Robust utility maximisation in markets with transaction costs

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Abstract We consider a continuous-time market with proportional transaction costs. Under appropriate assumptions, we prove the existence of optimal strategies for investors who maximise their worst-case utility over a class of possible models. We consider utility functions defined on either the positive axis or the whole real line.

Keywords Utility maximisation · Transaction cost · Model uncertainty

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1 Introduction

In this paper, the existence of solutions to the utility maximisation problem from terminal utility is studied in the presence of model ambiguity. We assume that investors prepare for the worst-case scenario in the sense that they take the infimum of utility functionals over the class of possible models before maximising over admissible investment strategies.

The literature on robust optimisation typically assumes that uncertainty is modelled by a family of prior measures $\mathcal{P}$ on some canonical space in which trajectories

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of the processes lie. Starting with Quenez [29] and Schied [34], the case in which $\mathcal{P}$ is dominated by a reference measure P_* has received ample treatment. In diffusion settings, this corresponds to uncertainty in the drift. Such an approach is not completely convincing since market participants may also be uncertain about the volatilities.

More recently, the non-dominated problem has also been studied in various contexts. For instance, Tevzadze et al. [35] investigated a compact set of possible drift and volatility coefficients and tackled the robust problem by solving an associated Hamilton–Jacobi–Bellman equation. In Matoussi et al. [24], where volatility coefficients are uncertain over a compact set and the drift is known, the theory of BSDEs is applied. Existence results in a fairly general class of models are available only in discrete time; see Nutz [27], Blanchard and Carassus [8], Neufeld and Šikić [26], Bartl [2], Bartl et al. [3] and Rásonyi and Meireles-Rodrigues [31]. A minimax result was established for bounded utilities in frictionless continuous-time markets in Denis and Kervarec [14].

As far as we know, our existence results below are the first to apply in a broad class of continuous-time models. We now summarise the principal ideas underlying our arguments. First, we work under proportional transaction costs. In this setting, strategies can be identified with finite-variation processes which we endow with a suitable convergence structure. Second, instead of a family of measures, we consider a parametrised family of stochastic processes on a fixed filtered probability space. Necessarily, instead of one portfolio value, we need to consider a family of possible values corresponding to the respective parameters. Third, the latter fact forces us to take the family of strategies as our domain of optimisation (unlike most of the optimal investment literature since Kramkov and Schachermayer [21], which prefers to optimise over a set of random variables, the terminal values of possible portfolios). Fourth, we exploit that the boundedness of terminal portfolio values in an appropriate sense implies boundedness of the strategies themselves (again, in an appropriate sense); this is false in continuous-time frictionless markets, but true in our setting. Fifth, we profit from a method first developed in Rásonyi [30] that verifies the supermartingale property of a putative optimiser, based on a lemma of Delbaen and Owari [13]. Because of the fourth point above, our techniques do not seem to be applicable in the continuous-time frictionless setting. See, however, the companion paper by Rásonyi and Meireles-Rodrigues [31] which treats *discrete-time* frictionless markets.

The robust model in this paper, similarly to those introduced in Biagini and Pınar [6], Neufeld and Nutz [25], Lin and Riedel [22], assumes that there is a parametrisation for the uncertain dynamics of risky assets. However, as we shall see below, no specific assumption is made about the parametrisation and an arbitrary index set is permitted. From a practical point of view, this approach is particularly tractable and easily implemented when it comes to calibration. For example, when estimating drift and volatility parameters for diffusion price processes, the results only give guesses (hopefully with some confidence sets) about the true values. Thus it is reasonable to parametrise ambiguity by considering suitable ranges which contain possible values for the coefficients being estimated.

From a mathematical point of view, the treatment of robust models in the present paper simplifies technical issues, as will become apparent from the proofs. Working on the same (filtered) probability space, instead of considering a family of measures,

gives us more flexibility by avoiding the canonical setting with problems concerning null events, filtration completion, etc. Measurable selection arguments, see Bouchard and Nutz [9], Biagini et al. [4] or Nutz [27], are not needed any more. Our approach can still incorporate most of the relevant model classes, and their laws on the path space do not need to be equivalent; see Sect. 2.

Compactness plays an important role in proving the existence of optimisers. Usually, the utility maximisation problem is transformed into an "abstract" version with random variables (the terminal wealth of admissible portfolios), and then convex compactness results in L^0, in particular, Komlós-type arguments, are applied successfully; see Kramkov and Schachermayer [21]. Unfortunately, the robust setting is unlikely to be lifted to "abstract" versions, since the uncertainty produces a whole collection of wealth processes. As a result, Komlós-type arguments on the space L^0 cannot be employed. Furthermore, the candidate dual problem in this setting does not, in general, admit a solution (see Bartl [2, Remark 2.3]) so that the usual approach of getting optimisers from solutions of dual problems seems inapplicable. Therefore, we are forced to work on the primal problem directly.

We are using two Komlós-type arguments. The first one is performed on the space of finite-variation processes (strategies), which gives a candidate for the optimiser, and the second is used in an Orlicz space context, to handle possible losses of trading when establishing the supermartingale property of the optimal wealth process, relying on Delbaen and Owari [13]. A crucial observation is that the utility of a portfolio is a sequentially upper semicontinuous function of the strategies (when the latter are equipped with a convenient convergence structure); see Guasoni [15] where the optimisation problem was viewed in a similar manner.

The paper is organised as follows. Section 2 introduces the robust market model and technical assumptions. Sections 3 and 4 study the existence of solutions for the robust utility maximisation problem when the utility functions are defined on $\mathbb{R}_+$ and $\mathbb{R}$, respectively. Ramifications are discussed in Sect. 5. Some preliminaries on finite-variation processes and on Orlicz space theory are presented in Sect. 5.

2 The market model

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)$ be a filtered probability space, where the filtration is assumed to be right-continuous and $\mathcal{F}_0$ coincides with the P-completion of the trivial sigma-algebra. We define by L^0 the set of all a.s.-equivalence classes of random variables and its positive cone by L^0_+.

Let Θ be a (non-empty) set, which is interpreted as the parametrisation of uncertainty. We consider a financial market consisting of a riskless asset $S^0_t = 1$ for all $t \in [0, T]$ and a risky asset, whose dynamics is unknown. To describe the latter, we consider a family $(S^\theta_t)_{t \in [0,T]}$, $\theta \in \Theta$, of adapted, positive processes with continuous trajectories which represent the possible price evolutions. No condition is imposed on Θ nor, for the moment, on the dynamics of the risky asset.

Remark 2.1 We now comment on the difference between our concept of model ambiguity and that of most previous papers, where a family of priors is considered on a canonical space.

Working on a given probability space and filtration amounts to fixing the information structure of the problem; the information flow is normally generated by a particular driving process (such as a multidimensional Brownian motion). Possible prices are then functionals of a parameter (finite- or infinite-dimensional, see Examples 2.2 and 2.4 below) and the driving noise. Strategies are functionals adapted to the given information flow.

Considering a family of probabilities, one has greater liberty in the sense that no common driving noise is required, but the choice of strategies is limited; they must be adapted functionals on the canonical space, i.e., they are functions of the price process. In our modelling, the controls are adapted to an information flow that may be strictly bigger than the natural filtration of any possible price process.

In a strictly formal sense, none of two the approaches is more general than the other; see also examples in [31]. Intuitively, the standard setting is the more general one, while ours seems more easily tractable and fits better with a practical calibration and/or statistical inference framework.

We illustrate by the following examples that the present setting is useful and contains interesting models from previous studies.

Example 2.2 In the robust Black–Scholes market model, the risky asset satisfies the SDE

$$dS_t^{(\mu,\sigma)} = S_t^{(\mu,\sigma)}(\mu\,dt + \sigma\,dW_t), \qquad S_0^{(\mu,\sigma)} = s_0 > 0,$$

where μ, σ are constants and W is a standard Brownian motion. The uncertainty is modelled by

$$\Theta = \{\theta = (\mu,\sigma) \in \mathbb{R}^2 : \underline{\mu} \leq \mu \leq \overline{\mu},\ \underline{\sigma} \leq \sigma \leq \overline{\sigma}\},$$

where $\underline{\mu} \leq \overline{\mu}$, $0 < \underline{\sigma} \leq \overline{\sigma}$ are given constants. The classical Black–Scholes model corresponds to the case $\underline{\mu} = \overline{\mu}$ and $\underline{\sigma} = \overline{\sigma}$. It is observed that the laws of S^{μ_1,σ_1}, S^{μ_2,σ_2} are singular when $\sigma_1 \neq \sigma_2$. If only volatility uncertainty is considered, the family of laws is mutually singular. See [22, 6] about treatments for similar models.

Remark 2.3 In the domain of robust finance, measurable selection techniques are often used; see e.g. [27]. This requires a certain measurability of the family of laws corresponding to various models. In our present approach, however, this is not a necessity. Let e.g. Θ' be a non-Borelian (or even non-analytic) subset of Θ in Example 2.2 above. Theorems 3.6 and 4.7 apply to the family of models S^θ, $\theta \in \Theta'$, too.

Example 2.4 In the above example, Θ was a subset of a finite-dimensional Euclidean space. One may easily fabricate similar examples where Θ is infinite-dimensional. For instance, let Θ consist of all pairs of predictable processes (μ_t, σ_t) such that for all $t \in [0, T]$, $\mu_t \in [\underline{\mu}, \overline{\mu}]$ a.s. and $\sigma_t \in [\underline{\sigma}, \overline{\sigma}]$ a.s., and consider the SDEs

$$dS_t^{(\mu,\sigma)} = S_t^{(\mu,\sigma)}(\mu_t\,dt + \sigma_t\,dW_t), \qquad S_0^{(\mu,\sigma)} = s_0 > 0,$$

for each $(\mu, \sigma) \in \Theta$.

The following example extends the robust Black–Scholes model and allows an external economic factor.

Example 2.5 This is a factor model which is inspired by [20], but much simplified. Let $\Theta \subseteq \mathbb{R}^{2 \times 2}$ be a set. The risky asset is governed by the SDE

$$dS_t^\theta = S_t^\theta \big((m(Y_t^\theta) + \sigma(\theta^{11} Y_t^\theta + \theta^{21}))dt + \sigma \, dW_t^1 \big), \qquad S_0^\theta = s_0 > 0,$$

and the factor process evolves according to

$$dY_t^\theta = \big(g(Y_t^\theta) + \langle \rho, \theta^{1 \cdot} Y_t^\theta + \theta^{2 \cdot} \rangle \big) dt + \rho_1 \, dW_t^1 + \rho_2 \, dW_t^2, \qquad Y_0^\theta = y_0,$$

where m, g are suitable functions, $W = (W^1, W^2)$ is a two-dimensional Brownian motion and $\rho = (\rho_1, \rho_2) \in \mathbb{R}^2$ a fixed parameter. The bracket $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^2$. Note that the original setting of [20] cannot be directly transferred to the present one as it involves a family of weak solutions of SDEs which are not necessarily realisable on our given stochastic basis.

The risky asset is traded under proportional transaction costs $\lambda \in (0, 1)$. More precisely, investors have to pay a higher (ask) price S^θ when buying the risky asset, but receive a lower (bid) price $(1 - \lambda) S^\theta$ when selling it.

Let $\mathcal{V}$ denote the family of nondecreasing, right-continuous functions on $[0, T]$ which are 0 at time 0. Let $\mathbf{V}$ denote the set of triplets $H = (H^\uparrow, H^\downarrow, H_0)$, where $H_t^\uparrow$, $H_t^\downarrow$, $t \in [0, T]$, are optional processes such that $H^\uparrow(\omega), H^\downarrow(\omega) \in \mathcal{V}$ for each $\omega \in \Omega$ and $H_0 \in \mathbb{R}$ (deterministic). The space $\mathbf{V}$ can be equipped with a convergence structure; see Sect. A.1 below for details.

Each trading strategy corresponds to an element $H \in \mathbf{V}$. In this formulation, $H^\uparrow$ denotes the cumulative amount of transfers from the riskless asset to the risky one and $H^\downarrow$ represents the transfers in the opposite direction; H_0 encodes the amount of initial transfer from the riskless asset to the risky one. Therefore the portfolio position in the risky asset at time t equals $\phi_t := H_0 + H_t^\uparrow - H_t^\downarrow$, $t \in [0, T]$, $\phi_{0-} := 0$.

For any $x \in \mathbb{R}$, we denote $x^+ := \max\{0, x\}$, $x^- := \max\{0, -x\}$. For an initial capital $x \in \mathbb{R}$, the dynamics of the cash account of an investor following the strategy H evolves according to

$$W_t^x(\theta, H) := x - H_0^+ S_0^\theta + H_0^- S_0^\theta (1 - \lambda) - \int_0^t S_u^\theta \, dH_u^\uparrow + \int_0^t (1 - \lambda) S_u^\theta \, dH_u^\downarrow,$$

for $t \in [0, T]$. The liquidation value is defined by

$$W_t^{x, \text{liq}}(\theta, H) := W_t^x(\theta, H) + \phi_t^+ (1 - \lambda) S_t^\theta - \phi_t^- S_t^\theta. \tag{2.1}$$

We introduce the definition of consistent price systems, which play a similar role to martingale measures in frictionless markets; see [19, 17, 16].

Definition 2.6 For each $\theta \in \Theta$, a λ-*consistent price system* (λ-CPS) for the model θ is a pair $(\tilde{S}^\theta, Q^\theta)$ of a probability measure $Q^\theta \approx P$ and a (càdlàg) Q^θ-local martin-

gale $\tilde{S}^\theta$ such that

$$(1 - \lambda)S_t^\theta \leq \tilde{S}_t^\theta \leq S_t^\theta \qquad \text{a.s., for each } t \in [0, T]. \tag{2.2}$$

A λ-*strictly consistent price system* (λ-SCPS) is a CPS such that the inequalities in (2.2) are strict.

We impose the existence of consistent price systems for every model S^θ. In Sect. 3, we need the following assumption in order to be able to use the results of [12].

Assumption 2.7 For each $\theta \in \Theta$ and for all $0 < \mu < \lambda$, the price process S^θ admits a μ-CPS.

This assumption is fulfilled if for every $\theta \in \Theta$, the process S^θ satisfies the no-arbitrage condition for μ-transaction cost for all $\mu > 0$; see [17]. See Example 4.6 for a risky asset violating Assumption 2.7.
    Clearly, if $0 < \mu < \lambda$, then a μ-CPS is also a λ-SCPS.

Lemma 2.8 *If $(\tilde{S}^\theta, Q^\theta)$ is a λ-strictly consistent price system, the random variable*

$$\delta(\theta) := \inf_{t \in [0, T]} \min\{\tilde{S}_t^\theta - (1 - \lambda)S_t^\theta, S_t^\theta - \tilde{S}_t^\theta\} \tag{2.3}$$

is almost surely strictly positive and $E^{Q^\theta}[\delta(\theta)] < \infty$.

Proof The argument follows that of [19, Lemma 3.6.4].   □

    Let

$$\mathcal{M}^\theta := \{dQ^\theta/dP : \ (\tilde{S}^\theta, Q^\theta) \text{ is a } \lambda\text{-CPS}\}.$$

For a consistent price system $(\tilde{S}^\theta, Q^\theta)$, we define the process

$$V_t^x(\theta, H) := W_t^x(\theta, H) + \phi_t \tilde{S}_t^\theta,$$

without emphasising the dependence of V on the specific consistent price system. It is easy to check that $W_t^{x,\text{liq}}(H) \leq V_t^x(\theta, H)$ a.s., for each $t \in [0, T]$.

3 Utility function on $\mathbb{R}_+$

Assumption 3.1 The utility function $U : (0, \infty) \to \mathbb{R}$ is nondecreasing and concave.

    Define the convex conjugate of U by

$$V(y) := \sup_{x > 0} (U(x) - xy), \qquad y > 0.$$

Admissible strategies are defined in a natural way, thanks to the domain of the utility function.

Definition 3.2 A strategy $H = (H^\uparrow, H^\downarrow, H_0) \in \mathbf{V}$ is *admissible* for initial capital $x > 0$ and for the model $\theta \in \Theta$ if for each $t \in [0, T]$,

$$W_t^{x,\mathrm{liq}}(\theta, H) \geq 0 \quad \text{a.s.}$$

Denote by $\mathcal{A}^\theta(x)$ the set of all admissible strategies for θ. Set

$$\mathcal{A}_0^\theta(x) := \{H \in \mathcal{A}^\theta(x) : \phi_T = H_0 + H_T^\uparrow - H_T^\downarrow = 0\}$$

and $\mathcal{A}(x) = \bigcap_{\theta \in \Theta} \mathcal{A}_0^\theta(x)$.

Remark 3.3 For each $H \in \mathcal{A}(x)$ and $\theta \in \Theta$,

$$W_T^{x,\mathrm{liq}}(\theta, H) = W_T^x(\theta, H) = V_T^x(\theta, H)$$

due to $\phi_T = 0$. We also see from (2.1) that at time $0 < t < T$, the liquidation value is neither concave nor convex in H. However, the condition $\phi_T = 0$ recovers concavity of the liquidation value with respect to H at time T. This is crucial for finding maximisers in the subsequent analysis.

Let $x > 0$. Note that $\mathcal{A}(x) \neq \emptyset$ since it contains the identically zero strategy. Our investors want to find the optimiser for

$$u(x) := \sup_{H \in \mathcal{A}(x)} \inf_{\theta \in \Theta} E\big[U\big(W_T^{x,\mathrm{liq}}(\theta, H)\big)\big]. \tag{3.1}$$

It is worth noting that maximising in H is a concave problem; however, minimising over Θ is *not* a convex problem.

For each $\theta \in \Theta$ and $x > 0$, we denote

$$\mathcal{C}^\theta(x) := \{X \in L_+^0 : X \leq W_T^{x,\mathrm{liq}}(\theta, H) \text{ for some } H \in \mathcal{A}^\theta(x)\}.$$

For each $y > 0$, the set of *supermartingale deflators* $\mathcal{B}^\theta(y)$ consists of the strictly positive processes $Y = (Y_t^0, Y_t^1)_{t \in [0,T]}$, $Y_0^0 = y$, such that $Y^1/Y^0 \in [(1-\lambda)S^\theta, S^\theta]$ and $W^x(\theta, H)Y^0 + \phi Y^1$ is a (càdlàg) supermartingale for all $H \in \mathcal{A}^\theta(x)$. Also, we define

$$\mathcal{D}^\theta(y) := \{Y_T^0 : (Y^0, Y^1) \in \mathcal{B}^\theta(y)\}.$$

The primal and dual value functions for the θ-model are

$$u^\theta(x) := \sup_{f \in \mathcal{C}^\theta(x)} E[U(f)], \qquad v^\theta(y) := \inf_{h \in \mathcal{D}^\theta(y)} E[V(h)].$$

The next lemma states that the sets $\mathcal{C}^\theta(x)$ and $\mathcal{D}^\theta(y)$ are polar to each other. It follows directly from [12, Proposition 2.9].

Lemma 3.4 *Fix $x, y > 0$. Let Assumption 2.7 be in force. A random variable $X \in L_+^0$ satisfies $X \in \mathcal{C}^\theta(x)$ if and only if $E[XY] \leq xy$ for all $Y \in \mathcal{D}^\theta(y)$. A random variable $Y \in L_+^0$ satisfies $Y \in \mathcal{D}(y)$ if and only if $E[XY] \leq xy$ for all $X \in \mathcal{C}^\theta(x)$.*

We impose a technical assumption.

Assumption 3.5 The dual value function $v^\theta(y)$, $y > 0$, is finite for *all* $\theta \in \Theta$.

Theorem 3.6 *Let $x > 0$. Under Assumptions* 2.7, 3.1, 3.5, *the robust utility maximisation problem* (3.1) *admits a solution, i.e., there is $H^* \in \mathcal{A}(x)$ satisfying*

$$u(x) = \inf_{\theta \in \Theta} E\big[U\big(W_T^{x,\text{liq}}(\theta, H^*)\big)\big].$$

When U is bounded from above, the same conclusion holds assuming only that there exists (*at least*) *one $\tilde{\theta} \in \Theta$ for which there exists a λ-SCPS.*

Proof If U is constant, there is nothing to prove. Otherwise, by adding a constant to U, we may assume that $U(\infty) > 0 > \lim_{x \to 0} U(x)$.

Notice that $U(\infty) > 0$ and

$$u^\theta(x) \geq U(x) \tag{3.2}$$

imply $\liminf_{x \to \infty} u^\theta(x)/x \geq 0$. From Lemma 3.4, trivially,

$$u^\theta(x) \leq v^\theta(y) + xy \tag{3.3}$$

for all $y > 0$. Fixing y, we obtain $\limsup_{x \to \infty} u^\theta(x)/x \leq y$, and sending y to zero gives

$$\lim_{x \to \infty} \frac{u^\theta(x)}{x} = 0. \tag{3.4}$$

After these preparations, we turn to the main arguments. Assumption 3.5, (3.3) and (3.2) imply that $u^\theta(x)$ is finite for each θ and so is $u(x)$. Let $H^n \in \mathcal{A}(x)$, $n \in \mathbb{N}$, be a maximising sequence, i.e.,

$$\inf_{\theta \in \Theta} E\big[U\big(W_T^{x,\text{liq}}(\theta, H^n)\big)\big] \uparrow u(x) \qquad \text{as } n \to \infty.$$

Let us fix for the moment $\theta \in \Theta$ and a μ-CPS $(\tilde{S}^\theta, Q^\theta)$ with $0 < \mu < \lambda$. First, we prove that the process

$$V_t^x(\theta, H^n) = W_t^x(\theta, H^n) + \phi_t^n \tilde{S}_t^\theta$$

is a Q^θ-supermartingale for all n. Indeed, Itô's formula gives

$$dV_t^x(\theta, H^n) = -S_t^\theta \, dH_t^{n,\uparrow} + (1 - \lambda)S_t^\theta \, dH_t^{n,\downarrow} + \tilde{S}_t^\theta \, d\phi_t^n + \phi_{t-}^n \, d\tilde{S}_t^\theta$$

$$= (\tilde{S}_t^\theta - S_t^\theta) \, dH_t^{n,\uparrow} + \big((1 - \lambda)S_t^\theta - \tilde{S}_t^\theta\big) \, dH_t^{n,\downarrow} + \phi_{t-}^n \, d\tilde{S}_t^\theta.$$

Admissibility of H^n implies

$$(H_0^n)^+(S_0^\theta - \tilde{S}_0^\theta) + \int_0^t (S_u^\theta - \tilde{S}_u^\theta)\, dH_u^{n,\uparrow} + \int_0^t \left(\tilde{S}_u^\theta - (1-\lambda)S_u^\theta\right) dH_u^{n,\downarrow}$$

$$+ \left(\int_0^t \phi_{u-}^n\, d\tilde{S}_u^\theta\right)^-$$

$$\leq x + (H_0^n)^-\left(S_0^\theta(1-\lambda) - \tilde{S}_0^\theta\right) + \left(\int_0^t \phi_{u-}^n\, d\tilde{S}_u^\theta\right)^+. \tag{3.5}$$

In particular, we obtain

$$\left(\int_0^t \phi_{u-}^n\, d\tilde{S}_u^\theta\right)^- \leq x + (H_0^n)^- S_0^\theta(1-\lambda) \tag{3.6}$$

for every $t \in [0, T]$, and therefore $\int_0^t \phi_{u-}^n\, d\tilde{S}_u^\theta$, $t \in [0, T]$, is a Q^θ-supermartingale; see [1, Corollary 3.5]. It follows that $V_t^x(\theta, H^n)$, $t \in [0, T]$, is also a Q^θ-supermartingale.

We claim that $\sup_n (H_0^n)^-$ is finite. If this were not the case, then along a subsequence n_k, $k \in \mathbb{N}$, we should have $(H_0^{n_k})^- \to \infty$, $k \to \infty$, and $(H_0^{n_k})^+ = 0$, $k \in \mathbb{N}$. Taking Q^θ-expectations in (3.5), we should get

$$0 \leq x + \lim_{k\to\infty} (H_0^{n_k})^-\left(S_0^\theta(1-\lambda) - \tilde{S}_0^\theta\right) = -\infty,$$

a contradiction. Hence the supremum is indeed finite.

Furthermore, from the supermartingale property of $\int_0^t \phi_{u-}^n\, d\tilde{S}_u^\theta$, $t \in [0, T]$, and from (3.6),

$$\sup_n E^{Q^\theta}\left[\left(\int_0^T \phi_{u-}^n\, d\tilde{S}_u^\theta\right)^+\right] \leq x + \sup_n (H_0^n)^- S_0^\theta(1-\lambda)$$

follows. Using (2.3), we deduce from (3.5) that

$$\sup_n E^{Q^\theta}\left[(H_0^n)^+\delta(\theta) + \int_0^T \delta(\theta)\left(dH_u^{n,\uparrow} + dH_u^{n,\downarrow}\right)\right]$$

$$\leq \sup_n E^{Q^\theta}\left[(H_0^n)^+(S_0^\theta - \tilde{S}_0^\theta) + \int_0^T (S_u^\theta - \tilde{S}_u^\theta)\, dH_u^{n,\uparrow} + \left(\tilde{S}_u^\theta - (1-\lambda)S_u^\theta\right) dH_u^{n,\downarrow}\right]$$

$$< \infty.$$

Apply Lemma A.1 with the choice $dQ/dQ^\theta := \delta(\theta)/E^{Q^\theta}[\delta(\theta)]$. It implies that there exist convex weights $\alpha_j^n \geq 0$, $j = n, \ldots, M(n)$, with $\sum_{j=n}^{M(n)} \alpha_j^n = 1$ such that $\tilde{H}^n := \sum_{j=n}^{M(n)} H^n \to H^*$ in $\mathbf{V}$. Since the utility function is concave, we obtain that $\tilde{H}^n$, $n \in \mathbb{N}$, is also a maximising sequence as

$$\inf_{\theta\in\Theta} E\left[U\left(W_T^{x,\text{liq}}(\theta, \tilde{H}^n)\right)\right] \geq \inf_{\theta\in\Theta} E\left[U\left(W_T^{x,\text{liq}}(\theta, H^n)\right)\right] \longrightarrow u(x) \qquad \text{as } n \to \infty.$$

We now prove that the sequence $U^+(W_T^{x,\text{liq}}(\theta, \tilde{H}^n))$, $n \in \mathbb{N}$, is uniformly integrable for each $\theta \in \Theta$. Suppose by contradiction that the sequence is not uniformly integrable for some θ. Then one can find disjoint sets $A_n \in \mathcal{F}$, $n \in \mathbb{N}$, and a constant $\alpha > 0$ such that

$$E\big[U^+\big(W_T^{x,\text{liq}}(\theta, \tilde{H}^n)\big)1_{A_n}\big] \geq \alpha \qquad \text{for } n \geq 1.$$

Set $w^n = \sum_{i=1}^n W_T^{x,\text{liq}}(\theta, \tilde{H}^i)1_{\{W_T^{x,\text{liq}}(\theta, \tilde{H}^i) \geq u_0\}}1_{A_i}$, where u_0 is chosen such that it satisfies $U(u_0) = 0$. It is immediate that

$$E[U(w^n)] = \sum_{i=1}^n E\big[U^+\big(W_T^{x,\text{liq}}(\theta, \tilde{H}^i)\big)1_{A_i}\big] \geq n\alpha.$$

In addition, for any $h \in \mathcal{D}^\theta(1)$, the supermartingale property shows that $E[hw^n] \leq nx$. Consequently, we obtain $w^n \in C^\theta(nx)$ by Lemma 3.4. We compute

$$\frac{u^\theta(nx)}{nx} \geq \frac{E[U(w^n)]}{nx} \geq \frac{\alpha}{x} > 0,$$

and passing to the limit when $n \to \infty$ contradicts (3.4). Thus $U^+(W_T^{x,\text{liq}}(\theta, \tilde{H}^n))$, $n \in \mathbb{N}$, is indeed uniformly integrable.

Since $\tilde{H}^n \to H^*$ in $\mathbf{V}$, $W_T^{x,\text{liq}}(\theta, \tilde{H}^n) \to W_T^{x,\text{liq}}(\theta, H^*)$ almost surely by Remark A.2. So Fatou's lemma and uniform integrability imply

$$\limsup_{n \to \infty} \Big(\inf_{\theta \in \Theta} E\big[U\big(W_T^{x,\text{liq}}(\theta, \tilde{H}^n)\big)\big]\Big) \leq \inf_{\theta \in \Theta} \limsup_{n \to \infty} E\big[U\big(W_T^{x,\text{liq}}(\theta, \tilde{H}^n)\big)\big]$$

$$\leq \inf_{\theta \in \Theta} E\big[U\big(W_T^{x,\text{liq}}(\theta, H^*)\big)\big],$$

which proves that H^* is an optimiser. It remains to check that $H^* \in \mathcal{A}(x)$. For each θ, $W_t^{x,\text{liq}}(\theta, H^*) \geq 0$ a.s., for Lebesgue-almost every t, by Remark A.2; so we get admissibility of H^* since $t \to W_t^{x,\text{liq}}$ is a.s. right-continuous.

In the case where U is bounded from above, it is enough to perform the first part of the above argument for $\tilde{\theta}$, obtain H^* and then simply invoke Fatou's lemma to complete the proof. □

Remark 3.7 In the classical theory where there is no uncertainty, i.e., when Θ contains only one element, the existence result holds assuming the finiteness of $u(x)$ only. This condition, however, does not suffice to find optimisers in the robust problem. Indeed, the finiteness of $u(x)$ makes the robust problem well posed, compactness gives a candidate for the optimiser, but this is still not enough to prove that the candidate is indeed an optimiser. To complete the proof, it is necessary to have upper semicontinuity of the expected utility when considered as a function of the strategy variable. In [27], a counterexample (in which $u(x)$ is finite, but one could not find an optimiser) is given in the nondominated case. The author's argument exploits precisely the lack of upper semicontinuity property in *one* model. Furthermore, [27] gives a sufficient

condition to have upper semicontinuity, namely the integrability of the positive part of the utility function under *every* possible model; see [27, Theorem 2.2] and also [8] for further developments. In our approach, upper semicontinuity follows from the finiteness of the dual value function for *every* model.

4 Utility functions on $\mathbb{R}$

Assumption 4.1 The utility function $U : \mathbb{R} \to \mathbb{R}$ is bounded from above, nondecreasing, concave and $U(0) = 0$. Define the convex conjugate of U by

$$V(y) := \sup_{x \in \mathbb{R}} \big(U(x) - xy \big), \qquad y > 0.$$

We also assume that

$$\lim_{x \to -\infty} \frac{U(x)}{x} = \infty, \tag{4.1}$$

$$\limsup_{y \to \infty} \frac{V(2y)}{V(y)} < \infty. \tag{4.2}$$

Remark 4.2 Under (4.1), the function V takes finite values and $V(y) > 0$ for y large enough; hence (4.2) makes sense. The condition $U(0) = 0$ is used only to simplify calculations. Condition (4.1) is mild and so is (4.2); indeed, as shown in [32, Corollary 4.2(i)], for every utility function U with reasonable asymptotic elasticity, its conjugate V satisfies (4.2). The studies [11, 23] assumed a smooth U which is strictly concave on its entire domain; we need neither smoothness nor strict concavity of U.

As discussed in [7, 33], the choice of admissible trading strategies is a delicate issue in the context of utility maximisation with utility functions defined on the real line. A common approach is to consider strategies whose wealth processes are bounded uniformly from below by a constant. This choice, however, turns out to be restrictive and fails to contain optimisers. In frictionless markets, [33] proved that for a utility function having reasonable asymptotic elasticity, the optimal investment process is a supermartingale under each martingale measure Q such that $E[V(dQ/dP)]$ is finite. We thus use the supermartingale property to define admissibility, just like in [28, 10].

To begin with, we define

$$\mathcal{M}_V^\theta = \{Q^\theta : (\tilde{S}^\theta, Q^\theta) \text{ is a } \lambda\text{-consistent price system, } E[V(dQ^\theta/dP)] < \infty\},$$

the set of local martingale measures in consistent price systems for the θ-model with finite generalised relative entropy.

Definition 4.3 We define

$$\mathcal{A}^\theta(x) := \{H \in \mathbf{V} : \phi_T = 0, \ V^x(\theta, H) \text{ is a } Q^\theta\text{-supermartingale}$$

$$\text{for each } \lambda\text{-consistent price system } (\tilde{S}^\theta, Q^\theta) \text{ with } Q^\theta \in \mathcal{M}_V^\theta\}$$

and set $\mathcal{A}(x) := \bigcap_{\theta \in \Theta} \mathcal{A}^\theta(x)$.

The optimisation problem becomes

$$u(x) = \sup_{H \in \mathcal{A}(x)} \inf_{\theta \in \Theta} E\big[U\big(W_T^{x,\mathrm{liq}}(\theta, H)\big)\big].$$

Assumption 4.4 For each $\theta \in \Theta$, the price process S^θ admits a λ-SCPS $(Q^\theta, \tilde{S}^\theta)$ such that $Q^\theta \in \mathcal{M}_V^\theta$.

Remark 4.5 Unlike in [11, 12, 23] and in Sect. 3, we do not impose in the present section the existence of consistent price systems for *every* transaction cost coefficient $0 < \mu < \lambda$; we only stipulate Assumption 4.4. The following example shows that it is quite possible to have CPSs for relatively large λ without having them for arbitrarily small μ. In this example, there is an *obvious arbitrage*, in the language of [17], which persists (ceases) with sufficiently small (large) transaction costs.

Example 4.6 Let us consider

$$S_t = 1 + t + \frac{1}{2\pi} \arctan W_t, \qquad t \in [0, 1].$$

If $\lambda < 3/7$, then $(1 - \lambda)S_1 > 1 = S_0$ a.s.; therefore, there is no consistent price system. If $\lambda \geq 2/3$, then

$$S_t(1 - \lambda) \leq 3/4 \leq S_t, \qquad t \in [0, T].$$

In other words, $(P, \tilde{S} \equiv 3/4)$ is a consistent price system.

Theorem 4.7 *Under Assumptions 4.1 and 4.4, there exists a strategy $H^* \in \mathcal{A}(x)$ such that*

$$u(x) = \inf_{\theta \in \Theta} E\big[U\big(W_T^{x,\mathrm{liq}}(\theta, H^*)\big)\big].$$

Proof We adapt certain techniques of [30]. Our arguments bring novelties even in the case where Θ is a singleton (i.e., without model uncertainty). Define $\Phi^*(x) = -U(-x)$ for $x \geq 0$. Its conjugate (in the sense of Sect. A.2 below) is

$$\Phi(y) := \begin{cases} 0, & \text{if } 0 \leq y \leq \beta, \\ V(y) - V(\beta), & \text{if } y > \beta, \end{cases}$$

where β is the left derivative of U at 0; see [5]. Note that Φ, Φ^* are Young functions and Φ is of class Δ_2 by (4.2).

Let $H^n \in \mathcal{A}(x)$, $n \in \mathbb{N}$, be a maximising sequence, i.e.,

$$\inf_{\theta \in \Theta} E\big[U\big(W_T^{x,\mathrm{liq}}(\theta, H^n)\big)\big] \uparrow u(x) \geq U(x). \tag{4.3}$$

First, for all $\theta \in \Theta$, it holds that

$$\sup_n E\big[U^-\big(W_T^{x,\mathrm{liq}}(\theta, H^n)\big)\big] < \infty. \tag{4.4}$$

Indeed, let us assume that there exists $\theta \in \Theta$ such that (4.4) does not hold, or equivalently, there exists a subsequence $n_k = n_k^\theta$, $k \in \mathbb{N}$, such that

$$E\big[U^-\big(W_T^{x,\text{liq}}(\theta, H^{n_k})\big)\big] > k.$$

Let us denote by C an upper bound of U; then

$$E\big[U\big(W_T^{x,\text{liq}}(\theta, H^{n_k})\big)\big] \le C - E\big[U^-\big(W_T^{x,\text{liq}}(\theta, H^{n_k})\big)\big] \longrightarrow -\infty \qquad \text{as } k \to \infty$$

which contradicts (4.3). Hence (4.4) indeed holds.

Consider a λ-strictly consistent price system $(\tilde{S}^\theta, Q^\theta)$. Fenchel's inequality gives

$$U\big(V_T^x(\theta, H^n)\big) - V(dQ^\theta/dP) \le (dQ^\theta/dP)V_T^x(\theta, H^n)$$

and therefore

$$(dQ^\theta/dP)\big(V_T^x(\theta, H^n)\big)^- \le \Big(U\big(V_T^x(\theta, H^n)\big) - V(dQ^\theta/dP)\Big)^-. \qquad (4.5)$$

From (4.4) and (4.5), we deduce that

$$\sup_n E^{Q^\theta}\big[\big(V_T^x(\theta, H^n)\big)^-\big] < \infty. \qquad (4.6)$$

Itô's formula gives

$$\begin{aligned}
dV_t^x(\theta, H^n) &= -S_t^\theta\, dH_t^{n,\uparrow} + (1-\lambda)S_t^\theta\, dH_t^{n,\downarrow} + \tilde{S}_t^\theta\, d\phi_t^n + \phi_{t-}^n\, d\tilde{S}_t^\theta \\
&= (\tilde{S}_t^\theta - S_t^\theta)\, dH_t^{n,\uparrow} + \big((1-\lambda)S_t^\theta - \tilde{S}_t^\theta\big)\, dH_t^{n,\downarrow} + \phi_{t-}^n\, d\tilde{S}_t^\theta.
\end{aligned}$$

This implies that

$$\begin{aligned}
(H_0^n)^+&(S_0^\theta - \tilde{S}_0^\theta) + \int_0^t (S_u^\theta - \tilde{S}_u^\theta)\, dH_u^{n,\uparrow} + \int_0^t \big(\tilde{S}_u^\theta - (1-\lambda)S_u^\theta\big)\, dH_u^{n,\downarrow} \\
&+ \left(\int_0^t \phi_{u-}^n\, d\tilde{S}_u^\theta\right)^- \\
&\le x + (H_0^n)^-\big(S_0^\theta(1-\lambda) - \tilde{S}_0^\theta\big) + \big(V_t^x(\theta, H^n)\big)^- + \left(\int_0^t \phi_{u-}^n\, d\tilde{S}_u^\theta\right)^+.
\end{aligned}$$

In particular,

$$\left(\int_0^t \phi_{u-}^n\, d\tilde{S}_u^\theta\right)^- \le x + (H_0^n)^- S_0^\theta(1-\lambda) + \big(V_t^x(\theta, H^n)\big)^-. \qquad (4.7)$$

For each n, the process $V^x(\theta, H^n)$ is a Q^θ-supermartingale; so there exists a Q^θ-martingale which dominates the right-hand side of (4.7) and also the left-hand side of the same expression. [1, Corollary 3.5] implies that $\int_0^t \phi_{u-}^n\, d\tilde{S}_u^\theta$, $t \in [0, T]$, is

a Q^θ-supermartingale. We get $\sup_n (H_0^n)^- < \infty$ in the same way as in the proof of Theorem 3.6. Consequently, (4.6), (4.7) and the boundedness of $(H_0^n)^-$, $n \in \mathbb{N}$, give

$$\sup_n E^{Q^\theta} \left[\left(\int_0^T \phi_{t-}^n d\tilde{S}_t^\theta \right)^+ \right] < \infty.$$

Noting that $(\tilde{S}^\theta, Q^\theta)$ is a λ-strictly consistent price system, we obtain from the above arguments that

$$\sup_n E^{Q^\theta} \left[(H_0^n)^+ \delta(\theta) + \delta(\theta) \int_0^T (dH_t^{n,\uparrow} + dH_t^{n,\downarrow}) \right]$$

$$\leq \sup_n E^{Q^\theta} \left[(H_0^n)^+ (S_0^\theta - \tilde{S}_0^\theta) + \int_0^T (S_t^\theta - \tilde{S}_t^\theta) dH_t^{n,\uparrow} + (\tilde{S}_t^\theta - (1-\lambda)S_t^\theta) dH_t^{n,\downarrow} \right]$$

$$< \infty.$$

Lemma A.1 implies the existence of convex weights $\alpha_j^n \geq 0$, $j = n, \ldots, M(n)$, with $\sum_{j=n}^{M(n)} \alpha_j^n = 1$ such that $\tilde{H}^n := \sum_{j=n}^{M(n)} \alpha_j^n H^n \to H^*$ in **V**. Since the utility function is concave, $\tilde{H}^n$, $n \in \mathbb{N}$, is also a maximising sequence.

We now prove that $H^* \in \mathcal{A}(x)$; in other words, the process $V^x(\theta, H^*)$ is a Q^θ-supermartingale, for each $Q^\theta \in \mathcal{M}_V^\theta$ and each $\theta \in \Theta$. To do so, it suffices to control the negative part of $V^x(\theta, H^*)$. It should be emphasised that (4.6) is not enough for our purposes and a stronger statement using Orlicz space theory is needed (see Sect. A.2). Using concavity of U and linearity of $V^x(\theta, \cdot)$, we get from (4.4) that

$$\sup_n E[U^-(V_T^x(\theta, \tilde{H}^n))] < \infty. \tag{4.8}$$

Applying Lemma A.3 to the sequence of random variables in (4.8), we obtain convex weights $\alpha_j'^n \geq 0$, $n \leq j \leq M(n)$, with $\sum_{j=n}^{M(n)} \alpha_j'^n = 1$ such that

$$Z^n := \sum_{j=n}^{M(n)} \alpha_j'^n (V_T^x(\theta, \tilde{H}^n))^-$$

satisfy

$$L := \left\| \sup_n Z^n \right\|_{\Phi^*} < \infty. \tag{4.9}$$

By the Fenchel inequality and (4.9),

$$E^{Q^\theta} \left[\sup_n Z^n \right] = L E^{Q^\theta} \left[\frac{\sup_n Z^n}{L} \right]$$

$$\leq L E \left[\Phi \left(\frac{dQ^\theta}{dP} \right) \right] + L E \left[\Phi^* \left(\frac{\sup_n Z^n}{L} \right) \right] < \infty \tag{4.10}$$

for each $Q^\theta \in \mathcal{M}_V^\theta$. When $L = 0$, we have that $E^{Q^\theta}[\sup_n Z^n] = 0$ trivially. Now we define

$$\overline{H}^n := \sum_{j=n}^{M(n)} \alpha_j'^n \tilde{H}^n,$$

which is also a maximising sequence. Using the fact that the negative part of a supermartingale is a submartingale, we get $V_t^x(\theta, \overline{H}^n)^- \leq E^{Q^\theta}[V_T^x(\theta, \overline{H}^n)^- | \mathcal{F}_t]$ and thus

$$\sup_n V_t^x(\theta, \overline{H}^n)^- \leq \sup_n E^{Q^\theta}[V_T^x(\theta, \overline{H}^n)^- | \mathcal{F}_t].$$

Taking expectations on both sides of the above inequality, we obtain

$$E^{Q^\theta}\left[\sup_n V_t^x(\theta, \overline{H}^n)^-\right] \leq E^{Q^\theta}\left[\sup_n E^{Q^\theta}[V_T^x(\theta, \overline{H}^n)^- | \mathcal{F}_t]\right]$$
$$\leq E^{Q^\theta}\left[E^{Q^\theta}\left[\sup_n V_T^x(\theta, \overline{H}^n)^- | \mathcal{F}_t\right]\right]$$
$$= E^{Q^\theta}\left[\sup_n V_T^x(\theta, \overline{H}^n)^-\right]$$
$$\leq E^{Q^\theta}\left[\sup_n Z^n\right] < \infty,$$

using (4.10). Since the random variable $\sup_n(V_t^x(\theta, \overline{H}^n))^-$ is an upper bound of the sequence $V_t^x(\theta, \overline{H}^n)^-$, $n \in \mathbb{N}$, this proves uniform integrability of that sequence under Q^θ at any time $t \in [0, T]$.

Clearly, $\overline{H}^n \to H^*$ in $\mathbf{V}$ and therefore $V_t^x(\theta, \overline{H}^n) \to V_t^x(\theta, H^*)$ a.s., for every $t \in [0, T] \setminus Z$ where Z has Lebesgue measure 0; see Remark A.2. Also,

$$\left(V_t^x(\theta, H^*)\right)^- \leq E^{Q^\theta}\left[\sup_n V_T^x(\theta, \overline{H}^n)^- \Big| \mathcal{F}_t\right], \qquad t \in [0, T], \tag{4.11}$$

where the latter process is a martingale, and hence $(V_t^x(\theta, H^*))_{t \in [0,T]}$ is uniformly integrable. Let $0 \leq s \leq t < T$ be both in $[0, T] \setminus Z$. Noting the supermartingale property, Fatou's lemma yields

$$E^{Q^\theta}[V_t^x(\theta, H^*) | \mathcal{F}_s] = E^{Q^\theta}\left[\liminf_{n \to \infty} V_t^x(\theta, \overline{H}^n) \Big| \mathcal{F}_s\right]$$
$$\leq \liminf_{n \to \infty} E^{Q^\theta}[V_t^x(\theta, \overline{H}^n) | \mathcal{F}_s]$$
$$\leq \liminf_{n \to \infty} V_s^x(\theta, \overline{H}^n) = V_s^x(\theta, H^*).$$

The same argument works for $t = T$, too. Now it extends to arbitrary $t \in [0, T]$ by using Fatou's lemma and (4.11). Finally, it extends to arbitrary $s \in [0, T]$ by the backward martingale convergence theorem and by right-continuity of $s \to V_s^x(\theta, H^*)$. This means that $V^x(\theta, H^*)$ is a Q^θ-supermartingale and therefore $H^* \in \mathcal{A}(x)$.

Since U is bounded from above, by Fatou's lemma,

$$\limsup_{n\to\infty}\left(\inf_{\theta\in\Theta} E\big[U\big(W_T^{x,\text{liq}}(\theta,\overline{H}^n)\big)\big]\right) \le \inf_{\theta\in\Theta}\limsup_{n\to\infty} E\big[U\big(W_T^{x,\text{liq}}(\theta,\overline{H}^n)\big)\big]$$

$$\le \inf_{\theta\in\Theta} E\big[U\big(W_T^{x,\text{liq}}(\theta,H^*)\big)\big],$$

which proves the optimality of H^*.   □

Remark 4.8 We can compare our approach to that of [30] where in a general setting, supermartingale portfolio processes and their terminal values are considered, relying on [13]. In order to get optimal strategies, a certain Fatou-closure property of such terminal values is used. It is known that the proof of such a property (see e.g. [30, Lemma 4.1]) is rather subtle and does not construct the optimal strategy simply as a convex combination of an approximating sequence of strategies. In other words, only an optimal terminal value is obtained there, without obtaining an optimal strategy. When model ambiguity is present, this approach is doomed to fail since there does not seem to exist a method that provides Fatou-closedness "uniformly in the family of possible models". For this reason, the techniques of [30] cannot cope with model uncertainty in markets with or without friction.

5 Conclusions

It is possible to extend our results in Sect. 4. One could treat the multi-asset "conic" framework of [19]; unbounded utilities could also be incorporated along the lines of [30, Theorem 3.12]; random endowments (or random utilities) can be added at little cost since we do not consider the dual problem at all. These extensions, however, require no essential new ideas while they would considerably complicate the presentation. Our emphasis here is on introducing a new approach, and not on striving for the utmost generality.

Admitting jumps in the price process leads to a more involved class of strategies. The treatment of that setting is a direction of research worth pursuing in the future.

Appendix

A.1 Finite-variation processes

Let $\mathcal{V}$ denote the family of nondecreasing, right-continuous functions on $[0, T]$ which are 0 at 0. Let r_k, $k \in \mathbb{N}$, be an enumeration of $D := (\mathbb{Q} \cap [0, T]) \cup \{T\}$ with $r_0 = T$. For $f, g \in \mathcal{V}$, define

$$\rho(f, g) := \sum_{k=0}^{\infty} 2^{-k} |f(r_k) - g(r_k)|.$$

The series converges since $|f(r_k) - g(r_k)| \leq f(T) + g(T)$, and it defines a metric. The corresponding Borel σ-field is denoted by $\mathcal{G}$.

Let $\mathbf{V}$ denote the set of triplets $H = (H^\uparrow, H^\downarrow, H_0)$ where $H_t^\uparrow$, $H_t^\downarrow$, $t \in [0, T]$, are optional processes such that $H^\uparrow(\omega), H^\downarrow(\omega) \in \mathcal{V}$ for each $\omega \in \Omega$ and $H_0 \in \mathbb{R}$ (deterministic). Considered as mappings $H^\uparrow, H^\downarrow : (\Omega, \mathcal{F}) \to (\mathcal{V}, \mathcal{G})$, they are measurable, by the definition of the metric ρ. We identify elements of $\mathbf{V}$ when they coincide (as functions in t) outside a P-null set. We say that a sequence (H^n) in $\mathbf{V}$ is *convergent* to some $H \in \mathbf{V}$ if $H^{n,\uparrow} \to H^\uparrow$ and $H^{n,\downarrow} \to H^\downarrow$ a.s. in $\mathcal{V}$ as $n \to \infty$ and also $H_0^n \to H_0$ (in the topology of $\mathbb{R}$).

Convex-compactness-type results for finite-variation processes have been introduced in various forms in the literature. The following result is very similar to [18, Lemma 3.5].

Lemma A.1 *Let $H^n \in \mathbf{V}$, $n \in \mathbb{N}$, be such that*

$$\sup_{n \in \mathbb{N}} (E^Q[H_T^{n,\uparrow} + H_T^{n,\downarrow}] + |H_0^n|) < \infty$$

for some $Q \approx P$. Then there are $H \in \mathbf{V}$ and convex weights $\alpha_j^n \geq 0$, $j = n, \ldots, M(n)$, with $\sum_{j=n}^{M(n)} \alpha_j^n = 1$, $n \in \mathbb{N}$, such that

$$\tilde{H}^n := \sum_{j=n}^{M(n)} \alpha_j^n H^j \longrightarrow H \qquad in \ \mathbf{V}.$$

It follows that for P-almost every $\omega \in \Omega$,

$$\tilde{H}_t^{n,\uparrow}(\omega) \longrightarrow H_t^\uparrow(\omega) \quad and \quad \tilde{H}_t^{n,\downarrow}(\omega) \longrightarrow H_t^\downarrow(\omega)$$

at $t = T$ and at each t which is a continuity point of both $H^\uparrow(\omega)$ and $H^\downarrow(\omega)$.

Proof Recall that $D = ([0, T] \cap \mathbb{Q}) \cup \{T\}$. By assumption, the sequence $H_T^{n,\uparrow}$, $n \in \mathbb{N}$, is bounded in $L^1(Q)$ for some $Q \approx P$; so we use the Komlós theorem together with a diagonalisation procedure to obtain sequences of convex weights α_j^n such that

$$\tilde{H}_t^{n,\uparrow} \longrightarrow H_t^\uparrow, \qquad t \in D, \tag{A.1}$$

for some $\mathcal{F}_t$-measurable random variables $H_t^\uparrow$, on an event $\tilde{\Omega}$ with $P[\tilde{\Omega}] = 1$. Since the limiting process, if it exists, is nondecreasing and right-continuous, we set

$$H_t^\uparrow = \lim_{q\downarrow t, q\in\mathbb{Q}} H_q^\uparrow, \qquad t \in [0, T).$$

We prove that for $\omega \in \tilde{\Omega}$,

$$\tilde{H}_t^{n,\uparrow}(\omega) \longrightarrow H_t^\uparrow(\omega) \tag{A.2}$$

for each $t \in [0, T)$ that is a continuity point of the function $s \to H_s^\uparrow(\omega)$. Fix $\varepsilon > 0$. Using continuity at t of $H^\uparrow$, we find two rational numbers q_1, q_2 such that $q_1 < t < q_2$ and $H_{q_2}^\uparrow(\omega) - H_{q_1}^\uparrow(\omega) < \varepsilon$. From (A.1), there exists $N = N(\omega)$ such that

$$|\tilde{H}_{q_2}^{n,\uparrow}(\omega) - H_{q_2}^\uparrow(\omega)| < \varepsilon, \quad |\tilde{H}_{q_1}^{n,\uparrow}(\omega) - H_{q_1}^\uparrow(\omega)| < \varepsilon, \qquad \forall n \geq N.$$

We estimate, for all $n \geq N$,

$$|\tilde{H}_{q_2}^{n,\uparrow}(\omega) - \tilde{H}_{q_1}^{n,\uparrow}(\omega)| \leq |\tilde{H}_{q_2}^{n,\uparrow}(\omega) - H_{q_2}^\uparrow(\omega)| + |H_{q_2}^\uparrow(\omega) - H_{q_1}^\uparrow(\omega)|$$
$$+ |H_{q_1}^\uparrow(\omega) - \tilde{H}_{q_1}^{n,\uparrow}(\omega)|$$
$$\leq 3\varepsilon.$$

Therefore, using monotonicity of $\tilde{H}^{n,\uparrow}$, we obtain for all $n \geq N(\omega)$ that

$$|\tilde{H}_t^{n,\uparrow}(\omega) - H_t^\uparrow(\omega)| \leq |\tilde{H}_t^{n,\uparrow}(\omega) - \tilde{H}_{q_2}^{n,\uparrow}(\omega)| + |\tilde{H}_{q_2}^{n,\uparrow}(\omega) - H_{q_2}^\uparrow(\omega)|$$
$$+ |H_{q_2}^\uparrow(\omega) - H_t^\uparrow(\omega)|$$
$$\leq |\tilde{H}_{q_1}^{n,\uparrow}(\omega) - \tilde{H}_{q_2}^{n,\uparrow}(\omega)| + |\tilde{H}_{q_2}^{n,\uparrow}(\omega) - H_{q_2}^\uparrow(\omega)|$$
$$+ |H_{q_2}^\uparrow(\omega) - H_{q_1}^\uparrow(\omega)|$$
$$\leq 5\varepsilon.$$

Notice that (A.2) also holds for $t = T$. The same argument can be repeated for the sequence $\tilde{H}^{n,\downarrow}$, $n \in \mathbb{N}$, and also $H_0^n \to H_0$ can be guaranteed with some $H_0 \in \mathbb{R}$ by extracting a further subsequence. □

Remark A.2 The above proof shows that if $f_n \to f$ in $\mathcal{V}$ as $n \to \infty$, then $f_n(x)$ tends to $f(x)$ in every continuity point x of f. Consequently, for any continuous $g : [0, T] \to \mathbb{R}$, we have $\int_0^T g(t) \, df_n(t) \to \int_0^T g(t) \, df(t)$ as $n \to \infty$, where the integration is meant with respect to the measures induced by f_n, f.

As a consequence, for the sequence $(\tilde{H}^n)$ constructed in Lemma A.1 above, we have

$$W_t^{x,\text{liq}}(\theta, \tilde{H}^n)(\omega) \longrightarrow W_t^{x,\text{liq}}(\theta, H)(\omega)$$

and $V_t^x(\theta, \tilde{H}^n)(\omega) \to V_t^x(\theta, H)(\omega)$ as $n \to \infty$, almost surely, in $t = T$ and in every t which is a continuity point of both $H^\uparrow(\omega)$, $H^\downarrow(\omega)$, and in particular for Lebesgue-a.e. t. Fubini's theorem thus implies that there is a set Z of zero Lebesgue measure

(excluding T) such that for all $t \in [0, T] \setminus Z$, $W_t^{x,\mathrm{liq}}(\theta, \tilde{H}^n) \to W_t^{x,\mathrm{liq}}(\theta, H)$ and $V_t^x(\theta, \tilde{H}^n) \to V_t^x(\theta, H)$ hold P-almost surely.

A.2 Orlicz spaces

We call $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ a *Young function* if it is convex with $\Phi(0) = 0$ and $\lim_{x \to \infty} \Phi(x)/x = \infty$. The set

$$L^\Phi := \{X \in L^0 : E[\Phi(\gamma|X|)] < \infty \text{ for some } \gamma > 0\}$$

is a Banach space with the norm

$$\|X\|_\Phi := \inf\{\gamma > 0 : X \in \gamma B_\Phi\},$$

where $B_\Phi := \{X \in L^0 : E[\Phi(|X|)] \leq 1\}$ is the unit ball of L^Φ. Define the conjugate function $\Phi^*(y) := \sup_{x \geq 0}(xy - \Phi(x))$, $y \in \mathbb{R}_+$. This is also a Young function and $(\Phi^*)^* = \Phi$. We say that Φ is of class Δ_2 if

$$\limsup_{x \to \infty} \frac{\Phi(2x)}{\Phi(x)} < \infty.$$

We recall [13, Corollary 3.10], a compactness result which is used to handle the losses of trading strategies in this paper.

Lemma A.3 *Let Φ be a Young function of class Δ_2 and let ξ_n, $n \geq 1$, be a norm-bounded sequence in L^{Φ^*}. Then there are convex weights $\alpha_j^n \geq 0$, $n \leq j \leq M(n)$, with $\sum_{j=n}^{M(n)} \alpha_j^n = 1$ such that*

$$\xi_n' := \sum_{j=n}^{M(n)} \alpha_j^n \xi_j$$

converges almost surely to some $\xi \in L^{\Phi^*}$ as $n \to \infty$, and $\sup_n |\xi_n'|$ is in L^{Φ^*}.

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