



A rigorous derivation of mean-field models describing 2D micro phase separation

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Received: 9 July 2019 / Accepted: 31 December 2019 / Published online: 15 February 2020
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Abstract

We study the free boundary problem describing the micro phase separation of diblock copolymer melts in the regime that one component has small volume fraction ρ such that the micro phase separation results in an ensemble of small disks of one component. We consider the two dimensional case in this paper, whereas the three dimensional case was already considered in Niethammer and Oshita (Calc Var PDE 39:273–305, 2010). Starting from the free boundary problem restricted to disks we rigorously derive the heterogeneous mean-field equations on a time scale of the order of $\mathcal{R}^3 \ln(1/\rho)$, where \mathcal{R} is the mean radius of disks. On this time scale, the evolution is dominated by coarsening and stabilization of the radii of the disks, whereas migration of disks becomes only relevant on a larger time scale.

Mathematics Subject Classification 35B27 · 35R35 · 82D60

1 Introduction

Diblock copolymer molecules consist of subchains of two different type of monomers, say A - and B -monomers. The different type of subchains tend to segregate, and hence the phase separation take place. However since the subchains are chemically bonded, the two subchains mix on a macroscopic scale, while on a molecular scale, A - and B -subchains still segregate and the micro-domains are formed. This is called micro phase separation. For more physical background on this phenomenon we refer to [2,8].

In the strong segregation regime, energetically favorable configurations have been characterized in the Ohta–Kawasaki theory [18] by minimizers of an energy functional, which is in the two dimensional case of the form

Communicated by J. Ball.

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$$E(\Omega) = \mathcal{H}^1(\partial\Omega) + \frac{\sigma}{2} \int_{(0,L)^2} |(-\Delta)^{-1/2}(\chi - \rho)|^2 dx. \tag{1}$$

Here $(0, L)^2 \subset \mathbb{R}^2$ is the domain covered by the copolymers, $\Omega \subset [0, L]^2$ denotes the region covered by, say, A -monomers, $\rho = \frac{|\Omega|}{L^2} \in (0, 1)$ the average density, $\sigma \in \mathbb{R}_+ = (0, \infty)$ is a parameter related to the polymerization index, χ is the characteristic function of Ω , and \mathcal{H}^1 denotes one dimensional Hausdorff measure.

The first term in the energy prefers large blocks of monomers, the second favors a very fine mixture. Competition between these terms leads to minimizers of E which represent micro phase separation.

Starting with the pioneering work [15], where the Ohta–Kawasaki theory is formulated on a bounded domain as a singularly perturbed problem and the limiting sharp interface problem is identified, there has been a large body of analytical work. Minimizers of the energy functionals have been characterized in [1,3,4,20], the existence/stability of stationary solutions has been investigated in [16,17,19,21] and a time dependent model has been considered in [7,9]. The mean field models in the three dimensional case have been derived in [6,10,12].

We consider the gradient flow of the energy, which is a standard way to set up a model for the evolution of the copolymer configuration that decreases energy and preserves the volume fraction. Then the evolution equation becomes the following extension of the Mullins–Sekerka evolution for phase separation in binary alloys [11]. The normal velocity v of the interface $\partial\Omega = \partial\Omega(t)$ satisfies

$$v = [\nabla w \cdot \vec{n}] \quad \text{on } \partial\Omega, \tag{2}$$

where $[\nabla w \cdot \vec{n}]$ denotes the jump of the normal component of the gradient of the potential across the interface. Here \vec{n} denotes the outer normal to Ω and

$$[f] = \lim_{x \notin \Omega, x \rightarrow \partial\Omega} f(x) - \lim_{x \in \Omega, x \rightarrow \partial\Omega} f(x).$$

The potential w is for each time determined via

$$-\Delta w = 0 \quad \text{in } (0, L)^2 \setminus \partial\Omega, \tag{3}$$

$$w = \kappa + \sigma(-\Delta)^{-1}(\chi - \rho) \quad \text{on } \partial\Omega, \tag{4}$$

where κ is the curvature of $\partial\Omega$. We are interested in the case that the volume of $\Omega(t)$ is preserved in time and can thus impose Neumann or periodic boundary conditions for w on $\partial(0, L)^2$. In what follows we will consider a periodic setting and hence always require that the potential w is $(0, L)^2$ -periodic. For local well-posedness of this evolution see [5].

The evolution defined by (2)–(4) has a formal interpretation as a gradient flow of the energy (1) on a Riemannian manifold. Indeed, consider the manifold of subsets of the 2-dimensional flat torus \mathbb{T} of length L with fixed volume, that is $\mathcal{M} := \{\Omega \subset \mathbb{T}; |\Omega| = L^2\rho\}$, whose tangent space $T_\Omega\mathcal{M}$ at an element $\Omega \in \mathcal{M}$ is described by all kinematically admissible normal velocities of $\partial\Omega$, that is,

$$T_\Omega\mathcal{M} = \left\{ v: \partial\Omega \rightarrow \mathbb{R}; \int_{\partial\Omega} v dS = 0 \right\}.$$

The Riemannian structure is given by the following metric tensor on the tangent space:

$$g_\Omega(v^1, v^2) = \int_{\mathbb{T}} \nabla w^1 \cdot \nabla w^2 dx, \tag{5}$$

where $w^\alpha : \mathbb{T} \rightarrow \mathbb{R}$ ($\alpha = 1, 2$) solves

$$\begin{aligned} -\Delta w^\alpha &= 0 && \text{in } \mathbb{T} \setminus \partial\Omega, \\ [\nabla w^\alpha \cdot \vec{n}] &= v^\alpha && \text{on } \partial\Omega \end{aligned} \tag{6}$$

for $v^\alpha \in T_{\Omega(t)}\mathcal{M}$ ($\alpha = 1, 2$). The gradient flow of the energy (1) is now the dynamical system where at each time the velocity is the element of the tangent space in the direction of steepest descent of the energy. In other words, v is such that

$$g_{\Omega(t)}(v, \tilde{v}) = -\langle DE(\Omega(t)), \tilde{v} \rangle \tag{7}$$

for all $\tilde{v} \in T_{\Omega(t)}\mathcal{M}$. Choosing $\tilde{v} = v$ we immediately obtain the energy estimate associated with each gradient flow, which is

$$\int_0^T g_{\Omega(t)}(v, v) dt + E(\Omega(T)) = E(\Omega(0)) \quad \text{for all } T > 0. \tag{8}$$

In what follows we consider the micro phase separation in the two dimensional case in the regime where the fraction of A -monomers is much smaller than the one of B -monomers. In this case A -phase consists of an ensemble of many small approximately circular particles. We reduce the evolution to the gradient flow on circular particles.

For that purpose we define the submanifold $\mathcal{N} \subset \mathcal{M}$ of all sets Ω which are the union of disjoint balls $\Omega = \bigcup_i B_{R_i}(X_i)$, where the centers $\{X_i\}_i$ and the radii $\{R_i\}_i$ are variables. Hence \mathcal{N} can be identified with an open subspace of the hypersurface $\{\mathbf{Y} = \{R_i, X_i\}_i; (R_i, X_i) \in \mathbb{R}_+ \times \mathbb{T}, \pi \sum_i R_i^2 = L^2 \rho\}$ in \mathbb{R}^{3N} , where N is the number and $i = 1, \dots, N$ an enumeration of the particles with centers in the torus \mathbb{T} . Since the normal velocity v satisfies $v = \frac{dR_i}{dt} + \frac{dX_i}{dt} \cdot \vec{n}$ on $\partial B_{R_i}(X_i)$, the tangent space can be identified with the hyperplane

$$T_{\mathbf{Y}}\mathcal{N} = \left\{ \mathbf{Z} = \sum_i \left(V_i \frac{\partial}{\partial R_i} + \xi_i \cdot \frac{\partial}{\partial X_i} \right); (V_i, \xi_i) \in \mathbb{R} \times \mathbb{R}^2, \sum_i R_i V_i = 0 \right\} \subset \mathbb{R}^{3N},$$

such that V_i describes the rate of change of the radius of particle i and ξ_i the rate of change of its center. We use the abbreviation $\mathbf{Z} = \{V_i, \xi_i\}_i$ for $\mathbf{Z} = \sum_i (V_i \frac{\partial}{\partial R_i} + \xi_i \cdot \frac{\partial}{\partial X_i})$.

The metric tensor is then given by

$$g_{\mathbf{Y}}(\mathbf{Z}^1, \mathbf{Z}^2) = \int_{\mathbb{T}} \nabla w^1 \cdot \nabla w^2 dx,$$

where the function $w^\alpha : \mathbb{T} \rightarrow \mathbb{R}$ solves

$$\begin{aligned} -\Delta w^\alpha &= 0 && \text{in } \mathbb{T} \setminus \cup_i \partial B_{R_i}(X_i), \\ [\nabla w^\alpha \cdot \vec{n}] &= V_i^\alpha + \xi_i^\alpha \cdot \vec{n} && \text{on } \partial B_{R_i}(X_i). \end{aligned} \tag{9}$$

for $\mathbf{Z}^\alpha = \{V_i^\alpha, \xi_i^\alpha\}_i \in T_{\mathbf{Y}}\mathcal{N}$, $\alpha = 1, 2$. For the following it will be convenient to split the metric tensor into the radial and shift part respectively. For any $\{V_i, \xi_i\}_i$, we write

$$w = u + \phi$$

where u and ϕ are harmonic in- and outside the particles and where

$$\begin{aligned} [\nabla u \cdot \vec{n}] &= V_i && \text{on } \partial B_{R_i}(X_i), \\ [\nabla \phi \cdot \vec{n}] &= \xi_i \cdot \vec{n} && \text{on } \partial B_{R_i}(X_i). \end{aligned} \tag{10}$$

We consider the energy $E(\mathbf{Y}) = E_{\text{surf}}(\mathbf{Y}) + \sigma E_{\text{nl}}(\mathbf{Y})$, where

$$E_{\text{surf}}(\mathbf{Y}) = 2\pi \sum_i R_i \quad \text{and} \quad E_{\text{nl}}(\mathbf{Y}) = \int_{\mathbb{T}} |\nabla \mu|^2 dx$$

with $\mu : \mathbb{T} \rightarrow \mathbb{R}$ solving $-\Delta \mu = \chi_{\cup B_{R_i}} - \rho$. We obtain the differentials of the energies in the direction of a tangent vector $\tilde{\mathbf{Z}} = \{\tilde{V}_i, \tilde{\xi}_i\}_i$ as

$$\langle DE_{\text{surf}}(\mathbf{Y}), \tilde{\mathbf{Z}} \rangle = 2\pi \sum_i \tilde{V}_i$$

and

$$\langle DE_{\text{nl}}(\mathbf{Y}), \tilde{\mathbf{Z}} \rangle = -2 \int_{\mathbb{T}} \nabla \mu \cdot \nabla \tilde{w} dx = 2 \sum_i \int_{\partial B_{R_i}(X_i)} \mu (\tilde{V}_i + \tilde{\xi}_i \cdot \vec{n}) dS.$$

Here $\tilde{w} : \mathbb{T} \rightarrow \mathbb{R}$ is a function w^α satisfying (9) for $\mathbf{Z}^\alpha = \tilde{\mathbf{Z}}$. The integration by parts yields

$$g_{\mathbf{Y}}(\mathbf{Z}, \tilde{\mathbf{Z}}) = - \sum_i \int_{\partial B_{R_i}(X_i)} w (\tilde{V}_i + \tilde{\xi}_i \cdot \vec{n}) dS.$$

From now on we consider an arrangement of particles as described above which evolves according to the gradient flow equation. This means that for any $t \geq 0$, it holds for $\mathbf{Z}(t) = \frac{d}{dt} \mathbf{Y}(t)$ that

$$g_{\mathbf{Y}}(\mathbf{Z}, \tilde{\mathbf{Z}}) = -\langle DE(\mathbf{Y}), \tilde{\mathbf{Z}} \rangle,$$

that is,

$$\sum_i \int_{\partial B_{R_i}(X_i)} w (\tilde{V}_i + \tilde{\xi}_i \cdot \vec{n}) dS = 2\pi \sum_i \tilde{V}_i + 2\sigma \sum_i \int_{\partial B_{R_i}(X_i)} \mu (\tilde{V}_i + \tilde{\xi}_i \cdot \vec{n}) dS \quad (11)$$

for all $\tilde{\mathbf{Z}} \in T_{\mathbf{Y}}\mathcal{N}$. Since $\tilde{\mathbf{Z}}$ is an arbitrary element of the tangent space we conclude from (11) that w satisfies

$$\frac{1}{2\pi R_i} \int_{\partial B_{R_i}(X_i)} (w - \frac{1}{R_i} - 2\sigma \mu) dS = \lambda(t) \quad (12)$$

and

$$\int_{\partial B_{R_i}(X_i)} (w - 2\sigma \mu) \vec{n} dS = 0 \quad (13)$$

for all i such that $R_i > 0$, with a Lagrange parameter $\lambda(t)$ that ensures volume conservation. Equations (12) and (13) are the analogue of (4) in the restricted setting.

Our aim is to identify the evolution in the limit of vanishing volume fraction of particles. More precisely, we consider a sequence of systems characterized by the parameter

$$\varepsilon := \left(\ln \left(\frac{d}{\mathcal{R}} \right) \right)^{-1/2} \quad (14)$$

in the limit $\varepsilon \rightarrow 0$. Here d is defined by

$$d^2 \sum_i 1 = L^2 \quad (15)$$

and \mathcal{R} by

$$\mathcal{R}^2 \sum_i 1 = \sum_i R_i(0)^2. \quad (16)$$

Then $\frac{1}{d^2}$ denotes the number density of particles, and $\pi \mathcal{R}^2$ the average volume of particles. Here and throughout this paper we use the abbreviation $\sum_i = \sum_{i:R_i>0}$.

Our main result informally says that when $L \sim L_{sc}$, with

$$L_{sc}^2 := d^2 \ln(d/\mathcal{R}) \sim d^2 \ln(1/\rho), \tag{17}$$

on the time scale of order $\mathcal{R}^3 \ln(1/\rho)$, the number density of particles with radius r and center x , denoted by $v = v(t, r, x)$ (suitably normalized), satisfies

$$\partial_t v + \partial_r \left(\frac{1}{r^2} \{r\psi - 1 - \sigma r^3\} v \right) = 0, \tag{18}$$

where $\psi = \psi(t, x)$ satisfies for each t that

$$-\Delta \psi + 2\pi \psi \int_0^\infty v \, dr = 2\pi \left(\int_0^\infty \frac{1}{r} v \, dr + \frac{\sigma}{L^2} \int_{\mathbb{T}} \int_0^\infty r^2 v \, dr \, dy \right) \text{ in } \mathbb{T} \tag{19}$$

in the limit $\varepsilon \rightarrow 0$. Here σ is also suitably normalized.

We remark that on the other hand, in the case that $L \ll L_{sc}$, that is, in the very dilute case, one obtains a homogeneous version where ψ is constant in space, and is replaced by $\lambda(t)$. More precisely that the number density of particles with radius r , denoted by $v(t, r)$ (suitably normalized), satisfies

$$\partial_t v + \partial_r \left(\frac{1}{r^2} (\lambda r - 1 - \sigma r^3) v \right) = 0 \tag{20}$$

with

$$\lambda(t) = \frac{\int_0^\infty \frac{1}{r} v \, dr + \sigma \int_0^\infty r^2 v \, dr}{\int_0^\infty v \, dr}. \tag{21}$$

2 The result

In this section, we will introduce suitably rescaled variables, state the precise assumption on our initial particle arrangement, and present the statement of our main result.

We assume from now on that $L = L_{sc}$ for the ease of presentation, and we will rescale the spatial variables by L_{sc} such that

$$L_{sc} = L = 1 \text{ and hence } d = \varepsilon, \mathcal{R} = \varepsilon \exp(-1/\varepsilon^2) =: \alpha_\varepsilon.$$

Notice that $\rho = \pi \alpha_\varepsilon^2 \varepsilon^{-2}$ and $\ln(1/\rho) \sim \varepsilon^{-2}$. We introduce $\hat{R}_i, \hat{t}, \hat{V}_i, \hat{\xi}, \hat{w}, \hat{\sigma}$ and $\hat{\mu}$ via

$$\begin{aligned} R_i(t) &= \alpha_\varepsilon \hat{R}_i(\hat{t}), & t &= \alpha_\varepsilon^3 \ln(1/\rho) \hat{t}, & w(t, x) &= \alpha_\varepsilon^{-1} \hat{w}(\hat{t}, x), \\ V_i(t) &= \frac{1}{\alpha_\varepsilon^2 \ln(1/\rho)} \hat{V}_i(\hat{t}) \sim \frac{\varepsilon^2}{\alpha_\varepsilon^2} \hat{V}_i(\hat{t}), & \xi_i(t) &= \frac{\varepsilon}{\alpha_\varepsilon^2} \hat{\xi}_i(\hat{t}), \\ \sigma &= \frac{1}{\alpha_\varepsilon^3 \ln(1/\rho)} \hat{\sigma} \sim \frac{\varepsilon^2}{\alpha_\varepsilon^3} \hat{\sigma}, & \mu(t, x) &= \frac{\alpha_\varepsilon^2}{\varepsilon^2} \hat{\mu}(\hat{t}, x). \end{aligned}$$

From now on we only deal with the rescaled quantities and drop the hats in the notation.

We denote the joint distribution of particle centers and radii at a given time t by $\nu_t^\varepsilon \in (C_p^0)^*$, which is given by

$$\int \zeta \, d\nu_t^\varepsilon = \sum_i \varepsilon^2 \zeta(R_i(t), X_i(t)) \text{ for } \zeta \in C_p^0, \tag{22}$$

where C_p^0 stands for the space of continuous functions on $\mathbb{R}_+ \times \mathbb{T}$ with compact support contained in $\mathbb{R}_+ \times \mathbb{T}$. Here \mathbb{T} denotes the unit flat torus, and $\mathbb{R}_+ = (0, \infty)$. Note that since $\zeta(r, x) = 0$ for $r = 0$, particles which have vanished do not enter the distribution. The natural space for v_i^ε and its limit v_i is the space $(C_p^0)^*$ of Borel measures on $\mathbb{R}_+ \times \mathbb{T}$.

We are now going to make the assumptions on our initial particle arrangement precise. Notice first, that in view of (15) and (16) we have

$$\int dv_0^\varepsilon = \sum_i \varepsilon^2 = 1 \quad \text{and} \quad \int r^2 dv_0^\varepsilon = \sum_i \varepsilon^2 R_i^2(0) = 1. \tag{23}$$

It follows immediately, that

$$\int r dv_0^\varepsilon = \sum_i \varepsilon^2 R_i(0) \leq 1, \tag{24}$$

that is the surface energy of the initial particle arrangement is finite.

Furthermore it is natural to assume that initially the nonlocal energy is uniformly bounded in ε , that is

$$\int_{\mathbb{T}} |\nabla \mu^\varepsilon(0, x)|^2 dx \leq C, \tag{25}$$

where C is independent of ε and where $\mu^\varepsilon(0, x)$ satisfies $-\Delta \mu^\varepsilon(0, \cdot) = \frac{\varepsilon^2}{\alpha_\varepsilon^2} \chi_{U_i B_i(0)} - \pi$ and $\int_{\mathbb{T}} \mu^\varepsilon dx = 0$.

We will see later [cf. (84)], that the nonlocal energy controls $\sum_i \varepsilon^2 R_i^4$. Hence, finiteness of the nonlocal energy initially also implies $\sum_i \varepsilon^2 R_i^4(0) \leq C$. For our analysis we need a little more than this. We need a certain tightness assumption which ensures, that not too much mass is contained in very large particles as $\varepsilon \rightarrow 0$. More precisely, we assume that

$$\sup_\varepsilon \sum_{R_i \geq M} \varepsilon^2 R_i^4(0) \rightarrow 0 \quad \text{as } M \rightarrow \infty. \tag{26}$$

Finally, we assume that initially particles are well separated in the sense that we assume that there is $\gamma > 0$, such that

$$\left\{ B_{2\gamma\varepsilon}(X_i(0)) \right\}_i \quad \text{are disjoint.} \tag{27}$$

In accordance with the notation in (22) we will use in the following the abbreviation $\int \zeta dv_t := \int_0^\infty \int_{\mathbb{T}} \zeta(r, x) dv_t(r, x)$ for $v_t \in (C_p^0)^*$. Otherwise the domain of integration is specified.

The natural space for potentials of diffusion fields is $H^1(\mathbb{T})$. Furthermore we will denote by $\mathring{H}^1(\mathbb{T})$ the subspace of $H^1(\mathbb{T})$ of functions with mean value zero.

We can now state our main result which informally says that v_i^ε converges as $\varepsilon \rightarrow 0$ to a weak solution of (18)–(19).

Theorem 2.1 *Let $T > 0$ be given and assume that the assumptions in Sect. 2 are satisfied. Then there exists a subsequence, again denoted by $\varepsilon \rightarrow 0$, and a weakly continuous map $[0, T] \ni t \mapsto v_t \in (C_p^0)^*$ with*

$$\int \zeta dv_t^\varepsilon \rightarrow \int \zeta dv_t \quad \text{uniformly in } t \in [0, T] \quad \text{for all } \zeta \in C_p^0,$$

$\int r^2 dv_t = 1$ for all $t \in [0, T]$. Furthermore, there exists a measurable map $(0, T) \ni t \mapsto \psi(t) \in H^1(\mathbb{T})$ such that (18) and (19) hold in the following weak sense

$$\frac{d}{dt} \int \zeta dv_t = \int \partial_r \zeta \frac{1}{r^2} \left(r \psi(t, x) - 1 - \sigma r^3 \right) dv_t \tag{28}$$

distributionally on $(0, T)$ for all $\zeta \in C_p^0$ with $\partial_r \zeta \in C_p^0$. Here

$$\int_{\mathbb{T}} \nabla \psi(t, x) \cdot \nabla \zeta - 2\pi \sigma \zeta dx + 2\pi \int \zeta \left(\psi(t, x) - \frac{1}{r} \right) dv_t = 0 \tag{29}$$

for all $\zeta \in H^1(\mathbb{T})$ and almost all $t \in (0, T)$.

The proof of Theorem 2.1 goes similarly to the approach for the three dimensional case in [12]. However in contrast to the three dimensional case we need to estimate $1/R_i$ term in the proof that the tightness property is preserved in time (see Lemmas 3.9 and 3.10) since the Lagrange multiplier diverges when particles disappear.

3 Proof of Theorem 2.1

We can deduce Theorem 2.1 by the homogenization of Rayleigh Principle (see Theorem 3.11). This will be obtained from the homogenization of metric tensor (Lemmas 3.5, 3.6) and the limit of the differential of the energy (Lemma 3.8). Also we need some a-priori estimates, which are given by a series of lemmas. The proof of Lemmas in this section will be given in Sect. 4. For readers convenience we will not abbreviate the arguments.

3.1 Gradient flow structure

In rescaled variables the submanifold \mathcal{N} turns into

$$\mathcal{N}^\varepsilon = \left\{ \mathbf{Y}^\varepsilon = \{R_i, X_i\}_i ; \sum_i \varepsilon^2 R_i^2 = 1 \right\}$$

and the tangent space

$$T_{\mathbf{Y}^\varepsilon} \mathcal{N}^\varepsilon = \left\{ \tilde{\mathbf{Z}}^\varepsilon = \sum_i \left(\tilde{V}_i \frac{\partial}{\partial R_i} + \varepsilon \alpha_\varepsilon \ln(1/\rho) \tilde{\xi}_i \cdot \frac{\partial}{\partial X_i} \right) ; \sum_i R_i \tilde{V}_i = 0 \right\}.$$

We use the abbreviation $\tilde{\mathbf{Z}}^\varepsilon = \{\tilde{V}_i, \tilde{\xi}_i\}_i$ for $\tilde{\mathbf{Z}}^\varepsilon = \sum_i \left(\tilde{V}_i \frac{\partial}{\partial R_i} + \varepsilon \alpha_\varepsilon \ln(1/\rho) \tilde{\xi}_i \cdot \frac{\partial}{\partial X_i} \right)$, and regard $\{\tilde{V}_i, \tilde{\xi}_i\}_i$ as the component of a tangent vector $\tilde{\mathbf{Z}}^\varepsilon$ with respect to a basis

$$\left\{ \frac{\partial}{\partial R_i}, \varepsilon \alpha_\varepsilon \ln(1/\rho) \frac{\partial}{\partial X_i} \right\}_i.$$

We will always denote by $\mathbf{Z}^\varepsilon = \{V_i, \xi_i\}_i$ the direction of steepest descent. Recall that $V_i = \frac{dR_i}{dt}$, but $\xi_i = (\varepsilon \alpha_\varepsilon \ln(1/\rho))^{-1} \frac{dX_i}{dt}$. The notation $\tilde{\mathbf{Z}}^\varepsilon$ will be used for an arbitrary element of the tangent space. Furthermore we use the abbreviation $B_i := B_{\alpha_\varepsilon R_i}(X_i)$.

We define the energy in rescaled variables as

$$E_\varepsilon(\mathbf{Y}^\varepsilon) = E_{\text{surf},\varepsilon}(\mathbf{Y}^\varepsilon) + \sigma E_{\text{nl},\varepsilon}(\mathbf{Y}^\varepsilon)$$

$$E_{\text{surf},\varepsilon}(\mathbf{Y}^\varepsilon) = 2\pi \sum_i \varepsilon^2 R_i, \quad E_{\text{nl},\varepsilon}(\mathbf{Y}^\varepsilon) = \int_{\mathbb{T}} |\nabla \mu^\varepsilon|^2 dx,$$

where $\mu^\varepsilon = \mu^\varepsilon(t, x)$ solves $-\Delta\mu^\varepsilon = \frac{\varepsilon^2}{\alpha_\varepsilon} \chi_{\cup B_i} - \pi$ and $\int_{\mathbb{T}} \mu^\varepsilon dx = 0$, and the metric tensor for $\tilde{\mathbf{Z}}^\varepsilon \in T_{\mathbf{Y}^\varepsilon \mathcal{N}^\varepsilon}$ is computed via

$$g_{\mathbf{Y}^\varepsilon}(\tilde{\mathbf{Z}}^\varepsilon, \tilde{\mathbf{Z}}^\varepsilon) = \int_{\mathbb{T}} |\nabla \tilde{w}^\varepsilon|^2 dx = \int_{\mathbb{T}} |\nabla \tilde{u}^\varepsilon + \nabla \tilde{\phi}^\varepsilon|^2 dx, \tag{30}$$

where $\tilde{w}^\varepsilon = \tilde{u}^\varepsilon + \tilde{\phi}^\varepsilon$,

$$\begin{aligned} \int_{\mathbb{T}} \nabla \tilde{u}^\varepsilon \cdot \nabla \zeta dx + \sum_i \int_{\partial B_i} \frac{\varepsilon^2}{\alpha_\varepsilon} \tilde{V}_i \zeta dS &= 0, \\ \int_{\mathbb{T}} \nabla \tilde{\phi}^\varepsilon \cdot \nabla \zeta dx + \sum_i \int_{\partial B_i} \frac{\varepsilon}{\alpha_\varepsilon} \tilde{\xi}_i \cdot \vec{n} \zeta dS &= 0 \end{aligned} \tag{31}$$

for all $\zeta \in \dot{H}^1(\mathbb{T})$. Notice that the potentials are only determined up to additive constants. In what follows we fix this constant by requiring that $\int_{\mathbb{T}} \tilde{u}^\varepsilon dx = \int_{\mathbb{T}} \tilde{\phi}^\varepsilon dx = 0$. For the steepest descent directions $\mathbf{Z}^\varepsilon = \{V_i, \xi_i\}_i$, we define the potentials $w^\varepsilon, u^\varepsilon, \phi^\varepsilon$ analogously.

Equations (12) and (13) for the direction of steepest descent, turn into

$$\frac{1}{|\partial B_i|} \int_{\partial B_i} (u^\varepsilon + \phi^\varepsilon - 2\sigma \mu^\varepsilon) dS = \frac{1}{R_i} + \lambda^\varepsilon(t) \tag{32}$$

for some $\lambda^\varepsilon(t) \in \mathbb{R}$ and

$$\int_{\partial B_i} (u^\varepsilon + \phi^\varepsilon - 2\sigma \mu^\varepsilon) \vec{n} dS = 0 \tag{33}$$

for all i such that $R_i > 0$. Here and in what follows we abbreviate, with some abuse of notations, for a disk $B_R(X)$ the perimeter by $|\partial B_R|$ and its area by $|B_R|$. Now the energy estimate (8) reads

$$\begin{aligned} \int_0^{t_1} \int_{\mathbb{T}} |\nabla w^\varepsilon|^2 dx dt + 4\pi \sum_i \varepsilon^2 \frac{R_i^2}{2}(t_1) + \sigma \int_{\mathbb{T}} |\nabla \mu^\varepsilon(t_1)|^2 dx \\ = 4\pi \sum_i \varepsilon^2 \frac{R_i^2}{2}(0) + \sigma \int_{\mathbb{T}} |\nabla \mu^\varepsilon(0)|^2 dx \end{aligned} \tag{34}$$

for all $t_1 > 0$. Finally, the Rayleigh principle says that \mathbf{Z}^ε satisfies

$$\begin{aligned} \int_0^T \beta(t) \left(\frac{1}{2} g_{\mathbf{Y}^\varepsilon(t)}(\mathbf{Z}^\varepsilon, \mathbf{Z}^\varepsilon) + \langle DE_\varepsilon(\mathbf{Y}^\varepsilon(t)), \mathbf{Z}^\varepsilon \rangle \right) dt \\ \leq \int_0^T \beta(t) \left(\frac{1}{2} g_{\mathbf{Y}^\varepsilon(t)}(\tilde{\mathbf{Z}}^\varepsilon, \tilde{\mathbf{Z}}^\varepsilon) + \langle DE_\varepsilon(\mathbf{Y}^\varepsilon(t)), \tilde{\mathbf{Z}}^\varepsilon \rangle \right) dt \end{aligned} \tag{35}$$

for all $\tilde{\mathbf{Z}}^\varepsilon \in T_{\mathbf{Y} \mathcal{N}}$ and nonnegative $\beta \in C^\infty([0, T])$.

3.2 A priori estimates and weak limits

It follows from definitions (15), (16) and the facts that volume of particles is conserved and the number can only decrease, that

$$\int dv_t^\varepsilon \leq 1 \quad \text{for all } t \in [0, T], \tag{36}$$

$$\int r^2 dv_t^\varepsilon = 1 \quad \text{for all } t \in [0, T]. \tag{37}$$

On the other hand the uniform bound on the energy in (34) implies the following.

Lemma 3.1

$$\int r^4 dv_t^\varepsilon \leq C \quad \text{for all } t \in [0, T]. \tag{38}$$

Next to v_t^ε , we introduce signed Borel measures $\rho_t^\varepsilon \in (C_p^0)^*$ and $\psi_t^\varepsilon \in ((C_p^0)^2)^*$ on $\mathbb{R}_+ \times \mathbb{T}$ via

$$\begin{aligned} \int \zeta d\rho_t^\varepsilon &= \varepsilon^2 \sum_i \zeta(R_i(t), X_i(t)) V_i(t) \quad \text{for } \zeta \in C_p^0, \\ \int \eta \cdot d\psi_t^\varepsilon &= \varepsilon^2 \sum_i \eta(R_i(t), X_i(t)) \cdot \xi_i(t) \quad \text{for } \eta \in (C_p^0)^2. \end{aligned}$$

The measures satisfies $\partial_t v^\varepsilon + \partial_r \rho^\varepsilon + \varepsilon \alpha_\varepsilon \ln(1/\rho) \operatorname{div} \psi^\varepsilon = 0$ in the sense of distributions, that is

$$\int_0^T \left(\partial_t \beta(t) \int \zeta dv_t^\varepsilon + \beta(t) \left\{ \int \partial_r \zeta d\rho_t^\varepsilon + \varepsilon \alpha_\varepsilon \ln(1/\rho) \int \nabla_x \zeta \cdot d\psi_t^\varepsilon \right\} \right) dt = 0 \tag{39}$$

for all $\zeta \in C_p^0 \cap C^\infty$ and $\beta \in C_0^\infty([0, T])$. As will be shown in Sect. 4, we have

$$\mathcal{D}^\varepsilon := \int_0^T 2\pi \sum_i \varepsilon^2 R_i^2 (V_i^2 + \frac{1}{4} |\xi_i|^2) dt \leq (1 + o(1)) \int_0^T \int_{\mathbb{T}} |\nabla w^\varepsilon|^2 dx dt \leq C, \tag{40}$$

which yields in particular

$$\mathcal{D} := \liminf_{\varepsilon \rightarrow 0} \mathcal{D}^\varepsilon \leq \limsup_{\varepsilon \rightarrow 0} \mathcal{D}^\varepsilon < \infty. \tag{41}$$

Bounds (39) and (40) yield a weak Hölder regularity in t of $\{v_t^\varepsilon\}_t$:

$$\begin{aligned} \left| \int \zeta dv_{t_1}^\varepsilon - \int \zeta dv_{t_2}^\varepsilon \right| &\leq |t_1 - t_2|^{1/2} \\ &\left(\frac{\mathcal{D}^\varepsilon}{2\pi} \sup_t \int \left(|\partial_r \zeta|^2 + 4\varepsilon^2 \alpha_\varepsilon^2 (\ln \frac{1}{\rho})^2 |\nabla_x \zeta|^2 \right) \frac{1}{r^2} dv_t^\varepsilon \right)^{1/2}. \end{aligned} \tag{42}$$

Using Arzela–Ascoli’s Theorem, (36) and (42) imply that there exists a weakly continuous family $\{v_t\}_t$ of nonnegative Borel measures on $\mathbb{R}_+ \times \mathbb{T}$ such that for a subsequence

$$\int \zeta dv_t^\varepsilon \rightarrow \int \zeta dv_t \quad \text{uniformly in } t \in [0, T] \tag{43}$$

for ζ in a countable subset of $C_p^0 \cap C^\infty$. Again by (36), we see that we can extend the locally uniform convergence in (43) to all $\zeta \in C_p^0$. Obviously, the bound (36) is conserved

$$\int dv_t \leq 1 \quad \text{for all } t \in [0, T], \tag{44}$$

and due to (37) and (38) we have

$$\int r^2 dv_t = 1 \quad \text{for all } t \in [0, T], \tag{45}$$

$$\int r^4 dv_t \leq C \quad \text{for all } t \in [0, T]. \tag{46}$$

The uniform control of the signed Borel measures $\{r^2 d\rho_t^\varepsilon dt\}_\varepsilon$ and $\{r^2 d\psi_t^\varepsilon dt\}_\varepsilon$ on $\mathbb{R}_+ \times \mathbb{T} \times [0, T]$ implied by (40) ensures the weak convergence, where the limits can be regarded as bounded linear functionals on $L^2(r^2 dv_t dt)$ and $L^2(r^2 dv_t dt)^2$ respectively, and hence by Riesz Representation Theorem, there exist $v \in L^2(r^2 dv_t dt)$ and $\xi \in L^2(r^2 dv_t dt)^2$ with

$$\int_0^T \int (|v(t)|^2 + |\xi(t)|^2) r^2 dv_t dt \leq \frac{D}{2\pi}, \tag{47}$$

such that for a subsequence

$$\begin{aligned} \int_0^T \beta(t) \int \zeta d\rho_t^\varepsilon dt &\rightarrow \int_0^T \beta(t) \int \zeta v(t) dv_t dt, \\ \int_0^T \beta(t) \int \eta \cdot d\psi_t^\varepsilon dt &\rightarrow \int_0^T \beta(t) \int \eta \cdot \xi(t) dv_t dt, \end{aligned} \tag{48}$$

for all $\beta \in C^0([0, T])$, $\zeta \in C_p^0$ and $\eta \in (C_p^0)^2$, and

$$\int_0^T \beta \int |\xi(t)|^2 r^2 dv_t dt \leq \liminf_{\varepsilon \rightarrow 0} \int_0^T \beta \sum_i \varepsilon^2 R_i^2 |\xi_i|^2 dt \tag{49}$$

for all nonnegative $\beta \in C^0([0, T])$. Thus the limit of (39) is

$$\int_0^T \left(\partial_t \beta(t) \int \zeta dv_t + \beta(t) \int \partial_r \zeta v(t) dv_t \right) dt = 0 \tag{50}$$

for all $\zeta \in C_p^0 \cap C^\infty$ and $\beta \in C_0^\infty([0, T])$.

Definition 3.2 For $\tilde{\mathbf{Z}}^\varepsilon$ satisfying $\int_0^T g_{\mathbf{Y}^\varepsilon}(\tilde{\mathbf{Z}}^\varepsilon, \tilde{\mathbf{Z}}^\varepsilon) dt \leq C$, we can define $\tilde{v} \in L^2(r^3 dv_t dt)$ and $\tilde{\xi} \in L^2(r^3 dv_t dt)^2$ analogously. We say that $\tilde{\mathbf{Z}}^\varepsilon$ converges weakly to $(\tilde{v}, \tilde{\xi})$.

We are going to show below that for any $T > 0$, particles do not collide on a time interval $[0, T]$ for sufficiently small ε . More precisely

Lemma 3.3 *For any $T > 0$ we can find $\varepsilon_0 > 0$ such that*

$$\{B_{\gamma\varepsilon}(X_i(t))\}_i \quad \text{are disjoint for all } t \in [0, T], \tag{51}$$

for all $\varepsilon \in (0, \varepsilon_0]$.

Thus it follows that the marginal of ν_t with respect to x has a bounded Lebesgue density. Hence it follows from (46) that the functional

$$\langle L(t), \zeta \rangle := \int \zeta r^2 d\nu_t - \int_{\mathbb{T}} \zeta dx, \quad \zeta \in H^1(\mathbb{T})$$

is an element of $H^{-1}(\mathbb{T}) = (H^1(\mathbb{T}))^*$ for all $t \in [0, T]$. Hence $K(t, \cdot) \in H^1(\mathbb{T})$ is uniquely determined via (29) up to additive constants.

In order to prove Lemma 3.3 we show the following.

Lemma 3.4 (slow motion of the particle centers) *As long as (51) is satisfied, we have*

$$|\xi_i(t)| \leq C \frac{\alpha_\varepsilon}{\varepsilon^2} \left(\int_{\mathbb{T}} |\nabla \phi^\varepsilon|^2 + |\nabla u^\varepsilon|^2 + |\nabla \mu^\varepsilon|^2 dx \right)^{1/2}.$$

3.3 Homogenization of the metric tensor

We identify the Γ -limit for the metric tensor and provide the necessary results to pass to the limit in the metric tensor. The following is a lower semicontinuity result.

Lemma 3.5 (lower semicontinuity) *For all nonnegative $\beta = \beta(t) \in C^\infty([0, T])$ we have*

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \int_0^T \beta g_{\mathbf{Y}^\varepsilon}(\mathbf{Z}^\varepsilon, \mathbf{Z}^\varepsilon) dt &= \liminf_{\varepsilon \rightarrow 0} \int_0^T \beta \int_{\mathbb{T}} |\nabla w^\varepsilon|^2 dx dt \\ &= \liminf_{\varepsilon \rightarrow 0} \int_0^T \beta \int_{\mathbb{T}} |\nabla u^\varepsilon|^2 + 2\nabla u^\varepsilon \cdot \nabla \phi^\varepsilon + |\nabla \phi^\varepsilon|^2 dx dt \\ &\geq \int_0^T \beta \int_{\mathbb{T}} |\nabla u|^2 dx dt + 2\pi \int_0^T \beta \int (|v|^2 + \frac{1}{4}|\xi|^2) r^2 d\nu_t dt, \end{aligned} \tag{52}$$

where for almost all t the function $u(t, \cdot) \in \dot{H}^1(\mathbb{T})$ is determined via

$$\int_{\mathbb{T}} \nabla u(t) \cdot \nabla \zeta dx + 2\pi \int \zeta r v(t) d\nu_t = 0 \tag{53}$$

for all $\zeta \in \dot{H}^1(\mathbb{T})$.

Furthermore we show that for any tangent vector of the limit manifold $(\tilde{v}, \tilde{\xi})$, there exists an approximating sequence along which the metric tensor is continuous.

Lemma 3.6 (construction) *For any $\tilde{v} \in L^2(r^2 d\nu_t dt)$ with $\int \tilde{v} r^2 d\nu_t = 0$ for almost all t and any $\tilde{\xi} \in L^2(r^2 d\nu_t dt)^3$ there exists a sequence $\tilde{\mathbf{Z}}^\varepsilon$ with $\tilde{\mathbf{Z}}^\varepsilon \in T_{\mathbf{Y}^\varepsilon} \mathcal{N}^\varepsilon$ such that $\tilde{\mathbf{Z}}^\varepsilon$ weakly converges to $(\tilde{v}, \tilde{\xi})$, and*

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \int_0^T \beta g_{\mathbf{Y}^\varepsilon}(\tilde{\mathbf{Z}}^\varepsilon, \tilde{\mathbf{Z}}^\varepsilon) dt \\ \leq \int_0^T \beta \int_{\mathbb{T}} |\nabla \tilde{u}|^2 dx dt + 2\pi \int_0^T \beta \int (|\tilde{v}|^2 + \frac{1}{4}|\tilde{\xi}|^2) r^2 d\nu_t dt \end{aligned} \tag{54}$$

for all nonnegative $\beta = \beta(t) \in C^\infty([0, T])$ and with $\tilde{u}(t, \cdot) \in \dot{H}^1(\mathbb{T})$ determined via

$$\int_{\mathbb{T}} \nabla \tilde{u}(t) \cdot \nabla \zeta dx + 2\pi \int \zeta r \tilde{v}(t) d\nu_t = 0 \tag{55}$$

for all $\zeta \in \dot{H}^1(\mathbb{T})$.

Note that the contribution from the drift term and the radial part do not interact in the limit $\varepsilon \rightarrow 0$.

3.4 The limit of the differential of the energy

We identify the limit of the differential of the energy. To that aim we identify the limit of the potentials μ^ε so as to prove the convergence for the nonlocal part of the energy.

Note that

$$\langle DE_{nl,\varepsilon}(\mathbf{Y}^\varepsilon), \tilde{\mathbf{Z}}^\varepsilon \rangle = -2 \int_{\mathbb{T}} \nabla \mu^\varepsilon \cdot \nabla \tilde{w}^\varepsilon dx.$$

Here, as also in the homogenization of the metric tensor, the key idea is that the potentials can be represented as a sum of monopoles, which represent the self-interaction of particles, plus a slowly varying field, which represents the interaction between different particles.

We set $l = \gamma\varepsilon$, where $\gamma > 0$ is as in (51). We write $\mu^\varepsilon = \sum_i \tilde{\mu}_i + \tilde{\mu}^\varepsilon$ with

$$\tilde{\mu}_i(x) = \begin{cases} \frac{\varepsilon^2}{4\alpha_\varepsilon^2} (\alpha_\varepsilon^2 R_i^2 - |x - X_i|^2) + \frac{\varepsilon^2 R_i^2}{2} \ln \frac{l}{\alpha_\varepsilon R_i} & : x \in B_i \\ \frac{\varepsilon^2 R_i^2}{2} \ln \frac{l}{|x - X_i|} & : x \in B_l(X_i) \setminus B_i \\ 0 & : \text{elsewhere} \end{cases} \quad (56)$$

Here $\tilde{\mu}_i$ is chosen such that $-\Delta \tilde{\mu}_i = \frac{\varepsilon^2}{\alpha_\varepsilon^2}$ in B_i and $\Delta \tilde{\mu}_i = 0$ in $B_l(X_i) \setminus B_i$. Since

$$\nabla \tilde{\mu}_i(x) = \begin{cases} -\frac{\varepsilon^2(x - X_i)}{2\alpha_\varepsilon^2} & : x \in B_i \\ -\frac{\varepsilon^2 R_i^2(x - X_i)}{2|x - X_i|^2} & : x \in B_l(X_i) \setminus B_i \\ 0 & : \text{elsewhere,} \end{cases} \quad (57)$$

it holds that

$$[\nabla \tilde{\mu}_i \cdot \vec{n}] = 0 \text{ on } \partial B_i \text{ and } [\nabla \tilde{\mu}_i \cdot \vec{n}] = \frac{\varepsilon^2 R_i^2}{2l} \text{ on } \partial B_l(X_i). \quad (58)$$

The slowly varying field $\tilde{\mu}^\varepsilon$ converges strongly to K and this enables us to pass to the limit in the differential of the energy.

Lemma 3.7

$$\sup_{t \in [0, T]} \|\nabla \tilde{\mu}^\varepsilon - \frac{1}{2} \nabla K\|_{L^2(\mathbb{T})} \rightarrow 0$$

as $\varepsilon \rightarrow 0$. Here $K(t, \cdot) \in \dot{H}^1(\mathbb{T})$ satisfies for each t

$$\int_{\mathbb{T}} \nabla K \cdot \nabla \zeta dx = 2\pi \left(\int r^2 \zeta dv_t - \int_{\mathbb{T}} \zeta dx \right) \quad (59)$$

for all $\zeta \in H^1(\mathbb{T})$.

Lemma 3.8 Assume that $\tilde{\mathbf{Z}}^\varepsilon$ satisfies

$$\int_0^T g_{\mathbf{Y}^\varepsilon}(\tilde{\mathbf{Z}}^\varepsilon, \tilde{\mathbf{Z}}^\varepsilon) dt \leq C \quad (60)$$

and converges weakly to $(\tilde{v}, \tilde{\xi})$. Then for all $\beta \in C^\infty([0, T])$,

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \beta \langle DE_{nl,\varepsilon}(\mathbf{Y}^\varepsilon), \tilde{\mathbf{Z}}^\varepsilon \rangle dt = \int_0^T \beta \langle DE_{nl}(v_t), (\tilde{v}, \tilde{\xi}) \rangle dt.$$

where

$$\langle DE_{nl}(v_t), (\tilde{v}, \tilde{\xi}) \rangle = 2\pi \int (r^3 + rK(t, x))\tilde{v} dv_t.$$

3.4.1 Tightness

In order to prove Lemma 3.8 we first need to show that the tightness property (26) is preserved in time so that no mass is lost at infinity in the limit $\varepsilon \rightarrow 0$.

Lemma 3.9 (tightness) *For any $t > 0$ we have*

$$\sum_{R_i \geq M} \varepsilon^2 R_i^4(t) \rightarrow 0 \quad \text{as } M \rightarrow \infty \text{ uniformly in } \varepsilon.$$

This lemma is crucial to our proof. The proof is much more difficult than three dimensional case. In fact, the main idea of the proof is to show by asymptotics that, at least in some average sense, V_i satisfies approximately $R_i^2 V_i \sim u R_i - 1 - \sigma R_i^3$ where u is the Lagrange multiplier that ensures the volume conservation. In contrast to the three dimensional case, the Lagrange multiplier may diverge. However we can still show the following a-priori estimate, and thus, at least on average, $V_i \leq 0$, if R_i is sufficiently large, and no mass can escape to infinity as $\varepsilon \rightarrow 0$, from which one deduces Lemma 3.9.

Lemma 3.10 *For any $T > 0$, there exist constants $C_T > 0$ and $\varepsilon_1 > 0$ such that*

$$\begin{aligned} & \int_0^T \left(\int \frac{1}{1 - \varepsilon^2 \ln r} \frac{1}{r^2} dv_t^\varepsilon \right)^{1/2} dt \\ & \leq C \left(\int_0^T \left(1 + \int_{\mathbb{T}} |\nabla(w^\varepsilon - 2\sigma\mu^\varepsilon)|^2 dx \right) dt \right)^{1/2} \leq C_T \end{aligned} \tag{61}$$

for all $\varepsilon \in (0, \varepsilon_1]$.

3.5 Homogenization of Rayleigh principle

The main task that remains to be done now is to determine the equation for the velocity function v . It will be characterized as the minimizer in the Rayleigh principle. Thus our task is to characterize the limits of \mathbf{Z}^ε that satisfy (35).

Notice that we can use Lemma 3.8 with $\mathbf{Z}^\varepsilon - \tilde{\mathbf{Z}}^\varepsilon$ where \mathbf{Z}^ε is the direction of steepest descent and $\tilde{\mathbf{Z}}^\varepsilon$ is as in Lemma 3.6. The main result from which one deduces Theorem 2.1 is the following.

Theorem 3.11 *For all nonnegative $\beta \in C^\infty([0, T])$ we have*

$$\begin{aligned} & \int_0^T \beta \left(\frac{1}{2} \int_{\mathbb{T}} |\nabla u|^2 dx + 2\pi \int \frac{1}{2} (|v|^2 + \frac{1}{4} |\xi|^2) r^2 dv_t \right. \\ & \quad \left. + \langle DE(v_t), (v, \xi) - (\tilde{v}, \tilde{\xi}) \rangle \right) dt \\ & \leq \int_0^T \beta \left(\frac{1}{2} \int_{\mathbb{T}} |\nabla \tilde{u}|^2 dx + 2\pi \int \frac{1}{2} (|\tilde{v}|^2 + \frac{1}{4} |\tilde{\xi}|^2) r^2 dv_t \right) dt \end{aligned} \tag{62}$$

for all $\tilde{v} \in L^2(r^2 dv_t dt)$ and $\tilde{\xi} \in L^2(r^2 dv_t dt)^2$ such that $\int r \tilde{v} dv_t = 0$ for almost all t and such that $(v - \tilde{v})(t, x, \cdot) = 0$ in a neighborhood of $r = 0$. Here

$$\langle DE(v_t), (\tilde{v}, \tilde{\xi}) \rangle = 2\pi \int \left(1 + \sigma \left(r^3 + rK(t, x) \right) \right) \tilde{v} dv_t$$

for \tilde{v} which vanish in a neighborhood of $r = 0$, and $u(t, \cdot), \tilde{u}(t, \cdot), K(t, \cdot) \in \mathring{H}^1(\mathbb{T})$ are determined for a.a. t via (53), (55), and (59) respectively.

The Euler–Lagrange equation for (62) becomes $r^2 v = r(u - \sigma K - \lambda) - 1 - \sigma r^3$ with λ being a Lagrange multiplier that ensures the constraint $\int r v dv_t = 0$. Setting $\psi(t, x) \equiv u - \sigma K - \lambda$, we can then derive (28)–(29). The proof is basically straightforward and goes similarly to the one in Chapter 6 of [13]. We omit the details here.

Notice that in the formulation (62) we need that $v - \tilde{v}$ has compact support in the r -variable. This is due to the fact that we cannot guarantee that the term $\int r v dv_t$ which appears in the differential of the surface energy is well-defined.

4 Proof of Lemmas

Proof of Lemma 3.4 We set $l = \gamma \varepsilon$, where γ is as in (51). Notice that due to (51) and $\sum_i \varepsilon^2 R_i^2 = 1$, the balls $\{B_l(X_i)\}_i$ are disjoint. For given $\{\xi_i\}_i$ define

$$\phi_i(x) = \begin{cases} -\frac{\varepsilon}{2} \left(\frac{1}{\alpha_\varepsilon} - \frac{\alpha_\varepsilon R_i^2}{l^2} \right) (x - X_i) \cdot \xi_i & : x \in B_i \\ -\frac{\varepsilon \alpha_\varepsilon R_i^2}{2} \left(\frac{1}{|x - X_i|^2} - \frac{1}{l^2} \right) (x - X_i) \cdot \xi_i & : x \in B_l(X_i) \setminus B_i \\ 0 & : \text{elsewhere.} \end{cases} \tag{63}$$

For further use we collect some properties of ϕ_i . It is easily checked that ϕ_i is continuous in \mathbb{T} , harmonic in $B_l(X_i) \setminus \partial B_i$ and satisfies

$$\begin{aligned} [\nabla \phi_i \cdot \vec{n}] &= \frac{\varepsilon}{\alpha_\varepsilon} \xi_i \cdot \vec{n} \quad \text{on } \partial B_i, \\ [\nabla \phi_i \cdot \vec{n}] &= -\frac{\varepsilon \alpha_\varepsilon R_i^2}{l^2} \xi_i \cdot \vec{n} \quad \text{on } \partial B_l(X_i). \end{aligned} \tag{64}$$

Furthermore, $\phi_i = 0$ on $\partial B_l(X_i)$ and

$$\phi_i = -\frac{\varepsilon R_i}{2} (1 + o(1)) \xi_i \cdot \vec{n} \quad \text{on } \partial B_i. \tag{65}$$

Then

$$\begin{aligned} \frac{1}{2} |\xi_i|^2 &= \frac{1}{|\partial B_i|} \int_{\partial B_i} |\xi_i \cdot \vec{n}|^2 dS \\ &= \frac{1}{|\partial B_i|} \int_{\partial B_i} \frac{\alpha_\varepsilon}{\varepsilon} [\nabla \phi^\varepsilon \cdot \vec{n}] \xi_i \cdot \vec{n} dS \\ &= -\frac{2\alpha_\varepsilon}{\varepsilon^2 R_i |\partial B_i|} \int_{\partial B_i} [\nabla \phi^\varepsilon \cdot \vec{n}] \phi_i dS (1 + o(1)). \end{aligned} \tag{66}$$

Integration by parts yields

$$\begin{aligned}
 \int_{\partial B_i} [\nabla \phi^\varepsilon \cdot \vec{n}] \phi_i \, dS &= - \int_{B_l(X_i) \setminus B_i} \nabla \phi^\varepsilon \cdot \nabla \phi_i \, dS \\
 &= \int_{\partial B_i} [\nabla \phi_i \cdot \vec{n}] \phi^\varepsilon \, dS + \int_{\partial B_l(X_i)} [\nabla \phi_i \cdot \vec{n}] \phi^\varepsilon \, dS \tag{67} \\
 &= \frac{\varepsilon \xi_i}{\alpha_\varepsilon} \cdot \int_{\partial B_i} \phi^\varepsilon \vec{n} \, dS - \frac{\varepsilon \alpha_\varepsilon R_i^2}{l^2} \int_{\partial B_l(X_i)} \phi^\varepsilon \xi_i \cdot \vec{n} \, dS.
 \end{aligned}$$

Together with (66), (67) and (33) we find

$$\begin{aligned}
 |\xi_i|^2 &\leq C \left(\frac{1}{\varepsilon R_i} \xi_i \cdot \frac{1}{|\partial B_i|} \int_{\partial B_i} \phi^\varepsilon \vec{n} \, dS + \frac{\alpha_\varepsilon}{\varepsilon l^2} \int_{\partial B_l(X_i)} \phi^\varepsilon \xi_i \cdot \vec{n} \, dS \right) \\
 &\leq C \left(\frac{1}{\varepsilon R_i |\partial B_i|} |\xi_i \cdot \int_{\partial B_i} \mu^\varepsilon \vec{n} \, dS| + \frac{1}{\varepsilon R_i |\partial B_i|} |\xi_i \right. \\
 &\quad \left. \cdot \int_{\partial B_i} u^\varepsilon \vec{n} \, dS| + \frac{\alpha_\varepsilon}{\varepsilon l^2} \left| \int_{\partial B_l(X_i)} \phi^\varepsilon \xi_i \cdot \vec{n} \, dS \right| \right). \tag{68}
 \end{aligned}$$

We estimate the last term on the right hand side of (68) via

$$\begin{aligned}
 \frac{\alpha_\varepsilon}{\varepsilon l^2} \left| \int_{\partial B_l(X_i)} \phi^\varepsilon \xi_i \cdot \vec{n} \, dS \right| &= \frac{\alpha_\varepsilon}{\varepsilon l^2} \left| \int_{B_l(X_i)} \nabla \phi^\varepsilon \cdot \xi_i \, dx \right| \\
 &\leq |\xi_i| \frac{\alpha_\varepsilon}{\varepsilon l^2} \int_{B_l(X_i)} |\nabla \phi^\varepsilon| \, dx \tag{69} \\
 &\leq C |\xi_i| \frac{\alpha_\varepsilon}{\varepsilon l} \left(\int_{B_l(X_i)} |\nabla \phi^\varepsilon|^2 \, dx \right)^{1/2}.
 \end{aligned}$$

We can write $\mu^\varepsilon = \sum_i \mu_i + \bar{\mu}^\varepsilon$ with

$$\mu_i(x) = \begin{cases} \frac{\varepsilon^2}{4\alpha_\varepsilon^2} (\alpha_\varepsilon^2 R_i^2 - |x - X_i|^2) + \frac{\varepsilon^2 R_i^2}{2} \ln \frac{l}{\alpha_\varepsilon R_i} - \frac{\pi}{4} (l^2 - |x - X_i|^2) & : x \in B_i \\ \frac{\varepsilon^2 R_i^2}{2} \ln \frac{l}{|x - X_i|} - \frac{\pi}{4} (l^2 - |x - X_i|^2) & : x \in B_l(X_i) \setminus B_i \\ 0 & : \text{elsewhere.} \end{cases} \tag{70}$$

Notice that μ_i is continuous in \mathbb{T} , and satisfies $-\Delta \mu_i = \frac{\varepsilon^2}{\alpha_\varepsilon^2} - \pi$ in B_i , $-\Delta \mu_i = -\pi$ in $B_l(X_i) \setminus \bar{B}_i$. These imply that $\bar{\mu}^\varepsilon$, and hence $\nabla \bar{\mu}^\varepsilon$ also are harmonic in $\cup_i B_l(X_i)$. Then, due to the properties of μ_i and the mean value theorem, we have

$$\begin{aligned}
 \int_{\partial B_i} \mu^\varepsilon \vec{n} \, dS &= \int_{B_i} \nabla \mu^\varepsilon \, dx = \int_{B_i} \nabla \mu_i \, dx + \int_{B_i} \nabla \bar{\mu}^\varepsilon \, dx \\
 &= \nabla \mu_i(X_i) |B_i| + \int_{B_i} \nabla \bar{\mu}^\varepsilon \, dx = \int_{B_i} \nabla \bar{\mu}^\varepsilon \, dx \tag{71} \\
 &= \frac{|B_i|}{|B_l(X_i)|} \int_{B_l(X_i)} \nabla \bar{\mu}^\varepsilon \, dx.
 \end{aligned}$$

But

$$\int_{B_l(X_i)} \nabla \bar{\mu}^\varepsilon \, dx = \int_{\partial B_l(X_i)} \bar{\mu}^\varepsilon \vec{n} \, dS = \int_{\partial B_l(X_i)} \mu^\varepsilon \vec{n} \, dS = \int_{B_l(X_i)} \nabla \mu^\varepsilon \, dx,$$

so

$$\left| \int_{\partial B_i} \mu^\varepsilon \vec{n} \, dS \right| \leq C |B_i| l^{-1} \left(\int_{B_l(X_i)} |\nabla \mu^\varepsilon|^2 \, dx \right)^{1/2}.$$

Next we write $u^\varepsilon = \sum_i u_i + \bar{u}^\varepsilon$ where

$$u_i(x) = \begin{cases} R_i V_i \varepsilon^2 \ln \left(\frac{\alpha_\varepsilon R_i}{l} \right) \sim -R_i V_i & : x \in B_i \\ \varepsilon^2 R_i V_i \ln \left(\frac{|x - X_i|}{l} \right) & : x \in B_l(X_i) \setminus B_i \\ 0 & : \text{elsewhere} \end{cases} \tag{72}$$

It is easily checked that u_i is continuous in \mathbb{T} , harmonic in $B_l(X_i) \setminus \partial B_i$ and satisfies

$$[\nabla u_i \cdot \vec{n}] = \frac{\varepsilon^2}{\alpha_\varepsilon} V_i \quad \text{on } \partial B_i.$$

Then from a similar argument as above, we see that

$$\left| \int_{\partial B_i} u^\varepsilon \vec{n} \, dS \right| \leq C |B_i| l^{-1} \left(\int_{B_l(X_i)} |\nabla u^\varepsilon|^2 \, dx \right)^{1/2}.$$

Thus, in summary we find

$$|\xi_i| \leq C \frac{\alpha_\varepsilon}{\varepsilon l} \left(\int_{B_l(X_i)} |\nabla \mu^\varepsilon|^2 + |\nabla u^\varepsilon|^2 + |\nabla \phi^\varepsilon|^2 \, dx \right)^{1/2}. \tag{73}$$

This completes the proof of Lemma. □

Proof of Lemma 3.3 In Sect. 3.3 we will see that $\int_0^T \int_{\mathbb{T}} |\nabla \phi^\varepsilon|^2 + |\nabla u^\varepsilon|^2 \, dx \, dt \leq C$ (cf. (81)). Furthermore, due to (34), we have $\int_{\mathbb{T}} |\nabla \mu^\varepsilon|^2 \, dx \leq C$. Then the statement follows from Lemma 3.4 and

$$\begin{aligned} |X_i(t) - X_j(t)| &= |X_i(0) - X_j(0) + \frac{\alpha_\varepsilon}{\varepsilon^2} \int_0^t \xi_i(s) - \xi_j(s) \, ds| \\ &\geq |X_i(0) - X_j(0)| - \frac{\alpha_\varepsilon}{\varepsilon^2} \int_0^t (|\xi_i(s)| + |\xi_j(s)|) \, ds. \end{aligned}$$

□

Proof of Lemma 3.5 We can prove

$$\liminf_{\varepsilon \rightarrow 0} \int_0^T \beta \int_{\mathbb{T}} |\nabla u^\varepsilon|^2 \, dx \, dt \geq \int_0^T \beta \int_{\mathbb{T}} |\nabla u|^2 \, dx \, dt + 2\pi \int_0^T \beta \int |v|^2 r^2 \, dv_t \, dt$$

in the same way as in [13] since (51) is satisfied.

Hence, it remains to show that

$$\liminf_{\varepsilon \rightarrow 0} \int_0^T \beta \int_{\mathbb{T}} |\nabla \phi^\varepsilon|^2 \, dx \, dt \geq 2\pi \int \beta \int \frac{1}{4} |\xi|^2 r^2 \, dv_t \, dt \tag{74}$$

and

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \beta \int_{\mathbb{T}} \nabla u^\varepsilon \cdot \nabla \phi^\varepsilon \, dx \, dt = 0. \tag{75}$$

Step 1: Monopoles. Our goal is to construct a good approximation of ϕ^ε which is based on cutting off the single monopole solutions. To that aim we define ϕ_i as in (63) with γ is as in (51) and thus the balls $\{B_l(X_i)\}_i$ are disjoint. Using (64) and (65), we find

$$\begin{aligned} \int_{B_l(X_i)} |\nabla\phi_i|^2 dx &= - \int_{\partial B_i} \phi_i \frac{\varepsilon}{\alpha_\varepsilon} \xi_i \cdot \bar{n} dS \\ &= \frac{1}{2}(1 + o(1)) \frac{\varepsilon^2 R_i}{\alpha_\varepsilon} \int_{\partial B_i} |\xi_i \cdot \bar{n}|^2 dS \\ &= \frac{1}{2}(1 + o(1)) 2\pi \varepsilon^2 R_i^2 \int_{\partial B_i} |\xi_i \cdot \bar{n}|^2 dS \\ &= \frac{\pi}{2} \varepsilon^2 R_i^2 |\xi_i|^2 (1 + o(1)). \end{aligned} \tag{76}$$

Step 2: A lower bound. We will show that for any given $\delta > 0$ we have

$$\int_{\mathbb{T}} |\nabla\phi^\varepsilon|^2 dx \geq (1 - \delta) \frac{\pi}{2} \sum_i \varepsilon^2 R_i^2 |\xi_i|^2 \tag{77}$$

if ε is sufficiently small. Indeed, due to the fact that $\{B_l(X_i)\}_i$ are disjoint we have

$$\int_{\mathbb{T}} |\nabla\phi^\varepsilon|^2 dx \geq \sum_i \int_{B_l(X_i)} |\nabla\phi^\varepsilon|^2 dx.$$

On the other hand

$$\int_{B_l(X_i)} |\nabla\phi^\varepsilon|^2 dx \geq \inf_{\psi} \int_{B_l(X_i)} |\psi|^2 dx,$$

where

$$\int_{B_l(X_i)} \psi \cdot \nabla\zeta + \int_{\partial B_i} \frac{\varepsilon}{\alpha_\varepsilon} \xi_i \cdot \bar{n} \zeta dS = 0$$

for all $\zeta \in C_0^\infty(B_l(X_i))$. From the corresponding Euler–Lagrange equation we see that the minimizer $\hat{\psi}$ is orthogonal to all divergence-free function and hence a gradient. We find that $\hat{\psi} = \nabla\phi_i$ and thus (77) follows from (76).

Step 3: Approximation of ϕ^ε . With

$$\hat{\phi}^\varepsilon(x) := \sum_i \phi_i(x), \tag{78}$$

we have

$$\int_{\mathbb{T}} |\nabla\phi^\varepsilon - \nabla\hat{\phi}^\varepsilon|^2 dx \leq C\varepsilon^k \sum_i \varepsilon^2 R_i^2 |\xi_i|^2 \tag{79}$$

for any $k > 0$. In fact, for $\bar{\phi}^\varepsilon := \phi^\varepsilon - \hat{\phi}^\varepsilon$ it follows from the definitions and (64) that

$$\int_{\mathbb{T}} \nabla\bar{\phi}^\varepsilon \cdot \nabla\zeta dx = \pm \sum_i \frac{\alpha_\varepsilon R_i^2}{l^2} \int_{\partial B_l(X_i)} \xi_i \cdot \bar{n} \zeta dS \tag{80}$$

for all $\zeta \in \dot{H}^1(\mathbb{T})$. We now define $L_\varepsilon \in (H^1(\mathbb{T}))^*$ via

$$\langle L_\varepsilon, \zeta \rangle := \sum_i \frac{\alpha_\varepsilon R_i^2}{l^2} \int_{\partial B_l(X_i)} \zeta \xi_i \cdot \bar{n} dS.$$

We observe that due to $\int_{\partial B_l(X_i)} \xi_i \cdot \vec{n} \, dS = 0$ we can write

$$\langle L_\varepsilon, \zeta \rangle = \sum_i \frac{\alpha_\varepsilon R_i^2}{l^2} \int_{\partial B_l(X_i)} \left(\zeta - \int_{\partial B_l(X_i)} \zeta \right) \xi_i \cdot \vec{n} \, dS.$$

We can then estimate

$$\begin{aligned} |\langle L_\varepsilon, \zeta \rangle| &\leq C \sum_i \frac{\alpha_\varepsilon R_i^2}{l^2} \left(\int_{\partial B_l(X_i)} |\xi_i \cdot \vec{n}|^2 \, dS \right)^{1/2} \left(\int_{\partial B_l(X_i)} \left| \zeta - \int_{\partial B_l(X_i)} \zeta \right|^2 \, dS \right)^{1/2} \\ &\leq C \sum_i \frac{\alpha_\varepsilon R_i^2}{l^2} \left(\pi l |\xi_i|^2 \right)^{1/2} \left(C l^2 \int_{B_l(X_i)} |\nabla \zeta|^2 \, dx \right)^{1/2}, \end{aligned}$$

where the last estimate follows from Trace Theorem and Poincaré’s inequality. This gives

$$\begin{aligned} |\langle L_\varepsilon, \zeta \rangle| &\leq C \sum_i \frac{\alpha_\varepsilon R_i^2}{l^2} l^{3/2} |\xi_i| \left(\int_{B_l(X_i)} |\nabla \zeta|^2 \, dx \right)^{1/2} \\ &\leq C \left(\sum_i \frac{\alpha_\varepsilon^2 R_i^2}{\varepsilon^3} \varepsilon^2 R_i^2 |\xi_i|^2 \right)^{1/2} \left(\sum_i \int_{B_l(X_i)} |\nabla \zeta|^2 \, dx \right)^{1/2}. \end{aligned}$$

Due to the simple estimate $R_i \leq C\varepsilon^{-1/2}$ we find

$$\sup_{\zeta \neq 0} \frac{|\langle L_\varepsilon, \zeta \rangle|}{\|\nabla \zeta\|_{L^2(\mathbb{T})}} \leq C\alpha_\varepsilon \varepsilon^{-2} \left(\sum_i \varepsilon^2 R_i^2 |\xi_i|^2 \right)^{1/2} \leq C\varepsilon^k \left(\sum_i \varepsilon^2 R_i^2 |\xi_i|^2 \right)^{1/2} \quad \text{for any } k > 0,$$

which proves (79).

Step 4: Bounds on the individual terms.

$$\int_0^T \int_{\mathbb{T}} |\nabla u^\varepsilon|^2 + |\nabla \phi^\varepsilon|^2 \, dx \, dt \leq 2 \int_0^T \int_{\mathbb{T}} |\nabla w^\varepsilon|^2 \, dx \, dt \leq C. \tag{81}$$

We write

$$\int_{\mathbb{T}} \nabla u^\varepsilon \cdot \nabla \phi^\varepsilon \, dx = \int_{\mathbb{T}} \nabla u^\varepsilon \cdot \nabla \bar{\phi}^\varepsilon \, dx + \int_{\mathbb{T}} \nabla u^\varepsilon \cdot \nabla \hat{\phi}^\varepsilon \, dx.$$

Now ∇u^ε is orthogonal to $\nabla \hat{\phi}^\varepsilon$ since

$$\int_{\mathbb{T}} \nabla u^\varepsilon \cdot \nabla \hat{\phi}^\varepsilon \, dx = - \sum_i \frac{\varepsilon^3 V_i}{\alpha_\varepsilon} \int_{\partial B_i} \phi_i \, dS,$$

but $\int_{\partial B_i} \phi_i \, dS = 0$. Thus

$$\int_{\mathbb{T}} \nabla u^\varepsilon \cdot \nabla \phi^\varepsilon \, dx = \int_{\mathbb{T}} \nabla u^\varepsilon \cdot \nabla \bar{\phi}^\varepsilon \, dx. \tag{82}$$

Furthermore, for some small $\delta > 0$, using (77),

$$\begin{aligned} \int_{\mathbb{T}} \nabla u^\varepsilon \cdot \nabla \bar{\phi}^\varepsilon \, dx &\geq -\delta \|\nabla u^\varepsilon\|_{L^2(\mathbb{T})}^2 - C_\delta \|\nabla \bar{\phi}^\varepsilon\|_{L^2(\mathbb{T})} \\ &\geq -\delta \|\nabla u^\varepsilon\|_{L^2(\mathbb{T})}^2 - C_\delta \varepsilon^4 \sum_i \varepsilon^3 R_i^3 |\xi_i|^2 \\ &\geq -\delta \|\nabla u^\varepsilon\|_{L^2(\mathbb{T})}^2 - C_\delta \varepsilon^4 \|\nabla \phi^\varepsilon\|_{L^2(\mathbb{T})}. \end{aligned}$$

We choose $\delta = \frac{1}{4}$ and then ε so small such that $C_\delta \varepsilon^4 \leq \frac{1}{4}$. Then (81) follows from (34) and (82).

Step 5: Lower semicontinuity. Since we have now established (81), we obtain the existence of the weak limit $\tilde{\xi}$ as explained before in Sect. 3.2, and hence the assertion (74) follows from (49) and (77).

Step 6: The mixed term vanishes in the limit.

As an immediate consequence of (77), (79), (81) and (82) we find

$$\int_0^T \|\nabla \tilde{\phi}^\varepsilon\|_{L^2(\mathbb{T})}^2 dt \leq C\varepsilon^4$$

and thus that (75) is valid. □

Proof of Lemma 3.6 We need only to show that for given $\tilde{\xi}$ as above we can find $\tilde{\phi}^\varepsilon$ such that

$$\limsup_{\varepsilon \rightarrow 0} \int_0^T \beta \int_{\mathbb{T}} |\nabla \tilde{\phi}^\varepsilon|^2 dx dt \leq 2\pi \int_0^T \beta \int \frac{1}{4} |\tilde{\xi}|^2 r^2 dv_t dt$$

and

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \beta \int_{\mathbb{T}} \nabla \tilde{u}^\varepsilon \cdot \nabla \tilde{\phi}^\varepsilon dx dt = 0$$

since the corresponding result for \tilde{u}^ε can be proved in the same way as in [13]. The proof is in fact quite similar to the proof of Lemma 3.5, since there the minimization property of $\{\xi_i\}_i$ is not used in the construction, which is henceforth quite general.

Step 1: Construction for smooth $\tilde{\xi}$. We first assume that $\tilde{\xi}$ is smooth. Then we define $\tilde{\xi}_i := \tilde{\xi}(X_i)$ and construct ϕ_i as in (63) for $\xi_i = \tilde{\xi}_i$, and a corresponding $\hat{\phi}^\varepsilon$ as in (78). The property (76) implies

$$\int_{\mathbb{T}} |\nabla \hat{\phi}^\varepsilon|^2 dx = \frac{\pi}{2} \sum_i \varepsilon^2 |\tilde{\xi}_i|^2 R_i^2 (1 + o(1))$$

and hence

$$\begin{aligned} \int_0^T \beta \int_{\mathbb{T}} |\nabla \hat{\phi}^\varepsilon|^2 dx dt &= \pi \int_0^T \beta \int \frac{1}{2} |\tilde{\xi}|^2 r^2 dv_t^\varepsilon dt (1 + o(1)) \\ &\rightarrow \pi \int_0^T \beta \int \frac{1}{2} |\tilde{\xi}|^2 r^2 dv_t dt \end{aligned}$$

as $\varepsilon \rightarrow 0$. Here we used (43) and that $\tilde{\xi}$ has compact support.

Furthermore, we obtain also exactly as in the proof of Lemma 3.5 that $\tilde{\phi}^\varepsilon := \tilde{\phi}^\varepsilon - \hat{\phi}^\varepsilon$ converges to zero strongly in $L^2((0, T); H^1(\mathbb{T}))$, and as a consequence the mixed term vanishes in the limit.

Step 2: Construction for general $\tilde{\xi}$.

In order to finish the proof of the lemma we have to show that we can approximate $\tilde{\xi} \in L^2(r^2 dv_t dt)^3$ by smooth functions $\tilde{\xi}_n$ with compact support such that

$$\int_0^T \beta \int |\tilde{\xi}_n|^2 r^2 dv_t dt \rightarrow \int_0^T \beta \int |\tilde{\xi}|^2 r^2 dv_t dt$$

as $n \rightarrow \infty$. But this follows from a density argument. □

Proof of Lemma 3.1 We first prove (38). Set $l = \gamma\varepsilon$, where γ is as in (51). Due to the fact that $\{B_l(X_i)\}_i$ are disjoint we have

$$\int_{\mathbb{T}} |\nabla\mu^\varepsilon|^2 dx \geq \sum_i \int_{B_l(X_i)} |\nabla\mu^\varepsilon|^2 dx. \tag{83}$$

On the other hand, for each i ,

$$\int_{B_l(X_i)} |\nabla\mu^\varepsilon|^2 dx \geq \inf_{\psi} \int_{B_l(X_i)} |\psi|^2 dx,$$

where the infimum is taken over all $\psi \in (L^2(B_l(X_i)))^2$ which satisfy

$$\int_{B_l(X_i)} \psi \cdot \nabla\zeta dx = \int_{B_l(X_i)} \left(\frac{\varepsilon^2}{\alpha_\varepsilon} \chi_{B_i} - \pi\right) \zeta dx$$

for all $\zeta \in C_0^\infty(B_l(X_i))$. We see that the minimizer $\hat{\psi}$ is orthogonal to all divergence-free vector-valued functions. Hence $\hat{\psi} = \nabla\phi$, $\phi = \text{const.}$ on $\partial B_l(X_i)$, where $-\Delta\phi = \frac{\varepsilon^2}{\alpha_\varepsilon} \chi_{B_i} - \pi$.

We find that $\hat{\psi} = \nabla\mu_i$, where μ_i is defined as in (70), and we have

$$\begin{aligned} \int_{B_l(X_i)} |\hat{\psi}|^2 dx &= \int_{B_l(X_i)} |\nabla\mu_i|^2 dx \\ &\geq \int_{B_l(X_i) \setminus B_i} \frac{(\varepsilon R_i)^4}{4|x - X_i|^2} - \frac{\pi\varepsilon^2 R_i^2}{2} dx \\ &= 2\pi \frac{(\varepsilon R_i)^4}{4} \ln \frac{l}{\alpha_\varepsilon R_i} - \frac{\pi^2}{2} (l^2 - (\alpha_\varepsilon R_i)^2) (\varepsilon R_i)^2. \end{aligned}$$

Then it follows from (34), $\sum_i \varepsilon^2 R_i^2 = 1$ and (83) that

$$\sum_i \varepsilon^2 R_i^4(t) \leq C \quad \text{for all } t \in [0, T], \tag{84}$$

which proves (38). □

Proof of Lemma 3.10 We can prove this in a similar way as in [14, section 3.4] by making use of (32) and Lemma A.1 although we are using Neumann boundary condition on ∂B_i instead of Dirichlet boundary condition.

Note $0 < \frac{1}{1-\varepsilon^2 \ln R_i} \leq \frac{1}{1+\varepsilon^2 \ln \varepsilon} \leq \frac{2e}{2e-1}$ by $0 < R_i \leq 1/\varepsilon$ and $|B_l(X_i)|/|A_i| \leq 2$ for small ε with $l = \gamma\varepsilon$ and

$$A_i = \{x \in \mathbb{T}; \alpha_\varepsilon R_i < |x - R_i| < \gamma\varepsilon\}.$$

Setting $v^\varepsilon = u^\varepsilon + \phi^\varepsilon - 2\sigma\mu^\varepsilon - \lambda^\varepsilon$, we have

$$\begin{aligned} \int \frac{1}{1-\varepsilon^2 \ln r} \frac{1}{r^2} dv_i^\varepsilon &= \sum_i \frac{\varepsilon^2}{1-\varepsilon^2 \ln R_i} \frac{1}{R_i^2} \\ &\leq C \sum_i \frac{\varepsilon^2}{1-\varepsilon^2 \ln R_i} \left| \frac{1}{R_i} - \frac{1}{|A_i|} \int_{A_i} v^\varepsilon dx \right|^2 + C \sum_i \frac{\varepsilon^2}{1-\varepsilon^2 \ln R_i} \left| \frac{1}{|A_i|} \int_{A_i} v^\varepsilon dx \right|^2 \\ &\leq C \sum_i \frac{\varepsilon^2}{1-\varepsilon^2 \ln R_i} \left| \frac{1}{R_i} - \frac{1}{|A_i|} \int_{A_i} v^\varepsilon dx \right|^2 + C \sum_i \frac{1}{|A_i|} \left| \int_{A_i} v^\varepsilon dx \right|^2 \end{aligned}$$

since $|B_i(X_i)|/|A_i| \leq 2$ for small ε . Using

$$\left| \int_{A_i} v^\varepsilon dx \right|^2 \leq |A_i| \int_{A_i} (v^\varepsilon)^2 dx,$$

we see that

$$\int \frac{1}{1 - \varepsilon^2 \ln r} \frac{1}{r^2} dv_i^\varepsilon \leq C \int_{\mathbb{T}} |v^\varepsilon|^2 dx + C \sum_i \frac{\varepsilon^2}{1 - \varepsilon^2 \ln R_i} \left| \frac{1}{R_i} - \frac{1}{|A_i|} \int_{A_i} v^\varepsilon dx \right|^2. \tag{85}$$

Now we estimate the second term of the right hand side of (85). Note that $v^\varepsilon = u^\varepsilon + \phi^\varepsilon - 2\sigma\mu^\varepsilon - \lambda^\varepsilon$ satisfies

$$\int_{\partial B_i} v^\varepsilon - \frac{1}{R_i} dS = 0$$

for each i . Using Lemma A.1 for $f = v^\varepsilon - \frac{1}{R_i}$, we have

$$\left| \frac{1}{|A_i|} \int_{A_i} \left(\frac{1}{R_i} - v^\varepsilon \right) dx \right|^2 \leq C \frac{|B_i(X_i)|^2}{|A_i|^2} \ln \left(\frac{\gamma\varepsilon}{\alpha_\varepsilon R_i} \right) \int_{A_i} |\nabla v^\varepsilon|^2 dx,$$

and since $|B_i(X_i)|/|A_i| \leq 2$ for small ε ,

$$\sum_i \frac{\varepsilon^2}{1 - \varepsilon^2 \ln R_i} \left| \frac{1}{R_i} - \frac{1}{|A_i|} \int_{A_i} v^\varepsilon dx \right|^2 \leq C \int_{\mathbb{T}} |\nabla v^\varepsilon|^2 dx. \tag{86}$$

It follows from (85) and (86) that

$$\int \frac{1}{1 - \varepsilon^2 \ln r} \frac{1}{r^2} dv_i^\varepsilon \leq C \left(\int_{\mathbb{T}} |v^\varepsilon|^2 dx + \int_{\mathbb{T}} |\nabla v^\varepsilon|^2 dx \right). \tag{87}$$

We estimate the L^2 norm of v^ε .

$$\begin{aligned} \int_{\mathbb{T}} |v^\varepsilon|^2 dx &\leq C \int_{\mathbb{T}} |v^\varepsilon - \int_{\mathbb{T}} v^\varepsilon|^2 dx + C \left| \int_{\mathbb{T}} v^\varepsilon dx \right|^2 \\ &\leq C \int_{\mathbb{T}} |\nabla v^\varepsilon|^2 dx + C \left| \int_{\mathbb{T}} v^\varepsilon dx \right|^2. \end{aligned} \tag{88}$$

Using

$$\begin{aligned} \left| |A_i| \int_{\mathbb{T}} v^\varepsilon dx \right| &\leq \left| \int_{A_i} \left(\int_{\mathbb{T}} v^\varepsilon - v^\varepsilon \right) \right| + \left| \int_{A_i} v^\varepsilon \right| \\ &\leq \int_{A_i} \left| \int_{\mathbb{T}} v^\varepsilon - v^\varepsilon \right| + \left| \int_{A_i} v^\varepsilon \right| \\ &\leq C\varepsilon \left[\int_{A_i} \left| \int_{\mathbb{T}} v^\varepsilon - v^\varepsilon \right|^2 \right]^{1/2} + \left| \int_{A_i} v^\varepsilon \right|, \end{aligned} \tag{89}$$

we see that

$$\begin{aligned}
 & \left(\int \frac{r}{1 - \varepsilon^2 \ln r} dv_i^\varepsilon \right) \left| \int_{\mathbb{T}} v^\varepsilon \right| \\
 & \leq C \sum_i \frac{R_i}{1 - \varepsilon^2 \ln R_i} \left| |A_i| \int_{\mathbb{T}} v^\varepsilon dx \right| \\
 & \leq C \left(\sum_i \varepsilon^2 R_i^2 \right)^{1/2} \left(\sum_i \int_{A_i} \left| \int_{\mathbb{T}} v^\varepsilon - v^\varepsilon \right|^2 \right)^{1/2} + C \sum_i \frac{R_i}{1 - \varepsilon^2 \ln R_i} \left| \int_{A_i} v^\varepsilon \right| \\
 & \leq C \left(\int_{\mathbb{T}} |\nabla v^\varepsilon|^2 dx \right)^{1/2} + C \sum_i \frac{R_i}{1 - \varepsilon^2 \ln R_i} \left| \int_{A_i} v^\varepsilon \right|. \tag{90}
 \end{aligned}$$

We estimate the second term of the right hand side of (90).

$$\begin{aligned}
 & \sum_i \frac{R_i}{1 - \varepsilon^2 \ln R_i} \left| \int_{A_i} v^\varepsilon \right| \\
 & \leq \sum_i \frac{R_i |A_i|}{1 - \varepsilon^2 \ln R_i} \left| \frac{1}{|A_i|} \int_{A_i} v^\varepsilon - \frac{1}{R_i} \right| + \sum_i \frac{|A_i|}{1 - \varepsilon^2 \ln R_i} \\
 & \leq C \left(\sum_i \varepsilon^2 R_i^2 \right)^{1/2} \left(\sum_i \frac{\varepsilon^2}{1 - \varepsilon^2 \ln R_i} \left| \frac{1}{|A_i|} \int_{A_i} v^\varepsilon - \frac{1}{R_i} \right|^2 \right)^{1/2} + C \sum_i \varepsilon^2 \\
 & \leq C \left(1 + \left(\int_{\mathbb{T}} |\nabla v^\varepsilon|^2 dx \right)^{1/2} \right). \tag{91}
 \end{aligned}$$

Note that due to

$$1 = \left(\sum_i \varepsilon^2 R_i^2 \right)^2 \leq \left(\sum_i \varepsilon^2 R_i^3 (1 - \varepsilon^2 \ln R_i) \right) \left(\sum_i \frac{\varepsilon^2 R_i}{1 - \varepsilon^2 \ln R_i} \right), \tag{92}$$

$-\ln R_i \leq \frac{1}{c\alpha R_i^\alpha}$, $\sum_i \varepsilon^4 R_i^2 \leq C$ and $\sum_i \varepsilon^2 R_i^3 \leq C$, there is some c , independent of ε , such that

$$\sum_i \frac{\varepsilon^2 R_i}{1 - \varepsilon^2 \ln R_i} \geq c > 0.$$

Then it follows from (90) and (91) that

$$\left| \int_{\mathbb{T}} v^\varepsilon \right| \leq C \left(1 + \left(\int_{\mathbb{T}} |\nabla v^\varepsilon|^2 dx \right)^{1/2} \right). \tag{93}$$

Then by (88),

$$\int_{\mathbb{T}} |v^\varepsilon|^2 dx \leq C \left(\int_{\mathbb{T}} |\nabla v^\varepsilon|^2 dx + 1 \right).$$

Thus we obtain from (87),

$$\int \frac{1}{1 - \varepsilon^2 \ln r} \frac{1}{r^2} dv_i^\varepsilon \leq C \left(\int_{\mathbb{T}} |\nabla v^\varepsilon|^2 dx + 1 \right).$$

□

Proof of Lemma 3.9 *Step 1: An expression for V_i .* Similarly to ϕ_i in the proof of Lemma 3.4 we introduce a suitable test function v_i . Here it is the capacity potential of B_i with respect to $B_l(X_i)$, with $l = \gamma\varepsilon$. This gives that

$$v_i(x) = \begin{cases} a^\varepsilon \ln \frac{l}{\alpha_\varepsilon R_i} & : x \in B_i \\ a^\varepsilon \ln \frac{l}{|x-X_i|} & : x \in B_l(X_i) \setminus B_i \\ 0 & : \text{elsewhere} \end{cases}$$

with $a^\varepsilon = (\ln \frac{l}{\alpha_\varepsilon R_i})^{-1}$. With this definition we also have

$$[\nabla v_i \cdot \vec{n}] = -\frac{a^\varepsilon}{\alpha_\varepsilon R_i} \text{ on } \partial B_i \quad \text{and} \quad [\nabla v_i \cdot \vec{n}] = \frac{a^\varepsilon}{l} \text{ on } \partial B_l(X_i).$$

Then

$$\begin{aligned} 2\pi R_i V_i &= 2\pi R_i \frac{\alpha_\varepsilon}{\varepsilon^2} [\nabla u^\varepsilon \cdot \vec{n}] \\ &= \frac{1}{\varepsilon^2} \int_{\partial B_i} [\nabla u^\varepsilon \cdot \vec{n}] dS \\ &= \frac{1}{\varepsilon^2} \int_{\partial B_i} [\nabla u^\varepsilon \cdot \vec{n}] v_i dS \\ &= \frac{1}{\varepsilon^2} \int_{\partial B_i} [\nabla v_i \cdot \vec{n}] u^\varepsilon dS + \frac{1}{\varepsilon^2} \int_{\partial B_l(X_i)} [\nabla v_i \cdot \vec{n}] u^\varepsilon dS \\ &= -\frac{a^\varepsilon}{\varepsilon^2 \alpha_\varepsilon R_i} \int_{\partial B_i} u^\varepsilon dS + \frac{a^\varepsilon}{\varepsilon^2 l} \int_{\partial B_l(X_i)} u^\varepsilon dS. \end{aligned} \tag{94}$$

Due to (32) it follows that

$$R_i V_i = -\frac{a^\varepsilon}{\varepsilon^2} \left(\frac{1}{R_i} + \frac{1}{|\partial B_i|} \int_{\partial B_i} (2\sigma \mu^\varepsilon - \phi^\varepsilon) dx + \lambda^\varepsilon(t) \right) + \frac{a^\varepsilon}{2\pi \varepsilon^2 l} \int_{\partial B_l(X_i)} u^\varepsilon dS. \tag{95}$$

Using that $\mu^\varepsilon = \sum_i \mu_i + \bar{\mu}^\varepsilon$ with μ_i as in (70) we obtain

$$\begin{aligned} \frac{1}{|\partial B_i|} \int_{\partial B_i} \mu^\varepsilon dx &= \frac{1}{|\partial B_i|} \int_{\partial B_i} \mu_i dx + \frac{1}{|\partial B_i|} \int_{\partial B_i} \bar{\mu}^\varepsilon dx \\ &= \frac{\varepsilon^2 R_i^2}{2} \ln \frac{l}{\alpha_\varepsilon R_i} + O(\varepsilon^2) + \frac{1}{|\partial B_l(X_i)|} \int_{\partial B_l(X_i)} \bar{\mu}^\varepsilon dx \\ &= \frac{\varepsilon^2 R_i^2}{2} \ln \frac{l}{\alpha_\varepsilon R_i} + O(\varepsilon^2) + \frac{1}{|\partial B_l(X_i)|} \int_{\partial B_l(X_i)} \mu^\varepsilon dx. \end{aligned}$$

Similarly using that $\phi^\varepsilon = \sum_i \phi_i + \bar{\phi}^\varepsilon$ with ϕ_i as in (63) we obtain

$$\begin{aligned} \frac{1}{|\partial B_i|} \int_{\partial B_i} \phi^\varepsilon dx &= \frac{1}{|\partial B_i|} \int_{\partial B_i} \phi_i dx + \frac{1}{|\partial B_i|} \int_{\partial B_i} \bar{\phi}^\varepsilon dx \\ &= \frac{1}{|\partial B_l(X_i)|} \int_{\partial B_l(X_i)} \bar{\phi}^\varepsilon dx = \frac{1}{|\partial B_l(X_i)|} \int_{\partial B_l(X_i)} \phi^\varepsilon dx. \end{aligned}$$

It follows that

$$R_i V_i = -\frac{a^\varepsilon}{\varepsilon^2} \left(\frac{1}{R_i} + O(\varepsilon^2) + \lambda^\varepsilon(t) \right) - \sigma R_i^2 + \frac{a^\varepsilon}{2\pi \varepsilon^2 l} \int_{\partial B_l(X_i)} u^\varepsilon + \phi^\varepsilon - 2\sigma \mu^\varepsilon dS. \tag{96}$$

Step 2: A bound on $\int_{\partial B_l(X_i)} u^\varepsilon + \phi^\varepsilon - 2\sigma \mu^\varepsilon dS$.

We define $\psi(x) = \frac{1}{2}(l^2 - |x - X_i|^2)$ in $B_l(X_i)$ such that $-\Delta\psi = 2$ in $B_l(X_i)$, $-\nabla\psi = x - X_i$ in $B_l(X_i)$ and $\nabla\psi \cdot \vec{n} = -l$ on $\partial B_l(X_i)$.

Thus, abbreviating $v^\varepsilon = u^\varepsilon + \phi^\varepsilon - 2\sigma\mu^\varepsilon$, we have

$$\begin{aligned} \frac{1}{l} \int_{\partial B_l(X_i)} v^\varepsilon dS &= -\frac{1}{l^2} \int_{\partial B_l(X_i)} v^\varepsilon \nabla\psi \cdot \vec{n} dS \\ &= -\frac{1}{l^2} \int_{B_l(X_i)} \Delta\psi v^\varepsilon + \nabla\psi \cdot \nabla v^\varepsilon dx \\ &= \frac{3}{l^2} \int_{B_l(X_i)} v^\varepsilon - \frac{1}{l^2} \int_{B_l(X_i)} \nabla\psi \cdot \nabla v^\varepsilon dx. \end{aligned} \tag{97}$$

We multiply (97) by $R_i^m a^\varepsilon$ with $m \geq 0$, and sum over $i \in I$ for some $I \subset \{i : R_i > 0\}$. Then due to $a^\varepsilon \leq 2\varepsilon^2$, we find, using $l = \gamma\varepsilon$ and Hölder’s inequality, that

$$\begin{aligned} &\left| \sum_{i \in I} R_i^m \frac{a^\varepsilon}{l} \int_{\partial B_l(X_i)} v^\varepsilon dS \right| \\ &\leq C \left(\sum_{i \in I} R_i^m \int_{B_l(X_i)} |v^\varepsilon| + \sum_{i \in I} R_i^m \int_{B_l(X_i)} |\nabla\psi \cdot \nabla v^\varepsilon| dx \right) \\ &\leq C \left(\sum_{i \in I} R_i^m l^{5/3} \left(\int_{B_l(X_i)} |v^\varepsilon|^6 \right)^{1/6} + \sum_{i \in I} R_i^m l^2 \left(\int_{B_l(X_i)} |\nabla v^\varepsilon|^2 \right)^{1/2} \right) \\ &\leq C \left(\sum_{i \in I} \varepsilon^2 R_i^{6m/5} \right)^{5/6} \left(\int_{\mathbb{T}} |v^\varepsilon|^6 \right)^{1/6} + C \left(\sum_{i \in I} \varepsilon^4 R_i^{2m} \right)^{1/2} \left(\int_{\mathbb{T}} |\nabla v^\varepsilon|^2 \right)^{1/2}. \end{aligned}$$

Due to Sobolev embedding and Poincaré’s inequality (recall that v^ε has mean value zero) we finally find

$$\begin{aligned} &\left| \sum_{i \in I} R_i^m \frac{a^\varepsilon}{l} \int_{\partial B_l(X_i)} u^\varepsilon + \phi^\varepsilon - 2\sigma\mu^\varepsilon dS \right| \\ &\leq C \left(\int_{\mathbb{T}} |\nabla\mu^\varepsilon|^2 + |\nabla\phi^\varepsilon|^2 + |\nabla u^\varepsilon|^2 \right)^{1/2} \left\{ \left(\sum_{i \in I} \varepsilon^2 R_i^{6m/5} \right)^{5/6} + \left(\sum_{i \in I} \varepsilon^4 R_i^{2m} \right)^{1/2} \right\}. \end{aligned} \tag{98}$$

Step 3: A bound for $\lambda^\varepsilon(t)$. We go back to (96) to estimate $\lambda^\varepsilon(t)$. For that purpose we multiply (96) by ε^2 and sum over i . We find

$$\begin{aligned} 0 &= -\sum_i a^\varepsilon \left(\frac{1}{R_i} + O(\varepsilon^2) \right) - \sigma \sum_i \varepsilon^2 R_i^2 \\ &\quad - \lambda^\varepsilon(t) \sum_i a^\varepsilon + \sum_i \frac{a^\varepsilon}{2\pi l} \int_{\partial B_l(X_i)} u^\varepsilon + \phi^\varepsilon - 2\sigma\mu^\varepsilon dS. \end{aligned} \tag{99}$$

First notice that due to

$$\begin{aligned} 1 &= \left(\sum_i \varepsilon^2 R_i^2(t) \right)^5 \leq \left(\sum_i \varepsilon^2 \right) \left(\sum_i \varepsilon^4 R_i^4 \ln \frac{l}{\alpha_\varepsilon R_i} \right)^2 \left(\sum_i \frac{R_i}{\ln \frac{l}{\alpha_\varepsilon R_i}} \right)^2 \\ 1 &= \left(\sum_i \varepsilon^2 R_i^2(t) \right)^2 \leq \left(\sum_i \varepsilon^4 R_i^4 \ln \frac{l}{\alpha_\varepsilon R_i} \right) \left(\sum_i \frac{1}{\ln \frac{l}{\alpha_\varepsilon R_i}} \right) \end{aligned}$$

we have $\left(\sum_i \frac{1}{\ln \frac{1}{\alpha_\varepsilon R_i}}\right)^{-1} \leq \sum_i \varepsilon^4 R_i^4 \ln \frac{1}{\alpha_\varepsilon R_i} \leq C$ due to (84). As a consequence, we also have $\sum_i \frac{1}{\alpha_\varepsilon} \leq C$.

Taking $m = 0$ in (98), we find

$$\begin{aligned}
 |\lambda^\varepsilon(t)| &\leq C \left(\sum_i \frac{a^\varepsilon}{R_i} + \sum_i \varepsilon^2 (\varepsilon^2 + R_i^2) + \left(\int_{\mathbb{T}} |\nabla u^\varepsilon|^2 + |\nabla \mu^\varepsilon|^2 + |\nabla \phi^\varepsilon|^2 \right)^{1/2} \right) \\
 &\leq C \left(1 + \sum_i \frac{a^\varepsilon}{R_i} + \left(\int_{\mathbb{T}} |\nabla u^\varepsilon|^2 + |\nabla \mu^\varepsilon|^2 + |\nabla \phi^\varepsilon|^2 \right)^{1/2} \right).
 \end{aligned}
 \tag{100}$$

Using

$$\begin{aligned}
 \sum_i \frac{a^\varepsilon}{R_i} &\leq \sum_i \frac{\varepsilon^2}{1 - \varepsilon^2 \ln \frac{R_i}{\gamma}} \frac{1}{R_i} \leq \left(\sum_i \varepsilon^2 \right)^{1/2} \left(\sum_i \frac{\varepsilon^2}{1 - \varepsilon^2 \ln \frac{R_i}{\gamma}} \frac{1}{R_i^2} \right)^{1/2} \\
 &\leq C \left(\int \frac{1}{1 - \varepsilon^2 \ln r} \frac{1}{r^2} dv_i^\varepsilon \right)^{1/2},
 \end{aligned}$$

and Lemma 3.10, it follows [cf. (81) in Sect. 3.3] that

$$\int_0^T |\lambda^\varepsilon(t)| dt \leq C_T.
 \tag{101}$$

Step 4: Completion of Proof. Now we go back to (96), multiply with $\varepsilon^2 R_i^2$ and sum, but only over i such that $R_i \geq M$. We find

$$\begin{aligned}
 \sum_{R_i \geq M} \varepsilon^2 R_i^3 V_i &= \left(- \sum_{R_i \geq M} \varepsilon^2 R_i - \lambda^\varepsilon(t) \sum_{R_i \geq M} \varepsilon^2 R_i^2 \right. \\
 &\quad \left. + \sum_{R_i \geq M} \frac{\varepsilon^2 R_i^2}{2\pi l} \int_{\partial B_l(X_i)} u^\varepsilon + \phi^\varepsilon - 2\sigma \mu^\varepsilon dS \right) (1 + O(\varepsilon)) - \sigma \sum_{R_i \geq M} \varepsilon^2 R_i^4.
 \end{aligned}
 \tag{102}$$

Here we used $\frac{a^\varepsilon}{\varepsilon^2} = \frac{1}{1 + \varepsilon^2 \ln \frac{\gamma}{R_i}} = 1 + O(\varepsilon)$ as $\varepsilon \rightarrow 0$ for i such that $R_i \geq M$, $\varepsilon R_i \leq 1$ and $M > \gamma$.

The key feature of (102) is that we have a term of the form $-\sum_{R_i \geq M} \varepsilon^2 R_i^4$ on the right hand side and otherwise terms which converge to zero as $M \rightarrow \infty$ uniformly in ε . Now define

$$y(t) := \frac{1}{4} \sum_{R_i \geq M} \varepsilon^2 R_i^4(t)$$

such that it follows from (102) together with (98) with $m = 2$ that

$$\begin{aligned}
 y'(t) + 4\sigma y(t) &\leq C \left\{ (|\lambda^\varepsilon(t)| + 1) \sum_{R_i \geq M} \varepsilon^2 R_i^2 \right. \\
 &\quad \left. + \left(\int_{\mathbb{T}} |\nabla u^\varepsilon|^2 + |\nabla \mu^\varepsilon|^2 + |\nabla \phi^\varepsilon|^2 \right)^{1/2} \left(\left(\sum_{R_i \geq M} \varepsilon^2 R_i^{12/5} \right)^{5/6} + \varepsilon \left(\sum_{R_i \geq M} \varepsilon^2 R_i^4 \right)^{1/2} \right) \right\}.
 \end{aligned}
 \tag{103}$$

Now

$$\sum_{R_i \geq M} \varepsilon^2 R_i^2 \leq \frac{1}{M^2} \sum_{R_i \geq M} \varepsilon^2 R_i^4 \leq \frac{C}{M^2}, \tag{104}$$

$$\sum_{R_i \geq M} \varepsilon^2 R_i^{12/5} \leq \frac{1}{M^{8/5}} \sum_{R_i \geq M} \varepsilon^2 R_i^4 \leq \frac{C}{M^{8/5}}. \tag{105}$$

To estimate the last term on the right-hand side of (103) we recall that we have the simple estimate $R_i \leq C\varepsilon^{-1/2}$ such that

$$\varepsilon \left(\sum_{R_i \geq M} \varepsilon^2 R_i^4 \right)^{1/2} \leq \varepsilon^{1/2} \left(\sum_{R_i \geq M} \varepsilon^2 R_i^2 \right)^{1/2} \leq C \frac{\varepsilon^{1/2}}{M}, \tag{106}$$

$$\sum_{R_i \geq M} \varepsilon^4 R_i^4 = \varepsilon^2 \sum_{R_i \geq M} \varepsilon^2 R_i^4 \leq C \frac{\varepsilon}{M^2}. \tag{107}$$

Thus, (103)–(107) imply that we have an estimate of the form

$$y'(t) + 4\sigma y(t) \leq C \frac{f(t)}{M^\delta} \quad \text{with} \quad \int_0^T |f(t)| dt \leq C_T \quad \text{and some } \delta > 0. \tag{108}$$

A simple comparison argument now implies that indeed $y(t) \leq C(y(0) + M^{-\delta})$ and thus the statement of the lemma follows with (26). □

Proof of Lemma 3.7 We use an argument similar to Step 2 of the proof of [13, Lemma 5.4], and give only an outline of the proof. It follows from the definition of K (59) that

$$\int_{\mathbb{T}} \nabla K \cdot \nabla \zeta dx = 2\pi \left(\int r^2 \zeta dv_t - \int_{\mathbb{T}} \zeta dx \right) = 2\pi \int r^2 \zeta dv_t \tag{109}$$

for all $\zeta \in \dot{H}^1(\mathbb{T})$. Furthermore since $\Delta \tilde{\mu}^\varepsilon = \pi$ in $\mathbb{T} \setminus \cup_i (\partial B_l(X_i) \cup \partial B_i)$, $[\nabla \tilde{\mu}^\varepsilon \cdot \vec{n}] = 0$ on ∂B_i , and $[\nabla \tilde{\mu}^\varepsilon \cdot \vec{n}] = -\frac{\varepsilon^2 R_i^2}{2l}$ on $\partial B_l(X_i)$, we see that $\tilde{\mu}^\varepsilon$ solves

$$\begin{aligned} \int_{\mathbb{T}} \nabla \tilde{\mu}^\varepsilon \cdot \nabla \zeta dx &= \sum_i \int_{\partial B_l(X_i)} [\nabla \tilde{\mu}^\varepsilon \cdot \vec{n}] \zeta dS - \pi \int_{\mathbb{T}} \zeta dx \\ &= \sum_i \int_{\partial B_l(X_i)} \frac{lR_i^2}{2\gamma^2} \zeta dS \end{aligned}$$

for all $\zeta \in \dot{H}^1(\mathbb{T})$. We define

$$\langle L_\varepsilon(t), \zeta \rangle := \sum_i \int_{\partial B_l(X_i)} \frac{lR_i^2}{\gamma^2} \zeta dS, \quad \langle L(t), \zeta \rangle := 2\pi \int r^2 \zeta dv_t$$

for $\zeta \in \dot{H}^1(\mathbb{T})$. Then $L_\varepsilon(t)$ and $L(t)$ are elements of $\dot{H}^{-1}(\mathbb{T}) = (\dot{H}^1(\mathbb{T}))^*$ for all $t \in [0, T]$. Since $\langle L_\varepsilon(t), \zeta \rangle = 2 \int_{\mathbb{T}} \nabla \tilde{\mu}^\varepsilon \cdot \nabla \zeta dx$ and $\langle L(t), \zeta \rangle = \int_{\mathbb{T}} \nabla K \cdot \nabla \zeta dx$, we only need to prove that $L_\varepsilon \rightarrow L$ strongly in $C^0([0, T]; \dot{H}^{-1}(\mathbb{T}))$.

In order to prove this, define

$$\langle \tilde{L}_\varepsilon(t), \zeta \rangle := \sum_i \int_{B_l(X_i)} \frac{2R_i^2}{\gamma^2} \zeta dx.$$

Then we see that $\tilde{L}_\varepsilon(t) \in (L^4(\mathbb{T}))^* \cong L^{4/3}(\mathbb{T})$ and satisfies $\|\tilde{L}_\varepsilon(t)\|_{L^{4/3}} \leq C$ for all $t \in [0, T]$ and small $\varepsilon > 0$. Here L^2 estimate does not work. Furthermore for all $t_1, t_2 \in [0, T]$ we have

$$\|\tilde{L}_\varepsilon(t_1) - \tilde{L}_\varepsilon(t_2)\|_{L^{4/3}} \leq C|t_1 - t_2|^{1/2}(\mathcal{D}^\varepsilon)^{1/2}.$$

These mean that $\tilde{L}_\varepsilon \in C^0([0, T]; L^{4/3}(\mathbb{T}))$ is uniformly bounded and equicontinuous. It follows from the compact embedding of $L^{4/3}(\mathbb{T})$ into $\dot{H}^{-1}(\mathbb{T})$ and the generalized Arzela-Ascoli Theorem that \tilde{L}_ε is relatively compact in $C^0([0, T]; \dot{H}^{-1}(\mathbb{T}))$.

Since we can deduce that $\tilde{L}_\varepsilon(t) \rightharpoonup L(t)$ weakly in $(C^0_\rho)^*$ for each $t \in [0, T]$, and that $L_\varepsilon - \tilde{L}_\varepsilon \rightarrow 0$ in $C^0([0, T]; \dot{H}^{-1}(\mathbb{T}))$, the proof of Lemma is completed. \square

Proof of Lemma 3.8 The differential of the nonlocal energy is given by

$$\begin{aligned} \langle DE_{\text{nl},\varepsilon}(\mathbf{Y}^\varepsilon), \tilde{\mathbf{Z}}^\varepsilon \rangle &= -2 \int_{\mathbb{T}} \nabla \mu^\varepsilon \cdot \nabla \tilde{w}^\varepsilon \, dx \\ &= -2 \int_{\mathbb{T}} \nabla \mu^\varepsilon \cdot (\nabla \tilde{u}^\varepsilon + \nabla \tilde{\phi}^\varepsilon) \, dx \end{aligned} \tag{110}$$

with \tilde{u}^ε and $\tilde{\phi}^\varepsilon$ determined via (31).

Define u_i as in (72), however we take $V_i = \tilde{V}_i$, and set $\tilde{u}^\varepsilon = \sum_i u_i + \tilde{u}^\varepsilon$. Note that

$$\nabla u_i = \begin{cases} \varepsilon^2 R_i \tilde{V}_i \frac{x-X_i}{|x-X_i|^2} & : \alpha_\varepsilon R_i < |x - X_i| < l \\ 0 & : \text{otherwise} \end{cases} . \tag{111}$$

Also we use the same notations $\tilde{\mu}_i, \tilde{\mu}^\varepsilon$ as in Sect. 3.4.

We write

$$\begin{aligned} &\int_0^T \beta \int_{\mathbb{T}} \nabla \tilde{u}^\varepsilon \cdot \nabla \mu^\varepsilon \, dx \, dt \\ &= \int_0^T \beta \int_{\mathbb{T}} \nabla \tilde{u}^\varepsilon \cdot \left(\nabla \sum_j \tilde{\mu}_j + \nabla \frac{K}{2} \right) \, dx \, dt + \int_0^T \beta \int_{\mathbb{T}} \nabla \tilde{u}^\varepsilon \cdot \nabla \left(\tilde{\mu}^\varepsilon - \frac{K}{2} \right) \, dx \, dt \\ &= - \int_0^T \beta \sum_i \int_{\partial B_i} \frac{\varepsilon^2 \tilde{V}_i}{\alpha_\varepsilon} \sum_j \tilde{\mu}_j \, dS \, dt - \int_0^T \frac{\beta}{2} \sum_i \int_{B_i(X_i)} \nabla u_i \cdot \nabla K \, dx \, dt \\ &\quad + \int_0^T \frac{\beta}{2} \int_{\mathbb{T}} \nabla \tilde{u}^\varepsilon \cdot \nabla K \, dx \, dt + \int_0^T \beta \int_{\mathbb{T}} \nabla \tilde{u}^\varepsilon \cdot \nabla \left(\tilde{\mu}^\varepsilon - \frac{K}{2} \right) \, dx \, dt. \end{aligned} \tag{112}$$

It follows from the condition (60) that

$$(1 - o(1))4\pi \int_0^T \sum_i \varepsilon^2 R_i^2 \tilde{V}_i^2 \, dt \leq \int_0^T \int_{\mathbb{T}} |\nabla \tilde{u}^\varepsilon|^2 \, dx \, dt \leq C \tag{113}$$

and that \tilde{u}^ε converges to \tilde{u} strongly in $L^2((0, T) \times \mathbb{T})$, where \tilde{u} is determined via (53). (See the proof of [13, Lemma 5.4] for details.) Using

$$\left| \sum_{R_i \geq M} \varepsilon^2 R_i^3 \tilde{V}_i \right| \leq \left(\sum_{R_i \geq M} \varepsilon^2 R_i^4 \right)^{1/2} \left(\sum_i \varepsilon^2 R_i^2 \tilde{V}_i^2 \right)^{1/2} ,$$

(113), Lemma 3.9, $|R_i \ln 1/R_i| \leq C$ for $R_i \ll 1$ and $\varepsilon R_i \leq 1$, we see that

$$\begin{aligned}
 - \int_0^T \beta \sum_i \int_{\partial B_i} \frac{\varepsilon^2 \tilde{V}_i}{\alpha_\varepsilon} \sum_j \tilde{\mu}_j \, dS \, dt &= - \int_0^T \beta \sum_i \int_{\partial B_i} \frac{\varepsilon^2 \tilde{V}_i}{\alpha_\varepsilon} \tilde{\mu}_i \, dS \, dt \\
 &= - \int_0^T \beta \sum_i \pi \varepsilon^4 R_i^3 \tilde{V}_i \ln \frac{l}{\alpha_\varepsilon R_i} \, dt \\
 &\rightarrow -\pi \int_0^T \beta \int r^3 \tilde{v} \, dv_t \, dt \tag{114}
 \end{aligned}$$

as $\varepsilon \rightarrow 0$. In addition, we obtain

$$\begin{aligned}
 \lim_{\varepsilon \rightarrow 0} \int_0^T \beta \int_{\mathbb{T}} \nabla \tilde{u}^\varepsilon \cdot \nabla K \, dx \, dt &= \int_0^T \beta \int_{\mathbb{T}} \nabla \tilde{u} \cdot \nabla K \, dx \, dt \\
 &= -2\pi \int_0^T \beta \int r K(t, x) \tilde{v} \, dv_t \, dt, \tag{115}
 \end{aligned}$$

and it follows from Lemma 3.7 that

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \beta \int_{\mathbb{T}} \nabla \tilde{u}^\varepsilon \cdot \nabla \left(\tilde{\mu}^\varepsilon - \frac{K}{2} \right) \, dx \, dt = 0. \tag{116}$$

Finally from $\sum_i |B_l(X_i)| \leq C\gamma^2 \sum_i \varepsilon^2 \leq C\gamma^2$ and (111),

$$\begin{aligned}
 \left| \sum_i \int_{B_l(X_i)} \nabla u_i \cdot \nabla K \, dx \right| &\leq \sum_i \left(\int_{B_l(X_i)} |\nabla u_i|^2 \, dx \right)^{1/2} \left(\int_{B_l(X_i)} |\nabla K|^2 \, dx \right)^{1/2} \\
 &\leq \left(\sum_i \int_{B_l(X_i)} |\nabla u_i|^2 \, dx \right)^{1/2} \left(\sum_i \int_{B_l(X_i)} |\nabla K|^2 \, dx \right)^{1/2} \\
 &\leq C\omega(C\gamma^2), \tag{117}
 \end{aligned}$$

where

$$\omega(z) := \sup_{|E| \leq z} \left(\int_E |\nabla K|^2 \, dx \right)^{1/2}. \tag{118}$$

Since we can choose γ arbitrarily small, we obtain

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \beta \int_{\mathbb{T}} \nabla \tilde{u}^\varepsilon \cdot \nabla \mu^\varepsilon \, dx \, dt = -\pi \int_0^T \beta \int (r^3 + rK(x)) \tilde{v} \, dv_t \, dt$$

from (112), (114), (115), (116) and (117).

Similarly we can prove that $\int_0^T \beta \int_{\mathbb{T}} \nabla \tilde{\phi}^\varepsilon \cdot \nabla \mu^\varepsilon \, dx \, dt \rightarrow 0$ as $\varepsilon \rightarrow 0$. This completes the proof. □

Acknowledgements Open Access funding provided by Projekt DEAL.

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A Appendix

Lemma A.1 $A = \{x \in \mathbb{R}^2; a < |x| < b\}$. *There holds*

$$\left| \frac{1}{\pi b^2} \int_A f \, dx \right|^2 \leq \frac{1}{2\pi} \left(\ln \frac{b}{a} \right) \int_A |\nabla f|^2 \, dx \tag{119}$$

for all $f \in H^1(A)$ such that $\int_{\partial B_a} f \, dS = 0$.

Proof Using the polar coordinate (r, θ) , we write $f = f(r, \theta)$. Let

$$g(r) = \int_0^{2\pi} f(r, \theta) \, d\theta$$

for $r \in [a, b]$. Then by

$$g'(s) = \int_0^{2\pi} f_r(s, \theta) \, d\theta,$$

and

$$g'(s)^2 \leq 2\pi \int_0^{2\pi} f_r(s, \theta)^2 \, d\theta,$$

for $s \in [a, b]$, we get

$$\int_a^b g'(s)^2 s \, ds \leq 2\pi \int_a^b s \int_0^{2\pi} f_r(s, \theta)^2 \, d\theta \, ds \leq 2\pi \|\nabla f\|_2^2.$$

Note that $g(r) = \int_a^r g'(s) \, ds$ by $g(a) = 0$. Hence

$$\begin{aligned} |g(r)| &= \left| \int_a^r g'(s) \, ds \right| \leq \left(\int_a^r g'(s)^2 s \, ds \right)^{1/2} \left(\int_a^r \frac{1}{s} \, ds \right)^{1/2} \\ &\leq \sqrt{2\pi} \|\nabla f\|_2 \left(\ln \frac{r}{a} \right)^{1/2}. \end{aligned}$$

We find

$$\begin{aligned} \left| \int_A f \, dx \right| &= \left| \int_a^b g(r)r \, dr \right| \leq \sqrt{2\pi} \|\nabla f\|_2 \int_a^b r \left(\ln \frac{r}{a} \right)^{1/2} \, dr \\ &\leq \sqrt{2\pi} \|\nabla f\|_2 \frac{b^2}{2} \left(\ln \frac{b}{a} \right)^{1/2}. \end{aligned}$$

□

Remark: Similarly we can also show that

$$\lim_{\varepsilon \rightarrow 0} E(\mathbf{Y}^\varepsilon) = E(v_t), \quad \text{uniformly in } t \in [0, T],$$

where

$$E(v_t) = 2\pi \int \left(r + \sigma \frac{r^4}{4} \right) dv_t + \frac{\sigma}{4} \int_{\mathbb{T}} |\nabla K|^2 \, dx.$$

This statement is of interest in itself, but since we do not need it here we omit the proof for the sake of brevity.

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