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CORRECTION

Correction to: Yield curve shapes and the asymptotic short rate distribution in affine one-factor models

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I should like to thank Ralf Korn for alerting me to an error in the original paper [2]. The error concerns the threshold at which the yield curve in an affine short rate model changes from normal (strictly increasing) to humped (endowed with a single maximum). In particular, it is not true that this threshold is the same for the forward curve and for the yield curve, as claimed in [2]. Below, the correct mathematical expression for the threshold is given, supplemented with a self-contained and corrected proof.

1 Setting

In [2], affine short rate models for bond pricing were considered, i.e., models where the risk-neutral short rate process $r = (r_t)_{t\geq 0}$ is given by an affine process in the sense of [1]. The process r takes values in a state space D, which is either $[0, \infty)$ or $\mathbb{R}$. In this setting, the price at time t of a zero-coupon bond with time to maturity x, denoted by $P(t, t + x)$, is of the form

$$P(t, t + x) = \exp\big(A(x) + r_t B(x)\big),$$

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where A and B satisfy the generalized Riccati differential equations

$$\begin{aligned} \partial_x A(x) &= F\big(B(x)\big), & A(0) &= 0, \\ \partial_x B(x) &= R\big(B(x)\big) - 1, & B(0) &= 0. \end{aligned} \tag{1.1}$$

The functions F and R are of Lévy–Khintchine form and their parametrization is in one-to-one correspondence with the infinitesimal generator of r; cf. [2, Sect. 2]. Derived from the bond price are the *yield curve*

$$Y(x, r_t) := -\frac{\log P(t, t+x)}{x} = -\frac{A(x)}{x} - r_t \frac{B(x)}{x}$$

and the *forward curve*

$$f(x, r_t) := -\partial_x \log P(t, t+x) = -A'(x) - r_t B'(x).$$

The first objective of [2] was to derive the long-term yield and long-term forward rate. It was shown that the equation $R(c) = 1$ has at most a single negative solution c, and that under mild conditions,

$$b_{\text{asymp}} := \lim_{x\to\infty} Y(x, r_t) = \lim_{x\to\infty} f(x, r_t) = -F(c)$$

if such a solution exists; cf. [2, Theorem 3.7]. We remark that $\lambda := -\frac{1}{c} > 0$ was called *quasi-mean-reversion* of r in [2], with the convention that $\lambda = 0$ if no negative solution c exists. The second objective of [2] was to characterize all possible shapes of the yield and the forward curve. Recall that in common terminology, the yield or the forward curve is called

- *normal* if it is a strictly increasing function of x,
- *inverse* if it is a strictly decreasing function of x,
- *humped* if it has exactly one local maximum and no local minimum in $(0, \infty)$.

Finally, we recall the technical condition [2, Condition 3.1] in slightly rephrased form. The condition is necessary to guarantee finite bond prices when negative values of the short rate are allowed.

Condition 1.1 We assume that r is regular and conservative. If r has state space $D = \mathbb{R}$, which necessarily implies that R is of the linear form $R(x) = \beta x$ (cf. [1]), we require that

$$F(x) < \infty \qquad \text{for all } x \in \begin{cases} (1/\beta, 0], & \text{if } \beta < 0, \\ (-\infty, 0], & \text{else.} \end{cases}$$

2 Corrections to results

Theorem 3.1 in [2] should be replaced by the following corrected version.

Theorem 2.1 *Let the risk-neutral short rate be given by a one-dimensional affine process* $(r_t)_{t\geq 0}$ *satisfying Condition* 1.1 *and with quasi-mean-reversion* $-1/c = \lambda > 0$. *In addition, suppose that* $F \neq 0$ *and that at least one of* F *and* R *is nonlinear. Then the following hold*:

1. *The yield curve* $Y(\cdot, r_t)$ *can only be normal, inverse or humped.*
2. *Define*

$$b_{\text{y-norm}} := \frac{1}{c}\int_c^0 \frac{F(u)-F(c)}{R(u)-1}\,du,$$

$$b_{\text{inv}} := \begin{cases} -\frac{F'(0)}{R'(0)}, & \text{if } R'(0) < 0, \\ +\infty, & \text{if } R'(0) \geq 0. \end{cases}$$

   The yield curve is normal if $r_t \leq b_{\text{y-norm}}$, *humped if* $b_{\text{y-norm}} < r_t < b_{\text{inv}}$, *and inverse if* $r_t \geq b_{\text{inv}}$.

Remark 2.2 The correction only concerns the expression for $b_{\text{y-norm}}$, which was called b_{norm} in [2] and erroneously given as $b_{\text{norm}} = -F'(c)/R'(c)$. All other parts of the theorem are the same as in [2, Theorem 3.1].

Corollary 3.11 in [2] should be replaced by the following result.

Theorem 2.3 *Define* b_{inv} *as in Theorem* 2.1 *and set*

$$b_{\text{fw-norm}} := -\frac{F'(c)}{R'(c)}.$$

Under the conditions of Theorem 2.1, *the following hold*:

1. *The forward curve* $f(\cdot, r_t)$ *can only be normal, inverse or humped.*
2. *The forward curve is normal if* $r_t \leq b_{\text{fw-norm}}$, *humped if* $b_{\text{fw-norm}} < r_t < b_{\text{inv}}$, *and inverse if* $r_t \geq b_{\text{inv}}$.

Remark 2.4 We have intentionally renamed the result from corollary to theorem, since the correction changes the logical structure of the proof. Note that the above result is equivalent to [2, Corollary 3.11] up to the notational change from b_{norm} to $b_{\text{fw-norm}}$. Note that $b_{\text{y-norm}} \neq b_{\text{fw-norm}}$ in general, while in [2] it was erroneously claimed that $b_{\text{y-norm}} = b_{\text{fw-norm}}$.

Corollary 3.12 in [2] should be replaced by the following result.

Corollary 2.5 *Under the conditions of Theorem* 2.1, *it holds that*

$$b_{\text{fw-norm}} < b_{\text{y-norm}} < b_{\text{asymp}} < b_{\text{inv}}. \tag{2.1}$$

In addition, the state space D *of the short rate process satisfies*

$$D \cap (b_{\text{y-norm}}, b_{\text{inv}}) \neq \emptyset.$$

The error also affects [2, Fig. 1], where the expression for b_{norm} should be replaced by the correct value of $b_{\text{y-norm}}$. It also affects the application section [2, Sect. 4], where the values of b_{norm} and b_{inv} are calculated in different models. The corrections to [2, Sect. 4] are as follows.

In the *Vasiček model*, the short rate is given by

$$dr_t = -\lambda(r_t - \theta)\,dt + \sigma\,dW_t, \quad r_0 \in \mathbb{R},$$

with $\lambda, \theta, \sigma > 0$. This leads to the parametrization

$$F(u) = \lambda\theta u + \frac{\sigma^2}{2}u^2,$$
$$R(u) = -\lambda u.$$

By direct calculation, we obtain

$$b_{\text{y-norm}} = \theta - \frac{3\sigma^2}{4\lambda^2},$$
$$b_{\text{fw-norm}} = \theta - \frac{\sigma^2}{\lambda^2}.$$

Note that the value of $b_{\text{y-norm}}$ is now consistent with the results of [3, p. 186].

In the *Cox–Ingersoll–Ross model*, the short rate is given by

$$r_t = -a(r_t - \theta)\,dt + \sigma\sqrt{r_t}\,dW_t, \quad r_0 \in [0, \infty),$$

with $a, \theta, \sigma > 0$. This leads to the parametrization

$$F(u) = a\theta u,$$
$$R(u) = -\frac{\sigma^2}{2}u^2 - au.$$

By direct calculation, we obtain

$$b_{\text{y-norm}} = \frac{2a\theta}{\gamma - a}\log\frac{2\gamma}{a + \gamma},$$
$$b_{\text{fw-norm}} = \frac{a\theta}{\gamma},$$

where $\gamma := \sqrt{2\sigma^2 + a^2}$.

In the *gamma model*, the short rate is given by an Ornstein–Uhlenbeck-type process, driven by a compound Poisson process with intensity λk and exponentially distributed jump heights of mean $1/\theta$; see [2, Sect. 4.4] for details. In this model, we have

$$F(u) = \frac{\lambda\theta k u}{1 - \theta u}, \qquad R(u) = -\lambda u,$$

and by direct calculation, we obtain

$$b_{\text{y-norm}} = \frac{k\lambda}{1+\theta/\lambda}\log(1+\theta/\lambda),$$
$$b_{\text{fw-norm}} = \frac{k\theta}{(1+\theta/\lambda)^2}.$$

Since the resulting expressions are quite involved, we omit the calculations for the extended CIR model [2, Eq. (4.7)].

3 Corrected proofs

To prepare for the corrected proofs, we collect the following properties from [2, Sects. 2 and 3.1], which hold for the functions F, R, B and for the state space D under the assumptions of Theorem 2.1:

(P1) F is either strictly convex or linear; the same holds for R. Both functions are continuously differentiable on the interior of their effective domains.

(P2) The function B is strictly decreasing with limit $\lim_{x\to\infty} B(x) = c$.

(P3) $F(0) = R(0) = 0$ and $R'(c) < 0$. In addition, $F'(0) > 0$ if $D = [0,\infty)$.

(P4) Either
 (a) $D = [0,\infty)$, or
 (b) $D = \mathbb{R}$ and $R(u) = u/c$ with $c < 0$.

Note that Theorem 2.1 assumes that at least one of F and R is nonlinear. Together with (P1), this implies

(P1′) At least one of F and R is strictly convex.

In addition, we introduce the following terminology. Let $f:(0,\infty)\to\mathbb{R}$ be a continuous function. The *zero set* of f is $Z := \{x\in(0,\infty): f(x)=0\}$. The *sign sequence* of Z is the sequence of signs $\{+,-\}$ that f takes on the complement of Z, ordered by the natural order on $\mathbb{R}$. For example, the function x^2-1 on $(0,\infty)$ has the finite sign sequence $(-+)$; the function $\sin x$ has the infinite sign sequence $(+-+-\cdots)$. An obvious, but important property is the following: Let $g:(0,\infty)\to(0,\infty)$ be a *positive* continuous function. Then fg has the same zero set and the same sign sequence as f.

Proof of Theorem 2.3 From the Riccati equations (1.1), we can write the derivative of the forward curve as

$$\partial_x f(x,r_t) = -B'(x)\underbrace{\Big(F'\big(B(x)\big) + r_t R'\big(B(x)\big)\Big)}_{=:k(x)}. \tag{3.1}$$

Note that by (P2), the factor $-B'(x)$ is strictly positive, and hence $\partial_x f$ has the same sign sequence as k. We distinguish cases (a) and (b) as in (P4).

(a) Assume that $r_t \in D = [0, \infty)$. By (P2), $B(x)$ is strictly decreasing, and by (P1′), either F' or R' is strictly increasing. Thus if $r_t > 0$, it follows that $k(x)$ is a strictly decreasing function. If $r_t = 0$, then k is either strictly decreasing (if F' is strictly convex) or k is constant (if F is linear). By (P1), these are the only possibilities. In addition, the case $F = 0$ is ruled out by the assumptions.

(b) Assume that $r_t \in D = \mathbb{R}$. In this case, $R(u) = u/c$, and hence $R'(u) = 1/c$ is constant and F' is strictly increasing, by (P1′). We conclude that k is strictly decreasing.

In any case, k is either strictly decreasing or constant and non-zero. Thus the sign sequence of k can be completely characterized by its initial value $k(0)$ and its asymptotic limit as x tends to infinity. Let us first show that

$$k(0) \leq 0 \quad \Longleftrightarrow \quad r_t \geq b_{\text{inv}} = \begin{cases} -\frac{F'(0)}{R'(0)}, & \text{if } R'(0) < 0, \\ +\infty, & \text{if } R'(0) \geq 0. \end{cases} \tag{3.2}$$

Because we have $k(0) = F'(0) + r_t R'(0)$, the assertion follows immediately if $R'(0) < 0$. Consider the complementary case $R'(0) \geq 0$. This rules out case (b) in (P4), and hence we may assume that $D = [0, \infty)$. Since $F'(0) > 0$ by (P3), (3.2) follows. Next we show that

$$\lim_{x \to \infty} k(x) \geq 0 \quad \Longleftrightarrow \quad r_t \leq b_{\text{fw-norm}} = -\frac{F'(c)}{R'(c)}. \tag{3.3}$$

This follows immediately from $\lim_{x \to \infty} k(x) = F'(c) + r_t R'(c)$ and $R'(c) < 0$, by (P3). Combining (3.2) with (3.3) and using that k is either strictly decreasing or constant and non-zero, we obtain

$$\begin{aligned} r_t \geq b_{\text{inv}} \quad &\Longleftrightarrow \quad k \text{ has sign sequence } (-), \\ r_t \leq b_{\text{fw-norm}} \quad &\Longleftrightarrow \quad k \text{ has sign sequence } (+), \\ r_t \in (b_{\text{fw-norm}}, b_{\text{inv}}) \quad &\Longleftrightarrow \quad k \text{ has sign sequence } (+-). \end{aligned} \tag{3.4}$$

Since $\partial_x f$ has the same sign sequence as k, these statements can be directly translated into monotonicity properties of f. In the first case, the forward curve f is strictly decreasing, i.e., inverse; in the second case, it is strictly increasing, i.e., normal. In the third case, it is strictly increasing up to the unique zero of k and then strictly decreasing, i.e., humped. No other cases are possible. □

Proof of Theorem 2.1 From the Riccati equations (1.1), we can write the derivative of the yield curve as

$$\partial_x Y(x, r_t) = \frac{1}{x^2}\big(A(x) + r_t B(x)\big) - \frac{1}{x}\Big(F\big(B(x)\big) + r_t\Big(R\big(B(x)\big) - 1\Big)\Big).$$

Multiplying by the positive function x^2, we see that $\partial_x Y(x, r_t)$ has the same zero set and the same sign sequence as

$$M(x) := \Big(A(x) - xF\big(B(x)\big)\Big) + r_t\Big(B(x) - x\Big(R\big(B(x)\big) - 1\Big)\Big).$$

The derivative of M is given by

$$M'(x) := -xB'(x)\Big(F'\big(B(x)\big) + r_t R'\big(B(x)\big)\Big) = -xB'(x)k(x),$$

with k as in (3.1). Note that by (P2), the factor $-xB'(x)$ is strictly positive, and hence M' has the same sign sequence as k, which was already analyzed in (3.4). Since $M(0) = 0$, we can conclude that

$$\begin{aligned} r_t \geq b_{\text{inv}} \quad &\Longrightarrow \quad M \text{ has sign sequence } (-), \\ r_t \leq b_{\text{fw-norm}} \quad &\Longrightarrow \quad M \text{ has sign sequence } (+), \\ r_t \in (b_{\text{fw-norm}}, b_{\text{inv}}) \quad &\Longrightarrow \quad M \text{ has sign sequence } (+-) \text{ or } (+). \end{aligned} \tag{3.5}$$

Essentially, the mistake in [2] was to ignore the possible sign sequence $(+)$ **in the third case.** Not repeating the same mistake, we take a closer look at the third case and note that the sign sequence of M is $(+-)$ if and only if

$$\lim_{x\to\infty} M(x) < 0.$$

Decomposing $M(x) = L_1(x) + r_t L_2(x)$, it remains to study the asymptotic properties of L_1 and L_2. We have

$$\begin{aligned} L_1(x) = A(x) - xF\big(B(x)\big) &= \int_0^x \Big(F\big(B(s)\big) - F\big(B(x)\big)\Big)ds \\ &= \int_0^{B(x)} \frac{F(u) - F(B(x))}{R(u) - 1}du \quad \xrightarrow{x\to\infty} \quad \int_0^c \frac{F(u) - F(c)}{R(u) - 1}du. \end{aligned}$$

In addition,

$$\begin{aligned} L_2(x) = B(x) - x\Big(R\big(B(x)\big) - 1\Big) &= \int_0^x \Big(R\big(B(s)\big) - R\big(B(x)\big)\Big)ds \\ &= \int_0^{B(x)} \frac{R(u) - R(B(x))}{R(u) - 1}du \quad \xrightarrow{x\to\infty} \quad \int_0^c \frac{R(u) - 1}{R(u) - 1}du = c. \end{aligned}$$

Since $c < 0$, we conclude that

$$\lim_{x\to\infty} M(x) < 0 \quad \Longleftrightarrow \quad r_t > b_{\text{y-norm}} = \frac{1}{c}\int_c^0 \frac{F(u) - F(c)}{R(u) - 1}du.$$

By convexity of F and R and using that $c < 0$, we observe that

$$b_{\text{y-norm}} = \frac{1}{c}\int_c^0 \frac{F(u) - F(c)}{R(u) - 1}du \geq \frac{1}{c}\int_c^0 \frac{F'(c)}{R'(c)}du = -\frac{F'(c)}{R'(c)} = b_{\text{fw-norm}}.$$

Together with (3.5), this completes the proof. □

Proof of Corollary 2.5 Recall that $R(c) = 1$ and $c < 0$. By convexity of F and R, we have

$$\begin{aligned} F'(c) &\leq \frac{F(u) - F(c)}{u - c} \leq \frac{F(c)}{c} \leq F'(0), \\ R'(c) &\leq \frac{R(u) - 1}{u - c} \leq \frac{1}{c} \leq R'(0), \end{aligned} \tag{3.6}$$

for all $u \in (c, 0)$. Note that by (P1′), either F or R is strictly convex, so that strict inequalities must hold in either the first or the second line. If $R'(0) < 0$, then applying the strictly increasing transformation $x \mapsto -\frac{1}{x}$ to the second line in (3.6) and multiplying term by term with the first, we obtain

$$-\frac{F'(c)}{R'(c)} < -\frac{F(u) - F(c)}{R(u) - 1} < -\frac{F(c)}{c} < -\frac{F'(0)}{R'(0)}.$$

Applying the integral $\frac{1}{c} \int_0^c du$ to all terms, (2.1) follows. If $R'(0) \geq 0$, this approach is still valid for the first two inequalities in each line of (3.6), but not for the last one. However, in the case $R'(0) \geq 0$, we have set $b_{\text{inv}} = +\infty$ in (3.2), and the last inequality in (2.1) holds trivially. It remains to show that $D \cap (b_{\text{y-norm}}, b_{\text{inv}})$ is nonempty. F is a convex function and by Condition 1.1 finite at least on the interval $(c, 0)$. It follows that $F'(0) > -\infty$ and thus that $b_{\text{inv}} > -\infty$ in general. If $D = [0, \infty)$, then $F'(0) > 0$ by (P3) and hence $b_{\text{inv}} > 0$. Moreover $b_{\text{y-norm}} < b_{\text{asymp}} = -F(c) < \infty$, completing the proof. □

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