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Faces of simplices of invariant measures for actions of amenable groups

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Abstract We extend the result of Downarowicz (Israel J Math 165:189–210, 2008) to the case of amenable group actions, by showing that every face in the simplex of invariant measures on a zero-dimensional dynamical system with free action of an amenable group G can be modeled as the entire simplex of invariant measures on some other zero-dimensional dynamical system with free action of G. This is a continuation of our investigations from Frej and Huczek (Groups Geom Dyn 11:567–583, 2017), inspired by an earlier paper (Downarowicz in Israel J Math 156:93–110, 2006).

Keywords Invariant measure · Periodic measure · Choquet simplex · Amenable group · Group action · Symbolic system · Cantor space

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1 Introduction

Let X be a Cantor space i.e. a compact, metrizable, zero-dimensional perfect space, and let G be a countable amenable group acting on X via homeomorphisms φ_g, $g \in G$. Amenability of G means that there exists a sequence of finite sets $F_n \subset G$ (called a *Følner sequence*, or the sequence of Følner sets), such that for any $g \in G$ we have

$$\lim_{n\to\infty} \frac{|gF_n \triangle F_n|}{|F_n|} = 0,$$

where $gF = \{gf : f \in F\}$, $|\cdot|$ denotes the cardinality of a set, and $\triangle$ is the symmetric difference. The action of G is *free* if the equality $gx = x$ for any $g \in G$ and $x \in X$ implies that g is the neutral element of G. It is well known that one can represent the system (X, G) as an inverse limit $\varprojlim X_j \subset \prod_{j\in\mathbb{N}} X_j$ where each X_j is a group subshift on finitely many symbols i.e. a closed G-invariant subset of some ${\Lambda_j}^G$, $|\Lambda_j| < \infty$, with the action of G defined by $gx(h) = x(hg)$. Indeed, if $\mathcal{U} = \{U_j : j \in \mathbb{N}\}$ is a base for topology in X consisting of clopen sets then we define Λ_i to be the set of all elements of the cover of X by the sets of the form $V_1 \cap \cdots \cap V_i$, where either $V_j = U_j$ or $V_j = U_j^c$ for every $1 \le j \le i$. The space X_i is an image of X by the map $\pi_i : X \to \Lambda_i^G$ defined by the formula

$$\pi_i(x)(g) = \lambda_i \Leftrightarrow gx \in \lambda_i$$

The inverse system whose inverse limit is conjugate to (X, G) is then given by the sequence of the spaces X_i with bonding maps defined by coordinatewise inclusions. We will often refer to this inverse limit as a so called array system—an element of X in this interpretation is a map $x(\cdot,\cdot)$ on $G \times \mathbb{N}$, where $x(\cdot, j) \in X_j$. We will call such a map an array and from now on we will assume that our system is in array representation. By an (F, k)-*block* we mean a map $B\colon F \times [1, k] \to \bigcup_j \Lambda_j$, where F is a finite subset of G (which will occasionally be called the *shape* of a block), k is a positive integer and $[1, k]$ is an abbreviation for $\{1, \ldots, k\}$. If E is a subset of the domain of a block B then by $B[E]$ we will denote a restriction of B to E. By abuse of the notation, we will mean by $|B|$ the cardinality of the shape of B. We will use the same letter to denote both a block and a cylinder set induced by this block—the exact meaning is always clear from the context. A block B *occurs in* X if B is a restriction of some $x \in X$.

Let K be an abstract metrizable Choquet simplex, i.e. it is a compact convex set of a locally convex metric vector space, such that for each $v \in K$ there is a unique Borel probability measure supported on the set of extreme points of K with barycenter in v (see [9] for an exhaustive course on the theory of Choquet simplices). Following [3] we define:

Definition 1.1 1. An *assignment* on K is a function Φ defined on K such that for each $p \in K$, the value of $\Phi(p)$ is a measure-preserving group action $(X_p, \Sigma_p, \mu_p, G_p)$, where (X_p, Σ_p, μ_p) is a standard probability space.

2. Two assignments Φ on K and Φ' on K' are *equivalent* if there exists an affine homeomorphism $\pi : K \to K'$ such that $\Phi(p)$ and $\Phi'(\pi(p))$ are isomorphic for every $p \in K$.
3. If (X, G) is a continuous group action on a compact metric space X then the set of all G-invariant measures supported by X, endowed with the weak* topology of measures, is a Choquet simplex, and the assignment by identity $\Phi(\mu) = (X, Bor_X, \mu, G)$ (where Bor_X is the Borel sigma-field) is *the natural assignment* of (X, G).

By a *face* of a simplex S we mean a compact convex subset of S which is a simplex itself and whose extreme points are also the extreme points of S. If K is a face of a simplex $\mathcal{M}_G(X)$ of all G-invariant probability measures on X then by the *identity assignment* on K we mean the restriction of the natural assignment on $\mathcal{M}_G(X)$ to K.

In the current article we aim to prove the following:

Theorem 1.2 *Let X be a Cantor system with free action of an amenable group G and let K be a face in the simplex $\mathcal{M}_G(X)$ of G-invariant measures of X. There exists a Cantor system Y with free action of G, such that the natural assignment on Y is equivalent to the identity assignment on K.*

In case of actions of $\mathbb{Z}$ the theorem was proved in [3] (even with weaker assumptions; see Sect. 4) and the key tool used there was approximation of an arbitrary ergodic measure by a block (periodic) measure, i.e. a measure supported on a finite orbit. Density of periodic measures in the set of all invariant measures is usually a desired property and was proved to be true in various cases, e.g. for systems with specification property (see [1]). In case of a one-dimensional subshift one can construct a periodic measure by choosing a block B occurring in a system and uniformly distributing a probability mass on the orbit of a sequence obtained by periodic repetitions of B. Such a sequence need not be an element of a subshift (and the measure need not belong to its simplex of invariant measures), still it may give a useful approximation of a measure under consideration. For actions of groups other than $\mathbb{Z}$ (even $\mathbb{Z}^d$) this procedure usually cannot be performed, roughly saying, because of irregular shapes of blocks, and the notion of a block measure seems to be obscure. We devote the next section to implementing it in our setup, but before we proceed, we recall a few facts about Følner sequences.

In any amenable group there exists a Følner sequence with the following additional properties (see [6]):

1. $F_n \subset F_{n+1}$ for all n,
2. $e \in F_n$ for all n (e denotes the neutral element of G),
3. $\bigcup_{n\in\mathbb{N}} F_n = G$,
4. $F_n = F_n^{-1}$ for all n.

Following [10] we say that a Følner sequence F_n is *tempered* if for some $C > 0$ and all n,

$$\left|\bigcup_{k\leq n} F_k^{-1} F_{n+1}\right| \leq C|F_{n+1}|.$$

Proposition 1.3 ([8]) *Every Følner sequence F_n has a tempered subsequence.*

Standing assumption
Throughout this paper, we will assume that the Følner sequence which we use is tempered and has all the above properties.
We recall the pointwise ergodic theorem for amenable groups.

Theorem 1.4 ([8]) *Let G be an amenable group acting ergodically on a measure space (X, μ), and let F_n be a tempered Følner sequence. Then for any $f \in L^1(\mu)$,*

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{g\in F_n} f(gx) = \int f\, d\mu \qquad a.e.$$

If F and A are finite subsets of G and $0 < \delta < 1$, we say that F is (A, δ)*-invariant* if

$$\frac{|F \triangle AF|}{|F|} < \delta,$$

where $AF = \{af : a \in A, f \in F\}$. Clearly, if F is (A, δ)-invariant then it is also (A, δ')-invariant for all $\delta' > \delta$. Moreover, if F is simultaneously (A, δ)-invariant and (A', δ')-invariant then F is $(A \cup A', \delta + \delta')$-invariant. Observe that if A contains the neutral element of G, then (A, δ)-invariance is equivalent to the simpler condition

$$|AF| < (1 + \delta)\,|F|\,.$$

It is not hard to observe that if F is (A, δ)-invariant then

$$|\{f \in F : Af \cap F^c \neq \emptyset\}| < \delta|A||F| \tag{1.1}$$

and, equivalently,

$$|\{f \in F : Af \subset F\}| > |F|(1 - \delta|A|) \tag{1.2}$$

If (F_n) is a Følner sequence, then for every finite $A \subset G$ and every $\delta > 0$ there exists an N such that for $n > N$ the sets F_n are (A, δ)-invariant.

Definition 1.5 For $S \subset G$ and a finite, nonempty $F \subset G$ denote

$$\underline{D}_F(S) = \inf_{g\in G} \frac{|S \cap Fg|}{|F|}, \quad \overline{D}_F(S) = \sup_{g\in G} \frac{|S \cap Fg|}{|F|}.$$

If (F_n) is a Følner sequence then we define two values

$$\underline{D}(S) = \limsup_{n\to\infty} \underline{D}_{F_n}(S) \quad \text{and} \quad \overline{D}(S) = \liminf_{n\to\infty} \overline{D}_{F_n}(S),$$

which we call the *lower* and *upper Banach densities* of S, respectively.

Note that $\overline{D}(S) = 1 - \underline{D}(G \setminus S)$. We recall the following standard fact:

Lemma 1.6 *Regardless of the set S, the values of $\underline{D}(S)$ and $\overline{D}(S)$ do not depend on the Følner sequence, the limits superior and inferior in the definition are in fact limits, and moreover*

$$\underline{D}(S) = \sup\{\underline{D}_F(S) : F \subset G, F \text{ is finite}\} \quad \textit{and}$$
$$\overline{D}(S) = \inf\{\overline{D}_F(S) : F \subset G, F \text{ is finite}\} \geq \underline{D}(S).$$

For the proof see [5], Lemma 2.9.

Lemma 1.7 *Let (X, G) be a Cantor system in the array representation and let μ be an ergodic measure on X. Denote by e the neutral element of G. Let $\varphi : X \to X$, $\psi : X \to X$ be continuous maps which commute with the action of G.*

Then for μ-almost every x then

$$\mu(\{x \in X : \varphi(x)(e) \neq \psi(x)(e)\}) \leq \overline{D}(\{g \in G : \varphi(x)(g) \neq \psi(x)(g)\}).$$

Proof Denote

$$S(x) = \{g \in G : (\varphi(x))(g) \neq (\psi(x))(g)\}$$
$$B = \{x \in X : (\varphi(x))(e) \neq (\psi(x))(e)\}$$

Note that $(\varphi(x))(g) = \big(g\varphi(x)\big)(e) = \varphi(gx)(e)$ (and similarly for ψ), so $g \in S(x)$ is equivalent to $gx \in B$. Then,

$$\begin{aligned}\overline{D}_{F_n}(S(x)) &= \sup_{g \in G} \frac{|S(x) \cap F_n g|}{|F_n|}\\ &= \sup_{g \in G} \frac{1}{|F_n|} \sum_{h \in F_n g} \mathbb{1}_{S(x)}(h) \geq \frac{1}{|F_n|} \sum_{h \in F_n} \mathbb{1}_{S(x)}(h)\\ &= \frac{1}{|F_n|} \sum_{h \in F_n} \mathbb{1}_B(hx).\end{aligned}$$

Taking the lower limits we obtain by Theorem 1.4 that $\overline{D}(S(x)) \geq \mu(B)$.

2 Block measures

We will explicitly define a metric consistent with the weak* topology on the set of probability measures on X, represented as an array system. First, let $\mathcal{B}_k$ be the family of all blocks with domain $F_k \times [1, k]$, occurring in X, and let

$$d_k(\mu, \nu) = \frac{1}{|\mathcal{B}_k|} \sum_{B \in \mathcal{B}_k} |\mu(B) - \nu(B)|.$$

Now let

$$d(\mu, \nu) = \sum_{k=1}^{\infty} \frac{1}{2^k} d_k(\mu, \nu).$$

Note that we may assume that $\mathcal{B}_k$ consists only of blocks which yield cylinders of positive measure for some ergodic measure μ.

Lemma 2.1 *Let X be an array system and let (F_n) be a Følner sequence. For every $t \in \mathbb{N}$ there exists $\varepsilon_t > 0$ such that if μ and ν are ergodic measures and x, y satisfy:*

1. $\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{g\in F_n} \mathbb{1}_B(gx) = \mu(B)$ *for all blocks on $F_j \times [1, j]$, where $j \leq t+1$,*
2. $\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{g\in F_n} \mathbb{1}_B(gy) = \nu(B)$ *for all blocks on $F_j \times [1, j]$, where $j \leq t+1$,*
3. $\overline{D}(\{g \in G : x(g) \neq y(g)\}) < \varepsilon_t$,

then $d(\mu, \nu) < \frac{1}{2^t}$.

Proof We put $\varepsilon_t = \frac{1}{2^{t+2}(1+|F_{t+1}|)}$. Let n be large enough to ensure that

1. $\left|\frac{1}{|F_n|} \sum_{g\in F_n} \mathbb{1}_B(gx) - \mu(B)\right| < \varepsilon_t$ for all blocks B on $F_j \times [1, j]$, where $j \leq t+1$,
2. $\left|\frac{1}{|F_n|} \sum_{g\in F_n} \mathbb{1}_B(gy) - \nu(B)\right| < \varepsilon_t$ for all blocks B on $F_j \times [1, j]$, where $j \leq t+1$,
3. $|\{g \in G : x(g) \neq y(g)\}| < 2\varepsilon_t |F_n|$.

Then, for B as above

$$\begin{aligned}
|\mu(B) - \nu(B)| &\leq \left|\mu(B) - \frac{1}{|F_n|} \sum_{g\in F_n} \mathbb{1}_B(gx)\right| \\
&\quad + \frac{1}{|F_n|} \sum_{g\in F_n} \left|\mathbb{1}_B(gx) - \mathbb{1}_B(gy)\right| + \left|\frac{1}{|F_n|} \sum_{g\in F_n} \mathbb{1}_B(gy) - \nu(B)\right| \\
&< 2\varepsilon_t + \frac{1}{|F_n|} |\{g \in G : x(g) \neq y(g)\}| \cdot |F_{t+1}| \\
&< 2\varepsilon_t(1 + |F_{t+1}|) = \frac{1}{2^{t+1}}.
\end{aligned}$$

Thus, for $k \leq t+1$ we have

$$d_k(\mu, \nu) = \frac{1}{|\mathcal{B}_k|} \sum_{B\in\mathcal{B}_k} |\mu(B) - \nu(B)| < \frac{1}{2^{t+1}}$$

and

$$d(\mu, \nu) \le \sum_{k=1}^{t+1} \frac{1}{2^k} d_k(\mu, \nu) + \frac{1}{2^{t+1}} < \frac{1}{2^t}.$$

□

For the sake of convenience, we introduce a notion of "distance" between a block and a measure. Let B be a block occurring in X, with domain $F \times [1, k]$ for some $F \subset G$ and $k \in \mathbb{N}$. For any block C with domain $F_j \times [1, j]$, where $j \le k$, we can define the frequency of C in B in the following way: let

$$N_F(F_j) = \left|\left\{g \in F : F_j g \subset F\right\}\right|$$
$$N_B(C) = \left|\left\{g \in F : F_j g \subset F \text{ and } B[F_j g \times [1, j]] = C\right\}\right|$$

(by the equality $B[F_j g \times [1, j]] = C$ we understand that $B(fg, i) = C(f, i)$ for all $f \in F_j$ and $i \in [1, j]$).

If $N_F(F_j) > 0$ let

$$\mathsf{fr}_B(C) = \frac{N_B(C)}{N_F(F_j)}.$$

Otherwise let $\mathsf{fr}_B(C) = 0$.

We say that A is a $(1-\delta)$-*subset of* F if $A \subset F$ and $|A| \ge (1-\delta)|F|$. By a standard argument we can draw from the pointwise ergodic Theorem 1.4 the following corollary.

Lemma 2.2 *Let μ be an ergodic measure on X. For every ε and j we can find n and η such that if F is a $(1-\eta)$-subset of F_m, $m \ge n$, then for some block C with domain $F \times [1, m]$ occurring in X we have $|\mathsf{fr}_C(D) - \mu(D)| < \varepsilon$ for every block D with domain $F_i \times [1, i]$, $i = 1, \ldots, j$.*

Moreover, n and η can be chosen so that the union of all blocks C not satisfying the approximation rule has measure smaller than ε.

Proof Fix $\varepsilon > 0$ and $j \in \mathbb{N}$. Taking in Theorem 1.4 $f = \mathbb{1}_D$, where D is any fixed block with domain $F_i \times [1, i]$, $i = 1, \ldots, j$, we obtain that

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{g \in F_n} \mathbb{1}_D(gx) = \mu(D)$$

for almost every x. Since there are only finitely many blocks with such domain, we may assume that the above equality is satisfied simultaneously for all such blocks D on a subset of X having measure 1. Hence we can find n such that for every $m \ge n$ the inequality

$$\left|\frac{1}{|F_m|} \sum_{g \in F_m} \mathbb{1}_D(gx) - \mu(D)\right| < \frac{\varepsilon}{2} \tag{2.1}$$

holds, for all such D, on a set of measure X_ε at least $1-\varepsilon$. Pick x from this set. Additionally, increasing n we may demand that each F_m, $m \geq n$, is $(F_j, \frac{\varepsilon}{4|F_j|})$-invariant.

Fix $m \geq n$ and let C be a block which appears in x on the domain F_m. By (1.2),

$$|F_m| \geq N_{F_m}(F_i) > |F_m|\left(1 - \frac{\varepsilon}{4|F_j|}|F_i|\right) \geq |F_m|\left(1 - \frac{\varepsilon}{4}\right).$$

Furthermore, using (1.1)

$$\begin{aligned} N_C(D) &= \sum_{\{g\in F_m : F_i g \subset F_m\}} \mathbb{1}_D(gx) \\ &\leq \sum_{g\in F_m} \mathbb{1}_D(gx) \leq N_C(D) + |\{g \in F_m : F_i g \cap F_m^c \neq \emptyset\}| \\ &\leq N_C(D) + \frac{\varepsilon}{4|F_j|}|F_i||F_m| \leq N_C(D) + \frac{\varepsilon}{4}|F_m|. \end{aligned}$$

Hence,

$$\frac{N_C(D)}{N_{F_m}(F_i)}(1 - \frac{\varepsilon}{4}) \leq \frac{1}{|F_m|}\sum_{g\in F_m} \mathbb{1}_D(gx) \leq \frac{N_C(D) + \frac{\varepsilon}{4}|F_m|}{|F_m|}$$

implying that

$$\mathsf{fr}_C(D) - \frac{\varepsilon}{4} \leq \frac{1}{|F_m|}\sum_{g\in F_m} \mathbb{1}_D(gx) \leq \mathsf{fr}_C(D) + \frac{\varepsilon}{4}.$$

Combining it with (2.1) we obtain $|\mathsf{fr}_C(D) - \mu(D)| < \frac{3\varepsilon}{4}$ for blocks C having domain exactly equal to $F_m \times [1, m]$.

Now suppose that F is a $(1-\eta)$-subset of F_m and that C is a block which appears in x on the domain F, while C' is x restricted to F_m.

$$\frac{N_{C'}(D) - \eta|F_m|}{N_{F_m}(F_i)} \leq \frac{N_C(D)}{N_F(F_i)} \leq \frac{N_{C'}(D)}{N_{F_m}(F_i) - \eta|F_m|}$$

The left hand is greater than

$$\frac{N_{C'}(D) - \eta\frac{N_{F_m}(F_i)}{1-\varepsilon/4}}{N_{F_m}(F_i)} \geq \mathsf{fr}_{C'}(D) - \frac{\eta}{1-\varepsilon/4}$$

while the right hand is bounded from above by

$$\frac{N_{C'}(D)}{N_{F_m}(F_i) - \eta\frac{N_{F_m}(F_i)}{1-\varepsilon/4}} \leq \mathsf{fr}_{C'}(D) + \frac{\eta}{1-\varepsilon/4-\eta}.$$

Hence

$$|\mathsf{fr}_{C'}(D) - \mathsf{fr}_C(D)| < \frac{\eta}{1 - \varepsilon/4 - \eta}$$

which can be made smaller than $\frac{\varepsilon}{4}$ by choosing apprioprately small η, irrespective of m, C and D.

Finally, note that if $|\mathsf{fr}_C(D) - \mu(D)| < \varepsilon$ is not true for some D as above and $C = x[F]$, F being a $(1 - \eta$-subset of F_m, $m \geq n$, then x is not in X_ε, as defined by 2.1, which means that the union of all C for which the approximation fails is less than ε. □

We can now define the distance between a block and a measure: let

$$d_k(B, \nu) = \frac{1}{|\mathcal{B}_k|} \sum_{D \in \mathcal{B}_k} |\mathsf{fr}_B(D) - \nu(D)|,$$

and let

$$d(B, \nu) = \sum_{k=1}^{\infty} \frac{1}{2^k} d_k(B, \nu).$$

Remark 2.3 Let ε be a positive number. To ensure that $d(B, \nu) < \varepsilon$ it is enough to verify that if j satisfies $\sum_{k=j+1}^{\infty} \frac{1}{2^k} < \frac{\varepsilon}{2}$ then for any block $D \in \mathcal{B}_i$, where $i = 1, \ldots, j$,

$$|\mathsf{fr}_B(D) - \nu(D)| < \frac{\varepsilon}{2j}.$$

Indeed, in this case we have $d_k(B, \nu) < \frac{\varepsilon}{2j}$ and

$$\begin{aligned} d(B, \nu) &= \sum_{k=1}^{j} \frac{1}{2^k} d_k(B, \nu) + \sum_{k=j+1}^{\infty} \frac{1}{2^k} d_k(B, \nu) \\ &< \sum_{k=1}^{j} \frac{\varepsilon}{2j} + \sum_{k=j+1}^{\infty} \frac{1}{2^k} < \varepsilon. \end{aligned}$$

Lemma 2.4 *For any $\varepsilon > 0$ and any positive integer j there exists δ such that if F is an (F_j, δ)-invariant set and B is a block with domain $F \times [1, j]$, then there exists a probability measure μ_B such that*

$$|\mathsf{fr}_B(D) - \mu_B(D)| < \frac{\varepsilon}{2j}$$

for any block $D \in \mathcal{B}_i$, where $i = 1, \ldots, j$.

Consequently, for any $\varepsilon > 0$ and sufficiently large j there exists δ such that if F is an (F_j, δ)-invariant set and B is a block with domain $F \times [1, j]$, then there exists a probability measure μ_B such that $d(B, \mu_B) < \varepsilon$.

Proof Let us first observe that the second assertion follows from the first by Remark 2.3. Therefore, it suffices to prove the first statement.

Let $\Delta_j = \Lambda_1 \times \cdots \times \Lambda_j$. The full shift Δ_j^G is a Cantor set on which we have the uniform Bernoulli probability measure λ which assigns equal measures M_j to all cylinders with domain $F_j \times [1, j]$. We shall define μ_B by specifying its density f_B with respect to λ. f_B will be constant on cylinders with domain $F_j \times [1, j]$: on each such cylinder associated with a block C let $f_B(x) = \frac{1}{M_j}\mathsf{fr}_B(C)$. Obviously $d_j(\mu_B, B) = 0$.

We will now estimate $d_i(\mu_B, B)$ for $i < j$. Let D be any block from $\mathcal{B}_i$. Let $\mathcal{C}_j$ be the family of (distinct) blocks from $\mathcal{B}_j$ such that $D = \bigcup_{C \in \mathcal{C}_j} C$. We have:

$$\mu_B(D) = \sum_{C \in \mathcal{C}_j} \mu_B(C) = \sum_{C \in \mathcal{C}_j} \mathsf{fr}_B(C),$$

and we need to show that the latter quantity is close to $\mathsf{fr}_B(D)$. Since F is (F_j, δ)-invariant, the set $\left\{g : F_j g \subset F\right\}$ is a $(1 - \delta \left|F_j\right|)$-subset of F by (1.2) (note that $\left\{g : F_j g \subset F\right\}$ is indeed a subset of F, because F_j contains the neutral element). Consequently, the set $\{g : F_i g \subset F\}$ also is a $(1 - \delta \left|F_j\right|)$-subset of F, being a superset of the former. For $C \in \mathcal{C}_j$, let F_C be the set of g such that $F_j g \subset F$, $B[F_j g \times [1, j]] = C$ (and automatically $B[F_i g \times [1, i]] = D$). That way we can represent $\{g : F_i g \subset F, B[F_i g \times [1, i]] = D\}$ as the following disjoint sum:

$$\begin{aligned} &\{g : F_i g \subset F,\ B[F_i g \times [1, i]] = D\} \\ &\quad = \bigcup_{C \in \mathcal{C}_j} F_C \cup \left\{g : F_i g \subset F,\ F_j g \cap F^c \neq \emptyset,\ B[F_i g \times [1, i]] = D\right\}. \end{aligned}$$

Taking cardinalities and dividing by $N_F(F_i)$, we obtain

$$\begin{aligned} \mathsf{fr}_B(D) = &\sum_{C \in \mathcal{C}_j} \frac{N_F(F_j)}{N_F(F_i)} \mathsf{fr}_B(C) \\ &+ \frac{1}{N_F(F_i)} \left|\left\{g : F_i g \subset F,\ F_j g \cap F^c \neq \emptyset,\ B[F_i g \times [1, i]] = D\right\}\right| \end{aligned}$$

Since both F_j and F_i are $(1 - \delta \left|F_j\right|)$-subsets of F, we have $\frac{N_F(F_j)}{N_F(F_i)} \geq 1 - \delta \left|F_j\right|$ and

$$\begin{aligned} &\frac{1}{N_F(F_i)} \left|\left\{g : F_i g \subset F, F_j g \cap F^c \neq \emptyset,\ B[F_i g \times [1, i]] = D\right\}\right| \\ &\quad \leq \frac{1}{N_F(F_i)} \left|\left\{g \in F : F_j g \cap F^c \neq \emptyset\right\}\right| \\ &\quad \leq \frac{\delta \left|F_j\right| |F|}{(1 - \delta \left|F_j\right|)|F|} = \frac{\delta \left|F_j\right|}{1 - \delta \left|F_j\right|}. \end{aligned}$$

If δ is small enough then the expression can be arbitrarily close to 0, while $\frac{N_F(F_j)}{N_F(F_i)}$ can be arbitrarily close to 1, so we can assume that

$$|\mathsf{fr}_B(D) - \mu_B(D)| = \left|\mathsf{fr}_B(D) - \sum_{C \in \mathcal{C}_j} \mathsf{fr}_B(C_j)\right| < \frac{\varepsilon}{2j}.$$

□

Note that in the above lemma δ may be as small as we want.

Corollary 2.5 *Let X be a zero-dimensional dynamical system with the action of an amenable group G and let μ be an ergodic measure on X. For any $\varepsilon > 0$ and any sufficiently large j there exists $\delta > 0$ such that if F is a $(1-\delta)$-subset of F_m, where m is so large that F is (F_j, δ)-invariant, then there is a block C occurring in X, whose domain is F such that the measure μ_C (as defined in* Lemma 2.4*) satisfies $d(\mu_C, \mu) < \varepsilon$.*

Proof Choose δ and j from Lemma 2.4 with $\frac{\varepsilon}{2}$ replacing ε. Assume also that $\sum_{k=j+1}^{\infty} \frac{1}{2^k} < \frac{\varepsilon}{4}$. By Lemma 2.2, we can find a block C on a (F_j, δ)-invariant domain F, such that $|\mathsf{fr}_C(D) - \mu(D)| < \frac{\varepsilon}{4j}$ for any block D with domain $F_i \times [1, i]$, $i = 1, \dots, j$. By Remark 2.3 we see that $d(C, \mu) < \frac{\varepsilon}{2}$, and directly from Lemma 2.4 also $d(C, \mu_C) < \frac{\varepsilon}{2}$, therefore $d(\mu_C, \mu) < \varepsilon$. □

For actions of $\mathbb{Z}$, it is a well-known fact that if a sufficiently long block C is a concatenation of shorter blocks $B_1, B_2, \dots, B_n$ of equal length, then the probability measure μ_C (which for actions of $\mathbb{Z}$ can easily be assumed to be shift invariant) can be arbitrarily close to the arithmetic average of the measures μ_{B_i}. An analogous claim can be made for the action of any amenable group G; however the lack of a natural way to decompose a subset of G into smaller sets requires the use of quasitilings.

Definition 2.6 A (*static*) *quasitaling* of a group G is a family $\mathcal{T}$ of finite subsets of G (called *tiles*), for which there exist a family $\mathcal{S}(\mathcal{T}) = \{S_1, S_2, \dots, S_n\}$ of finite subsets of G (called *shapes*) and a family $\mathcal{C}(\mathcal{T}) = \{C_1, C_2, \dots, C_n\}$ of subsets of G (called *centers*), such that every $T \in \mathcal{T}$ has a unique representation $T = S_i c$ for some $i \in \{1, \dots, n\}$ and $c \in C_i$.

Note that every quasitiling can be seen as a symbolic element $\mathcal{T} \in \{0, 1, \dots, n\}^G$, such that $\mathcal{T}(g) = i$ if $g \in C_i$ for some i, and $\mathcal{T}(g) = 0$ otherwise.

Definition 2.7 A quasitiling $\mathcal{T}$ is:

1. *disjoint*, if the tiles are pairwise disjoint;
2. *α-covering*, if the union of all tiles has lower Banach density at least α.
3. *congruent* with a quasitiling $\mathcal{T}'$, if for any two tiles $T \in \mathcal{T}, T' \in \mathcal{T}'$ we have either $T \supset T'$ or $T \cap T' = \emptyset$.

Any (T, k)-block whose shape T belongs to a quasitiling $\mathcal{T}$ will be called a $(\mathcal{T}, k)$-*block*.

Let (X, G) be a topological dynamical system. Suppose we assign to every $x \in X$ a quasitiling $\mathcal{T}(x)$ of G, with the same set of shapes $S_1, \dots, S_n$ for all x. This induces a map $x \mapsto \mathcal{T}(x)$ which can be seen as a map from (X, G) into $\{0, 1, \dots, n\}^G$ with the shift action. If such a map is a factor map (i.e. if it is continuous and commutes with the dynamics), we call it a *dynamical quasitiling*. A dynamical quastiling is said to be disjoint and/or α-covering, if $\mathcal{T}(x)$ has the respective property for every x. Note that though the set of shapes is common, the collection of centers depends on x so $C_i(x)$ become functions assigning to each x a subset of G. We introduce the following new definition.

Definition 2.8 We will say that a dynamical quasitiling $\mathcal{T}$ consisting of shapes $\mathcal{S}(\mathcal{T}) = \{S_1, S_2, \dots, S_n\}$ and centers $\mathcal{C}(\mathcal{T}) = \{C_1, C_2, \dots, C_n\}$ has *restricted block distribution* if for every $x \in X$ any (T, k)-block B which occurs in x on some domain $S_i c$ may occur in x only on domains of this form for $c \in C_i(x)$.

Remark 2.9 Clearly, if $\mathcal{T}$ has restricted block distribution then for any block D occuring in x on some domain $S_i c_0$ and any B being a block occuring in x on a (disjoint) union F of tiles we have

$$\mathsf{fr}_B(D) = \frac{1}{N_F(S)} |\{c \in C_i(x) \cap F : x(S_i c) = D\}|.$$

Lemma 2.10 *For any $\varepsilon > 0$ there exist $j \in \mathbb{N}$ and $\delta > 0$ such that if $\mathcal{T}$ is a disjoint quasitiling by (F_j, δ)-invariant sets, and C is a block with domain $H \times [1, j]$ such that some disjoint union of tiles $T_1, T_2, \dots, T_n$ of $\mathcal{T}$ is a $(1-\delta)$-subset of H, then the probability measure μ_C is ε-close to the average of the measures associated with blocks over individual tiles, i.e. if we denote by B_i the block with domain $T_i \times [1, j]$,*

$$d\left(\mu_C, \frac{1}{\sum_{i=1}^n |T_i|} \sum_{i=1}^n |T_i| \, \mu_{B_i}\right) < \varepsilon.$$

Proof Applying Lemma 2.4, for any j there is δ_j such that for any block B with domain $F \times [1, j]$, where F is a (F_j, δ_j)-invariant set, and for any block D with domain $F_i \times [1, i]$, $i \leq j$, we have $|\mathsf{fr}_B(D) - \mu_B(D)| < \frac{\varepsilon}{8j}$. Let H be a subset of G and let $\mathcal{T}$ be a quasitiling of G by (F_j, δ)-invariant sets for some $\delta > 0$. Suppose that the union $\bigcup_{i=1}^n T_i$ is a $(1-\delta)$-subset of H for some pairwise disjoint tiles $T_1, T_2, \dots, T_n$ belonging to $\mathcal{T}$. For every $k \leq j$ let us define the set

$$E_k = \left\{h \in H : \forall i \;\; F_k h \cap T_i^c \neq \emptyset\right\}$$

Then

$$\begin{aligned} |E_k| &\leq \sum_{i=1}^n \left| \left\{h \in T_i : \forall i \;\; F_k h \cap T_i^c \neq \emptyset\right\} \right| + |H \setminus \bigcup_{i=1}^n T_i| \\ &\leq \sum_{i=1}^n \delta |F_k||T_i| + \delta|H| \leq \delta|H|(|F_k| + 1), \end{aligned}$$

hence $N_H(F_k) \geq |H| - |E_k| \geq |H|(1 - \delta(1 + |F_k|))$. Clearly, we can demand that $\delta < \delta_j$ (further restrictions will follow). Note that since each $T \in \mathcal{T}$ is (F_j, δ)-invariant, for any block B whose domain is a tile of $\mathcal{T}$ the measure μ_B is well-defined.

Now, let C be a block with domain $H \times [1, j]$, where H is $(1-\delta)$-tiled by $T_1, \dots, T_n$, and let $C[T_i] = B_i$. For any $k \leq j$ and for any block D with domain $F_k \times [1, k]$ we have:

$$N_C(D) = \sum_{i=1}^{n} N_{B_i}(D) + N_{E_k}(D)$$

Therefore, using the traingle inequality,

$$\begin{aligned}
&\left|\mathsf{fr}_C(D) - \frac{1}{\sum_{i=1}^n |T_i|} \sum_{i=1}^{n} \mathsf{fr}_{B_i}(D)\, |T_i|\right| \\
&= \left|\frac{\sum_{i=1}^n N_{B_i}(D) + N_{E_k}(D)}{N_H(F_k)} - \frac{1}{\sum_{i=1}^n |T_i|} \sum_{i=1}^{n} \mathsf{fr}_{B_i}(D)\, |T_i|\right| \\
&\leq \frac{N_{E_k}(D)}{N_H(F_k)} + \sum_{i=1}^{n} N_{B_i}(D) \cdot \left|\frac{1}{N_H(F_k)} - \frac{1}{\sum_{i=1}^n |T_i|}\right| \\
&\quad + \frac{1}{\sum_{i=1}^n |T_i|} \cdot \sum_{i=1}^{n} \left(N_{B_i}(D) \left|1 - \frac{|T_i|}{N_{T_i}(F_k)}\right|\right)
\end{aligned}$$

We can estimate that $\frac{N_{E_k}(D)}{N_H(F_k)} < \delta(|F_k| + 1)$, $\left|1 - \frac{|T_i|}{N_{T_i}(F_k)}\right| \leq \frac{\delta|F_k|}{1-\delta|F_k|}$ and $\left|\frac{1}{N_H(F_k)} - \frac{1}{\sum_{i=1}^n |T_i|}\right| \leq \frac{1}{N_H(F_k)} \frac{\delta(|F_k|+2)}{1-\delta}$, so the whole expression can be made smaller than $\frac{\varepsilon}{8j}$ by appropriate choice of a small δ.

Now, for every B_i we have $\mathsf{fr}_{B_i}(D)$ is approximately equal to $\mu_{B_i}(D)$ with error $\frac{\varepsilon}{8j}$, and this approximation is preserved by the weighted average we have obtained, therefore

$$\left|\mathsf{fr}_C(D) - \frac{1}{\sum_{i=1}^n |T_i|} \sum_{i=1}^{n} |T_i|\, \mu_{B_i}(D)\right| < \frac{\varepsilon}{4j}$$

If j is sufficiently large, Remark 2.3 implies that

$$d\left(C, \frac{1}{\sum_{i=1}^n |T_i|} \sum_{i=1}^{n} |T_i|\, \mu_{B_i}\right) < \frac{\varepsilon}{2},$$

and since $d(C, \mu_C) < \frac{\varepsilon}{2}$, we also have

$$d\left(\mu_C, \frac{1}{\sum_{i=1}^n |T_i|} \sum_{i=1}^{n} |T_i|\, \mu_{B_i}\right) < \varepsilon.$$

□

Remark 2.11 In the above lemma, we can increase j and decrease δ without spoiling the approximation error ε, because if $j' \geq j$ and $\delta' \leq \delta$ then $(F_{j'}, \delta')$-invariant set is also (F_j, δ)-invariant and a $(1-\delta')$-subset of any H is a $(1-\delta)$-subset of H.

The next two lemmas concerning the existence of quasitilings were proved in [4].

Lemma 2.12 ([4], Corollary 3.5) *Let G be an amenable group acting freely on a zero-dimensional metric space X. For any $\varepsilon > 0$, any finite $K \subset G$ and any $\delta > 0$ there exists a disjoint, $(1-\varepsilon)$-covering dynamical quasitiling $\mathcal{T}$ such that every shape of $\mathcal{T}$ is (K, δ)-invariant.*

Lemma 2.13 ([4], Lemma 3.6) *Let G be an amenable group acting freely on a zero-dimensional metric space X and let $\mathcal{T}_0$ be any disjoint, dynamical quasitiling of G. For any $\varepsilon > 0$, any finite $K \subset G$ and any $\delta > 0$ there exists a disjoint, $(1-\varepsilon)$-covering dynamical quasitiling $\mathcal{T}_1$ such that every shape of $\mathcal{T}_1$ is (K, δ)-invariant, and every tile of $\mathcal{T}_0$ is either a subset of some tile of $\mathcal{T}_1$ or is disjoint from all such tiles.*

The following lemma is analogous to the case of classical one-dimensional subshifts.

Lemma 2.14 *For every $\varepsilon > 0$ there exist J and δ such that if for some $j > J$ the set F is a (F_j, δ)-invariant and B is a (F, j)-block, then $d(B, \mathcal{M}_G(X)) < \varepsilon$ and $d(\mu_B, \mathcal{M}_G(X)) < \varepsilon$.*

Proof We will prove the assertion in the language of blocks, i.e. we will show that $d(B, \mathcal{M}_G(X)) < \varepsilon$. The assertion for measures will follow from Corollary 2.5.

Suppose that there is $\varepsilon > 0$ such that for every J and δ there exists an integer $j > J$ and a (F, j)-block $B_{J,\delta}$ on domain $D_{J,\delta} \times [1, j]$, which is (F_j, δ)-invariant and $d(B_{J,\delta}, \mathcal{M}_G(X)) \geq \varepsilon$. For $\frac{\varepsilon}{2}$ and every J we choose $\delta = \delta_J$ via Lemma 2.4 with the additional requirement that $\delta_J |F_j| < \frac{\varepsilon}{J}$. We denote $B_J = B_{J,\delta_J}$ and $D_J = D_{J,\delta_J}$. Let μ be a limit point of the sequence μ_{B_J}. Clearly, $d(\mu, \mathcal{M}_G(X)) \geq \varepsilon/2$.

We will obtain a contradiction by showing that μ is G-invariant. It suffices to prove that $\mu(C) = \mu(g(C))$ for every (F_i, i)-block C and every $g \in G$, where

$$g(C) = \{gx : x[F_i \times [1, i]] = C\} = \{x : x[F_i g^{-1} \times [1, i]] = C\}$$

Fix $\gamma > 0$, $g \in G$ and a cylinder set C on $F_i \times [1, i]$. Let J be large enough to ensure that

1. $|\mu(C) - \mu_{B_J}(C)| + |\mu(g(C)) - \mu_{B_J}(g(C))| < \frac{\gamma}{3}$ (note that C is clopen),
2. $\frac{\varepsilon}{J} < \frac{\gamma}{6}$,
3. The domain D_J is (F_j, δ_J)-invariant, $j \geq J$, where $F_j \supset F_i \cup F_i g^{-1} \cup \{g\}$.

Then,

$$\begin{aligned} |\mu(C) - \mu(gC)| \leq{} & |\mu(C) - \mu_{B_J}(C)| + |\mu_{B_J}(C) - \mathsf{fr}_{B_J}(C)| \\ & + |\mathsf{fr}_{B_J}(C) - \mathsf{fr}_{B_J}(g(C))| \\ & + |\mathsf{fr}_{B_J}(g(C)) - \mu_{B_J}(g(C))| + |\mu_{B_J}(g(C)) - \mu(g(C))| \end{aligned}$$

The first condition guarantees that the sum of the first and the last terms are less than $\frac{\gamma}{3}$. The choice of B_J was made with use of Lemma 2.4, so both the second and the fourth summands are smaller than $\frac{\varepsilon}{4j}$. By the second requirement above, their sum is again less than $\frac{\gamma}{3}$. We only need to show that the middle term is bounded by $\frac{\gamma}{3}$.

Since D_J is (F_j, δ_J)-invariant, it is also (F_i, δ_J)-invariant and $(F_i g^{-1}, \delta_J)$-invariant, which implies that

$$(1-\delta_J|F_i|)|D_J| \leq N_{D_J}(F_i) \leq |D_J|$$
$$(1-\delta_J|F_i|)|D_J| \leq N_{D_J}(F_i g^{-1}) \leq |D_J|.$$

In particular,

$$|N_{D_J}(F_i g^{-1}) - N_{D_J}(F_i)| \leq \delta_J |F_i||D_J|.$$

Note also that $B_J[F_i h \times [1, i]] = C$ if and only if $B_J[(F_i g^{-1})(gh) \times [1, i]] = C$. Thus, an occurrence of C in B_J 'at position h' yields the occurrence of $g(C)$ in B_J 'at gh', if only $gh \in D_J$. By a similar argument, occurrences of $g(C)$ force occurrences of C. The number of pairs (h, gh) such that only one of these elements belongs to D_J is smaller than $\delta_J|F_j||D_J|$, so

$$|N_{B_J}(C) - N_{B_J}(g(C))| < \delta_J|F_j||D_J| \leq \delta_J|F_j|\frac{N_{D_J}(F_i)}{1-\delta_J|F_i|}$$

Hence

$$|\mathsf{fr}_{B_J}(C) - \frac{N_{B_J}(g(C))}{N_{D_J}(F_i)}| \leq \frac{\varepsilon/J}{1-\varepsilon/J} < \frac{\gamma}{6}$$

On the other hand,

$$\begin{aligned}|\frac{N_{B_J}(g(C))}{N_{D_J}(F_i)} - \mathsf{fr}_{B_J}(g(C))| &\leq N_{B_J}(g(C)) \cdot \frac{|N_{D_J}(F_i g^{-1}) - N_{D_J}(F_i)|}{N_{D_J}(F_i)N_{D_J}(F_i g^{-1})} \\ &\leq \frac{\delta_J|F_i|}{1-\delta|F_i|} \leq \frac{\varepsilon/J}{1-\varepsilon/J} < \frac{\gamma}{6},\end{aligned}$$

which ends the proof. □

Finally, we prove our last tool.

Lemma 2.15 *Let X be a zero-dimensional dynamical system (in array form) with the shift action of an amenable group G, and let K be a face in the simplex $\mathcal{M}_G(X)$. For any $\delta > 0$ and $\varepsilon > 0$ there exist η and j such that if $\mathcal{T}$ is a disjoint, $(1-\eta)$-covering dynamical quasitiling by (F_j, η)-invariant sets, which has restricted block distribution and $\mathcal{B}$ denotes the family of all $(\mathcal{T}, j)$-blocks B such that $d(B, K) > \delta$, then $\sum_{B\in\mathcal{B}} \mu(B)\,|B| \leq \varepsilon$ for every $\mu \in K$ (by convention, the sum over the empty set is equal to zero).*

Proof Fix $\delta > 0$ and $\varepsilon > 0$. Let $\mathfrak{F}$ denote the (closed) complement of the open $\delta/2$-ball around K in $\mathcal{M}(X)$ (note that we use here the space of all probability measures, not the space of invariant measures). Obviously, $\{\mu_B : B \in \mathcal{B}\} \subset \mathfrak{F}$ for sufficiently large j. For every α, consider the set $V_\alpha \subset \mathcal{M}(X)$ consisting of measures μ with the following property: if $\mu = \int_{\mathcal{M}(X)} \nu d\xi$, and ξ is supported by the closed α-neighborhood of $\mathcal{M}_G(X)$, then $\xi(\mathfrak{F}) < \varepsilon$.

Claim 1 *V_α is an open set.*
We will prove that its complement V_α^c is closed. Let μ be the weak* limit of a sequence μ_k of elements of V_α^c. Then $\mu_k = \int_{\mathcal{M}(X)} \nu d\xi_k$ for some ξ_k supported by the closed α-neighborhood of $\mathcal{M}_G(X)$ with $\xi_k(\mathfrak{F}) \geq \varepsilon$. The sequence ξ_k has a subsequence which converges in the weak* topology to some measure ξ—let us assume that ξ_k itself is already convergent. By the portmanteau lemma, $\xi(\mathfrak{F}) \geq \lim_{k\to\infty} \xi_k(\mathfrak{F}) \geq \varepsilon$. By the same lemma, ξ assigns to the closed α-neighborhood of $\mathcal{M}_G(X)$ the value 1, so it is supported by this neighborhood.

The only thing left to show is the equality $\mu = \int_{\mathcal{M}(X)} \nu d\xi$. For any function $f \in C(X)$ the map $\nu \mapsto \nu(f) = \int f d\nu$ is a real continuous map of $\mathcal{M}(X)$. Therefore, by the definition of weak* convergence (used both in spaces $\mathcal{M}(X)$ and $\mathcal{M}(\mathcal{M}(X))$),

$$\mu(f) = \lim_{k\to\infty} \mu_k(f) = \lim_{k\to\infty} \int_{\mathcal{M}(X)} \nu(f) d\xi_k = \int_{\mathcal{M}(X)} \nu(f) d\xi,$$

which is the desired equality.

Claim 2 *If α is small enough then V_α contains K.*
If not then letting α tend to 0, we could find a measure in K that is a barycenter of a distribution ξ on $\mathcal{M}_G(X)$ with $\xi(\mathfrak{F}) \geq \varepsilon$. This is not possible.

Returning to the main proof, let γ be small enough that the open γ-neighborhood of K is contained in V_α. Using Lemma 2.10 choose η and j to obtain the the error of approximation equal $\gamma/2$ for any (F_j, η)-quasitiling (i.e. in the lemma $\gamma/2$ and η play the role of ε and δ, respectively). Let $\mathcal{T}$ be such a quasitiling. By Lemma 2.14 and Remark 2.11, making j large enough, we can also assume that every block with domain $S \times [1, j]$ (where S is a shape of $\mathcal{T}$) that occurs in X lies in the α-neighborhood of the set of invariant measures on X. Note that the union $\bigcup \mathcal{B}$ of the collection of all elements of $\mathcal{B}$ (as defined in the statement of the lemma) is clopen, and thus the function $\mu \mapsto \mu(\bigcup \mathcal{B})$ is continuous on the set $\mathcal{M}(X)$. Suppose that μ is an ergodic measure in K such that $\sum_{B\in\mathcal{B}} \mu(B)\,|B| > \varepsilon$. The function $\nu \mapsto \sum_{B\in\mathcal{B}} \nu(B)\,|B|$ is continuous, therefore if ν is close enough to μ, then $\sum_{B\in\mathcal{B}} \nu(B)\,|B| > \varepsilon$. In particular, by Corollary 2.5 we can find a block C occurring in X, such that $d(\mu_C, \mu) < \frac{\gamma}{2}$, and $\sum_{B\in\mathcal{B}} \mu_C(B)\,|B| > \varepsilon$. By Lemma 2.4 we can demand that $\mathsf{fr}_C B$ approximates each $B \in \mathcal{B}$ so well that also $\sum_{B\in\mathcal{B}} \mathsf{fr}_C(B)\,|B| > \varepsilon$. Note that by the restricted block distribution, for elements B of the tiling $\mathsf{fr}_C(B)$ is derived by calculating only the appropriate elements of the tiling of C. We can also assume that the union of tiles of $\mathcal{T}$ contained in the domain of C is a $(1-\eta)$-subset of C. By Lemma 2.10, μ_C is closer than $\frac{\gamma}{2}$ to $\nu = \frac{1}{\sum_{i=1}^n |B_i|} \sum_{i=1}^n |B_i|\,\mu_{B_i}$, where $B_1, \ldots, B_n$ are all $(\mathcal{T}, j)$-blocks occurring in C as elements of the tiling. For all i such that $B_i \in \mathcal{B}$, we have $\delta_{\mu_{B_i}}(\mathfrak{F}) = 1$, so for $\xi = \frac{1}{\sum_{i=1}^{n}|B_i|} \sum_{i=1}^{n} |B_i| \, \delta_{\mu_{B_i}}$ we have

$$\xi(\mathfrak{F}) \geq \frac{1}{\sum_{i=1}^{n} |B_i|} \sum_{i=1}^{n} |B_i| \, \mathbb{1}_{\mathcal{B}}(B_i) \geq \sum_{B \in \mathcal{B}} |B| \, \mathsf{fr}_C(B) > \varepsilon.$$

Since ν is in V_α (where such a decomposition should not exist), this is a contradiction. ◻

3 Proof of the main result

Proof (Proof of Theorem 1.2) Let K be a face in $\mathcal{M}_G(X)$. Recall that X is represented as an array system, i.e. it is a subset of $Z = \prod_{j \in \mathbb{N}} {\Lambda_j}^G$, where $|\Lambda_j| < \infty$. For every $t \in \mathbb{N}$ we choose ε_t so that the sequence is summable and that $2\varepsilon_t$ satisfies the hypotheses of Lemma 2.1.

By Lemma 2.12, we can construct a sequence $\mathcal{T}_t$ of disjoint, $(1 - \frac{1}{t})$-covering, dynamical tilings of X, whose shapes are all $(F_t, \frac{1}{t})$-invariant subsets of G. Moreover, by Lemma 2.13 we can assume that every tile of $\mathcal{T}_{t-1}$ is either a subset of some tile of $\mathcal{T}_t$ or is entirely disjoint from all such tiles.

Having fixed the sequence $\mathcal{T}_t$ we slightly change the array representation of X. We extend each alphabet Λ_j to a new alphabet Λ_j^*, doubling the number of symbols by adding to it for every symbol λ a copy of it with superscript $*$, namely λ^*. For every x we then add the stars to symbols $x(g, t)$ where $g \in \mathcal{C}(\mathcal{T}_t(x))$. Thereby, we have defined a conjugate representation X^* of X; we denote the conjugacy by ψ and the corresponding map on the space of measures by $\Psi : \mathcal{M}(X) \to \mathcal{M}(X^*)$, $\Psi(\mu) = \mu \circ \psi^{-1}$ (the map ψ is continuous, because the quasitilings are dynamical; it is invertible, because removing stars we obtain the original system). The set $K^* = \Psi(K)$ is a face of the simplex $\mathcal{M}_G(X^*)$, affinely homeomorphic to K. Each quasitiling $\mathcal{T}_t$ is carried to a dynamical quasitiling $\mathcal{T}_t^*$ of X^* by the rule $\mathcal{T}_t^*(\psi(x)) = \mathcal{T}_t(x)$. Moreover, it gains the property of restricted block distribution, because symbols marked with stars prevent blocks occurring on tiles Sc from occuring on positions not consistent with the tiling.

We will construct a sequence of maps $\phi_t : X^* \to Z^*$, $Z = \prod_{j \in \mathbb{N}} (\Lambda_j^*)^G$, which are all going to be invertible continuous maps commuting with the action of a group. Then we will prove that the sequence of maps $\Phi_t : \mathcal{M}(X^*) \to \mathcal{M}(Z^*)$, $\Phi_t(\mu) = \mu \circ \phi_t^{-1}$, converges uniformly on K^* to an affine homeomorphism Φ, while ϕ_t converge pointwise on a set of full measure to a map establishing an isomorphism between (X^*, μ) and some $(Y, \Phi(\mu))$ for each $\mu \in K^*$ [hence also between $(X, \Psi^{-1}(\mu))$ and $(Y, \Phi(\mu))$].

The construction will be inductive: let ϕ_0 be the identity map. Now, supposing we have constructed a map ϕ_{t-1}, let $X_{t-1} = \phi_{t-1}(X^*)$ (in particular, X_{t-1} is conjugate to X). Please, note that we may treat any tiling $\mathcal{T}_t^*$ as a tiling of X_{t-1}—we will use the same symbol to denote $\mathcal{T}_t^*$ transported to X_{t-1} by ϕ_{t-1}. By Lemma 2.14 there exist

J_t and $\delta_t > 0$ such that if F is a (F_k, δ_t)-invariant, $k \geq J_t$, and B is a (F, k)-block occurring in X_{t-1}, then the distance between μ_B and $\mathcal{M}_G(X_{t-1})$ is less than ε_t.

For sufficiently large n_t we can pick $k > J_t$ such that:

1. The tiling $\mathcal{T}^*_{n_t}$ consists of tiles whose shapes are (F_k, δ_t)-invariant,
2. For every shape S of $\mathcal{T}^*_{n_t}$ there exists a block B_S with domain S, such that μ_{B_S} is closer than ε_t to some $\mu \in \Phi_{t-1}(K^*)$ (by Corollary 2.5),
3. If $\mathcal{B}$ denotes the family of all $(\mathcal{T}^*_{n_t}, k)$-blocks B such that $d(B, \Phi_{t-1}(K^*)) > \delta_t$, then $\sum_{B\in\mathcal{B}} \mu(B)\,|B| < \varepsilon_t$ for every $\mu \in \Phi_{t-1}(K^*)$ (by Lemma 2.15).

We shall define an auxiliary map $\tilde{\phi}_t$ on X_{t-1} as follows: for any $x \in X_{t-1}$ and any tile T of $\mathcal{T}^*_{n_t}(x)$, let S be the shape of T and let $B = x[T \times [1, k]]$. If the distance between B and $\phi_{t-1}(K^*)$ is more than δ_t, replace $x[T \times [1, k]]$ with B_S. Otherwise, $\tilde{\phi}_t$ introduces no changes. By doing this for all $T \in \mathcal{T}_{n_t}(x)$, we obtain a new array, $\tilde{\phi}_t(x)$.

Observe that if x is in the support of any measure $\mu \in \Phi_{t-1}(K^*)$, then (by the third assumption) the union of tiles $T \in \mathcal{T}_{n_t}(x)$ such that $x[T \times [1, k]]$ is a block distant by more than δ_t from $\Phi_{t-1}(K^*)$ has upper Banach density less than, say, $2\varepsilon_t$, therefore $\tilde{\phi}_t(x)$ differs from x on a set of coordinates of density less than $2\varepsilon_t$. By Lemma 1.7 this means that the set of points $x \in X_{t-1}$ such that $\tilde{\phi}_t(x)$ differs from x in column e also has measure μ less than $2\varepsilon_t$ for any ergodic $\mu \in \Phi_{t-1}(K^*)$. The map Φ_{t-1} is affine so it takes ergodic measures to ergodic measures and for every $\mu \in K^*$ the ergodic decomposition of any $\Phi_{t-1}(\mu) \in \Phi_{t-1}(K^*)$ is induced by the ergodic decomposition of μ. Thus this measure is less than $2\varepsilon_t$ for any $\mu \in \Phi_{t-1}(K^*)$.

Now let $\phi_t = \tilde{\phi}_t \circ \phi_{t-1}$. Since ϕ_t makes no changes in rows with indices k and greater (and they allow us to determine the content of rows 0 through k), it is a conjugacy. Furthermore, let $X_t = \phi_t(X^*)$ and let ν be an ergodic measure in $\mathcal{M}_G(X_t) = \Phi_t(\mathcal{M}_G(X))$. By Corollary 2.5 for sufficiently large n there is $x \in X_t$ such that $\mu_{x[C]}$, $C = F_n \times [1, k]$, is ε_t-close to ν. By the construction of ϕ_t, every $(\mathcal{T}_{n_t}, k)$-block in x is closer than ε_t to some $\mu \in \Phi_{t-1}(K^*)$. If F_n is a set sufficiently far in the Følner sequence, then $x[C]$ is a block that is close to being a concatenation of $\mathcal{T}_{n_t}$ blocks (the union of tiles of $\mathcal{T}_{n_t}$ contained in F_n is a $(1-\delta_t)$-subset of F_n). Therefore, by Lemma 2.10 the measure μ_C differs by less than ε_t from $\frac{1}{\sum_{i=1}^n |B_i|} \sum_{i=1}^n |B_i|\,\mu_{x[B_i]}$. Since each $x[B_i]$ is ε_t-close to $\mu_{x[B_i]}$ the combination is $4\varepsilon_t$-close to measure in $\Phi_{t-1}(K^*)$.

We will show that the maps Φ_t converge uniformly on K^*. To this end, it suffices to uniformly estimate the distance between $\Phi_t(\mu)$ and $\Phi_{t-1}(\mu)$ for ergodic $\mu \in K^*$ by a summable sequence. By Lemma 2.15, for any $\mu \in K^*$ we have the estimate $\sum_{B\in\mathcal{B}} (\Phi_{t-1}(\mu))\,(B)\,|B| < \varepsilon_t$, where $\mathcal{B}$ denotes the family of all $\mathcal{T}^*_{n_t}$-blocks B such that $d(B, K^*) > \delta_t$. As we have already said, this implies that if $x \in X^*$, then the set of coordinates in $\phi_{t-1}(x)$ belonging to tiles of $\mathcal{T}^*_{n_t}$ that are domains of blocks from $\mathcal{B}$ has upper Banach density less than $2\varepsilon_t$. Since $\tilde{\phi}_t$ only makes any changes on these coordinates, $\phi_t(x)$ differs from $\phi_{t-1}(x)$ on a set of density less than $2\varepsilon_t$. If x is in the support of some invariant measure μ, then $\phi_{t-1}(x)$ and $\phi_t(x)$ are in the support of $\Phi_{t-1}(\mu)$ and $\Phi_t(\mu)$, respectively, and since the two points agree on a set of large upper Banach density, the measures are within distance less than $\frac{1}{2^t}$ (according to the choice of ε_t with use of Lemma 2.1).

This uniform convergence, together with the fact that $\Phi_t(\mathcal{M}_G(X^*))$ is within the $2\varepsilon_t$-neighborhood of $\Phi_{t-1}(K^*)$, implies that $\Phi(\mathcal{M}_G(X^*)) \subset \Phi(K^*)$, and since the other inclusion is obvious, the two sets are equal.

Now, define the set Y (which will support the desired assignment) as follows:

$$Y = \bigcap_{s=1}^{\infty} \overline{\bigcup_{t=s}^{\infty} X_t}.$$

Observe that Y is a closed, shift-invariant set, and that for any Følner set F and any $k \in \mathbb{N}$ every block with domain $F \times [1, k]$ in Y occurs in infinitely many of the sets X_t. It follows that every invariant measure on Y can be approximated by invariant measures on the X_t's, and thus the set of invariant measures on Y is contained in $\Phi(\mathcal{M}_G(X^*)) = \Phi(K^*)$. The other inclusion is generally true: for any weakly* convergent sequence of measures μ_t supported by X_t, the limit measure μ is always supported by $\bigcap_{s=1}^{\infty} \overline{\bigcup_{t=s}^{\infty} X_t}$. Therefore $\mathcal{M}_G(Y) = \Phi(K^*)$.

By Lemma 1.7 for every ergodic $\mu \in K^*$ the set of points $x \in X_{t-1}$ such that the column $x(e)$ is modified by $\tilde{\phi}_t$ has measure μ less than $2\varepsilon_t$, because $\tilde{\phi}_t$ commutes with the shift map and for any x in the support of μ the set of modified coordinates has upper Banach density less than $2\varepsilon_t$. If this bound works for all ergodic measures it works for all measures in K^*. Since the sequence ε_t is summable, the Borel-Cantelli lemma implies that for almost every $x \in X^*$ the columns $\phi_t(x)(e)$ are all equal from some point onwards. By shift-invariance, the same is true for $\phi_t(x)(g)$ for any g, so ultimately we conclude that if $\mu \in K^*$, then for μ-almost every $x \in X^*$ every coordinate of x is only changed finitely many times. This means that a limit point $\phi(x)$ is then well-defined, and this map ϕ is invertible (since every $\phi_t(x)$ retains the original contents of x in the bottom row). In other words ϕ is an isomorphism between the measure-theoretic dynamical systems (X^*, μ) and $(Y, \Phi(\mu))$. □

4 Concluding remarks

Firstly, we note that we can strengthen Theorem 1.2 by combining it with theorem 1.2 of [7], obtaining the following version:

Theorem 4.1 *Let X be a Cantor system with free action of an amenable group G and let K be a face in the simplex $\mathcal{M}_G(X)$ of G-invariant measures of X. There exists a Cantor system Y with* minimal *free action of G, such that the natural assignment on Y is equivalent to the identity assignment on K.*

Secondly, note that the result of this paper is not strictly a strengthening of the main theorem 4.1 in [3], since while we gain the result for actions of amenable groups, we add the requirement that the action be free, whereas the original result merely requires that the face in question contain no periodic measures. Unfortunately, it is very much unclear how the machinery used to deal with periodic points would transfer to the group case, which is why the matter of directly extending the result of [3] remains open.

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