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Radially symmetric solutions of elliptic PDEs with uniformly negative weight

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Abstract We consider the perturbed Hammerstein integral equation

$$y(t) = \gamma(t)H(\varphi(y)) + \lambda \int_0^1 G(t,s)f(s,y(s))\,\mathrm{d}s$$

in the case where it may hold that $f(t,y) < 0$, for each $(t,y) \in [0,1] \times [0,+\infty)$, and $\lim_{y\to\infty} f(t,y) = -\infty$; in other words, f may be a strictly negative function on its entire domain and may uniformly blow up to $-\infty$ as $y \to +\infty$. We apply our results, in part, to radially symmetric solutions of PDEs of the form

$$-\Delta u(\boldsymbol{x}) = \lambda a(|\boldsymbol{x}|)g(u(\boldsymbol{x}))$$

subject to nonlocal boundary conditions and show that this problem can possess a positive solution even if $\lim_{u\to\infty} g(u) = -\infty$. By using a nonstandard cone and attendant open set, these results are able to be guaranteed by imposing relatively straightforward conditions. In addition, our results apply to forcing terms f and g with polynomial growth at $+\infty$ of any degree. We demonstrate that, in principle, our results can be applied to ecological modeling with density-dependent growth and nonlocal boundary conditions.

Keywords Hammerstein integral equation · Negative weight · Coercivity · Radially symmetric solution · Density-dependent growth

Mathematics Subject Classification Primary 35B09 · 35J25 · 45G10 · 45M20 · 47H30; Secondary 34B10 · 47H07 · 92D40

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1 Introduction

In this paper, we consider the existence of at least one positive solution of the perturbed Hammerstein integral equation

$$y(t) = \gamma(t)H(\varphi(y)) + \lambda \int_0^1 G(t, s)f(s, y(s))\,\mathrm{d}s. \tag{1.1}$$

The specific conditions imposed on the various constituent parts of Eq. (1.1) will be detailed in Sect. 2, but essentially the functions γ, H, and f are continuous, whereas the functional φ is a linear functional, realized as the Stieltjes integral $\varphi(y) = \int_0^1 y(t)\,d\alpha(t)$, where α is of bounded variation on [0, 1] but not necessarily monotone. Under a well-known transformation, solutions of the perturbed Hammerstein integral Eq. (1.1) can correspond, for example, to radially symmetric solutions of the PDE

$$\begin{aligned} -\Delta u(\boldsymbol{x}) &= \lambda a(|\boldsymbol{x}|)g\big(u(\boldsymbol{x})\big), \quad |\boldsymbol{x}| \in [R_1, R_2] \\ u(\boldsymbol{x})\big|_{x\in\partial\mathcal{B}_{R_1}} &= 0 \\ u(\boldsymbol{x})\big|_{x\in\partial\mathcal{B}_{R_2}} &= H\big(\varphi(u)\big), \end{aligned} \tag{1.2}$$

for $0 < R_1 < R_2 < +\infty$ and $\boldsymbol{x} \in \mathbb{R}^n$, where the element $H\big(\varphi(u)\big)$ represents a nonlocal, possibly nonlinear boundary condition. We provide specific examples of this correspondence in Sects. 3 and 4. Problems such as (1.1)–(1.2) can arise in thermostat problems, beam deformation and displacement, and chemical reactor theory among other applications—see Infante and Pietramala [33,39] and Cabada et al. [4], for example.

The main contribution of this paper is to demonstrate that by using the nonstandard cone

$$\mathcal{K} := \big\{y \in \mathscr{C}\big([0, 1]\big)\colon y(t) \ge q(t)\|y\|,\ \varphi(y) \ge C_0\|y\|\big\}$$

for a constant $C_0 > 0$ to be determined later and a continuous function $q\colon [0, 1] \to [0, 1]$, together with the open set, for $\rho > 0$,

$$\widehat{V}_\rho := \big\{y \in \mathcal{K}\colon \varphi(y) < \rho\big\},$$

we can construct a new method for attacking problem (1.1) (and, thus, problem (1.2) by extension) in the case where

$$\lim_{y\to\infty} f(t, y) = -\infty.$$

We have previously utilized a cone similar to $\mathcal{K}$ together with the set $\widehat{V}_\rho$ to deduce existence results in the case of sign-changing Green's functions for perturbed Hammerstein integral equations [22,27] as well as for the positone problem [21,23].

We would like to remark briefly that problems similar to (1.2) (or, especially, the analogous problem in the ODEs setting) have been studied extensive in the local BCs setting within the context of so-called indefinite weight problems—see the recent article by Feltrin and Zanolin [13] and the references therein. In this context, the function $\boldsymbol{x} \mapsto a\big(|\boldsymbol{x}|\big)$ is the "weight." Once possible specialization of our results, therefore, is to the problem of (1.2) in which the weight a satisfies $a(t) < 0$ for all $t \in [0, +\infty)$ and in which $0 > a(t)g(u) \to -\infty$ for each t as $u \to +\infty$.

Because of the coercivity condition imposed on φ, by means of $\mathcal{K}$, as well as the open set $\widehat{V}_\rho$, the conditions we impose are simple, flexible, and relatively easy to verify—this will be demonstrated via some examples in Sects. 3 and 4. The primary assumption that we impose on the map $(t, y) \mapsto f(t, y)$ is that

$$\lim_{y\to\infty}\frac{f(t,y)}{y^{\kappa_0}}=0,$$

uniformly for $t\in[0,1]$, for some $\kappa_0>0$. We study first the case in which $\kappa_0=1$; this is the focus of Sects. 2 and 3, whereas in Sect. 4 we study the more general case in which $\kappa_0>1$. We divide the two cases since the latter introduces some additional technical complications.

Note that the condition $\lim_{y\to\infty} f(t,y)=-\infty$ has some relevance in the realm of biological modeling. Indeed, it is common in density-dependent ecological modeling (e.g., the classical logistic differential equation) to have forcing terms that satisfy this condition. Of course, in such problems only positive solutions are of relevance. Therefore, there is perhaps some interest in developing methods that can guarantee existence of positive solutions to problems even in spite of forcing terms possessing this property. For instance, if we consider a diffusion problem of the form $U_t=\Delta U(t,\boldsymbol{x})+f\big(t,U(t,\boldsymbol{x})\big)$, and then steady-state solutions in the one-dimensional setting correspond to solutions of (1.1). If U represents the density of some population, then forcing terms of the form $f(t,y):=y(1-y)$ or $f(t,y)=y(1-y)-\frac{y^2}{1+y^2}$ commonly occur in such density-dependent ecological modeling—see, for example, the well-known spruce budworm model [55]. In these cases, the forcing terms are of quadratic growth at $+\infty$ and, in addition, uniformly blow up to $-\infty$ as $y\to+\infty$. In spite of the quadratic growth, our results could be applied since we can take $\kappa_0=3$, for example.

In addition to the possible relevance of biological modeling, from a purely mathematical point of view, allowing the forcing term f to satisfy the condition $\lim_{y\to\infty} f(t,y)=-\infty$ as well as $f(t,y)<0$, for all (t,y), makes the analysis more challenging. Indeed, as we explain momentarily, accommodating these more general forcing terms requires discarding the "usual" approach to semipositone problems and inventing a new methodology for dealing with this type of problem.

More specifically, the typical analysis of semipositone problems, which can be traced back to Anuradha et al. [3], involves imposing the so-called semipositone condition, wherein one assumes that $f(t,y)\geq-\eta_0$ for some $\eta_0>0$—or, more generally, $f(t,y)\geq-u(t)$, for some nonnegative map $u\in L^1\big([0,1]\big)$, say, for each $y\in[0,+\infty)$. Then, the standard methodology in the nonlocal setting (see [18,19,25], for instance) is to create a modified problem of the form

$$y(t)=\gamma(t)H^*\big(\varphi(y-w)\big)+\lambda\int_0^1 G(t,s)\big[f\big(s,y^*(s)\big)+u(s)\big]\,\mathrm{d}s,$$

where $H^*(z):=H\big(\max\{0,z\}\big)$ and $y^*(t):=\max\big\{0,(y-w)(t)\big\}$ with $w(t):=\lambda\int_0^1 G(t,s)u(s)\,\mathrm{d}s$, and deduce that this problem has a positive solution under some collection of hypotheses. Finally, the correspondence $z(t):=(y-w)(t)$ provides a positive solution of the original problem—see, for example, [19, Theorem 3.1].

Here, by contrast, we create a different modification. Instead, we consider the problem

$$y(t)=\gamma(t)H^*\big(\varphi(y-w_y)\big)+\lambda\int_0^1 G(t,s)\big[f\big(s,y^*(s)\big)+u(s)+\varphi(y)\big]\,\mathrm{d}s, \tag{1.3}$$

where we put

$$y^*(t):=\max\big\{0,y(t)-w_y(t)\big\}$$

and

$$H^*(z):=H\big(\max\{0,z\}\big)$$

and

$$w_y(t) := \lambda \int_0^1 G(t,s)\big[u(s) + \varphi(y)\big]\,\mathrm{d}s.$$

Thus, the map $t \mapsto w_y(t)$ is now parameterized by the function y, which is in contrast to the classical approach. The key observation is that since φ is a coercive linear functional due to the cone $\mathcal{K}$, we can use this coercivity to control the quantity $f\big(s, y^*(s)\big)$ so that even if $\lim_{y\to\infty} f(t,y) = -\infty$, it can still hold that $f\big(s, y^*(s)\big) + u(s) + \varphi(y) \geq 0$, for each $s \in [0,1]$. In the end, it is then easy to show that the translation $z(t) := \big(y - w_y\big)(t)$ produces a positive solution, z, of problem (1.1) from a positive solution, y, of problem (1.3). Finally, by using the open set $\widehat{V}_\rho$ instead of a more classical open set such as either $\Omega_\rho := \big\{y \in \mathcal{K}\colon \|y\| < \rho\big\}$ or $V_\rho := \big\{y \in \mathcal{K}\colon \min_{t\in[a,b]} y(t) < \rho\big\}$ it turns out that we can achieve more refined results.

We demonstrate the aforementioned improvements in Sect. 3, but let us mention at this point that the results in this paper represent a significant improvement over those we presented in [24]. The main result of that article, namely [24, Theorem 3.1], requires numerous assumptions that we do not require here—see Remark 3.6 in Sect. 3. Moreover, while the results there were certainly applicable, our methodology here is far simpler to apply and yields better results. Finally, another upshot of the methodology we introduce here is that it should be readily applicable to numerous other problem utilizing negative forcing terms, and so, is much more widely applicable than the more narrowly structured results of [24].

To conclude the introduction we would like to mention some of the existing literature in the area of both semipositone problems and nonlocal boundary value problems. In particular, as we already mentioned, the paper by Anuradha et al. [3] is important as regards the semipositone problem. In more recent years, many papers have appeared on semipositone boundary value problems equipped with a variety of boundary conditions—see, for example, [1,8,18,19,29,48,58,62,74] and the references therein. In addition, a variety of works have appeared on radially symmetric solutions for elliptic PDEs, and some representative works include those by Cianciaruso et al. [7], Dhanya et al. [9], do Ó et al. [10–12], Herrón and Lopera [32], Infante and Pietramala [41] and Webb [63]. Of particular relevance to our work here, the papers [41,63] address nonlocal boundary conditions in the elliptic PDEs setting, whereas [7] addresses nonlocal, possibly nonlinear boundary conditions in the elliptic PDEs setting.

The paper by Cianciaruso et al. [7] is particularly interesting and relevant because not only do they treat radially symmetric solutions of an elliptic PDE with possibly nonlinear, nonlocal boundary conditions, but the condition they impose on the equivalent of our H in (1.1) is essentially imposed only along the boundary of a set—i.e., only for y satisfying $\|y\| = \rho$. It is thus interesting to compare this condition with the one we utilize in this work with our new set $\widehat{V}_\rho$. We feel that the $\widehat{V}_\rho$ set offers some advantages in that our existence results only involve checking the value of H at point and no other growth condition on H whatsoever—not even along some sort of topological boundary. In fact, in some cases (see, for example, [26, Remark 3.6]), the $\widehat{V}_\rho$-methodology can be superior to that of [7]. The same remarks can be made about the very recent work of Cabada et al. [5], which while couched in a somewhat more general context than [7], nonetheless employs the same basic framework inasmuch as the nonlocal elements are concerned.

More generally, as concerns nonlocal boundary value problems, numerous works have appeared over the past few years. In addition to the fundamental works of Webb and Infante [59–61], which addressed linear nonlocal boundary conditions, other interesting papers have also addressed linear nonlocal boundary conditions in a variety of settings, such as those by Graef and Webb [30], Infante et al. [35,36,38,40], Jankowski [42], Karakostas and Tsamatos [44,45], Karakostas [46], Webb [65] and Yang [70,71]. On the other hand, as concerns the setting of nonlocal, potentially nonlinear boundary conditions, relevant works include those by Anderson [2], Goodrich [14–17,20], Infante et al. [33,34,37,39], Kang et al. [43], Karakostas [46] and Yang [68,69]. In addition, a recent paper by Cianciaruso and Pietramala [6] addresses nonlocal, nonlinear boundary conditions in the context of (p_1, p_2)-Laplacian equations; it is worth noting that in [6] the authors impose uniform growth conditions on the nonlinear elements (i.e., the equivalent of H in this paper), whereas, as noted earlier, due to our use of the $\widehat{V}$-type set here we impose only pointwise conditions—in fact, none of these preceding papers imposes only pointwise conditions on the nonlinear boundary element.

In addition, since we couch our results here in the context of perturbed Hammerstein integral equations, we note that there have been many works in this area over the past many years, including contributions by Cabada et al. [4], Cianciaruso et al. [7], Goodrich [22,23], Lan et al. [49,51–53], Liu and Wu [54], Xu and Yang [67] and Yang [72]. It is also worth noting that the classical articles by Picone [57] and Whyburn [66] make for interesting reading on the historical trajectory of nonlocal boundary value problems.

Finally, we would like to highlight the recent article by Lan and Yang [50]. In this article, the authors develop a new fixed point index, which can handle BVPs with forcing terms satisfying $\lim_{y\to\infty} f(t, y) = -\infty$. And a distinguishing aspect of our results is that we allow the forcing term to be *always* strictly negative. In fact, because for "small" y we only require $f(t, y) \geq -u(t)$, for each $t \in [0, 1]$, with $u \in L^1\big([0, 1]\big)$, it is actually possible that our function f not only satisfies $f(t, y) < 0$ for all (t, y), but it may also be the case that $\big|f(t, y)\big|$ is quite large—in other words, there is no restriction on just "how negative" the value $f(t, y)$ may be for any particular pair (t, y). By contrast, the results of [50] require the forcing term to satisfy $f(t, 0) \geq 0$, a.e. $t \in [0, 1]$. Thus, in this sense, our results here have extra flexibility and applicability since, unlike [50], we do not even require the nonnegativity of the number $f(t, 0)$ for any $t \in [0, 1]$. Moreover, since our results here permit the use of possibly nonlocal, nonlinear boundary conditions, this allows for some additional flexibility since the methodology of [50] did not address nonlocal boundary conditions. Thus, our results here are more flexible in this sense, too.

2 Preliminary lemmata and notation

We begin by listing the structural and regularity assumptions that we make regarding problem (1.1). Note that throughout this work we denote by $\|\cdot\|$ the usual supremum norm on the space $\mathscr{C}\big([0, 1]\big)$. As mentioned in Sect. 1, for the time being we restrict ourselves, via condition (H3.1), to the case in which $\kappa_0 = 1$—i.e., f grows sublinearly at $+\infty$. In Sect. 4, we will consider the case in which any $\kappa_0 > 1$ is allowed.

H1: The functional φ has the form

$$\varphi(y) := \int_{[0,1]} y(t)\, \mathrm{d}\alpha(t),$$

where $\alpha\colon [0, 1] \to \mathbb{R}$ satisfy $\alpha \in BV\big([0, 1]\big)$. In addition, we let the constant $C_1 > 0$ satisfy

$$\big|\varphi(y)\big| \leq C_1 \|y\|,$$

for each $y \in \mathscr{C}\big([0,1]\big)$. Finally, letting $S_0 \subseteq [0,1]$ be a set of full measure on which $\mathcal{G}(s) := \sup_{t\in[0,1]} G(t,s) > 0$, we assume that the constant C_0 defined by

$$C_0 := \inf_{s\in S_0} \frac{1}{\mathcal{G}(s)} \int_0^1 G(t,s)\,\mathrm{d}\alpha(t)$$

satisfies $+\infty > C_0 > 0$.

H2: The functions $\gamma\colon [0,1] \to [0,+\infty)$ and $H\colon [0,+\infty) \to [0,+\infty)$ are continuous, and the function γ satisfies the inequality

$$\varphi(\gamma) \geq C_0\|\gamma\|.$$

H3: The function $f\colon [0,1] \times [0,+\infty) \to \mathbb{R}$ is continuous and satisfies the following two growth conditions.

(1) $\lim_{y\to+\infty} \frac{f(t,y)}{y} = 0$, uniformly for $t \in [0,1]$
(2) There exists a map $u \in L^1\big([0,1];[0,+\infty)\big)$ such that

$$f(t,y) \geq -u(t)$$

for each $t \in [0,1]$ and $y \in [0,R_0]$, where

$$R_0 := \inf\big\{R^* \in (0,+\infty)\colon f(t,y) \geq -C_0 y,\ \text{for all } y \geq R^*\big\}.$$

H4: The function $G\colon [0,1] \times [0,1] \to [0,+\infty)$ satisfies:

(1) $G \in L^1\big([0,1]\times[0,1]\big)$;
(2) for each $\tau \in [0,1]$ it holds that

$$\lim_{t\to\tau} |G(t,s) - G(\tau,s)| = 0,\ \text{a.e. } s \in [0,1]; \quad \text{and}$$

(3) $\mathcal{G}(s) := \sup_{t\in[0,1]} G(t,s) < +\infty$ for each $s \in [0,1]$.

H5: There exists a continuous function $q\colon [0,1] \to [0,1]$ with $\|q\| \neq 0$ such that

(1) $\gamma(t) \geq q(t)\|\gamma\|$, for each $t \in [0,1]$; and
(2) $q(t) \geq G(t,s) \geq q(t)\mathcal{G}(s)$, for each $(t,s) \in [0,1]\times[0,1]$.

Remark 2.1 Note that we

- Make no assumptions on the behavior of H either asymptotically or on nondegenerate intervals—only pointwise assumptions will be made (see Theorem 3.1);
- Make no growth assumptions on the behavior of $(t,y) \mapsto f(t,y)$ except when $y \to +\infty$; and
- Allow for forcing terms, f, such that not only can $f(t,y) \to -\infty$ as $y \to +\infty$ but also it can hold that $f(t,y) < 0$ for all $(t,y) \in [0,1]\times[0,+\infty)$ with no restriction at all on the size of the quantity $\big|f(t,y)\big|$.

We would also like to point out that the range of allowable values of the parameter λ is explicitly computable here.

Remark 2.2 For commonly occurring kernels, G, the function q is easy to calculate. For example, if

$$G(t,s) := \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1 \\ s(1-t), & 0 \leq s \leq t \leq 1 \end{cases},$$

then one may take $q(t) := \min\{t, 1-t\}$, for example.

We begin by mentioning some properties of the set $\widehat{V}_\rho$, which was described in Sect. 1. The proof of this lemma can be isolated from [23, Lemmata 2.9–2.10] with only minor additions and alterations, and so, we omit the proof.

Lemma 2.3 *Given numbers $\rho > 0$ and $\rho_2 > \rho_1 > 0$ each of the following is true.*

(1) *The set $\widehat{V}_\rho$ is open (relative to $\mathcal{K}$).*
(2) *The set $\widehat{V}_\rho$ is bounded.*
(3) $\widehat{V}_{\rho_2} \supset \widehat{V}_{\rho_1}$
(4) $\widehat{V}_{\rho_2} \setminus \overline{\widehat{V}}_{\rho_1} \neq \varnothing$
(5) *Assuming that $C_1 > C_0$, it holds that $\widehat{V}_\rho \subseteq \Omega_{\frac{\rho}{C_0}} \setminus \overline{\Omega}_{\frac{\rho}{C_1}} \neq \varnothing$.*
(6) *If $y \in \partial\widehat{V}_\rho$, then $\varphi(y) = \rho$.*

For future use, we define the operator $T\colon \mathcal{K} \to \mathcal{K}$ by

$$(Ty)(t) = \gamma(t)H^*\big(\varphi(y - w_y)\big) + \lambda \int_0^1 G(t, s)\Big[f\big(s, y^*(s)\big) + u(s) + \varphi(y)\Big]\, \mathrm{d}s. \tag{2.1}$$

Recall that

$$\mathcal{K} := \left\{ y \in \mathscr{C}\big([0, 1]\big)\colon y(t) \geq q(t)\|y\|,\ \varphi(y) \geq \underbrace{\left(\inf_{s\in S_0} \frac{1}{\mathcal{G}(s)} \int_0^1 G(t, s)\, \mathrm{d}\alpha(t)\right)}_{=:C_0} \|y\| \right\}.$$

Remark 2.4 We would like to point out that the cone $\mathcal{K}$ above is related to the cones introduced by Graef et al. [28], Ma and Zhong [56] and Webb [64]. In some sense, it is an extension and amalgamation of the ideas introduced in those articles and then suitably modified to suit our purposes here.

Note that $\mathcal{K} \neq \varnothing$, and it is not trivial—i.e., $\mathcal{K} \neq \{0\}$. These facts are due to assumptions (H1)–(H5) and, in particular, the assumptions on γ that ensure that $\gamma \in \mathcal{K}$. With this in mind, our next preliminary lemma is to demonstrate that $T(\mathcal{K}) \subseteq \mathcal{K}$.

Lemma 2.5 *Assume that conditions (H1)–(H5) are satisfied. Then, with the operator T defined as in (2.1) it holds that $T(\mathcal{K}) \subseteq \mathcal{K}$.*

Proof We first show that when $y \in \mathcal{K}$ it follows that $(Ty)(t) \geq q(t)\|Ty\|$ for each $t \in [0, 1]$. To this end, notice that, by definition, $\gamma(t)H^*\big(\varphi(y - w_y)\big) \geq 0$, for each $t \in [0, 1]$. So, we certainly deduce that

$$\gamma(t)H^*\big(\varphi(y - w_y)\big) \geq q(t)\|\gamma\|H^*\big(\varphi(y - w_y)\big). \tag{2.2}$$

At the same time, we claim that $f\big(s, y^*(s)\big) + u(s) + \varphi(y) \geq 0$, for each $s \in [0, 1]$. To see that this claim is true, note that by the definition of the number R_0, we know that if $y^*(s) \in [0, R_0]$, then it follows that $f\big(s, y^*(s)\big) + u(s) \geq 0$, whereas if $y^*(s) \in (R_0, +\infty)$, then it holds that

$$\begin{aligned} f\big(s, y^*(s)\big) + u(s) + \varphi(y) &\geq f\big(s, y^*(s)\big) + \varphi(y) \geq -C_0 y^*(s) + C_0\|y\| \\ &\geq -C_0 y(s) + C_0\|y\| \\ &\geq \big(C_0 - C_0\big)\|y\| \\ &= 0, \end{aligned}$$

using the fact that $\|y\| \geq \|y^*\|$, for if $y^*(s) > 0$, then $y(s) \geq y^*(s) > 0$ by the definition of the map $s \mapsto y^*(s)$. Thus, we conclude that $f\big(s, y^*(s)\big) + u(s) + \varphi(y) \geq 0$, for each $s \in [0, 1]$. So, by the properties imposed on the kernel G we may estimate

$$\begin{aligned}\int_0^1 G(t,s)\underbrace{\big[f\big(s,y^*(s)\big)+u(s)+\varphi(y)\big]}_{\geq 0}\,\mathrm{d}s &\geq \int_0^1 q(t)\mathcal{G}(s)\big[f\big(s,y^*(s)\big)+u(s)+\varphi(y)\big]\,\mathrm{d}s\\ &\geq q(t)\sup_{t\in[0,1]}\int_0^1 G(t,s)\big[f\big(s,y^*(s)\big)+u(s)+\varphi(y)\big]\,\mathrm{d}s.\end{aligned} \tag{2.3}$$

Then, upon putting (2.2)–(2.3) together we conclude that

$$(Ty)(t) \geq q(t)\|Ty\|,$$

for each $y \in \mathcal{K}$.

On the other hand, to see that $\varphi(Ty) \geq C_0\|Ty\|$ whenever $y \in \mathcal{K}$, we write

$$\begin{aligned}\varphi(Ty) &= \varphi(\gamma)H^*\big(\varphi(y-w_y)\big) + \lambda\int_0^1\int_0^1 G(t,s)\underbrace{\big[f\big(s,y^*(s)\big)+u(s)+\varphi(y)\big]}_{\geq 0}\,\mathrm{d}\alpha(t)\,\mathrm{d}s\\ &\geq C_0\|\gamma\|H^*\big(\varphi(y-w_y)\big)\\ &\quad + \lambda\int_0^1\left[\frac{1}{\mathcal{G}(s)}\int_0^1 G(t,s)\,\mathrm{d}\alpha(t)\right]\mathcal{G}(s)\big[f\big(s,y^*(s)\big)+u(s)+\varphi(y)\big]\,\mathrm{d}s\\ &\geq C_0\|\gamma\|H^*\big(\varphi(y-w_y)\big) + \lambda\int_0^1 C_0\mathcal{G}(s)\big[f\big(s,y^*(s)\big)+u(s)+\varphi(y)\big]\,\mathrm{d}s\\ &\geq C_0\|Ty\|.\end{aligned}$$

So, we conclude that $T(\mathcal{K}) \subseteq \mathcal{K}$, and this completes the proof. □

We next provide a sequence of three lemmata, which collectively ensure that a fixed point of the operator T can, in fact, be related back to a solution of the original integral Eq. (1.1), provided that certain conditions are imposed on λ and the norm of the fixed point. Our first of these three lemmata provides a condition so that $\varphi(y - w_y) \geq 0$ holds. This will be important in ensuring that solutions of the modified integral equation can, in fact, be related back to solutions of the original integral equation.

Lemma 2.6 *Suppose that conditions (H1)–(H5) are true. Assume that* λ *is selected so that*

$$0 < \lambda < \left(\int_0^1\int_0^1 G(t,s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right)^{-1}.$$

If $y \in \partial\widehat{V}_\rho$, *where* ρ *satisfies*

$$\rho > \left(\lambda\int_0^1\int_0^1 G(t,s)u(s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right)\left(1-\lambda\int_0^1\int_0^1 G(t,s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right)^{-1},$$

then $\varphi(y - w_y) \geq 0$.

Proof Let $y \in \partial \widehat{V}_\rho$. Then, recalling that $w_y := \lambda \int_0^1 G(t,s)\big[u(s)+\varphi(y)\big]\,\mathrm{d}s$, we calculate

$$
\begin{aligned}
\varphi(y-w_y) &= \varphi(y)-\varphi(w_y)\\
&= \rho - \lambda \int_0^1 \int_0^1 G(t,s)\big[u(s)+\varphi(y)\big]\,d\alpha(t)\,\mathrm{d}s\\
&= \rho - \lambda \int_0^1 \int_0^1 G(t,s)\big[u(s)+\rho\big]\,\mathrm{d}\alpha(t)\,\mathrm{d}s\\
&= \rho\left[1-\lambda\int_0^1\int_0^1 G(t,s)\,\mathrm{d}\alpha(t)\,ds\right] - \lambda \int_0^1\int_0^1 G(t,s)u(s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s.
\end{aligned}
\tag{2.4}
$$

Consequently, for $y \in \partial \widehat{V}_\rho$ we deduce that $\varphi(y-w_y) \geq 0$ provided that

$$
\rho > \left(\lambda \int_0^1\int_0^1 G(t,s)u(s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right)\left(1-\lambda\int_0^1\int_0^1 G(t,s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right)^{-1}.
$$

Since, by assumption we have that $-\lambda \int_0^1\int_0^1 G(t,s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s > 0$, the above quantity is well defined. And this completes the proof. $\square$

Lemma 2.7 *Suppose that conditions (H1)–(H5) are true. Assume that* $0 < \lambda < \frac{1}{C_1}$. *In addition, for the number* λ *so fixed, suppose that the number* ρ *satisfies*

$$
\rho > \frac{\lambda C_1}{1-\lambda C_1}\int_0^1 u(s)\,\mathrm{d}s.
$$

Then, whenever $y \in \mathcal{K}\setminus\widehat{V}_\rho$, *it follows that* $y^* \equiv y - w_y$.

Proof Recall that the definition of the map $t \mapsto y^*(t)$ is

$$
y^*(t) := \max\big\{0, y(t)-w_y(t)\big\} = \max\left\{0, y(t)-\lambda\int_0^1 G(t,s)\big[u(s)+\varphi(y)\big]\,\mathrm{d}s\right\}.
$$

Note that condition (H5) allows us to estimate

$$
\begin{aligned}
y(t)-\lambda\int_0^1 G(t,s)\big[u(s)+\varphi(y)\big]\,\mathrm{d}s &\geq q(t)\|y\| - \lambda\int_0^1 q(t)\big[u(s)+\varphi(y)\big]\,\mathrm{d}s\\
&\geq q(t)\left[\|y\| - \lambda\int_0^1 \big[u(s)+C_1\|y\|\big]\,\mathrm{d}s\right]\\
&= q(t)\left[\|y\|(1-\lambda C_1) - \lambda\int_0^1 u(s)\,\mathrm{d}s\right]\\
&\geq q(t)\left[\frac{\rho}{C_1}(1-\lambda C_1) - \lambda\int_0^1 u(s)\,\mathrm{d}s\right],
\end{aligned}
\tag{2.5}
$$

where to obtain the final inequality we have used both the fact that $y \in \mathcal{K}\setminus\widehat{V}_\rho$, by assumption, so that

$$
C_1\|y\| \geq \varphi(y) \geq \rho.
$$

and the fact that by the restriction on λ in the statement of this lemma, it follows that $1-\lambda C_1 > 0$. Thus, we conclude from the above remarks, the restriction on ρ, and (2.5) that

$$
y(t)-\lambda\int_0^1 G(t,s)\big[u(s)+\varphi(y)\big]\,\mathrm{d}s \geq 0,
$$

for each $t \in [0, 1]$. Consequently, by the definition of the map $t \mapsto y^*(t)$, we deduce that $y^* \equiv y - w_y$, as claimed. $\square$

Remark 2.8 Note that Lemma 2.7 actually implies something stronger than the conclusion as stated. In particular, since λ is strictly less than C_1^{-1}, it follows that the quantity

$$\frac{\rho}{C_1}(1-\lambda C_1) - \lambda \int_0^1 u(s)\, \mathrm{d}s,$$

which appears in (2.5), is actually *strictly* greater than zero. This means that

$$y(t) - w_y(t) > 0,$$

whenever $q(t) \neq 0$.

Our final preliminary lemma demonstrates that solutions of the modified integral equation can be related back to solutions of the original integral equation by means of the map $t \mapsto \big(y - w_y\big)(t)$.

Lemma 2.9 *Suppose that conditions (H1)–(H5) are true. Assume that*

$$0 < \lambda < \min\left\{\frac{1}{C_1}, \left(\int_0^1 \int_0^1 G(t,s)\, \mathrm{d}\alpha(t)\, \mathrm{d}s\right)^{-1}\right\}.$$

If y_0 is a positive solution of the modified integral Eq. (1.3) *satisfying $y_0 \in \mathcal{K}\backslash\widehat{V}_{\rho^*}$, where the number $\rho^* > 0$ is defined by*

$$\rho^* := \max\left\{\left(\lambda \int_0^1 \int_0^1 G(t,s)u(s)\, \mathrm{d}\alpha(t)\, \mathrm{d}s\right)\left(1 - \lambda \int_0^1 \int_0^1 G(t,s)\, \mathrm{d}\alpha(t)\, \mathrm{d}s\right)^{-1}, \frac{\lambda C_1}{1-\lambda C_1}\int_0^1 u(s)\, \mathrm{d}s\right\},$$

then the function $z\colon [0,1] \to [0,+\infty)$ defined by $z(t) := \big(y_0 - w_{y_0}\big)(t)$ is a solution of the original integral Eq. (1.1).

Proof Define the function $z\colon [0,1] \to [0,+\infty)$ by $z_0(t) := \big(y_0 - w_{y_0}\big)(t)$. Since y_0 solves the modified problem (1.3) we know that

$$y_0(t) = \gamma(t)H^*\big(\varphi(y_0 - w_{y_0})\big) + \lambda \int_0^1 G(t,s)\Big[f\big(s, y_0^*(s)\big) + u(s) + \varphi(y_0)\Big]\, \mathrm{d}s. \quad (2.6)$$

By Lemma 2.6, the fact that

$$\mathcal{K}\backslash\widehat{V}_{\rho^*} = \bigcup_{\widehat{\rho} > \rho^*} \partial \widehat{V}_{\widehat{\rho}},$$

and the condition imposed on λ in the statement of this present lemma, we know that $\varphi\big(y_0 - w_{y_0}\big) \geq 0$, so that, by definition,

$$H^*\big(\varphi(y_0 - w_{y_0})\big) = H\big(\varphi(y_0 - w_{y_0})\big) = H\big(\varphi(z)\big). \quad (2.7)$$

At the same time, since $y_0 \in \mathcal{K}\backslash\widehat{V}_{\rho^*}$ we deduce from Lemma 2.7 that, in fact,

$$y_0^* \equiv y_0 - w_{y_0} \equiv z. \quad (2.8)$$

Consequently, we deduce from estimates (2.6)–(2.8) that

$$\begin{aligned} z(t) &= y_0(t) - w_{y_0}(t) \\ &= \gamma(t) H\big(\varphi(z)\big) + \lambda \int_0^1 G(t,s)\Big[f\big(s, y_0^*(s)\big) + u(s) + \varphi(y_0)\Big]\,\mathrm{d}s \\ &\quad - \lambda \int_0^1 G(t,s)\big[u(s) + \varphi\,(y_0)\,\big]\,\mathrm{d}s \\ &= \gamma(t) H\big(\varphi(z)\big) + \lambda \int_0^1 G(t,s)\Big[f\big(s, \big(y_0 - w_{y_0}\big)(s)\big)\Big]\,\mathrm{d}s \\ &= \gamma(t) H\big(\varphi(z)\big) + \lambda \int_0^1 G(t,s) f\big(s, z(s)\big)\,\mathrm{d}s. \end{aligned}$$

Thus, z solves the original integral Eq. (1.1). Finally, since $\varphi(y_0) > \rho^*$, we know from a combination of Lemma 2.7 and Remark 2.8 that $z(t) = \big(y_0 - w_{y_0}\big)(t) > 0$ whenever $q(t) \neq 0$. Hence, since $\|q\| \neq 0$, it follows that $\|z\| \neq 0$. Thus, z is a nontrivial, positive solution of the original integral Eq. (1.1). And this completes the proof. □

We conclude this section by stating a fixed point result that we will utilize in Sects. 3 and 4. One may consult, for example, Guo and Lakshmikantham [31], Infante et al. [40], or Zeidler [73] for further details on these types of results.

Lemma 2.10 *Let D be a bounded open set and, with $\mathcal{K}$ a cone in a Banach space $\mathscr{X}$, suppose both that $D \cap \mathcal{K} \neq \emptyset$ and that $\overline{D} \cap \mathcal{K} \neq \mathcal{K}$. Let $D_1 \supseteq \{0\}$ be open in $\mathscr{X}$ with $\overline{D}_1 \subseteq D \cap \mathcal{K}$. Assume that $T : \overline{D} \cap \mathcal{K} \to \mathcal{K}$ is a compact map such that $Tx \neq x$ for $x \in \mathcal{K} \cap \partial D$. If $i_{\mathcal{K}}\,(T, D \cap \mathcal{K}) = 1$ and $i_{\mathcal{K}}\,(T, D_1 \cap \mathcal{K}) = 0$, then T has a fixed point in $(D \cap \mathcal{K}) \setminus \big(\overline{D_1 \cap \mathcal{K}}\big)$. Moreover, the same result holds if $i_{\mathcal{K}}(T, D \cap \mathcal{K}) = 0$ and $i_{\mathcal{K}}\,(T, D_1 \cap \mathcal{K}) = 1$.*

3 Results for sublinear growth

We begin by stating and proving our main existence theorem for problem (1.1). We then provide some examples of the application of this theorem. Note that we will use the following notation both in this section and in Sect. 4.

- For a continuous function $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$, a set $[a,b] \subseteq [0,1]$, and numbers $0 \le r_1 < r_2 \le +\infty$, we denote

  (1) $f^M_{[a,b]\times[r_1,r_2]} := \max_{(t,y)\in[a,b]\times[r_1,r_2]} f(t,y)$;
  (2) $f^m_{[a,b]\times[r_1,r_2]} := \min_{(t,y)\in[a,b]\times[r_1,r_2]} f(t,y)$; and
  (3) $|f|^M_{[a,b]\times[r_1,r_2]} := \max_{(t,y)\in[a,b]\times[r_1,r_2]} \big|f(t,y)\big|$.

Theorem 3.1 *Assume that conditions (H1)–(H5) are satisfied. Let $\lambda_0 > 0$ be defined by*

$$\lambda_0 := \min\left\{\left(\int_0^1\int_0^1 G(t,s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right)^{-1}, \frac{1}{C_1}\right\}.$$

Suppose that for fixed $\lambda \in \big(0, \lambda_0\big)$ that there exist numbers $\rho_2 > \rho_1 > \rho^*$, where ρ^* is the number from Lemma 2.9, such that each of

$$H\left(\rho_1\left[1 - \lambda\int_0^1\int_0^1 G(t,s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right] - \lambda\int_0^1\int_0^1 G(t,s)u(s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right) > \frac{\rho_1}{\varphi(\gamma)} \tag{3.1}$$

and

$$\frac{\varphi(\gamma)}{\rho_2}H\left(\rho_2\left[1-\lambda\int_0^1\int_0^1 G(t,s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right]-\lambda\int_0^1\int_0^1 G(t,s)u(s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right)$$
$$+\lambda\int_0^1\int_0^1 G(t,s)\left[\frac{|f|^M_{[0,1]\times\left[0,\frac{\rho_2}{C_0}\right]}}{\rho_2}+\frac{u(s)}{\rho_2}+1\right]\mathrm{d}\alpha(t)\,\mathrm{d}s<1 \tag{3.2}$$

is true. Then problem (1.1) *has at least one positive solution.*

Proof The operator T is evidently completely continuous. Therefore, we omit the proof of this fact.

So, first suppose for contradiction that there exists $y\in\partial\widehat{V}_{\rho_1}$ such that $y=Ty+\mu e$ for some $\mu\geq 0$, where we put $e(t):=\gamma(t)$. Note that this is an admissible choice of e since $\gamma\in\mathcal{K}$, by assumption. Then, by applying φ to both sides of the operator equation and using that $\varphi(y)=\rho_1$ we deduce that

$$\begin{aligned}\rho_1&\geq\varphi(\gamma)H^*\big(\rho_1-\varphi(w_y)\big)+\lambda\int_0^1\int_0^1 G(t,s)\big[f\big(s,y^*(s)\big)+u(s)+\rho_1\big]\,\mathrm{d}\alpha(t)\,\mathrm{d}s\\&=\varphi(\gamma)H\left(\rho_1\left[1-\lambda\int_0^1\int_0^1 G(t,s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right]-\lambda\int_0^1\int_0^1 G(t,s)u(s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right)\\&\quad\underbrace{+\lambda\int_0^1\int_0^1 G(t,s)\underbrace{\big[f\big(s,y^*(s)\big)+u(s)+\rho_1\big]}_{\geq 0}\,\mathrm{d}\alpha(t)\,\mathrm{d}s}_{\geq 0}\\&\geq\varphi(\gamma)H\left(\rho_1\left[1-\lambda\int_0^1\int_0^1 G(t,s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right]-\lambda\int_0^1\int_0^1 G(t,s)u(s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right),\end{aligned}\tag{3.3}$$

where we have used equality (2.4) from the proof of Lemma 2.6 and the fact that $f\big(s,y^*(s)\big)+u(s)+\varphi(y)\geq 0$, for each $s\in[0,1]$, as demonstrated in Lemma 2.5. We have, in addition, used the fact that since condition (H1) holds, we must have that $\int_0^1 G(t,s)\,\mathrm{d}\alpha(t)>0$, a.e. $s\in[0,1]$. Note also that

$$\begin{aligned}H^*\Big(\varphi\big(y-\varphi(w_y)\big)\Big)&=H^*\big(\varphi(y)-\varphi(w_y)\big)=H^*\big(\rho_1-\varphi(w_y)\big)\\&=H\big(\rho_1-\varphi(w_y)\big)\end{aligned}$$

since from the proof of Lemma 2.6 we know that $\varphi(y-w_y)=\rho_1-\varphi(w_y)\geq 0$, whence the above equality holds by the definition of the map $z\mapsto H^*(z)$. But then by condition (3.1) in the statement of the theorem we obtain from inequality (3.3) that $\rho_1>\rho_1$, which is a contradiction. Thus,

$$i_{\mathcal{K}}\left(T,\widehat{V}_{\rho_1}\right)=0. \tag{3.4}$$

Importantly, by the lower bound imposed on the number ρ_1 we note that

$$\mathcal{K}\backslash\widehat{V}_{\rho^*}\supseteq\mathcal{K}\backslash\widehat{V}_{\rho_1}.$$

Conversely, suppose for contradiction the existence of $y\in\partial\widehat{V}_{\rho_2}$ such that $\mu y=Ty$ for some $\mu\geq 1$. Again applying φ to both sides of the operator equation and using that $\varphi(y)=\rho_2$ we obtain that

$$\begin{aligned}\rho_2 &\le \varphi(\gamma)H^*\big(\rho_2 - \varphi(w_y)\big) + \lambda\int_0^1\int_0^1 G(t,s)\big[f\big(s,y^*(s)\big)+u(s)+\rho_2\big]\,\mathrm{d}\alpha(t)\,\mathrm{d}s\\ &= \varphi(\gamma)H\left(\rho_2\left[1-\lambda\int_0^1\int_0^1 G(t,s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right]-\lambda\int_0^1\int_0^1 G(t,s)u(s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right)\\ &\quad+\lambda\int_0^1\int_0^1 G(t,s)\big[f\big(s,y^*(s)\big)+u(s)+\rho_2\big]\,\mathrm{d}\alpha(t)\,\mathrm{d}s,\end{aligned}$$

which can thus be recast as

$$\begin{aligned}1 &\le \frac{\varphi(\gamma)}{\rho_2}H\left(\rho_2\left[1-\lambda\int_0^1\int_0^1 G(t,s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right]-\lambda\int_0^1\int_0^1 G(t,s)u(s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right)\\ &\quad+\lambda\int_0^1\int_0^1 G(t,s)\left[\frac{f\big(s,y^*(s)\big)}{\rho_2}+\frac{u(s)}{\rho_2}+1\right]\mathrm{d}\alpha(t)\,\mathrm{d}s,\end{aligned}\tag{3.5}$$

using that $f\big(s,y^*(s)\big)+u(s)+\rho_2\ge 0$, for each $s\in[0,1]$, and that $\int_0^1 G(t,s)\,\mathrm{d}\alpha(t)>0$, a.e. $s\in[0,1]$. Now, recall that by Lemma 2.7 and the lower bound imposed on ρ_1, we have that $y^*\equiv y-w_y$. As such, $y(t)\ge y(t)-w_y(t)=y^*(t)\ge 0$, for each $t\in[0,1]$. Moreover, since $y\in\partial\widehat{V}_{\rho_2}$, it follows that $\|y^*\|\le\|y\|\le\frac{\rho_2}{C_0}$. Putting all of this together, we see that from (3.5) follows the estimate

$$\begin{aligned}1 &\le \frac{\varphi(\gamma)}{\rho_2}H\left(\rho_2\left[1-\lambda\int_0^1\int_0^1 G(t,s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right]-\lambda\int_0^1\int_0^1 G(t,s)u(s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right)\\ &\quad+\lambda\int_0^1\int_0^1 G(t,s)\left[\frac{|f|^M_{[0,1]\times\left[0,\frac{\rho_2}{C_0}\right]}}{\rho_2}+\frac{u(s)}{\rho_2}+1\right]\mathrm{d}\alpha(t)\,\mathrm{d}s,\end{aligned}\tag{3.6}$$

where we again use that

$$\int_0^1 G(t,s)\,\mathrm{d}\alpha(t)>0,\quad \text{a.e. } s\in[0,1].$$

But then by condition (3.2) we obtain from (3.6) that $1<1$, and so,

$$i_{\mathcal{K}}\left(T,\widehat{V}_{\rho_2}\right)=1.\tag{3.7}$$

Finally, by (3.4) and (3.7) we deduce the existence of $y_0\in\widehat{V}_{\rho_2}\setminus\overline{\widehat{V}}_{\rho_1}$ such that $Ty_0=y_0$. Note that Lemma 2.3 ensures that $\widehat{V}_{\rho_2}\setminus\overline{\widehat{V}}_{\rho_1}\neq\varnothing$. Since we must have $\varphi(y_0)>\rho_1>\rho^*>0$, it follows from Lemma 2.9 that y_0 is, in fact, a solution of the original integral Eq. (1.1) and is, in fact, a nontrivial positive solution. And this completes the proof. $\square$

Remark 3.2 Obviously, we can swap the roles of the numbers ρ_1 and ρ_2 in the statement and proof of Theorem 3.1 and thereby obtain an obvious corollary, whose precise statement we omit. In particular, we can assume that $\rho_1>\rho_2>\rho^*$ instead.

If we are willing to strengthen the conditions on H somewhat, then we can recast Theorem 3.1 in the following manner. In particular, Corollary 3.3 indicates how Theorem 3.1 can be simplified somewhat if we assume that the map $z\mapsto H(z)$ is increasing.

Corollary 3.3 *Assume that conditions (H1)–(H5) are satisfied. In addition, assume that the map $z \mapsto H(z)$ is increasing. Let λ_0 be defined by*

$$\lambda_0 := \min\left\{\left(\int_0^1\int_0^1 G(t,s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right)^{-1}, \frac{1}{C_1}\right\}.$$

Suppose that for fixed $\lambda \in \left(0, \lambda_0\right)$ that there exist numbers $\rho_2 > \rho_1 > \rho^*$, where ρ^* is the number from Lemma 2.9, such that each of

$$H\left(\rho_1\left[1-\lambda\int_0^1\int_0^1 G(t,s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right]-\lambda\int_0^1\int_0^1 G(t,s)u(s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right) > \frac{\rho_1}{\varphi(\gamma)} \tag{3.8}$$

and

$$\frac{\varphi(\gamma)}{\rho_2}H\left(\rho_2\right)+\lambda\int_0^1\int_0^1 G(t,s)\left[\frac{|f|^M_{\left[0,\frac{\rho_2}{C_0}\right]}}{\rho_2}+\frac{u(s)}{\rho_2}+1\right]\mathrm{d}\alpha(t)\,\mathrm{d}s < 1 \tag{3.9}$$

is true. Then problem (1.1) has at least one positive solution.

Proof The proof that $i_{\mathcal{K}}\left(T, \widehat{V}_{\rho_1}\right) = 0$ does not change at all. On the other hand, the only change in the proof that $i_{\mathcal{K}}\left(T, \widehat{V}_{\rho_2}\right) = 1$ is that since

$$\rho_2\left[1-\lambda\int_0^1\int_0^1 G(t,s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right]-\lambda\int_0^1\int_0^1 G(t,s)u(s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s < \rho_2$$

and H is an increasing map, we deduce that

$$\begin{aligned}
\rho_2 &\leq \varphi(\gamma)H^*\big(\rho_2-\varphi(w_y)\big)+\lambda\int_0^1\int_0^1 G(t,s)\big[f\big(s,y^*(s)\big)+u(s)+\rho_2\big]\,\mathrm{d}\alpha(t)\,\mathrm{d}s\\
&= \varphi(\gamma)H\left(\rho_2\left[1-\lambda\int_0^1\int_0^1 G(t,s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right]-\lambda\int_0^1\int_0^1 G(t,s)u(s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right)\\
&\quad +\lambda\int_0^1\int_0^1 G(t,s)\big[f\big(s,y^*(s)\big)+u(s)+\rho_2\big]\,\mathrm{d}\alpha(t)\,\mathrm{d}s\\
&\leq \varphi(\gamma)H\left(\rho_2\right)+\lambda\int_0^1\int_0^1 G(t,s)\big[f\big(s,y^*(s)\big)+u(s)+\rho_2\big]\,\mathrm{d}\alpha(t)\,\mathrm{d}s.
\end{aligned}$$

And from this the proof of the corollary follows at once. □

We now provide an example in the context of radially symmetric solutions to elliptic PDEs.

Example 3.4 We consider the problem

$$\begin{aligned}
\Delta w &= \sqrt{w}+1, \quad |\boldsymbol{x}| \in [1,e]\\
w(\boldsymbol{x})\Big|_{\boldsymbol{x}\in\partial\mathcal{B}_1} &= 0\\
w(\boldsymbol{x})\Big|_{\boldsymbol{x}\in\partial\mathcal{B}_e} &= H\left(\frac{1}{2}w\left(\boldsymbol{x}e^{\frac{1}{2}}\right)-\frac{1}{6}w\left(\boldsymbol{x}e^{\frac{2}{3}}\right)\right)\Big|_{\boldsymbol{x}\in\partial\mathcal{B}_1},
\end{aligned}$$

where in (1.2) we have defined the maps $w \mapsto f(w)$, $\xi \mapsto h(\xi)$, and $z \mapsto H(z)$ as follows.

$$h(\xi) \equiv 1$$
$$H(z) := \begin{cases} \frac{1}{10}z, & 0 \leq z \leq 20 \\ 10z - 198, & z \geq 20 \end{cases}$$
$$f(w) := -\sqrt{w} - 1$$

Note that

$$\lim_{w\to\infty} f(w) = -\infty$$

and $f(w) < 0$, for all $w \geq 0$. Thus, this problem is not semipositone. Moreover, since $\lim_{w\to\infty} \frac{f(w)}{w} = 0$, we can select $\kappa_0 = 1$ here.

Note that since we are considering Dirichlet-type nonlocal boundary conditions in the above PDE, here we shall select the kernel $(t, s) \mapsto G(t, s)$ of our integral Eq. (1.1) to be

$$G(t,s) := \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1 \\ s(1-t), & 0 \leq s \leq t \leq 1 \end{cases}.$$

In addition, we put $\gamma(t) := 1 - t$ in (1.1). Then, preceding PDE can be related to the ODE

$$\begin{aligned} w''(t) + \phi(t) f\big(w(t)\big) &= 0, \quad \text{a.e. } t \in [0,1] \\ w(0) &= H\left(\frac{1}{2}w\left(\frac{1}{2}\right) - \frac{1}{6}w\left(\frac{1}{3}\right)\right) \\ w(1) &= 0, \end{aligned}$$

where $\phi(t) = \left(e^{1-t}\right)^2$ for $t \in [0, 1]$. A description of the process of deriving of the above ODE from the original elliptic PDE can be found, among other places, in [47,52,53]. So we consider the ODE

$$\begin{aligned} w''(t) + e^{2(1-t)}\big(-\sqrt{w(t)} - 1\big) &= 0, \quad \text{a.e. } t \in [0,1] \\ w(0) &= H\left(\frac{1}{2}w\left(\frac{1}{2}\right) - \frac{1}{6}w\left(\frac{1}{3}\right)\right) \\ w(1) &= 0. \end{aligned}$$

Routine calculations demonstrate that in this case we have both that

$$\int_0^1 \int_0^1 G(t,s)\, \mathrm{d}\alpha(t)\, \mathrm{d}s = \frac{19}{432}$$

and that

$$C_0 := \inf_{s\in(0,1)} \frac{1}{s(1-s)} \int_0^1 G(t,s)\, \mathrm{d}\alpha(t)\, \mathrm{d}s = \frac{5}{36}.$$

Moreover, we find that $C_1 := \frac{2}{3}$. In addition, due to the form of G above, we put $q(t) := \min\{t, 1-t\}$, and then notice that $\gamma(t) \geq q(t)\|\gamma\|$, for $t \in [0, 1]$. Finally, note that

$$\varphi(\gamma) = \int_0^1 (1-t)\, \mathrm{d}\alpha(t) = \frac{5}{36} \geq C_0\|\gamma\|.$$

With the preceding constants in hand and using the notation introduced earlier in this and the preceding section, we calculate that

$$R_0 = \inf\left\{R^* \in (0, +\infty)\colon -\sqrt{w} - 1 \geq -\frac{5}{36}w, \quad \text{for all } w \in \left[R^*, +\infty\right)\right\} \approx 65.448,$$

and so, we may put $u(t) \equiv 9.09$ here. With this choice of the map $t \mapsto u(t)$, we calculate

$$\lambda_0 = \min\left\{\frac{3}{2}, \frac{432}{19}\right\} = \frac{3}{2}.$$

This means that

$$\rho^* := \max\left\{\frac{19 \cdot 9.09\lambda}{432 - 19\lambda}, \frac{2\lambda}{3 - 2\lambda} \cdot 9.09\right\}.$$

Since we have chosen $\lambda = 1$ in the above PDE, we conclude that we may set $\rho^* := 18.18$.

With the preliminary calculations completed, we now will use Corollary 3.3 to show that the PDE has at least one positive (radially symmetric) solution. Notice that Corollary 3.3 is applicable since H is an increasing function. In addition, we will swap the roles of ρ_1 and ρ_2 in the statement of the corollary—i.e., we will take $\rho_1 > \rho_2 > \rho^*$, which, as remarked in Remark 3.2, is permissible and does not affect the application of the existence result. So, to see that the corollary is applicable, we simply notice that if we put $\rho_2 := 50$, then

$$\frac{5}{36\rho_2}(10\rho_2 - 198) + \frac{19}{432}\left(1 + \frac{9.09}{\rho_2} + \frac{1 + \sqrt{\frac{36}{5}\rho_2}}{\rho_2}\right) < 1,$$

which verifies condition (3.9), whereas if we put $\rho_1 := 100$, then

$$\frac{10}{432}\left(413\rho_1 - 9.09 \cdot \frac{19}{432}\right) - 198 > \frac{36}{5}\rho_1,$$

which verifies condition (3.8). Thus, Corollary 3.3 may be applied to the PDE to deduce the existence of at least one positive solution, w_0, satisfying the localization

$$w_0 \in \widehat{V}_{100} \backslash \overline{\widehat{V}}_{50}.$$

Remark 3.5 Observe that the boundary condition appearing in Example 3.4 is nonlocal and piecewise linear and affine since

$$\begin{aligned} &H\left(\frac{1}{2}w\left(\frac{1}{2}\right) - \frac{1}{6}w\left(\frac{1}{3}\right)\right) \\ &= \begin{cases} \frac{1}{10}\left[\frac{1}{2}w\left(\frac{1}{2}\right) - \frac{1}{6}w\left(\frac{1}{3}\right)\right], & \frac{1}{2}w\left(\frac{1}{2}\right) - \frac{1}{6}w\left(\frac{1}{3}\right) \in [0, 20] \\ 10\left[\frac{1}{2}w\left(\frac{1}{2}\right) - \frac{1}{6}w\left(\frac{1}{3}\right)\right] - 198, & \frac{1}{2}w\left(\frac{1}{2}\right) - \frac{1}{6}w\left(\frac{1}{3}\right) \in [20, +\infty) \end{cases} \\ &= \begin{cases} \frac{1}{20}w\left(\frac{1}{2}\right) - \frac{1}{60}w\left(\frac{1}{3}\right), & \frac{1}{2}w\left(\frac{1}{2}\right) - \frac{1}{6}w\left(\frac{1}{3}\right) \in [0, 20] \\ 5w\left(\frac{1}{2}\right) - \frac{5}{3}w\left(\frac{1}{3}\right) - 198, & \frac{1}{2}w\left(\frac{1}{2}\right) - \frac{1}{6}w\left(\frac{1}{3}\right) \in [20, +\infty) \end{cases}. \end{aligned}$$

Thus, strictly speaking, our nonlocal boundary conditions do not have to be nonlinear. Both piecewise linear and affine boundary conditions, for example, can be accommodated by the existence results presented in this section.

Remark 3.6 A comparison of the application of Corollary 3.3 to [24, Theorem 3.1] reveals that our methodology here is much simpler and more elegant. For example, to apply [24, Theorem 3.1] we would need to assume

- that γ satisfies a technical condition [24, Condition (H2.1)];
- that H is sublinear at infinity [24, Condition (H2.2)];
- that λ_0 satisfies a very technical condition [24, Condition (H7)] that while able to be checked is computationally complicated;

- a condition on the number R_0 of condition (H3) that is much more complicated [24, Condition (H6)]; and
- an additional auxiliary condition on the map $(t, s) \mapsto G(t, s)$ [24, Statement of Theorem 3.1].

All in all, our methodology here is significantly simpler. In addition and as pointed out in Sect. 1, because of this significant simplification, the methodology here should be able to applied to a much wider variety of problems than merely (1.1).

We conclude with a brief mention of the specialization of the preceding results to the special setting in which

$$f(t, y) := a(t)g(y),$$

which was mentioned toward the beginning of Sect. 1. In this case, we note that (H3.1)–(H3.2) become, respectively:

- $a(t)\left(\lim_{y\to\infty} \frac{g(y)}{y}\right) = 0$; and
- $a(t)g(y) \geq -u(t)$.

This allows, in, say, Theorem 3.1, for the map $t \mapsto a(t)$ to satisfy $a(t) < 0$ for all $t \in [0, 1]$ with no restriction on the size of $\left|a(t)\right|$. While this is not directly comparable to, say, Feltrin and Zanolin [13] (e.g., there local boundary conditions are considered, multiplicity of solutions are considered, and the weight changes sign); nonetheless, our results here provide a complementation to some of the known results in the nonpositive weight setting.

4 Results for more general polynomial growth

In this section, we demonstrate that if we are willing to let the admissible range of λ be defined in a more complicated manner, then we can accommodate more general polynomial growth at $+\infty$. In particular, we replace assumption (H3.1) with the new assumption (H3$'$.1) below; note that we still assume in condition (H3$'$) the other assumptions imposed in condition (H3).

H3$'$: The function $f\colon [0, 1] \times [0, +\infty) \to \mathbb{R}$ is continuous and satisfies the following two growth conditions.

(1) There exists a number $\kappa_0 \in (1, +\infty)$ such that

$$\lim_{y\to\infty} \frac{f(t, y)}{y^{\kappa_0}} = 0,$$

uniformly for $t \in [0, 1]$.

(2) There exists a map $u \in L^1\big([0, 1]; [0, +\infty)\big)$ such that

$$f(t, y) \geq -u(t)$$

for each $t \in [0, 1]$ and $y \in \left[0, R_0'\right]$, where

$$R_0' := \inf\left\{R^* \in (0, +\infty)\colon f(t, y) \geq -C_0^{\kappa_0} y^{\kappa_0},\ \text{for all } y \geq R^*\right\}.$$

In addition, instead of the operator T, which we utilized in Sect. 3, we will need in this context the slightly modified operator $T_{\kappa_0}\colon \mathcal{K} \to \mathcal{K}$ defined by

$$\big(T_{\kappa_0} y\big)(t) := \gamma(t)H^*\big(\varphi(y - w_y)\big) + \lambda \int_0^1 G(t, s)\Big[f\big(s, y^*(s)\big) + u(s) + \big[\varphi(y)\big]^{\kappa_0}\Big]\, \mathrm{d}s. \tag{4.1}$$

Of course, this means that we will also need to modify the map $w_y\colon [0,1] \to \mathbb{R}$ to

$$w_y(t) := \lambda \int_0^1 G(t,s)\Big[u(s) + \big[\varphi(y)\big]^{\kappa_0}\Big]\,\mathrm{d}s.$$

Note that the assumptions in Sect. 3 dealt with the case $\kappa_0 = 1$, whereas condition (H3$'$.1) generalizes this to allow for $\kappa_0 > 1$. Let us recall, as explained in Sect. 1, that the case in which $\kappa_0 > 2$ is particularly interesting, for many density-dependent ecological models fall into this category since their forcing terms possess quadratic polynomial growth at $+\infty$.

Since most of the calculations carry over to this more general case with only minor to moderate modification, we will not provide so many details but rather just highlight the relevant changes. Therefore, we next present the analogues of Lemmata 2.5–2.7 and 2.9, suitably modified for the case in which $\kappa_0 > 1$.

Lemma 4.1 *Assume that conditions (H1)–(H2), (H3$'$), and (H4)–(H5) are satisfied. Then, with the operator T_{κ_0} defined as in (4.1) it holds that $T_{\kappa_0}(\mathcal{K}) \subseteq \mathcal{K}$.*

Proof That $y \in \mathcal{K}$ implies that $\big(T_{\kappa_0}y\big)(t) \ge q(t)\|T_{\kappa_0}y\|$ for each $t \in [0,1]$ only changes slightly and only in the case when $y^*(s) \in \big(R_0', +\infty\big)$ for some $s \in [0,1]$. In particular, if $y^*(s) \in \big(R_0', +\infty\big)$, then

$$\begin{aligned} f\big(s, y^*(s)\big) + u(s) + \big[\varphi(y)\big]^{\kappa_0} &\ge f\big(s, y^*(s)\big) + \big[\varphi(y)\big]^{\kappa_0} \ge -C_0^{\kappa_0}\big[y^*(s)\big]^{\kappa_0} + C_0^{\kappa_0}\|y\|^{\kappa_0} \\ &\ge -C_0^{\kappa_0}\big[y(s)\big]^{\kappa_0} + C_0^{\kappa_0}\|y\|^{\kappa_0} \\ &\ge -C_0^{\kappa_0}\|y\|^{\kappa_0} + C_0^{\kappa_0}\|y\|^{\kappa_0} \\ &= 0. \end{aligned}$$

On the other hand, there is no change to the part of the proof that $\varphi\big(T_{\kappa_0}y\big) \ge C_0\big\|T_{\kappa_0}y\big\|$ whenever $y \in \mathcal{K}$. So, we conclude that $T_{\kappa_0}(\mathcal{K}) \subseteq \mathcal{K}$, and this completes the proof. □

Lemma 4.2 *Assume that conditions (H1)–(H2), (H3$'$), and (H4)–(H5) are satisfied. Let $\rho > 0$ be given and assume that $y \in \partial\widehat{V}_\rho$. If $\lambda := \lambda(\rho)$ satisfies*

$$0 < \lambda < \rho\left(\int_0^1\int_0^1 G(t,s)\big[\rho^{\kappa_0} + u(s)\big]\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right)^{-1},$$

then $\varphi(y - w_y) > 0$.

Proof Let $y \in \partial\widehat{V}_\rho$. Then, using the restriction on λ given in the statement of the lemma we calculate

$$\begin{aligned} \varphi(y - w_y) &= \varphi(y) - \varphi(w_y) \\ &= \rho - \lambda\int_0^1\int_0^1 G(t,s)\Big[u(s) + \big[\varphi(y)\big]^{\kappa_0}\Big]\,\mathrm{d}\alpha(t)\,\mathrm{d}s \\ &= \rho - \lambda\int_0^1\int_0^1 G(t,s)\big[u(s) + \rho^{\kappa_0}\big]\,\mathrm{d}\alpha(t)\,\mathrm{d}s \\ &> 0, \end{aligned} \tag{4.2}$$

which proves the claim and completes the proof. □

Remark 4.3 Unlike in Sects. 2 and 3, in which the choice of λ was entirely in terms of initial data, in this case the choice of λ will depend upon the selection of the number ρ. Nonetheless, the range of admissible values of λ remains computable as the examples in the sequel demonstrate.

Lemma 4.4 *Assume that conditions (H1)–(H2), (H3′), and (H4)–(H5) are satisfied. Let* $\rho > 0$ *be given and assume that* $y \in \partial\widehat{V}_{\rho}$. *If* $\lambda := \lambda(\rho)$ *satisfies*

$$0 < \lambda < \frac{\rho}{C_1}\left(\rho^{\kappa_0} + \int_0^1 u(s)\,\mathrm{d}s\right)^{-1},$$

then it follows that $y^* \equiv y - w_y$.

Proof Similar to inequality (2.5) from earlier, we observe that

$$\begin{aligned} y(t) - w_y(t) &\geq q(t)\|y\| - \lambda\int_0^1 q(t)\Big[u(s) + \big[\varphi(y)\big]^{\kappa_0}\Big]\,\mathrm{d}s \\ &\geq q(t)\left[\frac{\rho}{C_1} - \lambda\int_0^1 \big[u(s) + \rho^{\kappa_0}\big]\,\mathrm{d}s\right] \\ &\geq 0, \end{aligned} \tag{4.3}$$

where the final inequality in (4.3) follows from the upper bound imposed on λ in the statement of the lemma. And this completes the proof. □

Remark 4.5 Just as with Remark 2.8 earlier, from inequality (4.3) we see that because of the strict upper bound on λ in the statement of Lemma 4.4, it actually holds that $\big(y - w_y\big)(t) > 0$ for each t such that $q(t) \neq 0$.

Lemma 4.6 *Assume that conditions (H1)–(H2), (H3′), and (H4)–(H5) are satisfied. Suppose that there are numbers* $+\infty > \rho_2 > \rho_1 > 0$ *such that* y_0 *is a positive solution of the modified integral Eq. (4.1) satisfying* $y_0 \in \widehat{V}_{\rho_2}\setminus\overline{\widehat{V}}_{\rho_1}$. *If the number* $\lambda := \lambda\,(\rho)$ *satisfies*

$$\begin{aligned} 0 < \lambda < \min_{\rho_1\leq\rho\leq\rho_2}\Bigg\{&\frac{\rho}{C_1}\left(\rho^{\kappa_0} + \int_0^1 u(s)\,\mathrm{d}s\right)^{-1}, \\ &\rho\left(\int_0^1\int_0^1 G(t,s)\big[\rho^{\kappa_0} + u(s)\big]\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right)^{-1}\Bigg\}, \end{aligned}$$

then the function $z\colon [0,1] \to [0,+\infty)$ *defined by* $z(t) := \big(y_0 - w_{y_0}\big)(t)$ *is a solution of the original integral Eq. (1.1).*

Proof Since y_0 solves the modified problem (4.1), we know that

$$y_0(t) = \gamma(t)H^*\big(\varphi(y_0 - w_{y_0})\big) + \lambda\int_0^1 G(t,s)\Big[f\big(s, y_0^*(s)\big) + u(s) + \big[\varphi(y_0)\big]^{\kappa_0}\Big]\,\mathrm{d}s.$$

Consequently, we may write

$$\begin{aligned} z(t) &= \big(y_0 - w_{y_0}\big)(t) \\ &= \gamma(t)H^*\big(\varphi(z)\big) + \lambda\int_0^1 G(t,s)\Big[f\big(s, y_0^*(s)\big) + u(s) + \big[\varphi(y_0)\big]^{\kappa_0}\Big]\,\mathrm{d}s \\ &\quad \underbrace{-\,\lambda\int_0^1 G(t,s)\Big[u(s) + \big[\varphi\,(y_0)\,\big]^{\kappa_0}\Big]\,\mathrm{d}s}_{=w_{y_0}(t)} \\ &= \gamma(t)H\big(\varphi(z)\big) + \lambda\int_0^1 G(t,s)f\big(s, y_0^*(s)\big)\,\mathrm{d}s. \end{aligned} \tag{4.4}$$

Similar to the proof of Lemma 2.9, since $y_0 \in \widehat{V}_{\rho_2} \backslash \overline{\widehat{V}}_{\rho_1}$ and given the restrictions on λ, we see that Lemma 4.4 implies that $y_0^* \equiv y_0 - w_{y_0} \equiv z$. At the same time, by Lemma 4.2, the fact that

$$\widehat{V}_{\rho_2} \backslash \overline{\widehat{V}}_{\rho_1} = \bigcup_{\rho_1 < \widehat{\rho} < \rho_2} \partial \widehat{V}_{\widehat{\rho}},$$

and the condition imposed on λ in the statement of this lemma, we know that $\varphi\left(y_0 - w_{y_0}\right) \geq 0$. Consequently, from the preceding discussion and estimate (4.4) we deduce that z solves the original integral Eq. (1.1). Finally, just as in the proof of Lemma 2.9 again, we know from the preceding lemmata and remarks that, in fact, $\|z\| > 0$ so that z is a nontrivial, positive solution of (1.1). And this completes the proof. □

We next present the following corollary to Lemma 4.6. The corollary formalizes the observation that under certain conditions, the upper bound on λ given in the statement of Lemma 4.6 can be simplified.

Corollary 4.7 *Suppose that the number ρ_2 in the statement of Lemma 4.6 satisfies*

$$\rho_2^{\kappa_0} < \min\left\{\frac{1}{\kappa_0 - 1}\int_0^1 u(s)\,\mathrm{d}s,\right.$$
$$\left.\left(\int_0^1\int_0^1 G(t,s)u(s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right)\left((\kappa_0 - 1)\int_0^1\int_0^1 G(t,s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right)^{-1}\right\}.$$

Then the restriction on λ as given in Lemma 4.6 can be simplified to

$$0 < \lambda < \min\left\{\frac{\rho_1}{C_1}\left(\rho_1^{\kappa_0} + \int_0^1 u(s)\,\mathrm{d}s\right)^{-1}, \rho_1\left(\int_0^1\int_0^1 G(t,s)\left[\rho_1^{\kappa_0} + u(s)\right]\mathrm{d}\alpha(t)\,\mathrm{d}s\right)^{-1}\right\}.$$

Proof Consider the map $G\colon [0,+\infty) \to [0,+\infty)$ defined by

$$G(\rho) := \frac{\rho}{\alpha_1\left(\rho^{\kappa_0} + \alpha_2\right)},$$

where α_1 and α_2 are positive constants. A simple calculation demonstrates that

$$G'(\rho) = \frac{\alpha_1\left(\rho^{\kappa_0} + \alpha_2\right) - \rho\alpha_1\kappa_0\rho^{\kappa_0 - 1}}{\alpha_1^2\left(\rho^{\kappa_0} + \alpha_2\right)^2} = \frac{\alpha_1\left(\rho^{\kappa_0} + \alpha_2 - \kappa_0\rho^{\kappa_0}\right)}{\alpha_1^2\left(\rho^{\kappa_0} + \alpha_2\right)^2}.$$

Evidently, $G'(\rho) > 0$ if and only if $(1 - \kappa_0)\rho^{\kappa_0} > -\alpha_2$. Since $1 - \kappa_0 < 0$, we conclude that $G'(\rho) > 0$ if and only if

$$\rho^{\kappa_0} < \frac{\alpha_2}{\kappa_0 - 1}.$$

Thus, if this inequality holds for each $\rho \in [\rho_1, \rho_2]$, then

$$\min_{\rho_1 \leq \rho \leq \rho_2} G(\rho) = \frac{\rho_1}{\alpha_1\left(\rho_1^{\kappa_0} + \alpha_2\right)}.$$

On the other hand, consider the map $H\colon [0,+\infty) \to [0,+\infty)$ defined by

$$H(\rho) := \frac{\rho}{\alpha_3\rho^{\kappa_0} + \alpha_4}.$$

Then we calculate

$$H'(\rho) = \frac{\alpha_3\rho^{\kappa_0} + \alpha_4 - \kappa_0\alpha_3\rho^{\kappa_0}}{(\alpha_3\rho^{\kappa_0} + \alpha_4)^2}.$$

Clearly, then, $H'(\rho) > 0$ if and only if

$$\rho^{\kappa_0}\alpha_3\left(1-\kappa_0\right)+\alpha_4 > 0.$$

Thus, the condition $\rho^{\kappa_0} < \frac{\alpha_4}{\alpha_3(\kappa_0-1)}$ ensures that $H'(\rho) > 0$. And this implies that

$$\min_{\rho_1\le\rho\le\rho_2} H(\rho)=\frac{\rho_1}{\alpha_3\rho_1^{\kappa_0}+\alpha_4}$$

whenever $\rho^{\kappa_0} < \frac{\alpha_4}{\alpha_3(\kappa_0-1)}$ for each $\rho\in[\rho_1,\rho_2]$.

Finally, define α_i, $1\le i\le 4$, as follows.

$$\begin{aligned}
\alpha_1 &:= C_1\\
\alpha_2 &:= \int_0^1 u(s)\,\mathrm{d}s\\
\alpha_3 &:= \int_0^1\int_0^1 G(t,s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\\
\alpha_4 &:= \int_0^1\int_0^1 G(t,s)u(s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s
\end{aligned}$$

Notice that $\alpha_i > 0$, for each i. Then, the conclusion of the corollary follows immediately from the observation that $\rho_1^{\kappa_0} < \rho_2^{\kappa_0}$. □

Finally, we present an existence theorem, which is the analogy of Theorem 3.1.

Theorem 4.8 *Assume that conditions (H1)–(H2), (H3′), and (H4)–(H5) are satisfied. Suppose that there exist numbers $\rho_2 > \rho_1 > 0$ and a number $\lambda := \lambda\left(\rho_1,\rho_2\right) > 0$ such that each of*

$$H\left(\rho_1\left[1-\rho_1^{\kappa_0-1}\lambda\int_0^1\int_0^1 G(t,s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right]-\lambda\int_0^1\int_0^1 G(t,s)u(s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right) > \frac{\rho_1}{\varphi(\gamma)} \tag{4.5}$$

and

$$\begin{aligned}
&\frac{\varphi(\gamma)}{\rho_2}H\left(\rho_2\left[1-\rho_2^{\kappa_0-1}\lambda\int_0^1\int_0^1 G(t,s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right]-\lambda\int_0^1\int_0^1 G(t,s)u(s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right)\\
&\quad+\lambda\int_0^1\int_0^1 G(t,s)\left[\frac{|f|^M_{[0,1]\times\left[0,\frac{\rho_2}{C_0}\right]}}{\rho_2}+\frac{u(s)}{\rho_2}+\rho_2^{\kappa_0-1}\right]\mathrm{d}\alpha(t)\,\mathrm{d}s<1
\end{aligned}$$

and

$$\begin{aligned}
0<\lambda<\min_{\rho_1\le\rho\le\rho_2}&\left\{\frac{\rho}{C_1}\left(\rho^{\kappa_0}+\int_0^1 u(s)\,\mathrm{d}s\right)^{-1},\right.\\
&\left.\rho\left(\int_0^1\int_0^1 G(t,s)\left[\rho^{\kappa_0}+u(s)\right]\mathrm{d}\alpha(t)\,\mathrm{d}s\right)^{-1}\right\}
\end{aligned}$$

is true. Then problem (1.1) has at least one positive solution.

Proof Similar to the first part of the proof of Theorem 3.1, for $y \in \partial \widehat{V}_{\rho_1}$ and using (4.2) we find that

$$\begin{aligned}\rho_1 \geq \varphi(\gamma) H\left(\rho_1\left[1-\rho_1^{\kappa_0-1}\lambda \int_0^1\int_0^1 G(t,s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right]-\lambda\int_0^1\int_0^1 G(t,s)u(s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right)\\ \underbrace{+\lambda\int_0^1\int_0^1 G(t,s)\underbrace{\left[f\big(s,y^*(s)\big)+u(s)+\rho_1^{\kappa_0}\right]}_{\geq 0}\,\mathrm{d}\alpha(t)\,\mathrm{d}s}_{\geq 0}\\ \geq \varphi(\gamma) H\left(\rho_1\left[1-\rho_1^{\kappa_0-1}\lambda \int_0^1\int_0^1 G(t,s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right]-\lambda\int_0^1\int_0^1 G(t,s)u(s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right).\end{aligned} \tag{4.6}$$

Then, combining condition (4.5) with inequality (4.6) leads to a contradiction. Conversely, the calculation for $y \in \partial \widehat{V}_{\rho_2}$ changes in the obvious way, and so, we skip the details of that calculation. □

Of course, we can provide an analogy of Corollary 3.3 in this more general setting. We do so next. Since the proof of Corollary 4.9 is very similar to that of Corollary 3.3, we omit the proof. Furthermore, in light of Corollary 4.7 we can provide additional corollaries beyond Corollary 4.9. But we omit their statements.

Corollary 4.9 *Assume that conditions (H1)–(H2), (H3′), and (H4)–(H5) are satisfied. In addition, assume that the map $z \mapsto H(z)$ is increasing. Suppose that there exist numbers $\rho_2 > \rho_1 > 0$ and a number $\lambda := \lambda(\rho_1, \rho_2) > 0$ such that each of*

$$H\left(\rho_1\left[1-\rho_1^{\kappa_0-1}\lambda \int_0^1\int_0^1 G(t,s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right]-\lambda\int_0^1\int_0^1 G(t,s)u(s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right) > \frac{\rho_1}{\varphi(\gamma)}$$

and

$$\frac{\varphi(\gamma)}{\rho_2}H(\rho_2)+\lambda\int_0^1\int_0^1 G(t,s)\left[\frac{|f|^M_{[0,1]\times\left[0,\frac{\rho_2}{C_0}\right]}}{\rho_2}+\frac{u(s)}{\rho_2}+\rho_2^{\kappa_0-1}\right]\mathrm{d}\alpha(t)\,\mathrm{d}s < 1$$

and

$$\begin{aligned}0<\lambda<\min_{\rho_1\leq\rho\leq\rho_2}\Bigg\{\frac{\rho}{C_1}\left(\rho^{\kappa_0}+\int_0^1 u(s)\,\mathrm{d}s\right)^{-1},\\ \rho\left(\int_0^1\int_0^1 G(t,s)\big[\rho^{\kappa_0}+u(s)\big]\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right)^{-1}\Bigg\}\end{aligned}$$

is true. Then problem (1.1) *has at least one positive solution.*

We conclude with a couple examples to illustrate the use of the preceding existence theorem.

Example 4.10 For $\lambda > 0$ to be selected later, we consider the problem

$$\begin{aligned}\Delta w+\lambda h\big(|\boldsymbol{x}|\big)f(w)=0,\quad |\boldsymbol{x}|\in[1,e]\\ w(\boldsymbol{x})\Big|_{\boldsymbol{x}\in\partial\mathcal{B}_1}=0\\ w(\boldsymbol{x})\Big|_{\boldsymbol{x}\in\partial\mathcal{B}_e}=H\left(\frac{1}{2}w\left(\boldsymbol{x}e^{\frac{1}{2}}\right)-\frac{1}{6}w\left(\boldsymbol{x}e^{\frac{2}{3}}\right)\right)\Big|_{\boldsymbol{x}\in\partial\mathcal{B}_1}\end{aligned}$$

where we define the maps $w \mapsto f(w)$, $\xi \mapsto h(\xi)$, and $z \mapsto H(z)$ as follows.

$$
\begin{aligned}
h(\xi) &\equiv 1\\
H(z) &:= \begin{cases} \frac{1}{10}z, & 0 \le z \le 20\\ 10z - 198, & z \ge 20 \end{cases}\\
f(w) &:= -w^3 - 1
\end{aligned}
$$

Note that

$$\lim_{w\to\infty} f(w) = -\infty$$

and $f(w) < 0$, for all $w \ge 0$. Thus, this problem is not semipositone. Moreover, since $\lim_{w\to\infty} \frac{f(w)}{w^4} = 0$, we can select $\kappa_0 = 4$ here. Note that we are using the same nonlocal element as we did in Example 3.4.

In a manner similar to that utilized in Example 3.4, we consider here the ODE

$$
\begin{aligned}
w''(t) + \lambda e^{2(1-t)}\big(-w^3 - 1\big) &= 0, \quad \text{a.e. } t \in [0,1]\\
w(0) &= H\left(\frac{1}{2}w\left(\frac{1}{2}\right) - \frac{1}{6}w\left(\frac{1}{3}\right)\right)\\
w(1) &= 0.
\end{aligned}
$$

Since the nonlocal element in this problem is the same as in Example 3.4 and, furthermore, the kernel $(t,s) \mapsto G(t,s)$ is identical as well, we once again obtain the following.

$$
\begin{aligned}
&\int_0^1\int_0^1 G(t,s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s = \frac{19}{432}\\
&C_0 := \inf_{s\in(0,1)} \frac{1}{s(1-s)}\int_0^1 G(t,s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s = \frac{5}{36}\\
&C_1 := \frac{2}{3}\\
&\varphi(\gamma) := \frac{5}{36}
\end{aligned}
$$

Moreover, we have that

$$
\begin{aligned}
R_0' &:= \inf\left\{R^* \in (0,+\infty) \colon -w^3 - 1 \ge -\left(\frac{5}{36}\right)^4 w^4, \quad \text{for all } w \in \big[R^*, +\infty\big)\right\}\\
&\approx 2687.3856,
\end{aligned}
$$

from which it follows that here we may select $u(t) \equiv 1.942 \times 10^{10}$. So, in this case, we calculate that

$$0 < \lambda < \min_{\rho_1\le\rho\le\rho_2}\left\{\frac{3}{2}\rho\left(\rho^4 + 1.942\times 10^{10}\right)^{-1}, \rho\left(\frac{19}{432}\left(\rho^4 + 1.942\times 10^{10}\right)\right)^{-1}\right\},$$

where ρ_1 and ρ_2 will be selected momentarily.

So, using the conditions imposed by Corollary 4.9 we obtain the following inequalities.

$$H\left(\rho_1\left(1-\lambda\rho_1^3\frac{19}{432}\right)-\lambda\frac{19}{432}\cdot 1.942\times 10^{10}\right)>\frac{36}{5}\rho_1$$

$$\frac{5}{36\rho_2}H\left(\rho_2\right)+\lambda\frac{19}{432}\left[\frac{\left(\left(\frac{36}{5}\rho_2\right)^3+1\right)}{\rho_2}+\frac{1.942\times 10^{10}}{\rho_2}+\rho_2^3\right]<1$$

Routine computations show that these inequalities are satisfied for, respectively, $\rho_1 = 90$ and $\rho_2 = 50$. (Note that as in Example 3.4 here we swap the roles of ρ_1 and ρ_2 as stated in Theorem 4.8 in the sense that $\rho_1 > \rho_2 > 0$.) Moreover, for these choices of ρ_1 and ρ_2 we may select, for example, $\lambda \in \left(0, 3.861 \times 10^{-9}\right)$. In particular, this means that we deduce the existence of at least one positive solution, say w_0, to the PDE

$$\begin{aligned}
\Delta w+\lambda h\big(|\boldsymbol{x}|\big)f(w)&=0,\quad |\boldsymbol{x}|\in[1,e]\\
w(\boldsymbol{x})\Big|_{\boldsymbol{x}\in\partial\mathcal{B}_1}&=0\\
w(\boldsymbol{x})\Big|_{\boldsymbol{x}\in\partial\mathcal{B}_e}&=H\left(\frac{1}{2}w\left(\boldsymbol{x}e^{\frac{1}{2}}\right)-\frac{1}{6}w\left(\boldsymbol{x}e^{\frac{2}{3}}\right)\right)\Bigg|_{\boldsymbol{x}\in\partial\mathcal{B}_1}
\end{aligned}$$

whenever $0 < \lambda < 3.861 \times 10^{-9}$. Notice that w_0 satisfies the localization $w_0 \in \widehat{V}_{90} \backslash \overline{\widehat{V}}_{50}$.

Our next example, Example 4.11 demonstrates that the upper bound on λ can change by several orders of magnitude depending upon, for example, the map $(t, y) \mapsto f(t, y)$ selected.

Example 4.11 For $\lambda > 0$ to be selected later, we consider the problem

$$\begin{aligned}
\Delta w+\lambda h\big(|\boldsymbol{x}|\big)f(w)&=0,\quad |\boldsymbol{x}|\in[1,e]\\
w(\boldsymbol{x})\Big|_{\boldsymbol{x}\in\partial\mathcal{B}_1}&=0\\
w(\boldsymbol{x})\Big|_{\boldsymbol{x}\in\partial\mathcal{B}_e}&=H\left(\frac{1}{2}w\left(\boldsymbol{x}e^{\frac{1}{2}}\right)-\frac{1}{6}w\left(\boldsymbol{x}e^{\frac{2}{3}}\right)\right)\Bigg|_{\boldsymbol{x}\in\partial\mathcal{B}_1}
\end{aligned}$$

where we define the maps $w \mapsto f(w)$, $\xi \mapsto h(\xi)$, and $z \mapsto H(z)$ as follows.

$$\begin{aligned}
h(\xi)&\equiv 1\\
H(z)&:=\begin{cases}\frac{1}{10}z, & 0\le z\le 20\\ 10z-198, & z\ge 20\end{cases}\\
f(w)&:=-w^2-1
\end{aligned}$$

So, here we can select $\kappa_0 = 3$. Note that we are using the same nonlocal element as we did in the previous examples.

In this example, we have that

$$\begin{aligned}
R_0'&:=\inf\left\{R^*\in(0,+\infty):\ -w^2-1\ge-\left(\frac{5}{36}\right)^3w^3,\quad \text{for all } w\in\left[R^*,+\infty\right)\right\}\\
&\approx 373.2507,
\end{aligned}$$

from which it follows that here we may select $u(t) \equiv 139, 325$. In can then be shown that the conditions of Corollary 4.9 are satisfied if we put $\rho_2 := 50$ and $\rho_1 := 200$ here. With these choices of ρ_1 and ρ_2, straightforward computations allow us to conclude that the PDE has a positive solution, say, $w_0 \in \widehat{V}_{200} \backslash \widehat{\overline{V}}_{50}$, for each λ satisfying $0 < \lambda < 3.686 \times 10^{-5}$.

Remark 4.12 With H as in Example 4.11 consider the problem

$$\begin{aligned} \Delta w + \lambda w(1-w) &= 0, \quad |\boldsymbol{x}| \in [1, e] \\ w(\boldsymbol{x})\Big|_{\boldsymbol{x}\in\partial\mathcal{B}_1} &= 0 \\ w(\boldsymbol{x})\Big|_{\boldsymbol{x}\in\partial\mathcal{B}_e} &= H\left(\frac{1}{2}w\left(\boldsymbol{x}e^{\frac{1}{2}}\right) - \frac{1}{6}w\left(\boldsymbol{x}e^{\frac{2}{3}}\right)\right)\Big|_{\boldsymbol{x}\in\partial\mathcal{B}_1}. \end{aligned}$$

This is precisely the problem considered in Example 4.11 with the exception that the forcing term f is now $f(w) := w(1-w)$, which is a logistic-type forcing term. It can be shown that the results of Example 4.11 apply to this forcing term as well.

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