension of $Y_{\mathbb{G}}$ the *Iitaka dimension* of E, and denote it by $\kappa(X, E)$ or $\kappa(E)$. Mistretta and Urbinati showed that if E is strongly semiample, meaning that $S^m E$ is globally generated on X for some $m > 0$, then $\kappa(E) = \kappa(\det E)$ [2, Remark 4.5]. They then raised several questions on Iitaka dimensions for vector bundles which are not necessarily strongly semiample. Our second result, which is an inequality relating the Iitaka dimensions of E and $\mathcal{O}_{\mathbb{P}(E)}(1)$, can help answer some of those questions.

Theorem 4 *Let E be an AGG vector bundle of rank r on a projective variety X. Let $\pi : \mathbb{P}(E) \to X$ be the projective bundle of one-dimensional quotients of E, and let $\mathcal{O}_{\mathbb{P}(E)}(1)$ be the tautological quotient line bundle of $\pi^* E$ on $\mathbb{P}(E)$. Then*

$$\kappa(X, E) \geq \kappa\big(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1)\big) - r + 1.$$

Corollary 5 *Let E be an AGG vector bundle on a projective variety X. If E is big (meaning that $\mathcal{O}_{\mathbb{P}(E)}(1)$ is big), then $\kappa(E) = \dim X$.*

This answers Question 4.7 and 4.8 in [2].

Corollary 6 *Let E be an AGG vector bundle on a projective variety X. If E is big, then* $\det E$ *is big.*

This answers Question 4.6 in [2] affirmatively when E is big. Note that Corollary 6 is interesting in its own right since it is false without the AGG assumption: for example let E be a direct sum of line bundles

$$E = \mathcal{O}_X(D_1) \oplus \cdots \oplus \mathcal{O}_X(D_r).$$

Then E is big if and only if some nonnegative $\mathbb{Z}$-linear combination of the D_i is big [1, Lemma 2.3.2], whereas $\det E$ is big if and only if $D_1 + \cdots + D_r$ is big.

2 Proofs

Lemma 7 *Let E be a vector bundle on a projective variety X. If p and q are positive integers such that $S^p E$ and $S^q E$ are globally generated on an open subset U of X, then $S^{p+q} E$ is globally generated on U. Moreover, if $x, y \in U$ are points such that $\varphi_p(x) \neq \varphi_p(y)$, then $\varphi_{p+q}(x) \neq \varphi_{p+q}(y)$.*

Proof By assumption the evaluation maps

$$\mathrm{ev}_{p,x}\colon H^0(X, S^pE) \twoheadrightarrow S^pE_x \ \text{ and } \ \mathrm{ev}_{q,x}\colon H^0(X, S^qE) \twoheadrightarrow S^qE_x$$

are surjective for each $x \in U$. It thus follows from the commutative diagram

$$\begin{array}{ccc} H^0(X, S^pE) \otimes H^0(X, S^qE) & \xrightarrow{\mathrm{ev}_{p,x}\otimes \mathrm{ev}_{q,x}} & S^pE_x \otimes S^qE_x \\ \downarrow & & \downarrow \\ H^0(X, S^{p+q}E) & \xrightarrow{\mathrm{ev}_{p+q,x}} & S^{p+q}E_x \end{array}$$

that the evaluation map $H^0(X, S^{p+q}E) \to S^{p+q}E_x$ is surjective. Hence $S^{p+q}E$ is globally generated on U.

For each point $x \in U$, by definition

$$\varphi_p(x) = [\mathrm{ev}_{p,x}\colon H^0(X, S^pE) \twoheadrightarrow S^pE_x].$$

If $x, y \in U$ are points such that $\varphi_p(x) \neq \varphi_p(y)$, then there exists an element v in $H^0(X, S^pE)$ such that

$$\mathrm{ev}_{p,x}(v) = 0 \ \text{ and } \ \mathrm{ev}_{p,y}(v) \neq 0.$$

Pick any $w \in H^0(X, S^qE)$ such that

$$\mathrm{ev}_{q,x}(w) \neq 0 \ \text{ and } \ \mathrm{ev}_{q,y}(w) \neq 0.$$

Denote by $v \cdot w \in H^0(X, S^{p+q}E)$ the image of $v \otimes w \in H^0(X, S^pE) \otimes H^0(X, S^qE)$ under the multiplication map. It follows from the commutative diagram above that

$$\mathrm{ev}_{p+q,x}(v \cdot w) = 0 \ \text{ and } \ \mathrm{ev}_{p+q,y}(v \cdot w) \neq 0.$$

Hence $\varphi_{p+q}(x) \neq \varphi_{p+q}(y)$. □

Proof of Theorem 3 By the proof of [2, Theorem 4.4], for each $m \in \mathbf{N}(E)$ the rational map $\varphi_m\colon X \dashrightarrow Y_m$ factors through $\varphi'_{\mathbb{G}} = \varphi_{\mathbb{G}} \circ u_{\mathbb{G}}^{-1}\colon X \dashrightarrow Y_{\mathbb{G}}$. So to show that $\varphi_m = \varphi'_{\mathbb{G}}$ for sufficiently large m, it is enough to show that $\varphi'_{\mathbb{G}}(x) \neq \varphi'_{\mathbb{G}}(y)$ implies $\varphi_m(x) \neq \varphi_m(y)$ for all points x and y in some nonempty open subset of X.

Pick $m_1, \ldots, m_n \in \mathbf{N}(E)$ which generate $\mathbf{N}(E)$ as a semigroup. By [2, Theorem 4.4], $\varphi_{km_i} = \varphi'_{\mathbb{G}}$ for $k \gg 0$. Hence all sufficiently large $m \in \mathbf{N}(E)$ can be written as $m = \sum_{i=1}^n k_i m_i$, where $k_i \in \mathbb{N}$ and at least one of $\varphi_{k_i m_i} = \varphi'_{\mathbb{G}}$. It thus follows from Lemma 7 that $\varphi'_{\mathbb{G}}(x) \neq \varphi'_{\mathbb{G}}(y)$ implies $\varphi_m(x) \neq \varphi_m(y)$. □

Proof of Theorem 4 Let $m > 0$ be a positive integer such that S^mE is globally generated on a nonempty open subset U of X. We denote

$$V = H^0(X, S^mE) \ \text{ and } \ \mathbb{G}_m = \mathbb{G}\big(V, \sigma_m(r)\big).$$

Let Q_m be the tautological quotient bundle on $\mathbb{G}_m$. Then $\mathbb{P}(Q_m)$ is isomorphic to a two-step flag variety

$$\mathbb{P}(Q_m) = \{[V \twoheadrightarrow W_1 \twoheadrightarrow W_2] \mid \dim W_1 = \sigma_m(r), \dim W_2 = 1\},$$

so there are projection morphisms

$$\pi_1\colon \mathbb{P}(Q_m) \to \mathbb{G}_m \ \text{ and } \ \pi_2\colon \mathbb{P}(Q_m) \to \mathbb{P}(V).$$

Let $x \in U$ be a point. Then the evaluation map $H^0(X, S^m E) \twoheadrightarrow S^m E_x$ is surjective. By definition a point $y \in \pi^{-1}(x)$ is a one-dimensional quotient $[E_x \twoheadrightarrow L]$ of E_x. Since

$$H^0\big(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m)\big) = V = H^0(X, S^m E) \twoheadrightarrow S^m E_x \twoheadrightarrow S^m L = \mathcal{O}_{\mathbb{P}(E)}(m)_y,$$

the line bundle $\mathcal{O}_{\mathbb{P}(E)}(m)$ is globally generated on the open set $\widetilde{U} = \pi^{-1}(U) \subseteq \mathbb{P}(E)$. Let

$$\phi_m : \widetilde{U} \to \mathbb{P}(V), \quad y = [E_x \twoheadrightarrow L] \mapsto [V \twoheadrightarrow S^m L]$$

be the morphism defined by the global sections of $\mathcal{O}_{\mathbb{P}(E)}(m)$ on $\mathbb{P}(E)$, and let

$$\psi_m : \widetilde{U} \to \mathbb{P}(Q_m), \quad y = [E_x \twoheadrightarrow L] \mapsto [V \twoheadrightarrow S^m E_x \twoheadrightarrow S^m L].$$

Then there is a commutative diagram

$$\begin{array}{ccccc} & & \overset{\phi_m}{\frown\!\!\!\longrightarrow} & & \\ \widetilde{U} & \xrightarrow[\psi_m]{} & \mathbb{P}(Q_m) & \xrightarrow[\pi_2]{} & \mathbb{P}(V) \\ \pi \big\downarrow & & \big\downarrow \pi_1 & & \\ U & \xrightarrow[\varphi_m]{} & \mathbb{G}_m & & \end{array} .$$

For each point $x \in U$,

$$\pi^{-1}(x) = \mathbb{P}(E_x), \ \pi_1^{-1}\big(\varphi_m(x)\big) = \mathbb{P}(S^m E_x),$$

and

$$\psi_m\big|_{\pi^{-1}(x)} : \mathbb{P}(E_x) \to \mathbb{P}(S^m E_x), \quad [E_x \twoheadrightarrow L] \mapsto [S^m E_x \twoheadrightarrow S^m L]$$

is the mth Veronese embedding. Hence

$$\dim \varphi_m(U) = \dim \psi_m(\widetilde{U}) - r + 1 \geq \dim \phi_m(\widetilde{U}) - r + 1.$$

Since $\dim \varphi_m(U) = \kappa(X, E)$ and $\dim \phi_m(\widetilde{U}) = \kappa\big(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1)\big)$ for sufficiently large $m \in \mathbf{N}(E)$,

$$\kappa(X, E) \geq \kappa\big(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1)\big) - r + 1.$$

□

Corollary 5 follows immediately from Theorem 4. As for Corollary 6, applying the argument in the proof of [2, Theorem 3.4] to an open set $U \subseteq X$ where $S^m E$ is globally generated, one sees that the rational map $\varphi_{\mathbb{G}} : X \dashrightarrow Y_{\mathbb{G}}$ is a composition of the Iitaka fibration $\varphi_{\det E} : X \dashrightarrow Y_\infty$ followed by a dominant rational map $Y_\infty \dashrightarrow Y_{\mathbb{G}}$. It follows that

$$\kappa(\det E) \geq \kappa(E).$$

If E is big (and AGG), $\kappa(E) = \dim X$ by Corollary 5, and hence $\kappa(\det E) = \dim X$.

Acknowledgements The author gratefully acknowledges the support of MoST (Ministry of Science and Technology, Taiwan).

References

1. Lazarsfeld, R.: Positivity in Algebraic Geometry I–II, Ergeb. Math. Grenzgeb, vols. 48–49. Springer, Berlin (2004)
2. Mistretta, E., Urbinati, S.: Iitaka fibrations for vector bundles, IMRN, rnx239. https://doi.org/10.1093/imrn/rnx239