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Calculus of Variations

Local curvature estimates for the Laplacian flow

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Abstract

In this paper we give local curvature estimates for the Laplacian flow on closed G_2-structures under the condition that the Ricci curvature is bounded along the flow. The main ingredient consists of the idea of Kotschwar et al. (J Funct Anal 271(9):2604–2630, 2016) who gave local curvature estimates for the Ricci flow on complete manifolds and then provided a new elementary proof of Sesum's result (Sesum in Am J Math 127(6):1315–1324, 2005), and the particular structure of the Laplacian flow on closed G_2-structures. As an immediate consequence, this estimates give a new proof of Lotay and Wei's (Geom Funct Anal 27(1):165–233, 2017) result which is an analogue of Sesum's theorem. The second result is about an interesting evolution equation for the scalar curvature of the Laplacian flow of closed G_2-structures. Roughly speaking, we can prove that the time derivative of the scalar curvature $R_{g(t)}$ is equal to the Laplacian of $R_{g(t)}$, plus an extra term which can be written as the difference of two nonnegative quantities.

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1 Introduction

Let $\mathcal{M}$ be a smooth 7-manifold. The Laplacian flow for closed G_2-structures on $\mathcal{M}$ introduced by Bryant [1] is to study the torsion-free G_2-structures

$$\partial_t \varphi(t) = \Delta_{\varphi(t)}\varphi(t), \quad \varphi(0) = \varphi, \tag{1.1}$$

where $\Delta_{\varphi(t)}\varphi(t) = dd^*_{\varphi(t)}\varphi(t) + d^*_{\varphi(t)}d\varphi(t)$ is the Hodge Laplacian of $g_{\varphi(t)}$ and φ is an initial closed G_2-structure. Since $d\partial_t\varphi(t) = \partial_t d\Delta_{\varphi(t)}\varphi(t) = 0$, we see that the flow (1.1) preserves the closedness of $\varphi(t)$. For more background on G_2-structures, see Sect. 2. When

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$\mathcal{M}$ is compact, the flow (1.1) can be viewed as the gradient flow for the Hitchin functional introduced by Hitchin [18]

$$\mathscr{H} : [\overline{\varphi}]_+ \longrightarrow \mathbb{R}^+, \quad \varphi \longmapsto \frac{1}{7}\int_{\mathcal{M}} \varphi \wedge \psi = \int_{\mathcal{M}} *_{\varphi} 1. \tag{1.2}$$

Here $\overline{\varphi}$ is a closed G_2-structure on $\mathcal{M}$ and $[\overline{\varphi}]_+$ is the open subset of the cohomology class $[\overline{\varphi}]$ consisting of G_2-structures. Any critical point of $\mathscr{H}$ gives a torsion-free G_2-structure.

The study of Laplacian flows on some special 7-manifolds, Laplacian solitons, and other flows on G_2-structures can be found in [13–16,19,24,29,33,34,38,39].

Recently, Donaldson [7–10] studied the co-associative Kovalev-Lefschetz fibrations G_2-manifolds and G_2-manifolds with boundary.

1.1 Notions and conventions

To state the main results, we fix our notions used throughout this paper. Let $\mathcal{M}$ be as before a smooth 7-manifold. The space of smooth functions and the space of smooth vector fields are denoted respectively by $C^\infty(\mathcal{M})$ and $\mathfrak{X}(\mathcal{M})$. The space of k-tenors (i.e., $(0, k)$-covariant tensor fields) and k-forms on $\mathcal{M}$ are denoted, respectively, by $\otimes^k(\mathcal{M}) = C^\infty(\otimes^k(T^*\mathcal{M}))$ and $\wedge^k(\mathcal{M}) = C^\infty(\wedge^k(T^*\mathcal{M}))$. For any k-tensor field $\boldsymbol{T} \in \otimes^k(\mathcal{M})$, we locally have the expression $\boldsymbol{T} = \boldsymbol{T}_{i_1\cdots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k} =: \boldsymbol{T}_{i_1\cdots i_k} dx^{i_1\otimes\cdots\otimes i_k}$. A k-form α on $\mathcal{M}$ can be written in the *standard form* as $\alpha = \frac{1}{k!}\alpha_{i_1\cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} =: \frac{1}{k!}\alpha_{i_1\cdots i_k} dx^{i_1\wedge\cdots\wedge i_k}$, where $\alpha_{i_1\cdots i_k}$ is fully skew-symmetric in its indices. Using the standard forms, if we take the interior product $X \lrcorner \alpha$ of a k-form $\alpha \in \wedge^k(\mathcal{M})$ with a vector field $X \in \mathfrak{X}(\mathcal{M})$, we obtain the $(k-1)$-form $X \lrcorner \alpha = \frac{1}{(k-1)!} X^m \alpha_{m i_1\cdots i_{k-1}} dx^{i_1\wedge\cdots\wedge i_{k-1}}$ which is also in the standard form. In particular, consider the vector space $\otimes^2(\mathcal{M})$ of 2-tensors. For any 2-tensor $\boldsymbol{A} = \boldsymbol{A}_{ij} dx^{i\otimes j}$, define $\boldsymbol{A}^{\odot} := \frac{1}{2}(\boldsymbol{A}_{ij} + \boldsymbol{A}_{ji}) dx^{i\otimes j} \equiv \boldsymbol{A}^{\odot}_{ij} dx^{i\otimes j}$ and $\boldsymbol{A}^{\wedge} := \frac{1}{2}(\boldsymbol{A}_{ij} - \boldsymbol{A}_{ji}) dx^{i\otimes j} \equiv \boldsymbol{A}^{\wedge}_{ij} dx^{i\otimes j}$. Then $\boldsymbol{A}^{\odot}$ is an element of $\odot^2(\mathcal{M})$, the space of symmetric 2-tensors. Since[1] $dx^{i\wedge j} = dx^{i\otimes j} - dx^{j\otimes i}$, it follows that $\boldsymbol{A}^{\wedge} = \frac{1}{2}\boldsymbol{A}_{ij} dx^{i\wedge j}$. Define $\alpha^{\boldsymbol{A}} := \frac{1}{2}\alpha^{\boldsymbol{A}}_{ij} dx^{i\wedge j}$ with $\alpha^{\boldsymbol{A}}_{ij} := \boldsymbol{A}_{ij}$. Then we see that $\alpha^{\boldsymbol{A}} = \boldsymbol{A}^{\wedge} \in \wedge^2(\mathcal{M})$ and $\otimes^2(\mathcal{M}) = \odot^2(\mathcal{M}) \oplus \wedge^2(\mathcal{M})$.

A given Riemannian metric g on $\mathcal{M}$ determines two isomorphisms between vector fields and 1-forms: $\flat_g : \mathfrak{X}(\mathcal{M}) \longrightarrow \wedge^1(\mathcal{M})$ and $\sharp_g : \wedge^1(\mathcal{M}) \longrightarrow \mathfrak{X}(\mathcal{M})$, where, for every vector field $X = X^i \frac{\partial}{\partial x^i}$ and 1-form $\alpha = \alpha_i dx^i$, $\flat_g(X) = X^i g_{ij} dx^j \equiv X_j dx^j$ and $\sharp_g(\alpha) = \alpha_i g^{ij} \frac{\partial}{\partial x^j} \equiv \alpha^j \frac{\partial}{\partial x^j}$. Using these two natural maps, we can frequently raise or lower indices on tensors. The metric g also induces a metric on k-forms $g(dx^{i_1\wedge\cdots\wedge i_k}, dx^{j_1\wedge\cdots\wedge j_k}) = \det(g(dx^{i_a}, dx^{j_b})) = \sum_{\sigma\in\mathfrak{S}_7} \mathrm{sgn}(\sigma) g^{i_1 j_{\sigma(1)}} \cdots g^{i_k j_{\sigma(k)}}$ where $\mathfrak{S}_7$ is the group of permutations of seven letters and $\mathrm{sgn}(\sigma)$ denotes the sign (± 1) of an element σ of $\mathfrak{S}_7$. The inner product $\langle\cdot,\cdot\rangle_g$ of two k-forms $\alpha, \beta \in \wedge^k(\mathcal{M})$ now is given by $\langle\alpha, \beta\rangle_g = \frac{1}{k!}\alpha_{i_1\cdots i_k}\beta^{i_1\cdots i_k} = \frac{1}{k!}\alpha_{i_1\cdots i_k}\beta_{j_1\cdots j_k} g^{i_1 j_1}\cdots g^{i_k j_k}$.

Given two 2-tensors $\boldsymbol{A}, \boldsymbol{B} \in \otimes^2(\mathcal{M})$, with the forms $\boldsymbol{A} = \boldsymbol{A}_{ij} dx^{i\otimes j}$ and $\boldsymbol{B} = \boldsymbol{B}_{ij} dx^{i\otimes j}$. Define $\langle\langle \boldsymbol{A}, \boldsymbol{B}\rangle\rangle_g := \boldsymbol{A}_{ij}\boldsymbol{B}^{ij}$. There are two special cases which will be used later:

[1] In our convention, for any 2-form $\alpha = \frac{1}{2}\alpha_{ij} dx^{ij}$, we have

$$\alpha\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell}\right) = \frac{1}{2}\alpha_{ij}\left(dx^{i\otimes j} - dx^{j\otimes i}\right)\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell}\right) = \frac{1}{2}\alpha_{ij}\left(\delta^i_k\delta^j_\ell - \delta^j_k\delta^i_\ell\right) = \frac{1}{2}\left(\alpha_{k\ell} - \alpha_{\ell k}\right) = \alpha_{k\ell}$$

which justifies the notion $\alpha_{k\ell}$ as $\alpha(\partial/\partial x^k, \partial/\partial x^\ell)$. In general, for any k-form $\alpha = \frac{1}{k!}\alpha_{i_1\cdots i_k} dx^{i_1\wedge\cdots\wedge i_k}$ we have $\alpha_{i_1\cdots i_k} = \alpha(\partial/\partial x^{i_1}, \cdots, \partial/\partial x^{i_k})$, because $dx^{i_1\wedge\cdots\wedge i_k} = \sum_{\sigma\in\mathfrak{S}_k} \mathrm{sgn}(\sigma) dx^{i_{\sigma(1)}\otimes\cdots\otimes i_{\sigma(k)}}$.

(1) $\alpha = \frac{1}{2}\alpha_{ij}dx^{i\wedge j} \in \wedge^2(\mathcal{M})$ and $\boldsymbol{B} = \boldsymbol{B}_{ij}dx^{i\otimes j} \in \otimes^2(\mathcal{M})$. In this case, α can be written as a 2-tensor $\boldsymbol{A}^\alpha = \boldsymbol{A}^\alpha_{ij}dx^{i\otimes j}$ with $\boldsymbol{A}^\alpha_{ij} = \alpha_{ij}$. Then $\langle\langle\alpha, \boldsymbol{B}\rangle\rangle_g := \langle\langle\boldsymbol{A}^\alpha, \boldsymbol{B}\rangle\rangle_g = \alpha_{ij}\boldsymbol{B}^{ij}$.

(2) $\alpha = \frac{1}{2}\alpha_{ij}dx^{i\wedge j}$ and $\beta = \frac{1}{2}\beta_{ij}dx^{i\wedge j} \in \wedge^2(\mathcal{M})$. In this case, α, β can be both written as 2-tensors $\boldsymbol{A}^\alpha = \boldsymbol{A}^\alpha_{ij}dx^{i\otimes j}$ and $\boldsymbol{B}^\beta = \boldsymbol{B}^\beta_{ij}dx^{i\otimes j}$ with $\boldsymbol{A}^\alpha_{ij} = \alpha_{ij}$ and $\boldsymbol{B}^\beta_{ij} = \beta_{ij}$. Then $\langle\langle\alpha, \beta\rangle\rangle_g := \langle\langle\boldsymbol{A}^\alpha, \boldsymbol{B}^\beta\rangle\rangle_g = \alpha_{ij}\beta^{ij} = 2\langle\alpha, \beta\rangle_g$.

The norm of $\boldsymbol{A} \in \otimes^2(\mathcal{M})$ is defined by $||\boldsymbol{A}||^2_g := \langle\langle\boldsymbol{A}, \boldsymbol{A}\rangle\rangle_g = \boldsymbol{A}_{ij}\boldsymbol{A}^{ij}$, while the norm of $\alpha \in \wedge^k(\mathcal{M})$ is $|\alpha|^2_g := \langle\alpha, \alpha\rangle_g = \frac{1}{k!}\alpha_{i_1\cdots i_k}\alpha^{i_1\cdots i_k}$. In particular, $||X||^2_g = X_i X^i = |\flat_g(X)|^2_g$ and $||\alpha||^2_g = 2|\alpha|^2_g$, for any vector field $X \in \mathfrak{X}(\mathcal{M})$ and 2-form α.

The Levi–Civita connection associated to a given Riemannian metric g is denoted by ∇_g or simply ∇. Our convention on Riemann curvature tensor is $R^m_{ijk}\frac{\partial}{\partial x^m} := \mathrm{Rm}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^k} = (\nabla_i\nabla_j - \nabla_j\nabla_i)\frac{\partial}{\partial x^k}$ and $R_{ijk\ell} := R^m_{ijk}g_{m\ell}$. The Ricci curvature of g is given by $R_{jk} := R_{ijk\ell}g^{i\ell}$. We use dV_g and $*_g$ to denote the volume form and Hodge star operator, respectively, on $\mathcal{M}$ associated to a metric g and an orientation.

We use the standard notion $A * B$ to denote some linear combination of contractions of the tensor product $A \otimes B$ relative to the metric $g(t)$ associated the $\varphi(t)$. In Theorem 1.4 and its proof, all universal constants c, C below depend only on the given real number p.

1.2 Main results

Applying De Turck's trick and Hamilton's Nash-Moser inverse function theorem, Bryant and Xu [2] proved the following local time existence for (1.1).

Theorem 1.1 **(Bryant-Xu** [2]*) For a compact* 7*-manifold* $\mathcal{M}$*, the initial value problem (1.1) has a unique solution for a short time interval* $[0, T_{\max})$ *with the maximal time* $T_{\max} \in (0, \infty]$ *depending on* φ*.*

As in the Ricci flow, we can prove following results on the long time existence for the Laplacian flow (1.1).

Theorem 1.2 (Lotay-Wei [32]) *Let* $\mathcal{M}$ *be a compact* 7*-manifold and* $\varphi(t)$*,* $t \in [0, T)$*, where* $T < \infty$*, be a solution to the flow* (1.1) *for closed* G_2*-structures with associated metric* $g(t) = g_{\varphi(t)}$ *for each* t*.*

(a) *If the velocity of the flow satisfies*

$$\sup_{\mathcal{M}\times[0,T)} ||\Delta_{g(t)}\varphi(t)||_{g(t)} < \infty,$$

then the solution φ_t *can be extended past time* T*.*

(b) *If* $T = T_{\max}$*, then*

$$\limsup_{t\to T_{\max}} \max_{\mathcal{M}} \left(||\mathrm{Rm}_{g(t)}||^2_{g(t)} + ||\nabla_{g(t)}\boldsymbol{T}(t)||^2_{g(t)}\right) = \infty.$$

Here $\boldsymbol{T}(t)$ *is the torsion of* $\varphi(t)$ *[see* (2.14)*].*

In this paper, we give a new elementary proof of Theorem 1.2, based on the idea of [25] and the structure of the Eq. (1.1).

Theorem 1.3 *Let $\mathcal{M}$ be a compact 7-manifold and $\varphi(t)$, $t \in [0, T)$, where $T < \infty$, be a solution to the flow* (1.1) *for closed G_2-structures with associated metric $g(t) = g_{\varphi(t)}$ for each t. Suppose that*

$$K := \sup_{\mathcal{M}\times[0,T)} ||\mathrm{Ric}_{g(t)}||_{g(t)} < \infty, \quad \Lambda := \max_{\mathcal{M}} ||\mathrm{Rm}_{g(0)}||_{g(0)}.$$

Then

$$\sup_{\mathcal{M}\times[0,T)} ||\mathrm{Rm}_{g(t)}||_{g(t)} < \infty,$$

where the bound depends only on n, K, T and Λ.

When $\mathcal{M}$ is compact, the theorem immediately implies the part (a) in Theorem 1.2. Indeed, we shall show that [see (3.10) and (3.29)]

$$\sup_{\mathcal{M}\times[0,T)} ||\Delta_{g(t)}\varphi(t)||_{g(t)} < \infty \Longleftrightarrow \sup_{\mathcal{M}\times[0,T)} ||\mathrm{Ric}_{g(t)}||_{g(t)} < \infty.$$

In the compact case, Theorem 1.3 shows that, if the conclusion in part (a) does not hold, then $T = T_{\max}$ and $\sup_{\mathcal{M}\times[0,T_{\max})} ||\mathrm{Rm}_{g(t)}||_{g(t)} < \infty$ which implies the quantity $\sup_{\mathcal{M}\times[0,T_{\max})} (||\mathrm{Rm}_{g(t)}||^2_{g(t)} + ||\nabla_{g(t)}\boldsymbol{T}(t)||^2_{g(t)})$ is finite, since the norm $||\nabla_{g(t)}\boldsymbol{T}(t)||^2_{g(t)}$ can be controlled by $||\mathrm{Rm}_{g(t)}||^2_{g(t)}$ [see (3.58)]. However, by part (b) in Theorem 1.2, it is impossible. Therefore, the conclusion in part (a) is true.

As remarked in [25], to prove Theorem 1.3, it suffices to establish the following integral estimate.

Theorem 1.4 *Let $\mathcal{M}$ be a smooth 7-manifold and $\varphi(t)$, $t \in [0, T)$, where $T < \infty$, be a solution to the flow* (1.1) *for closed G_2-structures with associated metric $g(t) = g_{\varphi(t)}$ for each t. Assume that there exist constants $A, K > 0$ and a point $x_0 \in \mathcal{M}$ such that the geodesic ball $B_{g(0)}(x_0, A/\sqrt{K})$ is compactly contained in $\mathcal{M}$ and that*

$$|\mathrm{Ric}_{g(t)}|_{g(t)} \le K \quad \text{on } B_{g(0)}\left(x_0, \frac{A}{\sqrt{K}}\right) \times [0, T].$$

Then, for any $p \ge 5$, there exists $c = c(p) > 0$ so that

$$\begin{aligned} &\int_{B_{g(0)}(x_0, A/2\sqrt{K})} ||\mathrm{Rm}_{g(t)}||^p_{g(t)} dV_t \\ &\quad \le c(1+K)e^{cKT} \int_{B_{g(0)}(x_0, A/\sqrt{K})} ||\mathrm{Rm}_{g(0)}||^p_{g(0)} dV_{g(0)} \\ &\qquad + cK^p\left(1 + A^{-2p}\right)e^{cKT}\mathrm{Vol}_{g(t)}\left(B_{g(0)}\left(x_0, \frac{A}{\sqrt{K}}\right)\right) \end{aligned} \tag{1.3}$$

for all $t \in [0, T]$.

Now by the standard De Giorgi–Nash–Moser iteration (our manifold is compact and the Ricci curvature is uniformly bounded), under the condition in Theorem 1.4, we can prove

$$||\mathrm{Rm}_{g(T)}||_{g(T)}(x_0) \le d_1(d_2 + \Lambda_0), \tag{1.4}$$

where d_1, d_2 are constants depending on K, T, A, and

$$\Lambda_0 := \sup_{B_{g(0)}(x_0, A/\sqrt{K})} ||\mathrm{Rm}_{g(0)}||_{g(0)}.$$

Actually, this follows from the same argument in [25] by noting that

$$(\Delta_{g(t)} - \partial_t)||\mathrm{Rm}_{g(t)}||_{g(t)} \geq -c||\mathrm{Rm}_{g(t)}||^2_{g(t)}. \tag{1.5}$$

To verify (1.5), we use (2.26), (3.56) and (3.60) to deduce that

$$||\nabla_{g(t)}\boldsymbol{T}(t)||_{g(t)} \leq c||\mathrm{Rm}_{g(t)}||_{g(t)}$$

and

$$||\nabla^2_{g(t)}\boldsymbol{T}(t)||_{g(t)} \leq c||\nabla_{g(t)}\mathrm{Rm}_{g(t)}||_{g(t)} + c||\mathrm{Rm}_{g(t)}||^{3/2}_{g(t)}.$$

Then, by (3.23) and the Cauchy inequality

$$\begin{aligned} ||\nabla_{g(t)}\mathrm{Rm}_{g(t)}||^2_{g(t)} &\leq -\frac{1}{2}(\partial_t - \Delta_{g(t)})||\mathrm{Rm}_{g(t)}||^2_{g(t)} + c||\mathrm{Rm}_{g(t)}||^3_{g(t)} \\ &\quad + c||\mathrm{Rm}_{g(t)}||^{3/2}_{g(t)}||\nabla_{g(t)}\mathrm{Rm}_{g(t)}||_{g(t)} \\ &\leq -\frac{1}{2}(\partial_t - \Delta_{g(t)})||\mathrm{Rm}_{g(t)}||^2_{g(t)} \\ &\quad + c||\mathrm{Rm}_{g(t)}||^3_{g(t)} + ||\nabla_{g(t)}\mathrm{Rm}_{g(t)}||^2_{g(t)} \end{aligned}$$

which implies (1.5). Now the estimate (1.4) yields Theorem 1.3.

The analogue of Theorem 1.2 in the Ricci flow was proved by Hamilton [17] (for part (b)) and Sesum [37] (for part (a)). It is an open question (due to Hamilton, see [3]) that the Ricci flow will exist as long as the scalar curvature remains bounded. For the Kähler–Ricci flow [40] or type-I Ricci flow [11], this question was settled. For the general case, some partial result on Hamilton's conjecture was carried out in [3].

For the Ricci-harmonic flow introduce by List [30,31] (see also, [35,36]), the analogue of Theorem 1.2 was proved in [30,31] (see also, [35,36]) and [4] (see [28] for another proof). The author [26,27] extended Cao's result [3] to the Ricci-harmonic flow. The same Hamilton's conjecture was asked by the author in [26,27].

We can ask the same question for the Laplacian flow on closed G_2-structures. In [32] (see p. 171, line -6 to -3, or Open Problem (3) in p. 230), Lotay and Wei asked that whether the Laplacian flow on closed G_2-structures will exist as long as the torsion tensor or scalar curvature remains bounded. Let $g(t)$ be the associated metric of $\varphi(t)$. Then the evolution equation for g_t is given by

$$\partial_t g_{ij} = -2R_{ij} - \frac{4}{3}|\boldsymbol{T}(t)|^2_{g(t)}g_{ij} - 4\boldsymbol{T}_i{}^k\boldsymbol{T}_{kj}. \tag{1.6}$$

For the Laplacian flow on closed G_2-structures, the torsion $\boldsymbol{T}(t)$ is actually a 2-form for each t, hence we use the norm $|\cdot|_{g(t)}$ in (1.6). The standard formula for the scalar curvature $R_{g(t)}$ gives [see (3.15)]

$$\partial_t R_{g(t)} = \Delta_{g(t)}R_{g(t)} + 2||\mathrm{Ric}_{g(t)}||^2_{g(t)} - \frac{2}{3}R^2_{g(t)} + 4R_{ijk\ell}\boldsymbol{T}^{ik}\boldsymbol{T}^{j\ell} + 4(\nabla^j\boldsymbol{T}^{ik})(\nabla_i\boldsymbol{T}_{jk}). \tag{1.7}$$

Now the above mentioned open problem states that

$$\text{Is it ture that } \limsup_{t\to T_{\max}} R_{g(t)} = -\infty?$$

The "minus infinity" comes from the fact that along the Laplacian flow on closed G_2-structures the scalar curvature is always nonpositive [see (2.26)]. The following Proposition 1.5 is motivate to solve this problem, and starts from the basic evolution Eq. (1.7) where the last two terms on the right-hand side do not have good signature. However, using the closedness of $\varphi(t)$ [in particular, the identity (3.15)], we can prove the following interesting evolution equation for $R_{g(t)}$.

Proposition 1.5 *Let $\mathcal{M}$ be a smooth 7-manifold and $\varphi(t)$, $t \in [0,T)$, where $T \in (0,\infty]$, be a solution to the flow (1.1) for closed G_2-structures with associated metric $g(t) = g_{\varphi(t)}$ for each t. Then the scalar curvature $R_{g(t)}$ satisfies*

$$\begin{aligned}\partial_t R_{g(t)} &= \Delta_{g(t)} R_{g(t)} + \Big\{ 2\Big|\Big| R_{ij} + \frac{2}{3}|\boldsymbol{T}(t)|^2_{g(t)} g_{ij} \Big|\Big|^2_{g(t)} + \frac{1}{2}\Big|\Big| R_{ijab}R^{ij}{}_{mn} - \psi_{abmn}\Big|\Big|^2_{g(t)} \\ &\quad + \frac{1}{2}\Big|\Big| 2\boldsymbol{T}_{ia}\boldsymbol{T}_{jb}R^{ij}{}_{mn} - \psi_{abmn}\Big|\Big|^2_{g(t)} \\ &\quad + \frac{1}{2}||2\widehat{\boldsymbol{T}}_{am}\widehat{\boldsymbol{T}}_{bn} - \psi_{abmn}||^2_{g(t)} + 2||\widehat{\boldsymbol{T}}(t)||^2_{g(t)} \\ &\quad + 4||\nabla_{g(t)}\boldsymbol{T}(t)||^2_{g(t)} \Big\} - \Big\{ ||\mathrm{Rm}_{g(t)}||^2_{g(t)} + \frac{26}{9}R^2_{g(t)} + \frac{1}{2}\Big|\Big| R_{ijab}R^{ij}{}_{mn}\Big|\Big|^2_{g(t)} \\ &\quad + 2\Big|\Big| \boldsymbol{T}_{ia}\boldsymbol{T}_{jb}R^{ij}{}_{mn}\Big|\Big|^2_{g(t)} + 2||\widehat{\boldsymbol{T}}_{g(t)}||^4_{g(t)} + 210 \Big\}. \end{aligned} \tag{1.8}$$

Here $\widehat{\boldsymbol{T}}_{ij} = \boldsymbol{T}_i{}^k \boldsymbol{T}_{kj}$.

The evolution Eq. (1.8) can be written simply as

$$\partial_t R_{g(t)} = \Delta_{g(t)} R_{g(t)} + A(t) - B(t) \tag{1.9}$$

for some suitable time-dependent nonnegative functions $A(t)$ and $B(t)$. By the maximum principle we obtain

$$R_{\max}(0) + \int_0^t \max_{\mathcal{M}}[A(\tau) - B(\tau)]d\tau \geq R_{g(t)} \geq R_{\min}(0) + \int_0^t \min_{\mathcal{M}}[A(\tau) - B(\tau)]d\tau.$$

Here $R_{\max}(0) := \max_{\mathcal{M}} R_{g(0)}$ and $R_{\min}(0) := \min_{\mathcal{M}} R_{g(0)}$. Observe that the above well-arranged evolution equation can give us a weakly lower bound for $R_{g(t)}$, which can not prove or disprove the conjecture of Lotay and Wei.

We give an outline of the current paper. We review the basic theory in Sect. 2 about G_2-structures, G_2-decompositions of 2-forms and 3-forms, and general flows on G_2-structures. In Sect. 3, we rewrite results in Sect. 2 for closed G_2-structures, and the local curvature estimates will be given in the last subsection.

2 Basic theory of G_2-structures

In this section, we view some basic theory of G_2-structures, following [1,20–23,32]. Let $\{e_1, \ldots, e_7\}$ denote the standard basis of $\mathbb{R}^7$ and let $\{e^1, \ldots, e^7\}$ be its dual basis. Define the 3-form

$$\phi := e^{1\wedge 2\wedge 3} + e^{1\wedge 4\wedge 5} + e^{1\wedge 6\wedge 7} + e^{2\wedge 4\wedge 6} - e^{2\wedge 5\wedge 7} - e^{3\wedge 4\wedge 7} - e^{3\wedge 5\wedge 6},$$

where $e^{i\wedge j\wedge k} := e^i \wedge e^j \wedge e^k$. The subgroup G_2, which fixes ϕ, of $\mathbf{GL}(7,\mathbb{R})$ is the 14-dimensional Lie subgroup of $\mathbf{SO}(7)$, acts irreducibly on $\mathbb{R}^7$, and preserves the metric and orientation for which $\{e_1, \cdots, e_7\}$ is an oriented orthonormal basis. Note that G_2 also preserves the 4-form

$$*_\phi\phi = e^{4\wedge 5\wedge 6\wedge 7} + e^{2\wedge 3\wedge 6\wedge 7} + e^{2\wedge 3\wedge 4\wedge 5} + e^{1\wedge 3\wedge 5\wedge 7} - e^{1\wedge 3\wedge 4\wedge 6} - e^{1\wedge 2\wedge 5\wedge 6} - e^{1\wedge 2\wedge 4\wedge 7}.$$

where the Hodge star operator $*_\phi$ is determined by the metric and orientation.

For a smooth 7-manifold $\mathcal{M}$ and a point $x \in \mathcal{M}$, define as in [32]

$$\wedge^3_+(T^*_x\mathcal{M}) := \left\{\varphi_x \in \wedge^3(T^*_x\mathcal{M}) : \begin{array}{c} \mathsf{u}^*\phi = \varphi_x \text{ for some invertible} \\ \text{map } \mathsf{u} \in \mathrm{Hom}_{\mathbb{R}}(T_x\mathcal{M}, \mathbb{R}^7) \end{array}\right\}$$

and the bundle

$$\wedge^3_+(T^*\mathcal{M}) := \bigsqcup_{x\in\mathcal{M}} \wedge^3_+(T^*_x\mathcal{M}).$$

We call a section φ of $\wedge^3_+(T^*\mathcal{M})$ a *positive* 3*-form* on $\mathcal{M}$ or a G_2*-structure* on $\mathcal{M}$, and denote the space of positive 3-forms by $\wedge^3_+(\mathcal{M})$. The existence of G_2-structures is equivalent to the property that $\mathcal{M}$ is oriented and spin, which is equivalent to the vanishing of the first and second Stiefel–Whitney classes. From the definition of G_2-structures, we see that any $\varphi \in \wedge^3_+(\mathcal{M})$ uniquely determines a Riemannian metric g_φ and an orientation dV_φ, hence the Hodge star operator $*_\varphi$ and the associated 4-form

$$\psi := *_\varphi\varphi. \tag{2.1}$$

We also have the isomorphisms $\flat_\varphi := \flat_{g_\varphi}$ and $\sharp_\varphi := \sharp_{g_\varphi}$. For a given G_2-structure $\varphi \in \wedge^3_+(\mathcal{M})$, we denote by $\langle\cdot,\cdot\rangle_\varphi$, $\langle\langle\cdot,\cdot\rangle\rangle$, $|\cdot|_\varphi$, $||\cdot||_\varphi$, the corresponding inner products $\langle\cdot,\cdot\rangle_{g_\varphi}$, $\langle\langle\cdot,\cdot\rangle\rangle_{g_\varphi}$ and norms $|\cdot|_{g_\varphi}$, $||\cdot||_{g_\varphi}$.

Given a G_2-structure $\varphi \in \wedge^3_+(\mathcal{M})$. We say that φ is *torsion-free* if φ is parallel with respect to the metric g_φ. Equivalently, φ is torsion-free if and only if ${}^\varphi\nabla\varphi = 0$, where ${}^\varphi\nabla$ is the Levi–Civita connection of g_φ.

Theorem 2.1 (Fernández-Gray [12]) *The G_2-structure φ is torsion-free if and only if φ is both closed (i.e., $d\varphi = 0$) and co-closed (i.e., $d *_\varphi \varphi = d\psi = 0$).*

When $\mathcal{M}$ is compact, the above theorem says that a G_2-structure φ is torsion-free if and only if φ is harmonic with respect to the induces metric g_φ.

We say that a G_2-structure φ is *closed* (resp., *co-closed*) if $d\varphi = 0$ (resp., $d\psi = 0$). Theorem 2.1 can be restated as that a G_2-structure is torsion-free if and only if it is both closed and co-closed.

2.1 G_2-decompositions of $\wedge^2(\mathcal{M})$ and $\wedge^3(\mathcal{M})$

A G_2-structure φ induces splittings of the bundles $\wedge^k(T^*\mathcal{M})$, $2 \le k \le 5$, into direct summands, which we denote by $\wedge^k_\ell(T^*\mathcal{M},\varphi)$ with ℓ being the rank of the bundle. We let the space of sections of $\wedge^k_\ell(T^*\mathcal{M},\varphi)$ by $\wedge^k_\ell(\mathcal{M},\varphi)$. Define the natural projections

$$\pi^k_\ell : \wedge^k(\mathcal{M}) \longrightarrow \wedge^k_\ell(\mathcal{M},\varphi), \quad \alpha \longmapsto \pi^k_\ell(\alpha). \tag{2.2}$$

We mainly focus on the G_2–decompositions of $\wedge^2(\mathcal{M})$ and $\wedge^3(\mathcal{M})$. Recall that

$$\wedge^2(\mathcal{M}) = \wedge^2_7(\mathcal{M},\varphi) \oplus \wedge^2_{14}(\mathcal{M},\varphi), \tag{2.3}$$

$$\wedge^3(\mathcal{M}) = \wedge^3_1(\mathcal{M},\varphi) \oplus \wedge^3_7(\mathcal{M},\varphi) \oplus \wedge^3_{27}(\mathcal{M},\varphi). \tag{2.4}$$

Here each component is determined by

$$\wedge^2_7(\mathcal{M},\varphi) = \{X \lrcorner \varphi : X \in \mathfrak{X}(\mathcal{M})\} = \{\beta \in \wedge^2(\mathcal{M}) : *_\varphi(\varphi\wedge\beta) = 2\beta\},$$

$$\wedge^2_{14}(\mathcal{M},\varphi) = \{\beta \in \wedge^2(\mathcal{M}) : \psi\wedge\beta = 0\} = \{\beta \in \wedge^2(\mathcal{M}) : *_\varphi(\varphi\wedge\beta) = -\beta\},$$

$$\wedge^3_1(\mathcal{M},\varphi) = \{f\varphi : f \in C^\infty(\mathcal{M})\},$$
$$\wedge^3_7(\mathcal{M},\varphi) = \left\{*_\varphi(\varphi\wedge\alpha) : \alpha \in \wedge^1(\mathcal{M})\right\} = \{X \lrcorner \psi : X \in \mathfrak{X}(\mathcal{M})\},$$
$$\wedge^3_{27}(\mathcal{M},\varphi) = \{\eta \in \wedge^3(\mathcal{M}) : \eta\wedge\varphi = \eta\wedge\psi = 0\}.$$

For any 2-form $\beta = \frac{1}{2}\beta_{ij}dx^{i\wedge j} \in \wedge^2(\mathcal{M})$, its two components $\pi^2_7(\beta)$ and $\pi^2_{14}(\beta)$ are determined by

$$\pi^2_7(\beta) = \frac{\beta + *_\varphi(\varphi\wedge\beta)}{3} = \frac{1}{2}\left(\frac{1}{3}\beta_{ab} + \frac{1}{6}\beta^{\ell m}\psi_{\ell mab}\right)dx^{ab}, \tag{2.5}$$
$$\pi^2_{14}(\beta) = \frac{2\beta - *_\varphi(\varphi\wedge\beta)}{3} = \frac{1}{2}\left(\frac{2}{3}\beta_{ab} - \frac{1}{6}\beta^{\ell m}\psi_{\ell mab}\right)dx^{ab}. \tag{2.6}$$

To decompose 3-forms, recall two maps introduce by Bryant [1]

$$\mathsf{i}_\varphi : \odot^2(\mathcal{M}) \longrightarrow \wedge^3(\mathcal{M}), \quad \mathsf{j}_\varphi : \wedge^3(\mathcal{M}) \longrightarrow \odot^2(\mathcal{M}), \tag{2.7}$$

where

$$\begin{aligned}\mathsf{i}_\varphi(h) &:= h_{ij}g^{j\ell}dx^i \wedge \left(\frac{\partial}{\partial x^\ell}\lrcorner\varphi\right) = \frac{1}{2}h_{i\ell}\varphi^\ell{}_{jk}dx^{ijk}\\ &= \frac{1}{6}\left(h_{i\ell}\varphi^\ell{}_{jk} + h_{j\ell}\varphi_i{}^\ell{}_k + h_{k\ell}\varphi_{ij}{}^\ell\right)dx^{ijk}, \quad h = h_{ij}dx^{ij} \in \odot^2(\mathcal{M}),\end{aligned} \tag{2.8}$$

and

$$\left(\mathsf{j}_\varphi(\eta)\right)(X,Y) := *_\varphi\left((X\lrcorner\varphi)\wedge(Y\lrcorner\varphi)\wedge\eta\right). \tag{2.9}$$

Then i_φ is injective and is isomorphic onto $\wedge^3_1(\mathcal{M},\varphi)\oplus\wedge^3_{27}(\mathcal{M},\varphi)$, and j_φ is an isomorphism between $\wedge^3_1(\mathcal{M},\varphi)\oplus\wedge^3_{27}(\mathcal{M},\varphi)$ and $\odot^2(\mathcal{M})$. Moreover, for any 3-form $\eta \in \wedge^3(\mathcal{M})$, we have

$$\eta = \mathsf{i}_\varphi(h) + X\lrcorner\psi \tag{2.10}$$

for some symmetric 2-tensor $h \in \odot^2(\mathcal{M})$ and vector field $X \in \mathfrak{X}(\mathcal{M})$. Then

$$\begin{aligned}\eta &= h_i{}^\ell dx^i \wedge \left(\frac{\partial}{\partial x^\ell}\lrcorner\varphi\right) + X^\ell\left(\frac{\partial}{\partial x^\ell}\lrcorner\psi\right) = \frac{1}{2}h_i{}^\ell\varphi_{\ell jk}dx^{ijk} + \frac{1}{6}X^\ell\psi_{\ell ijk}dx^{ijk}\\ &= \frac{1}{6}\left(3h_i{}^\ell\varphi_{\ell jk} + X^\ell\psi_{\ell ijk}\right)dx^{ijk} = \frac{1}{6}\eta_{ijk}dx^{ijk}.\end{aligned}$$

Write h as $h_{ij} = \mathring{h}_{ij} + \frac{1}{7}\mathrm{tr}_\varphi(h)\,g_\varphi$, where $\mathring{h} \in \odot^2_0(\mathcal{M})$ is the trace-free part of h, one has

$$\eta = \underbrace{\frac{3}{7}\left(\mathrm{tr}_\varphi(h)\right)\varphi}_{\pi^3_1(\eta)} + \underbrace{\frac{1}{2}\mathring{h}_i{}^\ell\varphi_{\ell jk}dx^{ijk}}_{\pi^3_{27}(\eta)} + \underbrace{\frac{1}{6}X^\ell\psi_{\ell ijk}dx^{ijk}}_{\pi^3_7(\eta)}. \tag{2.11}$$

2.2 The torsion tensors of a G_2-structure

By Hodge duality we obtain the G_2-decompositions of 4-forms $\wedge^4(\mathcal{M}) = \wedge^4_1(\mathcal{M},\varphi)\oplus\wedge^4_7(\mathcal{M},\varphi)\oplus\wedge^4_{27}(\mathcal{M},\varphi)$ and 5-forms $\wedge^5(\mathcal{M}) = \wedge^5_7(\mathcal{M},\varphi)\oplus\wedge^5_{14}(\mathcal{M},\varphi)$, respectively. By definition, we can find forms $\tau_0 \in C^\infty(\mathcal{M})$, $\tau_1, \widetilde{\tau}_1 \in \wedge^1(\mathcal{M})$, $\tau_2 \in \wedge^2_{14}(\mathcal{M},\varphi)$, and $\tau_3 \in \wedge^3_{27}(\mathcal{M},\varphi)$ such that

$$d\varphi = \tau_0\psi + 3\tau_1\wedge\varphi + *_\varphi\tau_3, \quad d\psi = 4\widetilde{\tau}_1\wedge\psi - *_\varphi\tau_2. \tag{2.12}$$

Since $\tau_2 \in \wedge^2_{14}(\mathcal{M}, \varphi)$, it follows that $\tau_2 \wedge \varphi = - *_\varphi \tau_2$. Then (2.12) can be written as in the sense of Bryant [1]

$$d\varphi = \tau_0 \psi + 3\tau_1 \wedge \varphi + *_\varphi \tau_3, \quad d\psi = 4\widetilde{\tau}_1 \wedge \psi + \tau_2 \wedge \varphi. \tag{2.13}$$

It can be proved that $\tau_1 = \widetilde{\tau}_1$ (see [23]). We call τ_0 the *scalar torsion*, τ_1 the *vector torsion*, τ_2 the *Lie algebra torsion*, and τ_3 the *symmetric traceless torsion*. We also call $\boldsymbol{\tau}_\varphi := \{\tau_0, \tau_1, \tau_2, \tau_3\}$ the *intrinsic torsion forms* of the G_2-structure φ.

Recall that a G_2-structure φ is torsion-free if and only if $d\varphi = d\psi = 0$ by Theorem 2.1. From (2.12) we see that φ is torsion-free if and only if the intrinsic torsion forms $\boldsymbol{\tau}_\varphi \equiv= 0$; that is, $\tau_0 = \tau_1 = \tau_2 = \tau_3 = 0$.

Lemma 2.2 (Fernández-Gray, [12]) *For any $X \in \mathfrak{X}(\mathcal{M})$, the 3-form $\nabla_X\varphi$ lines in the space $\wedge^3_7(\mathcal{M}, \varphi)$. Therefore the covariant derivative $\nabla\varphi \in \wedge^1(\mathcal{M}) \otimes \wedge^3_7(\mathcal{M})$.*

Consequently, there exists a 2-tensor $\boldsymbol{T} = \boldsymbol{T}_{ij} dx^{i\otimes j}$, called the *full torsion tensor*, such that

$$\nabla_\ell \varphi = \boldsymbol{T}_\ell{}^n \psi_{nabc}. \tag{2.14}$$

Equivalently,

$$\boldsymbol{T}_{\ell m} = \frac{1}{24}(\nabla_\ell \varphi_{abc})\psi_m{}^{abc}. \tag{2.15}$$

Write

$$\tau_1 = (\tau_1)_i dx^i \in \wedge^1(\mathcal{M}), \tag{2.16}$$

$$\tau_2 = \frac{1}{2}(\tau_2)_{ab} dx^{ab} \in \wedge^2_{14}(\mathcal{M}), \tag{2.17}$$

$$\tau_3 = \frac{1}{2}(\tau_3)_i{}^\ell \varphi_{\ell ij} dx^{ijk} \in \wedge^3_{27}(\mathcal{M}, \varphi). \tag{2.18}$$

The associated 2-tensor $\boldsymbol{\tau}_3 := (\tau_3)_{ij} dx^{i\otimes j}$ of τ_3 lies in the space $\odot^2_0(\mathcal{M})$. With this convenience, the full torsion tensor $\boldsymbol{T}_{\ell m}$ is determined by

$$\boldsymbol{T}_{\ell m} = \frac{\tau_0}{4} g_{\ell m} - (\boldsymbol{\tau}_3)_{\ell m} - \left(\sharp_\varphi(\tau_1) \lrcorner \varphi\right)_{\ell m} - \frac{1}{2}(\tau_2)_{\ell m} \tag{2.19}$$

or as 2-tensors,

$$\boldsymbol{T} = \frac{\tau_0}{4} g_\varphi - \boldsymbol{\tau}_3 - \sharp_\varphi(\tau_1) \lrcorner \varphi - \frac{1}{2}\tau_2. \tag{2.20}$$

Here the 2-form $\sharp_\varphi(\tau_1) \lrcorner \varphi$ is defined by

$$\sharp_\varphi(\tau_1) \lrcorner \varphi = \frac{1}{2}\left(\sharp_\varphi(\tau_1) \lrcorner \varphi\right) dx^{a\wedge b} = \frac{1}{2}\left((\tau_1)_k \varphi^k{}_{ab}\right) dx^{a\wedge b}.$$

As an application, this gives another proof of Theorem 2.1.

For fixed indices i and j, set

$$R_{ij|k\ell} := R_{ijk\ell} \text{ is skew-symmetric in } k \text{ and } \ell, \tag{2.21}$$

where

$$R_{ij|\bullet\bullet} := \frac{1}{2} R_{ij|k\ell} dx^{k\ell} = \frac{1}{2} R_{ijk\ell} dx^{k\ell} \in \wedge^2(\mathcal{M}). \tag{2.22}$$

Then, according to (2.5) and (2.6)

$$R_{ijk\ell} = R_{ij|k\ell} = \left(\pi^2_7(R_{ij|\bullet\bullet})\right)_{k\ell} + \left(\pi^2_{14}(R_{ij|\bullet\bullet})\right)_{k\ell},$$

where

$$\left(\pi^2_7(R_{ij|\bullet\bullet})\right)_{k\ell} = \frac{1}{3}R_{ij|k\ell} + \frac{1}{6}R_{ij|ab}\psi^{ab}{}_{k\ell} \;=\; \frac{1}{3}R_{ijk\ell} + \frac{1}{6}R_{ijab}\psi^{ab}{}_{k\ell},$$
$$\left(\pi^2_{14}(R_{ij|\bullet\bullet})\right)_{k\ell} = \frac{2}{3}R_{ij|k\ell} - \frac{1}{6}R_{ij|ab}\psi^{ab}{}_{k\ell} \;=\; \frac{1}{3}R_{ijk\ell} - \frac{1}{6}R_{ijab}\psi^{ab}{}_{k\ell}.$$

Karigiannis [23] (see also the equivalent formula obtained by Bryant in [1]) proved that the Ricci curvature is given by

$$\begin{aligned} R_{jk} = R_{ijk\ell}g^{i\ell} \;&=\; 3\left(\pi^2_7(R_{ij|\bullet\bullet})\right)_{k\ell} g^{i\ell} \;=\; \frac{3}{2}\left(\pi^2_{14}(R_{ij|\bullet\bullet})\right)_{k\ell} g^{i\ell} \\ &= -\left(\nabla_i \boldsymbol{T}_{jm} - \nabla_j \boldsymbol{T}_{im}\right)\varphi^m{}_k{}^i - \boldsymbol{T}_j{}^i\boldsymbol{T}_{ik} + \left(\mathrm{tr}_\varphi \boldsymbol{T}\right)\boldsymbol{T}_{jk} + \boldsymbol{T}_{jb}\boldsymbol{T}_{ia}\psi^{iab}{}_k, \\ &= -\nabla_i\left(\boldsymbol{T}_j{}^n\varphi_{nk}{}^i\right) + \nabla_j\left(\boldsymbol{T}_i{}^n\varphi_{nk}{}^i\right) - \boldsymbol{T}_j{}^i\boldsymbol{T}_{ik} + \left(\mathrm{tr}_\varphi\boldsymbol{T}\right)\boldsymbol{T}_{jk} - \boldsymbol{T}_{jb}\boldsymbol{T}_{ia}\psi^{iab}{}_k. \end{aligned} \tag{2.23}$$

Cleyton and Ivanov [6] also derived a formula for the Ricci tensor for closed G_2-structures in terms of $d^*_\varphi\varphi$. Taking the trace of (2.23), we obtain Btyant's formula [1] for the scalar curvature

$$\begin{aligned} R &= -12\nabla^\ell(\tau_1)_\ell + \frac{21}{8}\tau_0^2 - ||\boldsymbol{\tau}_3||^2_\varphi + 5||\sharp_\varphi(\tau_1)\lrcorner\varphi||^2_\varphi - \frac{1}{4}||\tau_2||^2_\varphi, \\ &= -12\nabla^\ell(\tau_1)_\ell + \frac{21}{8}\tau_0^2 - ||\boldsymbol{\tau}_3||^2_\varphi + 30|\tau_1|^2_\varphi - \frac{1}{2}|\tau_2|^2_\varphi, \end{aligned} \tag{2.24}$$

For a closed G_2-structure, we have $\tau_0 = \tau_1 = \tau_3 = 0$ and then $R = -\frac{1}{4}||\tau_2||^2_\varphi \le 0$. On the other hand, we have $(\tau_2)_{ij} = -2\boldsymbol{T}_{ij}$ by (2.20). Thus the full torsion tensor $\boldsymbol{T}$ is actually a 2-form

$$\boldsymbol{T} = \frac{1}{2}\boldsymbol{T}_{ij}dx^{ij} \in \wedge^2(\mathcal{M}) \tag{2.25}$$

and the scalar curvature can be written in terms of T

$$R = -||\boldsymbol{T}||^2_\varphi = -2|\boldsymbol{T}|^2_\varphi \le 0. \tag{2.26}$$

Hence, for closed G_2-structures, scalar curvatures are always non-positive.

Finally, we mention a Bianchi type identity

$$\nabla_i\boldsymbol{T}_{j\ell} - \nabla_j\boldsymbol{T}_{i\ell} = -\frac{1}{2}R_{ijab}\varphi^{ab}{}_\ell - \boldsymbol{T}_{ia}\boldsymbol{T}_{jb}\varphi^{ab}{}_\ell = -\left(\frac{1}{2}R_{ijab} + \boldsymbol{T}_{ia}\boldsymbol{T}_{jb}\right)\varphi^{ab}{}_\ell. \tag{2.27}$$

The proof can be found in [23].

2.3 Basic theory of closed G_2-structures

Let $\wedge^3_{+,\bullet}(\mathcal{M}) \subset \wedge^3_+(\mathcal{M},\varphi)$ be the set of all closed G_2-structures on $\mathcal{M}$. If $\varphi \in \wedge^3_{+,\bullet}(\mathcal{M})$ is closed, i.e., $d\varphi = 0$, then τ_0, τ_1, τ_3 are all zero, so the only nonzero torsion form is

$$\boldsymbol{\tau} \equiv \tau_2 = \frac{1}{2}(\tau_2)_{ij}dx^{ij} = \frac{1}{2}\boldsymbol{\tau}_{ij}dx^{ij}. \tag{2.28}$$

According to (2.20) and (2.25), we have $\boldsymbol{T}_{ij} = -\frac{1}{2}\boldsymbol{\tau}_{ij}$ so that

$$\boldsymbol{T} \equiv \frac{1}{2}\boldsymbol{T}_{ij}dx^{ij} \quad \text{or equivalently} \quad \boldsymbol{T} = -\frac{1}{2}\boldsymbol{\tau}, \tag{2.29}$$

is a 2-form. Since $d\psi = \boldsymbol{\tau} \wedge \varphi = - *_\varphi \boldsymbol{\tau}$, we get $d_\varphi^* \boldsymbol{\tau} = *_\varphi d *_\varphi \boldsymbol{\tau} = - *_\varphi d^2 \psi = 0$ which is given in local coordinates by

$$\nabla^i \boldsymbol{\tau}_{ij} = 0 \tag{2.30}$$

For a closed G_2-structure φ, according to (2.23), the Ricci curvature is given by (in this case $\boldsymbol{T}_{ij}$ is a 2-form)

$$R_{jk} = \left(\nabla_j \boldsymbol{T}_{im} - \nabla_i \boldsymbol{T}_{jm}\right) \varphi^m{}_k{}^i - \boldsymbol{T}_j{}^i \boldsymbol{T}_{ik} + \boldsymbol{T}_{jb} \boldsymbol{T}_{ia} \psi^{iab}{}_k.$$

Since $\boldsymbol{\tau} \in \wedge^2_{14}(\mathcal{M}, \varphi)$ and $\boldsymbol{T}_{ij} = -\frac{1}{2}\boldsymbol{\tau}_{ij}$, it follows from [32] (see pp. 179–180) that

$$(\nabla_j \boldsymbol{T}_{im}) \varphi^m{}_k{}^i = 2\boldsymbol{T}_j{}^\ell \boldsymbol{T}_{\ell k}. \tag{2.31}$$

and therefore, for a closed G_2-structure φ, the Ricci curvature is given by

$$R_{jk} = -(\nabla_i \boldsymbol{T}_{jm}) \varphi_k{}^{im} - \boldsymbol{T}_j{}^i \boldsymbol{T}_{ik}. \tag{2.32}$$

Taking the trace of (2.32) yields (2.26). Moreover, the factor $\nabla_i \boldsymbol{T}_{jm}$ in (3.6) can be expressed as (see Proposition 2.4 in [32])

$$\begin{aligned} \nabla_i \boldsymbol{T}_{jk} = & -\frac{1}{4} R_{ijmn} \varphi_k{}^{mn} - \frac{1}{4} R_{kjmn} \varphi_i{}^{mn} + \frac{1}{4} R_{ikmn} \varphi_j{}^{mn} \\ & - \frac{1}{2} \boldsymbol{T}_{im} \boldsymbol{T}_{jn} \varphi_k{}^{mn} - \frac{1}{2} \boldsymbol{T}_{km} \boldsymbol{T}_{jn} \varphi_i{}^{mn} + \frac{1}{2} \boldsymbol{T}_{im} \boldsymbol{T}_{kn} \varphi_j{}^{mn}. \end{aligned} \tag{2.33}$$

If φ is a closed G_2-structure, Section 2.2 in [32] shows that $\pi^3_7(\Delta_\varphi \varphi) = 0$ and hence, according to (2.10),

$$\Delta_\varphi \varphi = \mathrm{i}_\varphi(h) \in \wedge^3_1(\mathcal{M}, \varphi) \oplus \wedge^3_{27}(\mathcal{M}, \varphi), \tag{2.34}$$

where

$$h_{ij} = \frac{1}{2} \nabla_m \boldsymbol{\tau}_{ni} \varphi_j{}^{mn} - \frac{1}{6} |\boldsymbol{\tau}|^2_\varphi g_{ij} - \frac{1}{4} \boldsymbol{\tau}_i{}^\ell \boldsymbol{\tau}_{\ell j} = -R_{ij} - \frac{2}{3} |\boldsymbol{T}|^2_\varphi g_{ij} - 2\boldsymbol{T}_i{}^k \boldsymbol{T}_{kj}. \tag{2.35}$$

Here $|\boldsymbol{T}|^2_\varphi = \frac{1}{2} \boldsymbol{T}_{k\ell} \boldsymbol{T}^{k\ell} = \frac{1}{2} ||\boldsymbol{T}||^2_\varphi$.

2.4 General flows on G_2-structures

For any family $\varphi(t)$ of G_2-structures, according to the decomposition (2.10), we can consider the general flow

$$\partial_t \varphi(t) = \mathrm{i}_{\varphi(t)}(h(t)) + X(t) \lrcorner \psi(t) \tag{2.36}$$

where $h(t) \in \odot^2(\mathcal{M})$ and $X(t) \in \mathfrak{X}(\mathcal{M})$. The general flow (2.36) locally can be written as

$$\partial_t \varphi_{ijk} = h_i{}^\ell \varphi_{\ell jk} + h_j{}^\ell \varphi_{i\ell k} + h_k{}^\ell \varphi_{ij\ell} + X^\ell \psi_{\ell ijk}. \tag{2.37}$$

We write for $g(t)$ and $dV_{g(t)}$ the metric and volume form associated to $\varphi(t)$, respectively.

Theorem 2.3 *Under the general flow (2.36), we have*

$$\partial_t g_{ij} = 2h_{ij}, \tag{2.38}$$

$$\partial_t g^{ij} = -2h^{ij}, \tag{2.39}$$

$$\partial_t dV_{g(t)} = \left(\mathrm{tr}_{g(t)} h(t)\right) dV_{g(t)}, \tag{2.40}$$

$$\partial_t \boldsymbol{T}_{pq} = \boldsymbol{T}_p{}^m h_{mq} - \boldsymbol{T}_p{}^m X^k \varphi_{kmq} - (\nabla_k h_{ip}) \varphi^{ki}{}_q + \nabla_p X_q. \tag{2.41}$$

These evolution equations can be found in [23].

3 Laplacian flows on closed G_2-structures

We now consider the Laplacian flow for closed G_2-structures

$$\partial_t\varphi(t) = \Delta_{\varphi(t)}\varphi(t) = \Delta_{g(t)}\varphi(t), \quad \varphi(0) = \varphi, \tag{3.1}$$

where $\Delta_{\varphi(t)}\varphi(t) = dd^*_{\varphi(t)}\varphi(t) + d^*_{\varphi(t)}d\varphi(t)$ is the Hodge Laplacian of $g_{\varphi(t)}$ and φ is an initial closed G_2-structure. The short time existence for (3.1) on compact manifolds was proved by Bryant and Xu [2], see also Theorem 1.1.

A criterion for the long time existence for the Lapalcian flow on compact manifolds was given in Theorem 1.2. In this section, we give a new elementary proof of Lotay-Wei's result in compact case.

3.1 Evolution equations along the Laplacian flow

Since the Laplacian flow (3.1) preserves the closedness of $\varphi(t)$, it follows from (3.10) that we have

$$\Delta_{\varphi(t)}\varphi(t) = \mathrm{i}_{\varphi(t)}(h(t)) \in \wedge^3_1(\mathcal{M}, \varphi(t)) \oplus \wedge^3_{27}(\mathcal{M}, \varphi(t)), \tag{3.2}$$

where

$$h_{ij} = -R_{ij} - \frac{2}{3}|\boldsymbol{T}(t)|^2_{g(t)}g_{ij} - 2\boldsymbol{T}_i{}^k\boldsymbol{T}_{kj}. \tag{3.3}$$

From Theorem 2.3, we see that the associated metric tensor $g(t)$ evolves by

$$\partial_t g_{ij} = 2h_{ij} = -2R_{ij} - \frac{4}{3}|\boldsymbol{T}(t)|^2_{g(t)}g_{ij} - 4\boldsymbol{T}_i{}^k\boldsymbol{T}_{kj}. \tag{3.4}$$

and the volume form $dV_{g(t)}$ evolves by

$$\begin{aligned}\partial_t dV_{g(t)} = (\mathrm{tr}_{g(t)}h(t))dV_{g(t)} &= \left(-R_{g(t)} - \frac{14}{3}|\boldsymbol{T}(t)|^2_{g(t)} + 4|\boldsymbol{T}(t)|^2_{g(t)}\right)dV_{g(t)} \\ = \left(2 - \frac{14}{3} + 4\right)|\mathbf{T}(t)|^2_{g(t)}dV_{g(t)} &= \frac{4}{3}|\boldsymbol{T}(t)|^2_{g(t)}dV_{g(t)}.\end{aligned} \tag{3.5}$$

Hence, along the flow (3.1), the volume of $g(t)$ is nondecreasing.

Introduce the following notions

$$\blacksquare_{g(t)} := \partial_t - \blacktriangle_{g(t)}, \quad |\cdot|_{g(t)} := |\cdot|_{\varphi(t)}, \quad \Delta_{g(t)} := \Delta_{\varphi(t)}, \tag{3.6}$$

where $\blacktriangle_{g(t)} := g^{ij}\nabla_i\nabla_j$ is the usual Laplacian of $g(t)$ and $\Delta_{g(t)}$ is the Hodge Laplacian of $g(t)$, and also the 2-tenor $\mathrm{Sic}_{g(t)}$ with components

$$S_{ij} := R_{ij} + \frac{2}{3}|\boldsymbol{T}(t)|^2_{g(t)}g_{ij} + 2\boldsymbol{T}_i{}^k\boldsymbol{T}_{kj} = -h_{ij}. \tag{3.7}$$

Then the evolution Eq. (3.4) can be written as

$$\partial_t g_{ij} = -2S_{ij}. \tag{3.8}$$

The trace of $\mathrm{Sic}_{g(t)}$ is exactly the scalar curvature, up to a multiplying constant,

$$S_{g(t)} := \mathrm{tr}_{g(t)}\mathrm{Sic}_{g(t)} = R_{g(t)} + \frac{14}{3}|\boldsymbol{T}(t)|^2_{g(t)} - 4|\boldsymbol{T}(t)|^2_{g(t)} = -\frac{4}{3}|\boldsymbol{T}(t)|^2_{g(t)} = \frac{2}{3}R_{g(t)}. \tag{3.9}$$

It was proved in [32] that

$$|\Delta_{g(t)}\varphi(t)|^2_{g(t)} = (\mathrm{tr}_{g(t)}h(t))^2 + 2||h(t)||^2_{g(t)} = \frac{16}{9}|\boldsymbol{T}(t)|^4_{g(t)} + 2||\mathrm{Sic}_{g(t)}||^2_{g(t)}. \tag{3.10}$$

This identity together with (2.26) shows that the boundedness of $\Delta_{g(t)}\varphi(t)$ is equivalent to the boundedness of $\mathrm{Ric}_{g(t)}$.

The evolution Eq. (2.41) implies that for the Laplacian flow on closed G_2-structures, the torsion T_{ij} evolves by evolves

$$\partial_t \boldsymbol{T}_{ij} = \boldsymbol{T}_i{}^k h_{kj} - (\nabla_m h_{ni})\varphi_j{}^{mn}. \tag{3.11}$$

Furthermore, we can prove

Proposition 3.1 *Under the flow (3.1), we have*

$$\begin{aligned}\blacksquare_{g(t)}\boldsymbol{T}_{ij} &= 3R_j{}^k\boldsymbol{T}_{ki} - R_i{}^k\boldsymbol{T}_{kj} - \frac{1}{2}R_{ijmk}\boldsymbol{T}^{mk} - \frac{1}{2}R_{mpi}{}^kR_{qk}\psi_j{}^{pqm} - \frac{2}{3}|\boldsymbol{T}(t)|^2_{g(t)}\boldsymbol{T}_{ij}\\ &\quad + \nabla_p\boldsymbol{T}_{qi}\left(\boldsymbol{T}^{pk}\varphi_{kj}{}^q - 2\boldsymbol{T}^{qk}\varphi_{kj}{}^p\right) - \frac{2}{3}\varphi_{ji}{}^m\nabla_m|\boldsymbol{T}(t)|^2_{g(t)} - 4T_i{}^k\boldsymbol{T}_k{}^m\boldsymbol{T}_{mj}.\end{aligned} \tag{3.12}$$

Proof See [32].

For a geometric flow $\partial_t g_{ij} = \eta_{ij}$, where η_{ij} is a family of symmetric 2-tensors, we have (e.g. see formula (2.66), (2.29), and (2.30) in [5])

$$\begin{aligned}\partial_t R^\ell_{ijk} &= \frac{1}{2}g^{\ell p}\Big(\nabla_i\nabla_j\eta_{kp} + \nabla_i\nabla_k\eta_{jp} - \nabla_i\nabla_p\eta_{jk}\\ &\qquad - \nabla_j\nabla_i\eta_{kp} - \nabla_j\nabla_k\eta_{ip} + \nabla_j\nabla_p\eta_{ik}\Big),\\ \partial_t R_{jk} &= \frac{1}{2}g^{pq}\left(\nabla_q\nabla_j\eta_{kp} + \nabla_q\nabla_k\eta_{jp} - \nabla_q\nabla_p\eta_{jk} - \nabla_j\nabla_k\eta_{qp}\right),\\ \partial_t R_{g(t)} &= -\blacktriangle_{g(t)}\mathrm{tr}_{g(t)}\eta(t) + \mathrm{div}_{g(t)}(\mathrm{div}_{g(t)}\eta(t)) - R_{ij}h^{ij},\end{aligned}$$

where $(\mathrm{div}_{g(t)}\eta(t))_j = \nabla^i\eta_{ij}$. Applying those evolution equations to $\eta_{ij} = -2R_{ij} - \frac{4}{3}|\boldsymbol{T}(t)|^2_{g(t)}g_{ij} - 4\boldsymbol{T}_i{}^k\boldsymbol{T}_{kj} = -2S_{ij}$ we have

$$\begin{aligned}\mathrm{tr}_{g(t)}\eta(t) &= -2R_{g(t)} - \frac{28}{3}|\boldsymbol{T}(t)|^2_{g(t)} + 8|\boldsymbol{T}(t)|^2_{g(t)} \;=\; \frac{8}{3}|\boldsymbol{T}(t)|^2_{g(t)},\\ (\mathrm{div}_{g(t)}\eta(t))_j &= -2\nabla^iR_{ij} - \frac{4}{3}\nabla_j|\boldsymbol{T}(t)|^2_{g(t)} - 4\nabla^i\widehat{\boldsymbol{T}}_{ij}\\ &= -\nabla_jR_{g(t)} - \frac{4}{3}\nabla_j|\boldsymbol{T}(t)|^2_{g(t)} - 4\nabla^i\widehat{\boldsymbol{T}}_{ij},\\ \mathrm{div}_{g(t)}(\mathrm{div}_{g(t)}\eta(t)) &= \nabla^j(\mathrm{div}_{g(t)}\eta(t))_j\\ &= -\blacktriangle_{g(t)}R_{g(t)} - \frac{4}{3}\blacktriangle_{g(t)}|\boldsymbol{T}(t)|^2_{g(t)} - 4\nabla^j\nabla^i\widehat{\boldsymbol{T}}_{ij},\end{aligned}$$

where the symmetric 2-tensor $\widehat{\boldsymbol{T}}(t)$ is given by

$$\widehat{\boldsymbol{T}}_{ij} := \boldsymbol{T}_{ik}\boldsymbol{T}^k{}_j. \tag{3.13}$$

Plugging those identities into the above evolution equation for $R_{g(t)}$, we get

$$\partial_t R_{g(t)} = -4\blacktriangle_{g(t)}|\boldsymbol{T}(t)|^2_{g(t)} - \blacktriangle_{g(t)}R_{g(t)} - 4\nabla^j\nabla^i\widehat{\boldsymbol{T}}_{ij}$$

$$- R^{ij}\left(-2R_{ij} - \frac{4}{3}|\boldsymbol{T}(t)|^2_{g(t)}g_{ij} - 4\widehat{\boldsymbol{T}}_{ij}\right)$$
$$= \blacktriangle_{g(t)}R_{g(t)} - 4\nabla^j\nabla^i\widehat{\boldsymbol{T}}_{ij} + 2||\mathrm{Ric}_{g(t)}||^2_{g(t)} + \frac{4}{3}|\boldsymbol{T}(t)|^2_{g(t)}R_{g(t)} + 4R^{ij}\widehat{\boldsymbol{T}}_{ij}$$

which implies

$$\blacksquare_{g(t)}R_{g(t)} = 2||\mathrm{Ric}_{g(t)}||^2_{g(t)} - \frac{2}{3}R^2_{g(t)} - 4\nabla^j\nabla^i\widehat{\boldsymbol{T}}_{ij} + 4\langle\langle\mathrm{Ric}_{g(t)}, \widehat{\boldsymbol{T}}(t)\rangle\rangle_{g(t)}. \tag{3.14}$$

Observe that the last two terms on the right-hand side of (3.22) are not determined of their signs. In the following, we shall use the identity

$$\nabla^i\boldsymbol{T}_{ij} = 0 \tag{3.15}$$

follows from from (2.29) and (2.30), to simplify those two terms. Using the identity (3.15), the term $\nabla^j\nabla^i\widehat{\boldsymbol{T}}_{ij}$ can be simplified as follows.

$$\nabla^j\nabla^i\widehat{\boldsymbol{T}}_{ij} = \nabla^j\nabla^i\left(\boldsymbol{T}_i{}^k\boldsymbol{T}_{kj}\right) = \nabla^j\left[(\nabla^i\boldsymbol{T}_i{}^k)\boldsymbol{T}_{kj} + \boldsymbol{T}_i{}^k(\nabla^i\boldsymbol{T}_{kj})\right]$$
$$= \boldsymbol{T}^{ik}(\nabla_j\nabla_i\boldsymbol{T}_k{}^j) - (\nabla^j\boldsymbol{T}^{ik})(\nabla_i\boldsymbol{T}_{jk}).$$

On the other hand, from the Ricci identity

$$\nabla_j\nabla_i\boldsymbol{T}_k{}^j = \nabla_i\nabla_j\boldsymbol{T}_k{}^j - R_{jik\ell}\boldsymbol{T}^{\ell j} - R_{ji}{}^{j\ell}\boldsymbol{T}_{k\ell} = R_{ijk\ell}\boldsymbol{T}^{\ell j} + R_{i\ell}\boldsymbol{T}_k{}^\ell,$$

we see that the evolution Eq. (3.14) is equivalent to

$$\blacksquare_{g(t)}R_{g(t)} = 2||\mathrm{Ric}_{g(t)}||^2_{g(t)} - \frac{2}{3}R^2_{g(t)} + 4R_{ijk\ell}\boldsymbol{T}^{ik}\boldsymbol{T}^{j\ell} + 4(\nabla^j\boldsymbol{T}^{ik})(\nabla_i\boldsymbol{T}_{jk}). \tag{3.16}$$

From (3.7) and (3.13) we can rewrite the term $||\mathrm{Ric}_{g(t)}||^2_{g(t)}$ in (3.16) in terms of $\mathrm{Sic}_{g(t)}$ according to the following relation:

$$||\mathrm{Sic}_{g(t)}||^2_{g(t)} = \left(R_{ij} + \frac{2}{3}|\boldsymbol{T}(t)|^2_{g(t)}g_{ij} + 2\widehat{\boldsymbol{T}}_{ij}\right)\left(R^{ij} + \frac{2}{3}|\boldsymbol{T}(t)|^2_{g(t)}g^{ij} + 2\widehat{\boldsymbol{T}}^{ij}\right)$$
$$= ||\mathrm{Ric}_{g(t)}||^2_{g(t)} + \frac{4}{3}|\boldsymbol{T}(t)|^2_{g(t)}R_{g(t)} + 4\langle\langle\mathrm{Ric}_{g(t)}, \widehat{\boldsymbol{T}}(t)\rangle\rangle_{g(t)}$$
$$+ \frac{28}{9}|\boldsymbol{T}(t)|^4_{g(t)} + \frac{8}{3}|\boldsymbol{T}(t)|^2_{g(t)}\mathrm{tr}_{g(t)}\widehat{\boldsymbol{T}}(t) + 4||\widehat{\boldsymbol{T}}(t)||^2_{g(t)}$$
$$= ||\mathrm{Ric}_{g(t)}||^2_{g(t)} - \frac{2}{3}R^2_{g(t)} + 4\langle\langle\mathrm{Ric}_{g(t)}, \widehat{\boldsymbol{T}}(t)\rangle\rangle_{g(t)}$$
$$+ \frac{7}{9}R^2_{g(t)} - \frac{4}{3}R^2_{g(t)} + 4||\widehat{\boldsymbol{T}}(t)||^2_{g(t)}$$
$$= ||\mathrm{Ric}_{g(t)}||^2_{g(t)} + 4||\widehat{\boldsymbol{T}}(t)||^2_{g(t)} + 4\langle\langle\mathrm{Ric}_{g(t)}, \widehat{\boldsymbol{T}}(t)\rangle\rangle_{g(t)} - \frac{11}{9}R^2_{g(t)},$$

where we used the identities $\mathrm{tr}_{g(t)}\widehat{\boldsymbol{T}}(t) = g^{ij}\boldsymbol{T}_{ik}\boldsymbol{T}^k{}_j = \boldsymbol{T}_{ik}\boldsymbol{T}^{ki} = -2|\boldsymbol{T}(t)|^2_{g(t)}$ and $R_{g(t)} = -2|\boldsymbol{T}(t)|^2_{g(t)}$. Replacing $R_{g(t)}$ by $S_{g(t)}$ according to the identity (3.9), we can rewrite (3.16) as

$$\blacksquare_{g(t)}S_{g(t)} = \frac{4}{3}||\mathrm{Sic}_{g(t)}||^2_{g(t)} - \frac{16}{3}||\widehat{\boldsymbol{T}}(t)||^2_{g(t)} - \frac{16}{3}\langle\langle\mathrm{Ric}_{g(t)}, \widehat{\boldsymbol{T}}(t)\rangle\rangle_{g(t)} + \frac{32}{27}R^2_{g(t)}$$
$$+ \frac{8}{3}R_{ijk\ell}\boldsymbol{T}^{ik}\boldsymbol{T}^{j\ell} + \frac{8}{3}(\nabla^j\boldsymbol{T}^{ik})(\nabla_i\boldsymbol{T}_{jk}).$$

Similarly, replacing $\langle\langle \mathrm{Ric}_{g(t)}, \widehat{\boldsymbol{T}}(t)\rangle\rangle_{g(t)}$ by $\langle\langle \mathrm{Sic}_{g(t)}, \widehat{\boldsymbol{T}}(t)\rangle\rangle_{g(t)}$ with respect to the identity

$$\begin{aligned}\langle\langle \mathrm{Sic}_{g(t)}, \widehat{\boldsymbol{T}}(t)\rangle\rangle_{g(t)} &= \left(R_{ij} + \frac{2}{3}|\boldsymbol{T}(t)|^2_{g(t)} g_{ij} + 2\widehat{\boldsymbol{T}}_{ij}\right)\widehat{\boldsymbol{T}}^{ij}\\ &= \langle\langle \mathrm{Ric}_{g(t)}, \widehat{\boldsymbol{T}}(t)\rangle\rangle_{g(t)} - \frac{1}{3}R^2_{g(t)} + 2||\widehat{\boldsymbol{T}}(t)||^2_{g(t)},\end{aligned}$$

we obtain the following evolution equation for $S_{g(t)}$,

$$\blacksquare_{g(t)} S_{g(t)} = \frac{4}{3}\left[||\mathrm{Sic}_{g(t)} - 2\widehat{\boldsymbol{T}}(t)||^2_{g(t)} - S^2_{g(t)}\right] + \frac{8}{3}\left[R_{ijk\ell}\boldsymbol{T}^{ik}\boldsymbol{T}^{j\ell} + (\nabla^j\boldsymbol{T}^{ik})(\nabla_i\boldsymbol{T}_{jk})\right]. \tag{3.17}$$

Next, we try to deal with the last bracket in (3.17), which contains two terms $R_{ijk\ell}\boldsymbol{T}^{ik}\boldsymbol{T}^{j\ell}$ and $(\nabla^j\boldsymbol{T}^{ik})(\nabla_i\boldsymbol{T}_{jk})$. Using (2.27) and (2.33), the term $(\nabla^j\boldsymbol{T}^{ik})(\nabla_i\boldsymbol{T}_{jk})$ is equal to

$$\begin{aligned}(\nabla^j\boldsymbol{T}^{ik})(\nabla_i\boldsymbol{T}_{jk}) &= \left[\nabla^i\boldsymbol{T}^{jk} + \left(\frac{1}{2}R^{ij}{}_{ab} + \boldsymbol{T}^i{}_a\boldsymbol{T}^j{}_b\right)\varphi^{kab}\right]\nabla_i\boldsymbol{T}_{jk}\\ &= ||\nabla_{g(t)}\boldsymbol{T}(t)||^2_{g(t)} + \frac{1}{2}\left(\frac{1}{2}R^{ij}{}_{ab} + \boldsymbol{T}^i{}_a\boldsymbol{T}^j{}_b\right)\\ &\quad\left[-\frac{1}{2}R_{ijmn}\varphi^{mn}{}_k\varphi^{kab} - \frac{1}{2}R_{kjmn}\varphi_i{}^{mn}\varphi^{kab}\right.\\ &\quad + \frac{1}{2}R_{ikmn}\varphi_j{}^{mn}\varphi^{kab} - \boldsymbol{T}_{im}\boldsymbol{T}_{jn}\varphi^{mn}{}_k\varphi^{kab}\\ &\quad\left.-\boldsymbol{T}_{km}\boldsymbol{T}_{jn}\varphi_i{}^{mn}\varphi^{kab} + \boldsymbol{T}_{im}\boldsymbol{T}_{kn}\varphi_j{}^{mn}\varphi^{kab}\right].\end{aligned}$$

By symmetry the term

$$\left(\frac{1}{2}R^{ij}{}_{ab} + \boldsymbol{T}^i{}_a\boldsymbol{T}^j{}_b\right)\left(-\frac{1}{2}R_{kjmn}\varphi_i{}^{mn}\varphi^{kab} + \frac{1}{2}R_{ikmn}\varphi_j{}^{mn}\varphi^{kab}\right)$$

is equal to, interchanging $i \leftrightarrow j$ and $a \leftrightarrow b$ in the second term,

$$\left(\frac{1}{2}R^{ij}{}_{ab} + \boldsymbol{T}^i{}_a\boldsymbol{T}^j{}_b\right)\left(-\frac{1}{2}R_{kjmn}\varphi_i{}^{mn}\varphi^{kab}\right) + \left(\frac{1}{2}R^{ji}{}_{ba} + \boldsymbol{T}^j{}_b\boldsymbol{T}^i{}_a\right)\left(\frac{1}{2}R_{jkmn}\varphi_i{}^{mn}\varphi^{kba}\right)$$

which is zero. Similarly, we have, by interchanging $m \leftrightarrow n$ and then $i \leftrightarrow j$, $a \leftrightarrow b$ in the first term,

$$\begin{aligned}&\left(\frac{1}{2}R^{ij}{}_{ab} + \boldsymbol{T}^i{}_a\boldsymbol{T}^j{}_b\right)\left(-\boldsymbol{T}_{km}\boldsymbol{T}_{jn}\varphi_i{}^{mn}\varphi^{kab} + \boldsymbol{T}_{im}\boldsymbol{T}_{kn}\varphi_j{}^{mn}\varphi^{kab}\right)\\ &= \left(\frac{1}{2}R^{ij}{}_{ab} + \boldsymbol{T}^i{}_a\boldsymbol{T}^j{}_b\right)\left(-\boldsymbol{T}_{kn}\boldsymbol{T}_{jm}\varphi_i{}^{nm}\varphi^{kab} + \boldsymbol{T}_{im}\boldsymbol{T}_{kn}\varphi_j{}^{mn}\varphi^{kab}\right)\\ &= \left(\frac{1}{2}R^{ij}{}_{ab} + \boldsymbol{T}^i{}_a\boldsymbol{T}^j{}_b\right)\left(-\boldsymbol{T}_{kn}\boldsymbol{T}_{im}\varphi_j{}^{nm}\varphi^{kba} + \boldsymbol{T}_{im}\boldsymbol{T}_{kn}\varphi_j{}^{mn}\varphi^{kab}\right) = 0.\end{aligned}$$

Therefore, using the identity $\varphi_{ijk}\varphi^k{}_{ab} = g_{ia}g_{jb} - g_{ib}g_{ja} + \psi_{ijab}$ (see [23]), we arrive at

$$\begin{aligned}(\nabla^j\boldsymbol{T}^{ik})(\nabla_i\boldsymbol{T}_{jk}) &= ||\nabla_{g(t)}\boldsymbol{T}(t)||^2_{g(t)}\\ &\quad - \frac{1}{2}\left(\frac{1}{2}R^{ij}{}_{ab} + \boldsymbol{T}^i{}_a\boldsymbol{T}^j{}_b\right)\left(\frac{1}{2}R_{ij}{}^{mn} + \boldsymbol{T}_i{}^m\boldsymbol{T}_j{}^n\right)\varphi_{mnk}\varphi^{kab}\end{aligned}$$

$$
\begin{aligned}
&= ||\nabla_{g(t)}\boldsymbol{T}(t)||^2_{g(t)} - \frac{1}{2}\left(\frac{1}{2}R^{ij}{}_{ab} + \boldsymbol{T}^i{}_a\boldsymbol{T}^j{}_b\right) \\
&\quad \cdot \left(\frac{1}{2}R_{ij}{}^{mn} + \boldsymbol{T}_i{}^m\boldsymbol{T}_j{}^n\right)\left(\delta^a_m\delta^b_n - \delta^b_m\delta^a_n + \psi_{mn}{}^{ab}\right) \\
&= ||\nabla_{g(t)}\boldsymbol{T}(t)||^2_{g(t)} - \frac{1}{8}\left(R_{ijab} + 2\boldsymbol{T}_{ia}\boldsymbol{T}_{jb}\right)\Big[\left(R^{ijab} + 2\boldsymbol{T}^{ia}\boldsymbol{T}^{jb}\right) \\
&\quad - \left(R^{ijba} + 2\boldsymbol{T}^{ib}\boldsymbol{T}^{ja}\right) + \left(R^{ijmn} + 2\boldsymbol{T}^{im}\boldsymbol{T}^{jn}\right)\psi_{mn}{}^{ab}\Big].
\end{aligned}
$$

Since, by our convention,

$$
\left(R_{ijab} + 2\boldsymbol{T}_{ia}\boldsymbol{T}_{jb}\right)\left(R^{ijab} + 2\boldsymbol{T}^{ia}\boldsymbol{T}^{jb}\right) = ||\mathrm{Rm}_{g(t)}||^2_{g(t)} + 4R_{ijab}\boldsymbol{T}^{ia}\boldsymbol{T}^{jb} + 4||\boldsymbol{T}(t)||^4_{g(t)}
$$

and

$$
\left(R_{ijab} + 2\boldsymbol{T}_{ia}\boldsymbol{T}_{jb}\right)\left(R^{ijba} + 2\boldsymbol{T}^{ib}\boldsymbol{T}^{ja}\right) = -||\mathrm{Rm}_{g(t)}||^2_{g(t)} - 4R_{ijab}\boldsymbol{T}^{ia}\boldsymbol{T}^{jb} + 4||\widehat{\boldsymbol{T}}(t)||^2_{g(t)},
$$

it follows that

$$
\begin{aligned}
(\nabla^j\boldsymbol{T}^{ik})(\nabla_i\boldsymbol{T}_{jk}) &= ||\nabla_t\boldsymbol{T}(t)||^2_{g(t)} + \frac{1}{8}\Big[-2||\mathrm{Rm}_t||^2_t - 8R_{ijab}\boldsymbol{T}^{ia}\boldsymbol{T}^{jb} - 4||\boldsymbol{T}(t)||^4_{g(t)} \\
&\quad + 4||\widehat{\boldsymbol{T}}(t)||^2_{g(t)} - \left(R_{ijab} + 2\boldsymbol{T}_{ia}\boldsymbol{T}_{jb}\right)\left(R^{ijmn} + 2\boldsymbol{T}^{im}\boldsymbol{T}^{jn}\right)\psi_{mn}{}^{ab}\Big]
\end{aligned}
$$

and (3.17) can be written as

$$
\begin{aligned}
\blacksquare_{g(t)}S_{g(t)} &= \frac{4}{3}\left|\left|\mathrm{Sic}_{g(t)} - 2\widehat{\boldsymbol{T}}(t)\right|\right|^2_{g(t)} + \frac{8}{3}||\nabla_{g(t)}\boldsymbol{T}(t)||^2_{g(t)} + \frac{4}{3}||\widehat{\boldsymbol{T}}(t)||^2_{g(t)} \\
&\quad - \frac{2}{3}||\mathrm{Rm}_{g(t)}||^2_{g(t)} - \frac{13}{3}S^2_{g(t)} \\
&\quad - \frac{1}{3}\left(R_{ijab} + 2\boldsymbol{T}_{ia}\boldsymbol{T}_{jb}\right)\left(R^{ijmn} + 2\boldsymbol{T}^{im}\boldsymbol{T}^{jn}\right)\psi_{mn}{}^{ab}.
\end{aligned}
\tag{3.18}
$$

Finally, we deal with the last term J on the right-hand side of (3.18). From the identity $\psi_{ijk\ell}\psi^{ijk\ell} = 168$, we find that

$$
\begin{aligned}
J &:= -\frac{1}{3}\left(R_{ijab} + 2\boldsymbol{T}_{ia}\boldsymbol{T}_{jb}\right)\left(R^{ijmn} + 2\boldsymbol{T}^{im}\boldsymbol{T}^{jn}\right)\psi_{mn}{}^{ab} \\
&= \frac{1}{3}\left(-R_{ij}{}^{ab}R^{ijmn}\psi_{mnab} - 4\boldsymbol{T}_i{}^a\boldsymbol{T}_j{}^bR^{ijmn}\psi_{mnab} - 4\boldsymbol{T}^a{}_i\boldsymbol{T}^{im}\boldsymbol{T}^b{}_j\boldsymbol{T}^{jn}\psi_{mnab}\right) \\
&= \frac{1}{3}\Big[\left|\left|R_{ij}{}^{ab}R^{ijmn} - \frac{1}{2}\psi^{abmn}\right|\right|^2_{g(t)} - \left|\left|R_{ij}{}^{ab}R^{ijmn}\right|\right|^2_{g(t)} - \frac{168}{4} \\
&\quad + \left|\left|2\boldsymbol{T}_i{}^a\boldsymbol{T}_j{}^bR^{ijmn} - \psi^{abmn}\right|\right|^2_{g(t)} - 4\left|\left|\boldsymbol{T}_i{}^a\boldsymbol{T}_j{}^bR^{ijmn}\right|\right|^2_{g(t)} - 168 \\
&\quad + \left|\left|2\widehat{\boldsymbol{T}}^{am}\widehat{\boldsymbol{T}}^{bn} - \psi^{mnab}\right|\right|^2_{g(t)} - 4||\widehat{\boldsymbol{T}}(t)||^4_{g(t)} - 168\Big].
\end{aligned}
$$

Plugging the expression for J into (3.18), we obtain

Proposition 3.2 *The scalar curvature $R_{g(t)}$ or $S_{g(t)}$ evolves by*

$$
\blacksquare_{g(t)}S_{g(t)} = \frac{4}{3}\left|\left|\mathrm{Sic}_{g(t)} - 2\widehat{\boldsymbol{T}}(t)\right|\right|^2_{g(t)} + \frac{8}{3}||\nabla_{g(t)}\boldsymbol{T}(t)||^2_{g(t)} - \frac{13}{3}S^2_{g(t)} - 126
$$

$$
\begin{aligned}
&+\frac{1}{3}\left|\left|R_{ijab}R^{ij}{}_{mn}-\psi_{abmn}\right|\right|^2_{g(t)}+\frac{4}{3}||\widehat{\boldsymbol{T}}(t)||^2_{g(t)}-\frac{4}{3}||\widehat{\boldsymbol{T}}(t)||^4_{g(t)}\\
&+\frac{1}{3}\left|\left|2\boldsymbol{T}_{ia}\boldsymbol{T}_{jb}R^{ij}{}_{mn}-\psi_{abmn}\right|\right|^2_{g(t)}+\frac{1}{3}\left|\left|2\widehat{\boldsymbol{T}}_{am}\widehat{\boldsymbol{T}}_{bn}-\psi_{abmn}\right|\right|^2_{g(t)}\\
&-\frac{2}{3}||\mathrm{Rm}_{g(t)}||^2_{g(t)}-\frac{1}{3}\left|\left|R_{ijab}R^{ij}{}_{mn}\right|\right|^2_{g(t)}-\frac{4}{3}\left|\left|\boldsymbol{T}_{ia}\boldsymbol{T}_{jb}R^{ij}{}_{mn}\right|\right|^2_{g(t)}.
\end{aligned}
\tag{3.19}
$$

Since $S_{g(t)}=\frac{2}{3}R_{g(t)}$, it follows from the above theorem that (1.8) holds true.

Before giving local curvature estimates for Laplacian flow in the next subsection, we derive evolution equations for $\mathrm{Ric}_{g(t)}$, $\mathrm{Rm}_{g(t)}$, and $\boldsymbol{T}(t)$ in different forms. Using the Lichnerowicz Laplacian

$$
\blacktriangle_{L,g(t)}\eta_{jk}:=\blacktriangle_{g(t)}\eta_{jk}-R_j{}^p\eta_{pk}-R_k{}^p\eta_{jp}+2R_{pjkq}h^{qp},
$$

we see that the evolution equation for R_{ij} can be written as

$$
\partial_t R_{jk}=-\frac{1}{2}\left[\blacktriangle_{L,g(t)}\eta_{jk}+\nabla_j\nabla_k\mathrm{tr}_{g(t)}\eta(t)+\nabla_j(d^*_{g(t)}\eta_t)_k+\nabla_k(d^*_{g(t)}\eta_t)_j\right],
$$

where $(d^*_{g(t)}\eta(t))_k:=-\nabla^j\eta_{jk}$. For $\eta_{ij}=-2R_{ij}-\frac{4}{3}||\boldsymbol{T}(t)||^2_{g(t)}g_{ij}-4\boldsymbol{T}_i{}^k\boldsymbol{T}_{kj}$ we have proved $\mathrm{tr}_{g(t)}\eta(t)=\frac{8}{3}||\boldsymbol{T}(t)||^2_{g(t)}$ and $(d^*_{g(t)}\eta(t))_j=\nabla_jR_{g(t)}+\frac{4}{3}\nabla_j||\boldsymbol{T}(t)||^2_{g(t)}+4\nabla^i\widehat{\boldsymbol{T}}_{ij}$ with $\widehat{\boldsymbol{T}}_{ij}=\boldsymbol{T}_i{}^k\boldsymbol{T}_{kj}$. Then

$$
\begin{aligned}
\partial_t R_{jk}&=\blacktriangle_{L,g(t)}\left(R_{jk}+\frac{2}{3}||\boldsymbol{T}(t)||^2_{g(t)}g_{jk}+2\widehat{\boldsymbol{T}}_{jk}\right)-\frac{1}{2}\nabla_j\left(\nabla_kR_{g(t)}+\frac{4}{3}\nabla_k||\boldsymbol{T}(t)||^2_{g(t)}\right.\\
&\quad\left.+4\nabla^i\widehat{\boldsymbol{T}}_{ik}\right)-\frac{4}{3}\nabla_j\nabla_k||\boldsymbol{T}(t)||^2_{g(t)}-\frac{1}{2}\nabla_k\left(\nabla_jR_t+\frac{4}{3}\nabla_j||\boldsymbol{T}(t)||^2_{g(t)}+4\nabla^i\widehat{\boldsymbol{T}}_{ij}\right)\\
&=\blacktriangle_{L,g(t)}\left(R_{jk}+\frac{2}{3}||\boldsymbol{T}(t)||^2_{g(t)}g_{jk}+2\widehat{\boldsymbol{T}}_{jk}\right)-2\nabla_j\nabla^i\widehat{\boldsymbol{T}}_{ik}\\
&\quad-2\nabla_k\nabla^i\widehat{\boldsymbol{T}}_{ij}-\frac{2}{3}\nabla_j\nabla_k||\boldsymbol{T}_{g(t)}||^2_{g(t)}.
\end{aligned}
$$

But the first term is equal to

$$
\begin{aligned}
\blacktriangle_{L,g(t)}\left(R_{jk}+\frac{2}{3}||\boldsymbol{T}(t)||^2_{g(t)}g_{jk}+2\widehat{\boldsymbol{T}}_{jk}\right)&=\blacktriangle_{g(t)}R_{jk}-2R_j{}^pR_{pk}+2R_{pjkq}R^{pq}\\
+\left[\frac{2}{3}\left(\blacktriangle_{g(t)}||\boldsymbol{T}(t)||^2_{g(t)}\right)g_{jk}+2\blacktriangle_{g(t)}\widehat{\boldsymbol{T}}_{jk}-2R_j{}^p\widehat{\boldsymbol{T}}_{pk}-2\widehat{\boldsymbol{T}}_j{}^pR^p{}_k+4R_{pjkq}\widehat{\boldsymbol{T}}^{pq}\right]&,
\end{aligned}
$$

we have

$$
\begin{aligned}
\blacksquare_{g(t)}R_{ij}=-2R_i{}^pR_{pj}+2R_{pijq}R^{pq}&+\left[\frac{2}{3}\left(\blacktriangle_{g(t)}||\boldsymbol{T}(t)||^2_{g(t)}\right)g_{ij}+2\blacktriangle_{g(t)}\widehat{\boldsymbol{T}}_{ij}\right.\\
&-2R_i{}^p\widehat{\boldsymbol{T}}_{pj}-2\widehat{\boldsymbol{T}}_i{}^pR_{pj}+4R_{pijq}\widehat{T}^{pq}-2\nabla_i\nabla^p\widehat{\boldsymbol{T}}_{pj}\\
&\left.-2\nabla_j\nabla^p\widehat{\boldsymbol{T}}_{pi}-\frac{2}{3}\nabla_i\nabla_j||\boldsymbol{T}(t)||^2_{g(t)}\right].
\end{aligned}
\tag{3.20}
$$

Consequently, the norm of $\mathrm{Ric}_{g(t)}$ satisfies

$$
\blacksquare_{g(t)}||\mathrm{Ric}_{g(t)}||^2_{g(t)}=-2||\nabla_{g(t)}\mathrm{Ric}_{g(t)}||^2_{g(t)}+\left[\frac{4}{3}R_{g(t)}\blacktriangle_{g(t)}||\boldsymbol{T}(t)||^2_{g(t)}\right.
$$

$$+ 8R^{k}{}_{ij}{}^{\ell}\widehat{\boldsymbol{T}}_{k\ell}R^{ij} + \frac{8}{3}||\mathrm{Ric}_{g(t)}||^2_{g(t)}||\boldsymbol{T}(t)||^2_{g(t)} + 4R_{kij\ell}R^{k\ell}R^{ij}$$
$$+4R^{ij}\blacktriangle_{g(t)}\widehat{\boldsymbol{T}}_{ij} - 8R^{ij}\nabla_i\nabla^k\widehat{\boldsymbol{T}}_{kj} - \frac{4}{3}R^{ij}\nabla_i\nabla_j||\boldsymbol{T}(t)||^2_{g(t)}\Bigg]. \tag{3.21}$$

The general formula (e.g. formula (2.66) in [5]) for R^{ℓ}_{ijk} gives

$$\begin{aligned}\partial_t R^{\ell}_{ijk} &= -\nabla_i\nabla_k R_j{}^{\ell} - \nabla_j\nabla^{\ell}R_{ik} + \nabla_i\nabla^{\ell}R_{jk} + \nabla_j\nabla_k R_i{}^{\ell} + R_{ijk}{}^{q}R_q{}^{\ell} + R_{ij}{}^{\ell q}R_{kp}\\ &\quad + 2R_{ijk}{}^{q}\widehat{\boldsymbol{T}}_q{}^{\ell} + 2R_{ij}{}^{\ell q}\widehat{\boldsymbol{T}}_{kp} - \frac{2}{3}\left(\nabla_i\nabla_k||\boldsymbol{T}(t)||^2_{g(t)}\right)g_j{}^{\ell} - 2\nabla_i\nabla_k\widehat{\boldsymbol{T}}_j{}^{\ell}\\ &\quad - 2\nabla_j\nabla^{\ell}\widehat{\boldsymbol{T}}_{ik} + 2\nabla_i\nabla^{\ell}\widehat{\boldsymbol{T}}_{jk} + 2\nabla_j\nabla_k\widehat{\boldsymbol{T}}_i{}^{\ell} - \frac{2}{3}\left(\nabla_j\nabla^{\ell}||\boldsymbol{T}(t)||^2_{g(t)}\right)g_{ik}\\ &\quad + \frac{2}{3}\left(\nabla_i\nabla^{\ell}||\boldsymbol{T}(t)||^2_{g(t)}\right)g_{jk} + \frac{2}{3}\left(\nabla_j\nabla_k||\boldsymbol{T}(t)||^2_{g(t)}\right)g_i{}^{\ell}.\end{aligned} \tag{3.22}$$

Hence, the evolution equation for $||\mathrm{Rm}_{g(t)}||^2_{g(t)}$ is given by

$$\begin{aligned}\partial_t||\mathrm{Rm}_{g(t)}||^2_{g(t)} &= \nabla^2_{g(t)}\mathrm{Ric}_{g(t)} * \mathrm{Rm}_{g(t)} + \mathrm{Ric}_{g(t)} * \mathrm{Rm}_{g(t)} * \mathrm{Rm}_{g(t)}\\ &\quad + \mathrm{Rm}_{g(t)} * \mathrm{Rm}_{g(t)} * \widehat{\boldsymbol{T}}(t) + \mathrm{Ric}_{g(t)} * \nabla^2_{g(t)}||\boldsymbol{T}(t)||^2_{g(t)}\\ &\quad + \mathrm{Rm}_{g(t)} * \nabla^2_{g(t)}\widehat{\boldsymbol{T}}(t) + \frac{8}{3}|\boldsymbol{T}(t)|^2_{g(t)}||\mathrm{Rm}_{g(t)}||^2_{g(t)}.\end{aligned} \tag{3.23}$$

Moreover, it was proved in [32] that

$$\begin{aligned}||\nabla_{g(t)}\mathrm{Rm}_{g(t)}||^2_{g(t)} &\le -\frac{1}{2}\blacksquare_{g(t)}||\mathrm{Rm}_{g(t)}||^2_{g(t)} + C_1||\mathrm{Rm}_{g(t)}||^3_{g(t)} + C_1||\mathrm{Rm}_{g(t)}||^{3/2}_{g(t)}\\ &\quad \cdot ||\nabla^2_{g(t)}\boldsymbol{T}(t)||_{g(t)} + C_1||\mathrm{Rm}_{g(t)}||_{g(t)}||\nabla_{g(t)}\boldsymbol{T}(t)||^2_{g(t)}\end{aligned} \tag{3.24}$$

where C_1 is some universal constant, and

$$\begin{aligned}\blacksquare_{g(t)}\boldsymbol{T}(t) &= \mathrm{Rm}_{g(t)} * \boldsymbol{T}(t) + \mathrm{Rm}_{g(t)} * \boldsymbol{T}(t) * \psi(t)\\ &\quad + \nabla_{g(t)}\boldsymbol{T}(t) * \boldsymbol{T}(t) * \varphi(t) + \boldsymbol{T}(t) * \boldsymbol{T}(t) * \boldsymbol{T}(t).\end{aligned} \tag{3.25}$$

Squaring (3.25) gives

$$\begin{aligned}||\nabla_{g(t)}\boldsymbol{T}(t)||^2_{g(t)} &\le -\frac{1}{2}\blacksquare_{g(t)}||\boldsymbol{T}(t)||^2_{g(t)} + C_2||\mathrm{Rm}_{g(t)}||_{g(t)}||\boldsymbol{T}(t)||^2_{g(t)}\\ &\quad + C_2||\nabla_{g(t)}\boldsymbol{T}(t)||_{g(t)}||\boldsymbol{T}(t)||^2_{g(t)} + C_2||\boldsymbol{T}(t)||^4_{g(t)}\end{aligned} \tag{3.26}$$

for another universal constant C_2 which may differs from C_1. The Cauchy-Schwartz inequality shows $2C_2||\nabla_{g(t)}\boldsymbol{T}(t)||_{g(t)}||\boldsymbol{T}(t)||^2_{g(t)} \le ||\nabla_{g(t)}\boldsymbol{T}(t)||^2_{g(t)} + C_2^2||\boldsymbol{T}(t)||^4_{g(t)}$, so that the evolution inequality (3.26) becomes

$$\begin{aligned}||\nabla_{g(t)}\boldsymbol{T}(t)||^2_{g(t)} &\le -\blacksquare_{g(t)}||\boldsymbol{T}(t)||^2_{g(t)}\\ &\quad + C_3||\mathrm{Rm}_{g(t)}||_{g(t)}||\boldsymbol{T}(t)||^2_{g(t)} + C_3||\boldsymbol{T}(t)||^4_{g(t)}.\end{aligned} \tag{3.27}$$

Here C_3 is a universal constant.

3.2 Main idea of proving Theorem 1.4

In this section, we consider the Laplacian flow (3.1) on $\mathcal{M} \times [0, T]$, where $T \in (0, T_{\max})$. From now on we always omit the time subscripts from all considered quantities. From (3.7), (3.21), (3.23), (3.24), and (3.27) we have

$$\begin{aligned}
\|\nabla \mathrm{Ric}\|^2 &= -\frac{1}{2}\blacksquare\|\mathrm{Ric}\|^2 + \mathrm{Ric} * \mathrm{Ric} * \mathrm{Rm} - \frac{1}{3}(\blacktriangle R)\, R - \frac{2}{3}\|\mathrm{Ric}\|^2 R \\
&\quad + 2\langle\langle \mathrm{Ric}, \blacktriangle \widehat{\boldsymbol{T}}\rangle\rangle + \frac{1}{3}\langle\langle \mathrm{Ric}, \nabla^2 R\rangle\rangle + \mathrm{Ric} * \widehat{\boldsymbol{T}} * \mathrm{Rm} + \mathrm{Ric} * \nabla^2 \widehat{\boldsymbol{T}}, \\
\|\nabla \mathrm{Rm}\|^2 &\le -\frac{1}{2}\blacksquare\|\mathrm{Rm}\|^2 + C\|\mathrm{Rm}\|^3 + C\|\mathrm{Rm}\|^{3/2}\|\nabla^2 \boldsymbol{T}\| + C\|\mathrm{Rm}\|\|\nabla \boldsymbol{T}\|^2, \\
\partial_t \|\mathrm{Rm}\|^2 &= \nabla^2 \mathrm{Ric} * \mathrm{Rm} + \mathrm{Ric} * \mathrm{Rm} * \mathrm{Rm} + \mathrm{Rm} * \mathrm{Rm} * \widehat{\boldsymbol{T}} \\
&\quad + \mathrm{Ric} * \nabla^2 \|\boldsymbol{T}\|^2 + \mathrm{Rm} * \nabla^2 \widehat{\boldsymbol{T}} + \frac{4}{3}\|\boldsymbol{T}\|^2\|\mathrm{Rm}\|^2, \\
\|\nabla \boldsymbol{T}\|^2 &\le -\blacksquare\|\boldsymbol{T}\|^2 + C\|\mathrm{Rm}\|\|\boldsymbol{T}\|^2 + C\|\boldsymbol{T}\|^4, \\
\partial_t dV &= \frac{2}{3}\|\boldsymbol{T}\|^2 dV, \quad R = -\|\boldsymbol{T}\|^2.
\end{aligned}$$

Choose an open domain Ω of $\mathcal{M}$ and assume that

$$\|\mathrm{Ric}\| \le K \tag{3.28}$$

on $\Omega \times [0, T]$, Then the torsion $\boldsymbol{T}$ satisfies[2] $\|\boldsymbol{T}\| \lesssim K^{1/2}$ and metrics $g(t)$ are all equivalent to $g(0)$. We also observe from (2.25) and (3.11) that

$$\|\mathrm{Ric}\| \lesssim 1 \Longleftrightarrow |\Delta\varphi| \lesssim 1 \tag{3.29}$$

and the following simple fact

$$\partial_t \|A\|^2 = \frac{p}{2}\|A\|^{p-2}\partial_t \|A\|^2 \tag{3.30}$$

for any tensor A.

Choose a Lipschitz function η with support in Ω (and independent of time t) and consider the quantity

$$\frac{d}{dt}\int \|\mathrm{Rm}\|^p \eta^{2p} dV, \quad \int := \int_{\mathcal{M}},$$

where $p \ge 5$. As in [28], we introduce the following "good" quantities

$$\begin{aligned}
A_1 &:= \int \|\mathrm{Rm}\|^p \eta^{2p} dV, \quad A_2 := \int \|\mathrm{Rm}\|^{p-1}\eta^{2p} dV, \\
A_3 &:= \int \|\mathrm{Rm}\|^{p-1}\|\nabla\eta\|^2\eta^{2p-1} dV, \quad A_4 := \int \|\mathrm{Rm}\|^{p-1}\|\nabla\eta\|^2\eta^{2p-2} dV
\end{aligned}$$

and also "bad" quantities

$$B_1 := \frac{1}{K}\int \|\nabla \mathrm{Ric}\|^2\|\mathrm{Rm}\|^{p-1}\eta^{2p} dV, \quad B_2 := \int \|\nabla \mathrm{Rm}\|^2\|\mathrm{Rm}\|^{p-3}\eta^{2p} dV.$$

We split the proof of Theorem 1.4 into four steps.

---

[2] Here $A \lesssim B$ means that $A \le CB$ for some positive constant C independent of t.

(a) In the first step, we can show that, see Lemma 3.3,

$$\frac{d}{dt}A_1 \leq B_1 + cKB_2 + cKA_4 + cKA_1 + cK^2A_2 + c\int \left(-\blacksquare||\boldsymbol{T}||^2\right)||\mathrm{Rm}||^{p-1}\eta^{2p}dV.$$

(b) In the second step, we can prove that the term

$$c\int \left(-\blacksquare||\boldsymbol{T}||^2\right)||\mathrm{Rm}||^{p-1}\eta^{2p}dV$$

is bounded from above by [see (3.42)]

$$B_1 + cKB_2 + cK^2A_2 + cKA_1 - \frac{d}{dt}\left[\int c(-R)||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right].$$

Observe that the above integral is nonnegative, since the scalar curvature R is nonpositive along the Laplacian flow on closed G_2-structures. Hence we obtain from the first step that, see Lemma 3.4,

$$\frac{d}{dt}A_1 \leq 2B_1 + cKB_2 + cKA_4 + cKA_1 + cK^2A_2 - \frac{d}{dt}\left[\int c(-R)||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right].$$

(c) In the next two steps, we estimate the bad terms B_1 and B_2. In the third step, B_1 is estimated by [see (3.52)]

$$B_1 \leq cKB_2 + cKA_4 + cKA_1 + cK^2A_2 - \frac{d}{dt}\left[\frac{1}{K}\int ||\mathrm{Rm}||^{p-1}||\mathrm{Ric}||^2\eta^{2p}dV + c\int(-R)||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right].$$

Then the second step can be simplified as, see Lemma 3.5,

$$\frac{d}{dt}A_1 \leq cKB_2 + cKA_4 + cKA_1 + cK^2A_2 - \frac{d}{dt}\left[\frac{1}{K}\int ||\mathrm{Rm}||^{p-1}||\mathrm{Ric}||^2\eta^{2p}dV + c\int(-R)||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right].$$

(d) Finally, we estimate the term B_2. In this step we shall use the assumption that $p \geq 5$ (a technical assumption). Using the inequality $||\nabla\boldsymbol{T}|| \lesssim ||\mathrm{Rm}||$ and $||\nabla^2\boldsymbol{T}|| \lesssim ||\nabla\mathrm{Rm}|| + ||\mathrm{Rm}||||\boldsymbol{T}|| + ||\nabla\boldsymbol{T}||||\boldsymbol{T}|| + ||\boldsymbol{T}||^3$, we can prove [see (3.62)]

$$B_2 \leq cA_4 + cA_1 - \frac{d}{dt}\left[\frac{1}{p-1}\int ||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right].$$

Plugging it into the third step, we arrive at, see Lemma 3.6,

$$\frac{d}{dt}(A_1 + cKA_2) \leq cK(A_1 + cKA_2) + cKA_4 - \frac{d}{dt}\left[\frac{c}{K}\int ||\mathrm{Rm}||^{p-1}||\mathrm{Ric}||^2\eta^{2p}dV + c\int(-R)||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right].$$

The proof of Theorem 1.4 As in [25,28], we choose a geodesic ball $\Omega := B_{g(0)}(x_0, \rho/\sqrt{K})$ and a cut-off function

$$\eta = \left(\frac{\rho/\sqrt{K} - d_{g(0)}(x_0, \cdot)}{\rho/\sqrt{K}} \right)_+ .$$

Then, for all $t \in [0, T]$,

$$e^{-cKt} g(0) \le g(t) \le e^{cKt} g(0), \quad ||\nabla_{g(t)}\phi||_{g(t)} \le e^{cKT} ||\nabla_{g(0)}\phi||_{g(0)} \le \frac{\sqrt{K} e^{cKT}}{\rho}.$$

Define

$$\begin{aligned} U &:= \int ||\mathrm{Rm}||^p \eta^{2p} dV + cK \int ||\mathrm{Rm}||^{p-1} \eta^{2p} dV \\ &\quad + \frac{c}{K} \int ||\mathrm{Rm}||^{p-1} ||\mathrm{Ric}||^2 \eta^{2p} dV + c \int (-R) ||\mathrm{Rm}||^{p-1} \eta^{2p} dV. \end{aligned} \tag{3.31}$$

Then (3.64) (see below) yields

$$U' \le cKU + cKA_4. \tag{3.32}$$

For A_4, using the Young inequality, we have

$$\begin{aligned} A_4 = \int ||\mathrm{Rm}||^{p-1} ||\nabla\eta||^2 \eta^{2p-2} dV \ &\le \int_{B_{g(0)}(x_0, \rho/\sqrt{K})} ||\mathrm{Rm}||^{p-1} \eta^{2p-2} K \rho^{-2} e^{cKT} dV \\ &\le \int_{B_{g(0)}(x_0, \rho/\sqrt{K})} \left[\frac{(||\mathrm{Rm}||^{p-1}\eta^{2p-2})^{p/(p-1)}}{\frac{p}{p-1}} + \frac{(K\rho^{-2}e^{cKT})^p}{p} \right] dV \\ &\le A_1 + K^p \rho^{-2p} p e^{cKT} \mathrm{Vol}_{g(t)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}} \right) \right) \\ &\le U + cK^p e^{cKT} \rho^{-2p} \mathrm{Vol}_{g(t)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}} \right) \right). \end{aligned}$$

Thus

$$U' \le cKU + cK^{p+1} e^{cKT} \rho^{-2p} \mathrm{Vol}_{g(t)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}} \right) \right).$$

As in the proof of [25], one can easily deduce from above that

$$\begin{aligned} \int_{B_{g(0)}(x_0, \frac{\rho}{2\sqrt{K}})} ||\mathrm{Rm}_{g(t)}||^p_{g(t)} dV_{g(t)} \ &\le \ c(1+K) e^{cKT} \int_{B_{g(0)}(x_0, \frac{\rho}{\sqrt{K}})} ||\mathrm{Rm}_{g(0)}||^p_{g(0)} dV_{g(0)} \\ &\quad + cK^p \left(1 + \rho^{-2p}\right) e^{cKT} \mathrm{Vol}_{g(t)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}} \right) \right). \end{aligned} \tag{3.33}$$

Indeed, writing $A := cK$ and $B := cK^{p+1} e^{cKT} \rho^{-2p}$, we get

$$U' \le AU + B \,\mathrm{Vol}_{g(t)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}} \right) \right)$$

and then

$$e^{-At} U(t) \le U(0) + \int_0^t B e^{-A\tau} \mathrm{Vol}_{g(\tau)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}} \right) \right) d\tau.$$

On the other hand, the estimate $e^{-cKt}g(0) \le g(t) \le e^{cKt}g(0)$ yields

$$\mathrm{Vol}_{g(\tau)}\left(B_{g(0)}\left(x_0, \frac{\rho}{\sqrt{K}}\right)\right) \le e^{cKT}\mathrm{Vol}_{g(t)}\left(B_{g(0)}\left(x_0, \frac{\rho}{\sqrt{K}}\right)\right).$$

Consequently,

$$U(t) \le e^{AT}\left[U(0) + \frac{B}{A}e^{cKT}\mathrm{Vol}_{g(t)}\left(B_{g(0)}\left(x_0, \frac{\rho}{\sqrt{K}}\right)\right)\right], \quad t \in [0, T].$$

At last, we estimate from (3.28) and Young's inequality

$$\begin{aligned}
U(0) = &\int_{\mathcal{M}} ||\mathrm{Rm}_{g(0)}||^p_{g(0)}\eta^{2p}dV_{g(0)} + cK\int_{\mathcal{M}} ||\mathrm{Rm}_{g(0)}||^{p-1}_{g(0)}\eta^{2p}dV_{g(0)} \\
&+ \frac{c}{K}\int_{\mathcal{M}} ||\mathrm{Rm}_{g(0)}||^{p-1}_{g(0)}||\mathrm{Ric}_{g(0)}||^2_{g(0)}\eta^{2p}dV_{g(0)} \\
&+ c\int_{\mathcal{M}} (-R_{g(0)})||\mathrm{Rm}_{g(0)}||^{p-1}_{g(0)}\eta^{2p}dV_{g(0)} \\
\le &\int_{\mathcal{M}} ||\mathrm{Rm}_{g(0)}||^p_{g(0)}\eta^{2p}dV_{g(0)} + cK\int_{\mathcal{M}} ||\mathrm{Rm}_{g(0)}||^{p-1}_{g(0)}\eta^{2p}dV_{g(0)} \\
\le &\int_{\mathcal{M}} ||\mathrm{Rm}_{g(0)}||^p_{g(0)}\eta^{2p}dV_{g(0)} + C\int_{\mathcal{M}}\left[\left(||\mathrm{Rm}_{g(0)}||^{p-1}_{g(0)}\eta^{2(p-1)}\right)^{\frac{p}{p-1}}dV_{g(0)}\right. \\
&\left. + \int_{\mathcal{M}} (K\eta^2)^p dV_{g(0)}\right] \\
\le &\ (1+K)\int_{\mathcal{M}} ||\mathrm{Rm}_{g(0)}||^p_{g(0)}\eta^{2p}dV_{g(0)} + CK^p\mathrm{Vol}_{g(0)}\left(B_{g(0)}\left(x_0, \frac{\rho}{K}\right)\right) \\
\le &\ C(1+K)\int_{\mathcal{M}} ||\mathrm{Rm}_{g(0)}||^p_{g(0)}\eta^{2p}dV_{g(0)} + CK^pe^{cKT}\mathrm{Vol}_{g(t)}\left(B_{g(0)}\left(x_0, \frac{\rho}{\sqrt{K}}\right)\right)
\end{aligned}$$

which implies (3.33).

As an immediate consequence of the inequality (3.33) we give another proof of the part (a) in Theorem 1.2.

3.3 Proving four steps (*a*) − (*d*)

We are going to carry out the above mentioned four steps. From (3.23) and the above evolution equations, we have

$$\begin{aligned}
&\frac{d}{dt}\int ||\mathrm{Rm}||^p\eta^{2p}dV \\
&= \int \left(\partial_t||\mathrm{Rm}||^p\right)\eta^{2p}dV + \int ||\mathrm{Rm}||^p\eta^{2p}\partial_t dV \\
&= \int \frac{p}{2}||\mathrm{Rm}||^{p-2}\left(\partial_t||\mathrm{Rm}||^2\right)\eta^{2p}dV + \int ||\mathrm{Rm}||^p\eta^{2p}\left(-\frac{2}{3}R\right)dV \\
&= \int \frac{p}{2}||\mathrm{Rm}||^{p-2}\begin{bmatrix} \nabla^2\mathrm{Ric} * \mathrm{Rm} + \mathrm{Ric} * \mathrm{Rm} * \mathrm{Rm} \\ + \mathrm{Rm} * \mathrm{Rm} * \widehat{\boldsymbol{T}} + \mathrm{Ric} * \nabla^2||\boldsymbol{T}||^2 \\ + \mathrm{Rm} * \nabla^2\widehat{\boldsymbol{T}} + \frac{4}{3}||\boldsymbol{T}||^2||\mathrm{Rm}||^2 \end{bmatrix}\eta^{2p}dV \\
&\quad - \frac{2}{3}\int R||\mathrm{Rm}||^p\eta^{2p}dV
\end{aligned}$$

$$
\begin{aligned}
&\le c\int ||\mathrm{Rm}||^{p-2}\Big[\nabla^2\mathrm{Ric}*\mathrm{Rm}+K||\mathrm{Rm}||^2+K||\mathrm{Rm}||^2+\nabla^2||\boldsymbol{T}||^2*\mathrm{Ric}\\
&\quad+\nabla^2\widehat{\boldsymbol{T}}*\mathrm{Rm}\Big]\eta^{2p}dV+cK\int ||\mathrm{Rm}||^p\eta^{2p}dV\\
&\le c\int ||\mathrm{Rm}||^{p-2}\left[\nabla^2\mathrm{Ric}*\mathrm{Rm}+\nabla^2||\boldsymbol{T}||^2*\mathrm{Ric}+\nabla^2\widehat{\boldsymbol{T}}*\mathrm{Rm}\right]\eta^{2p}dV\\
&\quad+cK\int ||\mathrm{Rm}||^p\eta^{2p}dV. \qquad (3.34)
\end{aligned}
$$

It was proved in [25] that the first integral in (3.34) is bounded by

$$
\begin{aligned}
c\int ||\mathrm{Rm}||^{p-2}\left(\nabla^2\mathrm{Ric}*\mathrm{Rm}\right)\eta^{2p}dV &\le \frac{1}{K}\int ||\nabla\mathrm{Ric}||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV\\
+cK\int ||\nabla\mathrm{Rm}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}dV&+cK\int ||\mathrm{Rm}||^{p-1}||\nabla\eta||^2\eta^{2p-2}dV. \qquad (3.35)
\end{aligned}
$$

Since $||\boldsymbol{T}||^2=-R$, the same inequality holds for the integral

$$
c\int ||\mathrm{Rm}||^{p-2}\left(\nabla^2||\boldsymbol{T}||^2*\mathrm{Ric}\right)\eta^{2p}dV.
$$

To deal with the last term in the bracket of (3.34), we use the same argument of [25] to conclude

$$
\begin{aligned}
c\int ||\mathrm{Rm}||^{p-2}\left(\nabla^2\widehat{\boldsymbol{T}}*\mathrm{Rm}\right)\eta^{2p}dV &= c\int\left(\nabla||\mathrm{Rm}||^{p-2}*\nabla\widehat{\boldsymbol{T}}*\mathrm{Rm}\right)\eta^{2p}dV\\
&\quad+c\int\left(||\mathrm{Rm}||^{p-2}*\nabla\widehat{\boldsymbol{T}}*\nabla\mathrm{Rm}\right)\eta^{2p}dV\\
&\quad+c\int\left(||\mathrm{Rm}||^{p-2}*\nabla\widehat{\boldsymbol{T}}*\mathrm{Rm}*\nabla\eta\right)\eta^{2p-1}dV\\
&\le c\int ||\mathrm{Rm}||^{p-2}||\nabla\mathrm{Rm}||||\nabla\widehat{\boldsymbol{T}}||\eta^{2p}dV\\
&\quad+c\int ||\mathrm{Rm}||^{p-2}||\nabla\widehat{\boldsymbol{T}}||||\nabla\mathrm{Rm}||\eta^{2p}dV\\
&\quad+c\int ||\mathrm{Rm}||^{p-1}||\nabla\widehat{\boldsymbol{T}}||||\nabla\eta||\eta^{2p-1}dV\\
&\le c\int ||\mathrm{Rm}||^{p-2}||\nabla\mathrm{Rm}||||\nabla\widehat{\boldsymbol{T}}||\eta^{2p}dV\\
&\quad+c\int ||\mathrm{Rm}||^{p-1}||\nabla\widehat{\boldsymbol{T}}||||\nabla\eta||\eta^{2p-1}dV.
\end{aligned}
$$

According to the Cauchy-Schwartz inequality, the first and second integrals are bounded by

$$
\begin{aligned}
&\int ||\mathrm{Rm}||^{p-2}||\nabla\mathrm{Rm}||||\nabla\widehat{\boldsymbol{T}}||\eta^{2p}dV\\
&\quad\le cK\int ||\nabla\mathrm{Rm}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}dV+\frac{1}{K}\int ||\nabla\widehat{\boldsymbol{T}}||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV
\end{aligned}
$$

and

$$
\int ||\mathrm{Rm}||^{p-1}||\nabla\widehat{\boldsymbol{T}}||||\nabla\eta||\eta^{2p-1}dV
$$

$$\leq \frac{1}{K}\int ||\nabla\widehat{\boldsymbol{T}}||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV + cK\int ||\mathrm{Rm}||^{p-1}||\nabla\eta||^2\eta^{2p-2}dV.$$

Hence we obtain

$$\begin{aligned} c\int ||\mathrm{Rm}||^{p-2}\left(\nabla^2\widehat{\boldsymbol{T}} * \mathrm{Rm}\right)\eta^{2p}dV \leq \frac{1}{K}\int ||\nabla\widehat{\boldsymbol{T}}||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV \\ + cK\int ||\nabla\mathrm{Rm}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}dV \\ + cK\int ||\mathrm{Rm}||^{p-1}||\nabla\eta||^2\eta^{2p-2}dV. \end{aligned} \tag{3.36}$$

Using $\widehat{\boldsymbol{T}} = \boldsymbol{T} * \boldsymbol{T}$ and $R = -||\boldsymbol{T}||^2$ yields

$$\begin{aligned} &\frac{1}{K}\int ||\nabla\widehat{\boldsymbol{T}}||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV \\ &\leq \frac{c}{K}\int ||\nabla\boldsymbol{T}||^2||\boldsymbol{T}||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV \ \leq\ c\int ||\nabla\boldsymbol{T}||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV \\ &\leq c\int\left(-\frac{1}{4}\blacksquare||\boldsymbol{T}||^2 + c||\mathrm{Rm}||||\boldsymbol{T}||^2 + c||\boldsymbol{T}||^4\right)||\mathrm{Rm}||^{p-1}\eta^{2p}dV \\ &= c\int\left(-\blacksquare||\boldsymbol{T}||^2\right)||\mathrm{Rm}||^{p-1}\eta^{2p}dV \\ &\quad + cK\int ||\mathrm{Rm}||^p\eta^{2p}dV + cK^2\int ||\mathrm{Rm}||^{p-1}\eta^{2p}dV. \end{aligned} \tag{3.37}$$

Hence, using (3.35), (3.36), and (3.37), we arrive at

Lemma 3.3 *One has*

$$\begin{aligned} A_1' \equiv \frac{d}{dt}A_1 \leq B_1 + cKB_2 + cKA_4 + cKA_1 + cK^2A_2 \\ + c\int\left(-\blacksquare||\boldsymbol{T}||^2\right)||\mathrm{Rm}||^{p-1}\eta^{2p}dV. \end{aligned} \tag{3.38}$$

In the following computations, we are mainly going to estimate or simplify the bad terms B_1, B_2, and also the term involving $-\blacksquare||\boldsymbol{T}||^2$. Integration by parts on the last integral in (3.38) and using $R = -||\boldsymbol{T}||^2$, we obtain

$$\begin{aligned} c\int\left(-\blacksquare||\boldsymbol{T}||^2\right)||\mathrm{Rm}||^{p-1}\eta^{2p}dV &= c\int\left((\partial_t - \Delta)R\right)||\mathrm{Rm}||^{p-1}\eta^{2p}dV \\ &= c\int(\partial_t R)||\mathrm{Rm}||^{p-1}\eta^{2p}dV \\ &\quad + c\int\left\langle\nabla R, \nabla\left(||\mathrm{Rm}||^{p-1}\eta^{2p}\right)\right\rangle dV \\ &= \frac{d}{dt}\left(c\int R||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right) \\ &\quad - c\int R\left(\partial_t||\mathrm{Rm}||^{p-1}\right)\eta^{2p}dV \\ &\quad - c\int R||\mathrm{Rm}||^{p-1}\eta^{2p}\partial_t dV \\ &\quad + c\int\left\langle\nabla R, ||\mathrm{Rm}||^{p-3}\mathrm{Rm} * \nabla\mathrm{Rm}\right\rangle\eta^{2p}dV \end{aligned}$$

$$
\begin{aligned}
&+ c\int \left\langle \nabla R, ||\mathrm{Rm}||^{p-1}\eta^{2p-1}\nabla\eta\right\rangle dV\\
\le\ & c\int ||\mathrm{Rm}||^{p-2}\langle \nabla R, \nabla \mathrm{Rm}\rangle \eta^{2p} dV\\
&+c\int ||\mathrm{Rm}||^{p-1}||\nabla R||||\nabla\eta||\eta^{2p-1}dV\\
&+c\int R^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV\\
&-c\int R\left(\partial_t||\mathrm{Rm}||^{p-1}\right)\eta^{2p}dV\\
&+\frac{d}{dt}\left(c\int R||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right).
\end{aligned}
$$

The first two integrals can be simplified by using the Cauchy–Schwarz inequality as follows:

$$
\begin{aligned}
&c\int ||\mathrm{Rm}||^{p-2}\langle \nabla R, \nabla \mathrm{Rm}\rangle \eta^{2p} dV\\
&\le c\int ||\nabla \mathrm{Ric}||||\nabla \mathrm{Rm}||||\mathrm{Rm}||^{p-2}\eta^{2p}dV\\
&\le\ c\int \left(||\nabla \mathrm{Rm}||||\mathrm{Rm}||^{\frac{p-3}{2}}\eta^{p}\right)\left(||\nabla \mathrm{Ric}||||\mathrm{Rm}||^{\frac{p-1}{2}}\eta^{p}\right)dV\\
&\le \frac{1}{50}B_1 + cKB_2
\end{aligned}
$$

and

$$
\begin{aligned}
&c\int ||\mathrm{Rm}||^{p-1}||\nabla R||||\nabla\eta||\eta^{2p-1}dV\\
&\le\ c\int ||\mathrm{Rm}||^{p-1}||\nabla \mathrm{Ric}||||\nabla\eta||\eta^{2p-1}dV\\
&\le\ c\int \left(||\mathrm{Rm}||^{\frac{p-1}{2}}||\nabla\eta||\eta^{p-1}\right)\left(||\mathrm{Rm}||^{\frac{p-1}{2}}||\nabla \mathrm{Ric}||\eta^{p}\right)dV\\
&\le \frac{1}{50}B_1 + cKA_4.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
c\int \left(-\blacksquare||\boldsymbol{T}||^2\right)||\mathrm{Rm}||^{p-1}\eta^{2p}dV \le\ & \frac{2}{50}B_1 + cKB_2 + cKA_4 + cK^2A_2\\
&+\frac{d}{dt}\left(c\int R||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right)\\
&-c\int R\left(\partial_t||\mathrm{Rm}||^{p-1}\right)\eta^{2p}dV. \qquad (3.39)
\end{aligned}
$$

Now, the second integral in (3.39) is equal to

$$
\begin{aligned}
-c\int R\left(\partial_t||\mathrm{Rm}||^{p-1}\right)\eta^{2p}dV\ &=\ c\int(-R)\,||\mathrm{Rm}||^{p-3}\left(\partial_t||\mathrm{Rm}||^2\right)\eta^{2p}dV\\
=\ c\int(-R)||\mathrm{Rm}||^{p-3}\Big[&\nabla^2\mathrm{Ric}*\mathrm{Rm} + \mathrm{Ric}*\mathrm{Rm}*\mathrm{Rm} + \mathrm{Rm}*\mathrm{Rm}*\widehat{\boldsymbol{T}}
\end{aligned}
$$

$$+\,\mathrm{Ric}*\nabla^2||\boldsymbol{T}||^2+\mathrm{Rm}*\nabla^2\widehat{\boldsymbol{T}}+\frac{4}{3}||\boldsymbol{T}||^2||\mathrm{Rm}||^2\Big]\eta^{2p}dV$$
$$\leq\ c\int(-R)||\mathrm{Rm}||^{p-3}\left[\nabla^2\mathrm{Ric}*\mathrm{Rm}-\mathrm{Ric}*\nabla^2R+\nabla^2\widehat{\boldsymbol{T}}*\mathrm{Rm}\right]\eta^{2p}dV+cK^2A_2.$$

Using the identity, where $p\geq 5$,

$$\nabla||\mathrm{Rm}||^{p-3}=\frac{p-3}{2}\left(||\mathrm{Rm}||^2\right)^{\frac{p-3}{2}-1}\nabla||\mathrm{Rm}||^2=||\mathrm{Rm}||^{p-5}\mathrm{Rm}*\nabla\mathrm{Rm}$$

we obtain

$$\begin{aligned}
&c\int(-R)||\mathrm{Rm}||^{p-3}\eta^{2p}(\nabla^2\mathrm{Ric}*\mathrm{Rm})dV\\
&=c\int(-R)||\mathrm{Rm}||^{p-3}\eta^{2p}(\nabla\mathrm{Ric}*\nabla\mathrm{Rm})dV\\
&\quad+c\int\left\{\nabla\left[(-R)||\mathrm{Rm}||^{p-3}\phi^{2p}\right]*\nabla\mathrm{Ric}*\mathrm{Rm}\right\}dV\\
&\quad=c\int(-R)||\mathrm{Rm}||^{p-3}\eta^{2p}(\nabla\mathrm{Ric}*\nabla\mathrm{Rm})dV\\
&\quad+c\int||\mathrm{Rm}||^{p-3}\eta^{2p}(\nabla R*\nabla\mathrm{Ric}*\mathrm{Rm})dV\\
&\quad+\ c\int(-R)\eta^{2p}\left(\nabla||\mathrm{Rm}||^{p-3}*\nabla\mathrm{Ric}*\mathrm{Rm}\right)dV\\
&\quad+\ c\int(-R)||\mathrm{Rm}||^{p-3}\eta^{2p-1}\left(\nabla\phi*\nabla\mathrm{Ric}*\mathrm{Rm}\right)dV\\
&\leq\ c\int||\mathrm{Rm}||^{p-2}\eta^{2p}||\nabla\mathrm{Ric}||||\nabla\mathrm{Rm}||dV\\
&\quad+c\int||\nabla\mathrm{Ric}||||\nabla R||||\mathrm{Rm}||^{p-2}\eta^{2p}dV\\
&\quad+\ c\int||\mathrm{Rm}||^{p-2}||\nabla\mathrm{Ric}||||\nabla\mathrm{Rm}||\eta^{2p}dV\\
&\quad+c\int||\mathrm{Rm}||^{p-1}\eta^{2p-1}||\nabla\eta||||\nabla\mathrm{Ric}||dV\\
&\leq\ c\int\left(||\nabla\mathrm{Ric}||||\mathrm{Rm}||^{\frac{p-1}{2}}\eta^p\right)\left(||\nabla\mathrm{Rm}||||\mathrm{Rm}||^{\frac{p-3}{2}}\eta^p\right)dV\\
&\quad+\ c\int\left(||\nabla\mathrm{Ric}||||\mathrm{Rm}||^{\frac{p-1}{2}}\eta^p\right)\left(||\nabla\phi||||\mathrm{Rm}||^{\frac{p-1}{2}}\eta^{p-1}\right)dV\\
&\quad\leq\frac{1}{50}B_1+cKB_2+cKA_4.
\end{aligned}$$

Similarly, we can prove

$$c\int(-R)||\mathrm{Rm}||^{p-3}\left(-\mathrm{Ric}*\nabla^2R\right)\eta^{2p}dV\leq\frac{1}{50}B_1+cKB_2+cKA_4.$$

Using $\nabla\widehat{\boldsymbol{T}}=\nabla\boldsymbol{T}*\boldsymbol{T}\leq c||\nabla\boldsymbol{T}||||\boldsymbol{T}||\leq cK^{1/2}||\nabla\boldsymbol{T}||$ yields

$$c\int(-R)||\mathrm{Rm}||^{p-3}\eta^{2p}\left(\nabla^2\widehat{\boldsymbol{T}}*\mathrm{Rm}\right)dV$$

$$
\begin{aligned}
&= c\int (-R)||\mathrm{Rm}||^{p-3}\eta^{2p}(\nabla\widehat{\boldsymbol{T}} * \nabla \mathrm{Rm})dV \\
&\quad + c\int \left\{\nabla\left[(-R)||\mathrm{Rm}||^{p-3}\eta^{2p}\right] * \nabla\widehat{\boldsymbol{T}} * \mathrm{Rm}\right\} dV \\
&= c\int (-R)||\mathrm{Rm}||^{p-3}\eta^{2p}(\nabla\widehat{T} * \nabla \mathrm{Rm})dV \\
&\quad + c\int ||\mathrm{Rm}||^{p-3}\eta^{2p}(\nabla R * \nabla\widehat{\boldsymbol{T}} * \mathrm{Rm})dV \\
&\quad + c\int (-R)\eta^{2p}\left(\nabla||\mathrm{Rm}||^{p-3} * \nabla\widehat{\boldsymbol{T}} * \mathrm{Rm}\right) dV \\
&\quad + c\int (-R)||\mathrm{Rm}||^{p-3}\eta^{2p-1}\left(\nabla\eta * \nabla\widehat{\boldsymbol{T}} * \mathrm{Rm}\right) dV \\
&\leq c\int \left(||\mathrm{Rm}||^{p-2}\eta^{2p}||\nabla \mathrm{Rm}||\right. \\
&\quad \left.+||\mathrm{Rm}||^{p-1}\eta^{2p-1}||\nabla\eta||\right)\left(K^{1/2}||\nabla\boldsymbol{T}||\right) dV \\
&\leq c\int \left(||\nabla \mathrm{Rm}||||\mathrm{Rm}||^{\frac{p-3}{2}}\eta\right)\left(||\nabla\boldsymbol{T}||K^{1/2}||\mathrm{Rm}||^{\frac{p-1}{2}}\eta^{p}\right) dV \\
&\quad + \int \left(||\nabla\eta||||\mathrm{Rm}||^{\frac{p-1}{2}}\eta^{p-1}\right)\left(||\nabla\boldsymbol{T}||K^{1/2}||\mathrm{Rm}||^{\frac{p-1}{2}}\eta^{p}\right) dV \\
&\leq \epsilon c\int ||\nabla\boldsymbol{T}||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV + \frac{cK}{\epsilon}B_2 + \frac{cK}{\epsilon}A_4.
\end{aligned}
$$

According to (3.39) we get

$$
\begin{aligned}
& c\int ||\nabla\boldsymbol{T}||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV \\
&\leq c\int \left(-\blacksquare||\boldsymbol{T}||^2\right)||\mathrm{Rm}||^{p-1}\eta^{2p}dV + cKA_1 + cK^2A_2 \\
&\leq \frac{2}{50}B_1 + cKB_2 + cKA_4 + cK^2A_2 + cKA_1 \\
&\quad + \frac{d}{dt}\left(c\int R||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right) - c\int R\left(\partial_t||\mathrm{Rm}||^{p-1}\right)\eta^{2p}dV \\
&\leq \frac{2}{50}B_1 + cKB_2 + cKA_4 + cK^2A_2 + cKA_1 \\
&\quad + \frac{d}{dt}\left(\int cR||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right) + c\int (-R)||\mathrm{Rm}||^{p-3}\left(\partial_t||\mathrm{Rm}||^2\right)\eta^{2p}dV.
\end{aligned}
$$

Hence

$$
\begin{aligned}
& c\int (-R)||\mathrm{Rm}||^{p-3}\left(\partial_t||\mathrm{Rm}||^2\right)\eta^{2p}dV \\
&\leq \frac{2}{50}B_1 + cKB_2 + cKA_4 + \frac{cK}{\epsilon}B_2 + \frac{cK}{\epsilon}A_4 \\
&\quad + \epsilon\left[\frac{2}{50}B_1 + cKB_2 + cKA_4 + cK^2A_2 + cKA_1\right. \\
&\quad \left.+\frac{d}{dt}\left(\int cR||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right)\right]
\end{aligned}
$$

$$+\epsilon c\int(-R)||\mathrm{Rm}||^{p-3}\left(\partial_t||\mathrm{Rm}||^2\right)\eta^{2p}dV.$$

Choosing $\epsilon=\frac{1}{2}$ yields

$$\begin{aligned}&\frac{c}{2}\int(-R)||\mathrm{Rm}||^{p-3}\left(\partial_t||\mathrm{Rm}||^2\right)\eta^{2p}dV\\&\quad\leq\ \frac{3}{50}B_1+cKB_2+cKA_4+cK^2A_2+cKA_1+\frac{d}{dt}\left(\int cR||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right)\end{aligned}$$

and

$$\begin{aligned}&c\int||\nabla\boldsymbol{T}||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV\\&\quad\leq\ \frac{8}{50}B_1+cKB_2+cKA_4+cK^2A_2+cKA_1+\frac{d}{dt}\left(\int 2cR||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right).\end{aligned}$$

Thus

$$\begin{aligned}c\int(-R)||\mathrm{Rm}||^{p-3}\left(\partial_t||\mathrm{Rm}||^2\right)\eta^{2p}dV\ &\leq\ \frac{3}{50}B_1+cKB_2\\&+cKA_4+cK^2A_2+cKA_1+\frac{d}{dt}\left(\int cR||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right)\end{aligned}\tag{3.40}$$

and

$$\begin{aligned}c\int||\nabla\boldsymbol{T}||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV\ &\leq\ \frac{8}{50}B_1+cKB_2\\&+cKA_4+cK^2A_2+cKA_1+\frac{d}{dt}\left(\int cR||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right)\end{aligned}\tag{3.41}$$

and

$$\begin{aligned}c\int\left(-\blacksquare||\boldsymbol{T}||^2\right)||\mathrm{Rm}||^{p-1}\eta^{2p}dV\ &\leq\ \frac{5}{50}B_1+cKB_2\\&+cK^2A_2+cKA_1+\frac{d}{dt}\left(\int cR||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right).\end{aligned}\tag{3.42}$$

From (3.38) and (3.42) we arrive at

Lemma 3.4 *One has*

$$\begin{aligned}A_1'&\leq 2B_1+cKB_2+cKA_4+cK^2A_2+cKA_1\\&\quad+\frac{d}{dt}\left(\int cR||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right).\end{aligned}\tag{3.43}$$

We next estimate B_1 and B_2. Actually, we shall see that B_1 can be estimated in terms of B_2. Hence the key step is to estimate B_2. For B_1, using

$$\begin{aligned}||\nabla\mathrm{Ric}||^2&=-\frac{1}{2}\blacksquare||\mathrm{Ric}||^2+\mathrm{Ric}*\mathrm{Ric}*\mathrm{Rm}-\frac{1}{3}(\blacktriangle R)\boldsymbol{T}-\frac{2}{3}R||\mathrm{Ric}||^2\\&\quad+2\langle\langle\mathrm{Ric},\blacktriangle\widehat{\boldsymbol{T}}\rangle\rangle+\frac{1}{3}\langle\langle\mathrm{Ric},\nabla^2R\rangle\rangle+\mathrm{Ric}*\widehat{\boldsymbol{T}}*\mathrm{Rm}+\mathrm{Ric}*\nabla^2\widehat{\boldsymbol{T}}.\end{aligned}$$

we obtain

$$B_1\leq\frac{1}{2K}\int||\mathrm{Rm}||^{p-1}\eta^{2p}\left(\blacktriangle-\partial_t\right)||\mathrm{Ric}||^2dV+cKA_1$$

$$
\begin{aligned}
&+\frac{1}{3K}\int(-R)||\mathrm{Rm}||^{p-1}\eta^{2p}\Delta R\,dV+\frac{2}{K}\int\langle\langle\mathrm{Ric},\blacktriangle\widehat{\boldsymbol{T}}\rangle\rangle||\mathrm{Rm}||^{p-1}\eta^{2p}dV\\
&+\frac{1}{3K}\int\langle\langle\mathrm{Ric},\nabla^2R\rangle\rangle||\mathrm{Rm}||^{p-1}\eta^{2p}dV+\frac{1}{K}\int||\mathrm{Rm}||^{p-1}\left(\mathrm{Ric}*\nabla^2\widehat{\boldsymbol{T}}\right)\eta^{2p}dV.
\end{aligned}
\tag{3.44}
$$

From the estimates $\nabla||\mathrm{Ric}||^2 \lesssim ||\mathrm{Ric}||||\nabla\mathrm{Ric}||$, $\nabla||\mathrm{Rm}||^{p-1} \lesssim ||\mathrm{Rm}||^{p-2}||\nabla\mathrm{Rm}||$, and $\partial_t||\mathrm{Rm}||^{p-1} = \frac{p-1}{2}||\mathrm{Rm}||^{p-3}\partial_t||\mathrm{Rm}||^2$, we have

$$
\begin{aligned}
&\int||\mathrm{Rm}||^{p-1}\eta^{2p}\left(\blacktriangle-\partial_t\right)||\mathrm{Ric}||^2dV\\
&=\int\nabla||\mathrm{Ric}||^2*\nabla\left(||\mathrm{Rm}||^{p-1}\eta^{2p}\right)dV-\int||\mathrm{Rm}||^{p-1}\eta^{2p}\left(\partial_t||\mathrm{Ric}||^2\right)dV\\
&=\int\left(\nabla||\mathrm{Ric}||^2*\nabla||\mathrm{Rm}||^{p-1}\right)\eta^{2p}dV+\int\left(\nabla||\mathrm{Ric}||^2*\nabla\eta\right)||\mathrm{Rm}||^{p-1}\eta^{2p-1}dV\\
&\quad-\frac{d}{dt}\left[\int||\mathrm{Rm}||^{p-1}\eta^{2p}||\mathrm{Ric}||^2dV\right]+\int\left(\partial_t||\mathrm{Rm}||^{p-1}\right)\eta^{2p}||\mathrm{Ric}||^2dV\\
&\quad+\int||\mathrm{Rm}||^{p-1}\eta^{2p}||\mathrm{Ric}||^2(\partial_t dV)\\
&\le cK\int||\nabla\mathrm{Ric}||||\nabla\mathrm{Rm}||||\mathrm{Rm}||^{p-2}\eta^{2p}dV+cK\int||\nabla\mathrm{Ric}||||\nabla\eta||||\mathrm{Rm}||^{p-1}\eta^{2p-1}dV\\
&\quad+c\int||\mathrm{Rm}||^{p-3}\left(\partial_t||\mathrm{Rm}||^2\right)\eta^{2p}||\mathrm{Ric}||^2dV+cK^2A_1\\
&\quad-\frac{d}{dt}\left[\int||\mathrm{Rm}||^{p-1}||\mathrm{Ric}||^2\eta^{2p}dV\right]\\
&\le cK\left(\frac{1}{50c}B_1+cKB_2\right)+cK\left(\frac{1}{50c}B_1+cKA_4\right)+cK^2A_1\\
&\quad+c\int||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}\left(\partial_t||\mathrm{Rm}||^2\right)dV-\frac{d}{dt}\left[\int||\mathrm{Rm}||^{p-1}||\mathrm{Ric}||^2\eta^{2p}dV\right]\\
&\le\frac{2}{50}KB_1+cK^2B_2+cK^2A_4+cK^2A_1\\
&\quad+c\int||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}\left(\partial_t||\mathrm{Rm}||^2\right)dV-\frac{d}{dt}\left[\int||\mathrm{Rm}||^{p-1}||\mathrm{Ric}||^2\eta^{2p}dV\right].
\end{aligned}
$$

Thus

$$
\begin{aligned}
\int||\mathrm{Rm}||^{p-1}\eta^{2p}\blacksquare||\mathrm{Ric}||^2dV\ \le\ &\frac{2}{50}KB_1+cK^2B_2+cK^2A_4+cK^2A_1\\
&+c\int||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}\left(\partial_t||\mathrm{Rm}||^2\right)dV\\
&-\frac{d}{dt}\left[\int||\mathrm{Rm}||^{p-1}||\mathrm{Ric}||^2\eta^{2p}dV\right].
\end{aligned}
\tag{3.45}
$$

Consider the term

$$
\begin{aligned}
c\int||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}\left(\partial_t||\mathrm{Rm}||^2\right)dV\ =\ &c\int||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}\\
\Big[\nabla^2\mathrm{Ric}*\mathrm{Rm}+\mathrm{Ric}*\mathrm{Rm}*\mathrm{Rm}+\mathrm{Rm}*\mathrm{Rm}*\widehat{\boldsymbol{T}}+\mathrm{Ric}*\nabla^2||\boldsymbol{T}||^2+\mathrm{Rm}*\nabla^2\widehat{\boldsymbol{T}}&
\end{aligned}
$$

$$+\frac{4}{3}||\boldsymbol{T}||^2||\mathrm{Rm}||^2\Big]dV \;\le\; c\int ||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}\Big[\nabla^2\mathrm{Ric}*\mathrm{Rm}-\nabla^2 R*\mathrm{Ric}$$
$$+\nabla^2\widehat{\boldsymbol{T}}*\mathrm{Rm}\Big]dV+cK^2A_2.$$

The three terms in the bracket can be estimated as follows. Firstly

$$\begin{aligned}
&c\int ||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}\left(\nabla^2\mathrm{Ric}*\mathrm{Rm}\right)dV\\
&=\; c\int ||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}\left(\nabla\mathrm{Ric}*\nabla\mathrm{Rm}\right)dV\\
&\quad+c\int\left\{\nabla\left[||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}\right]*\nabla\mathrm{Ric}*\mathrm{Rm}\right\}dV\\
&=\; c\int ||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}\left(\nabla\mathrm{Ric}*\nabla\mathrm{Rm}\right)dV\\
&\quad+c\int ||\mathrm{Rm}||^{p-3}\eta^{2p}\left(\nabla||\mathrm{Ric}||^2*\nabla\mathrm{Ric}*\mathrm{Rm}\right)dV\\
&\quad+c\int ||\mathrm{Ric}||^2\eta^{2p}\left(\nabla||\mathrm{Rm}||^{p-3}*\nabla\mathrm{Ric}*\mathrm{Rm}\right)dV\\
&\quad+c\int ||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p-1}\left(\nabla\eta*\nabla\mathrm{Ric}*\mathrm{Rm}\right)dV\\
&\le\; cK\int ||\mathrm{Rm}||^{p-2}\eta^{2p}||\nabla\mathrm{Ric}||||\nabla\mathrm{Rm}||dV+cK\int ||\mathrm{Rm}||^{p-1}\eta^{2p-1}||\nabla\mathrm{Ric}||||\nabla\eta||dV\\
&\le\; cK\left(\epsilon B_1+\frac{K}{\epsilon}B_2\right)+cK\left(\epsilon B_1+\frac{K}{\epsilon}A_4\right)\;\le\;\frac{1}{50}KB_1+cK^2B_2+cK^2A_4.
\end{aligned}$$

The same estimate holds for

$$c\int ||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}\left(-\nabla^2R*\mathrm{Ric}\right)dV.$$

Finally,

$$\begin{aligned}
&c\int ||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}\left(\nabla^2\widehat{\boldsymbol{T}}*\mathrm{Rm}\right)dV \;=\; c\int ||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}\\
&\left(\nabla\widehat{\boldsymbol{T}}*\nabla\mathrm{Rm}\right)dV+c\int\left\{\nabla\left(||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}\right)*\nabla\widehat{\boldsymbol{T}}*\mathrm{Rm}\right\}dV\\
&\le\; c\int ||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}\left(K^{1/2}||\nabla\boldsymbol{T}||||\nabla\mathrm{Rm}||\right)dV\\
&\quad+c\int\left(\nabla||\mathrm{Ric}||^2\right)||\mathrm{Rm}||^{p-3}\eta^{2p}||\nabla\widehat{\boldsymbol{T}}||||\mathrm{Rm}||dV\\
&\quad+c\int ||\mathrm{Rm}||^2\left(\nabla||\mathrm{Rm}||^{p-3}\right)\eta^{2p}||\nabla\widehat{\boldsymbol{T}}||||\mathrm{Rm}||dV\\
&\quad+c\int ||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p-1}||\nabla\eta||||\nabla\widehat{\boldsymbol{T}}||||\mathrm{Rm}||dV\\
&\le\; cK\int ||\mathrm{Rm}||^{p-2}\eta^{2p}\left(K^{1/2}||\nabla\boldsymbol{T}||||\nabla\mathrm{Rm}||\right)dV\\
&\quad+cK\int ||\mathrm{Rm}||^{p-1}\eta^{2p-1}\left(K^{1/2}||\nabla\eta||||\nabla\boldsymbol{T}||\right)dV
\end{aligned}$$

$$
\begin{aligned}
&\le K\left[cKB_2+\frac{cK}{\epsilon}A_4+\epsilon c\int||\nabla \boldsymbol{T}||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right]\\
&\le \frac{8}{50}KB_1+cK^2B_2+cK^2A_4+cK^3A_2+cK^2A_1+\frac{d}{dt}\left[cK\int R||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right]
\end{aligned}
$$

Therefore

$$
\begin{aligned}
c\int ||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}\left(\partial_t||\mathrm{Rm}||^2\right)dV \;&\le\; \frac{10}{50}KB_1+cK^2B_2+cK^2A_4+cK^3A_2\\
+cK^2A_1+cK\frac{d}{dt}\left[\int R||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right] \qquad (3.46)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{2K}\int ||\mathrm{Rm}||^{p-1}\eta^{2p}(\blacktriangle-\partial_t)||\mathrm{Ric}||^2dV \;&\le\; \frac{6}{50}B_1+cKB_2+cKA_4+cK^2A_2+cKA_1\\
&-\frac{1}{K}\frac{d}{dt}\left[\int ||\mathrm{Rm}||^{p-1}||\mathrm{Ric}||^2\eta^{2p}dV\right]+c\frac{d}{dt}\left[\int R||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right]\\
&\le \frac{6}{50}B_1+cKB_2+cKA_4+cK^2A_2+cKA_1\\
&-\frac{d}{dt}\left[\frac{1}{K}\int ||\mathrm{Rm}||^{p-1}||\mathrm{Ric}||^2\eta^{2p}dV+c\int(-R)||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right]. \qquad (3.47)
\end{aligned}
$$

In the following, we estimate the left four terms in (3.44). We start from terms involving the scalar curvature.

$$
\begin{aligned}
\frac{1}{3K}\int(-R)||\mathrm{Rm}||^{p-1}\eta^{2p}\Delta R\,dV \;&=\; -\frac{1}{3K}\int\nabla R\cdot\nabla\left[(-R)||\mathrm{Rm}||^{p-1}\eta^{2p}\right]dV\\
&= -\frac{1}{3K}\int\nabla R\cdot\Big[-\nabla R||\mathrm{Rm}||^{p-1}\eta^{2p}+(-R)\nabla||\mathrm{Rm}||^{p-1}\eta^{2p}\\
&\quad+2p(-R)||\mathrm{Rm}||^{p-1}\eta^{2p-1}\nabla\eta\Big]dV \;\le\; \frac{1}{3K}\int||\nabla R||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV\\
&\quad+\frac{c}{K}\int(-R)||\mathrm{Rm}||^{p-2}||\nabla R||||\nabla\mathrm{Rm}||\eta^{2p}dV\\
&\quad+\frac{c}{K}\int(-R)||\mathrm{Rm}||^{p-1}\eta^{2p-1}||\nabla R||||\nabla\eta||dV\\
&\le \frac{1}{3K}\int||\nabla R||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV\\
&\quad+\frac{1}{3K}\int||\nabla R||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV+cKB_2\\
&\quad+\frac{1}{3K}\int||\nabla R||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV+cKA_4\\
&\le \frac{1}{K}\int||\nabla R||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV+cKB_2+cKA_4. \qquad (3.48)
\end{aligned}
$$

The another term involving the scalar curvature can be estimated by

$$
\begin{aligned}
\frac{1}{3K}\int\langle\langle\mathrm{Ric},\nabla^2R\rangle\rangle||\mathrm{Rm}||^{p-1}\eta^{2p}dV \;&=\; -\frac{1}{3K}\int\nabla^jR\nabla^i\left[R_{ij}||\mathrm{Rm}||^{p-1}\eta^{2p}\right]dV\\
&= -\frac{1}{3K}\int\nabla^jR\Big[\frac{1}{2}\nabla_jR||\mathrm{Rm}||^{p-1}\eta^{2p}+R_{ij}\nabla^i||\mathrm{Rm}||^{p-1}\eta^{2p}
\end{aligned}
$$

$$
\begin{aligned}
&+ R_{ij}||\mathrm{Rm}||^{p-1}2p\eta^{2p-1}\nabla^i\eta\Big]dV \ \le\ -\frac{1}{6K}\int ||\nabla R||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV\\
&+\frac{c}{K}\int ||\mathrm{Ric}||||\nabla R||||\mathrm{Rm}||^{p-2}||\nabla \mathrm{Rm}||\eta^{2p}dV\\
&+\frac{c}{K}\int ||\nabla R||||\mathrm{Ric}||||\mathrm{Rm}||^{p-1}\eta^{2p-1}||\nabla\eta||dV\\
&\le\ -\frac{1}{6K}\int ||\nabla R||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV+\frac{1}{18K}\int ||\nabla R||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV+cKB_2\\
&+\frac{1}{18K}\int ||\nabla R||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV+cKA_4\ \le\ cKB_2+cKA_4. \qquad (3.49)
\end{aligned}
$$

Using (3.41) we obtain

$$
\begin{aligned}
\frac{2}{K}\int \langle\langle \mathrm{Ric}, \blacktriangle\widehat{\boldsymbol{T}}\rangle\rangle||\mathrm{Rm}||^{p-1}\eta^{2p}dV\ &=\ \frac{1}{K}\int\left(\mathrm{Ric}*\blacktriangle\widehat{\boldsymbol{T}}\right)||\mathrm{Rm}||^{p-1}\eta^{2p}dV\\
=\ &\frac{1}{K}\int\left(\nabla\mathrm{Ric}*\nabla\widehat{T}\right)||\mathrm{Rm}||^{p-1}\eta^{2p}dV+\frac{1}{K}\int \mathrm{Ric}*\nabla\widehat{\boldsymbol{T}}*\nabla\left(||\mathrm{Rm}||^{p-1}\eta^{2p}\right)dV\\
\le\ &\frac{c}{K}\int ||\nabla\mathrm{Ric}||||\nabla\widehat{\boldsymbol{T}}||||\mathrm{Rm}||^{p-1}\eta^{2p}dV+\frac{c}{K}\int ||\mathrm{Ric}||||\nabla\widehat{\boldsymbol{T}}||||\mathrm{Rm}||^{p-2}||\nabla\mathrm{Rm}||\eta^{2p}dV\\
&+\frac{c}{K}\int ||\mathrm{Ric}||||\nabla\widehat{\boldsymbol{T}}||||\mathrm{Rm}||^{p-1}\eta^{2p-1}||\nabla\eta||dV\\
\le\ &\frac{1}{50}B_1+c\int ||\nabla\boldsymbol{T}||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV+cKB_2\\
&+c\int ||\nabla\boldsymbol{T}||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV+cKA_4+c\int ||\nabla\boldsymbol{T}||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV\\
\le\ &\frac{1}{50}B_1+cKB_2+cKA_4+c\int ||\nabla\boldsymbol{T}||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV\\
\le\ &\frac{9}{50}B_1+cKB_2+cKA_4+cK^2A_2+cKA_1+\frac{d}{dt}\left[\int cR||\mathrm{Rm}||^{p-1}\eta^{2p}d\right]. \qquad (3.50)
\end{aligned}
$$

Similarly, we can prove

$$
\begin{aligned}
\frac{1}{K}\int\left(\mathrm{Ric}*\nabla^2\widehat{\boldsymbol{T}}\right)||\mathrm{Rm}||^{p-1}\eta^{2p}dV\ &=\ \frac{1}{K}\int\left(\nabla\mathrm{Ric}*\nabla\widehat{\boldsymbol{T}}\right)||\mathrm{Rm}||^{p-1}\eta^{2p}dV\\
&+\frac{1}{K}\int \mathrm{Ric}*\nabla\widehat{\boldsymbol{T}}*\nabla\left(||\mathrm{Rm}||^{p-1}\eta^{2p}\right)dV\ \le\ \frac{1}{K}\int\left(\nabla\mathrm{Ric}*\nabla\widehat{\boldsymbol{T}}\right)||\mathrm{Rm}||^{p-1}\eta^{2p}dV\\
&+\frac{c}{K}\int ||\mathrm{Ric}||||\nabla\widehat{\boldsymbol{T}}||||\mathrm{Rm}||^{p-2}||\nabla\mathrm{Rm}||\eta^{2p}dV\\
&+\frac{c}{K}\int ||\mathrm{Ric}||||\nabla\widehat{\boldsymbol{T}}||||\mathrm{Rm}||^{p-1}\eta^{2p-1}||\nabla\eta||dV\\
\le\ &\frac{c}{K}\int ||\nabla\mathrm{Ric}||||\nabla\widehat{\boldsymbol{T}}||||\mathrm{Rm}||^{p-1}\eta^{2p}dV+\frac{c}{K}\int ||\mathrm{Ric}||||\nabla\widehat{\boldsymbol{T}}||||\mathrm{Rm}||^{p-2}||\nabla\mathrm{Rm}||\eta^{2p}dV\\
&+\frac{c}{K}\int ||\mathrm{Ric}||||\nabla\widehat{\boldsymbol{T}}||||\mathrm{Rm}||^{p-1}\eta^{2p-1}||\nabla\eta||dV\\
\le\ &\frac{9}{50}B_1+cKB_2+cKA_4+cK^2A_2+cKA_1+\frac{d}{dt}\left[\int cR||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right]. \qquad (3.51)
\end{aligned}
$$

Plugging (3.45) and (3.48)–(3.51) into (3.44), and using (3.41) and $||\nabla R||^2 \leq cK||\nabla \boldsymbol{T}||^2$, we obtain

$$
\begin{aligned}
B_1 \leq\ & \frac{6}{50}B_1 + cKB_2 + cKA_4 + cK^2A_2 + cKA_1 \\
& - \frac{d}{dt}\left[\frac{1}{K}\int ||\mathrm{Rm}||^{p-1}||\mathrm{Ric}||^2\eta^{2p}dV + c\int(-R)||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right] \\
& + \frac{1}{K}\int ||\nabla R||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV + \frac{18}{50}B_1 - \frac{d}{dt}\left[c\int(-R)||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right] \\
\leq\ & \frac{32}{50}B_1 + cKB_2 + cKA_4 + cK^2A_2 + cKA_1 \\
& - \frac{d}{dt}\left[\frac{1}{K}\int ||\mathrm{Rm}||^{p-1}||\mathrm{Ric}||^2\eta^{2p}dV + c\int(-R)||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right].
\end{aligned}
$$

Thus

$$
\begin{aligned}
B_1 \leq\ & cKB_2 + cKA_4 + cK^2A_2 + cKA_1 \\
& - \frac{d}{dt}\left[\frac{1}{K}\int ||\mathrm{Rm}||^{p-1}||\mathrm{Ric}||^2\eta^{2p}dV + c\int(-R)||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right]
\end{aligned}
\tag{3.52}
$$

From (3.43) and (3.52), we can conclude that

Lemma 3.5 *One has*

$$
\begin{aligned}
A_1' \leq\ & cKB_2 + cKA_4 + cK^2A_2 + cKA_1 \\
& - \frac{d}{dt}\left[\frac{c}{K}\int ||\mathrm{Rm}||^{p-1}||\mathrm{Ric}||^2\eta^{2p}dV + c\int(-R)||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right].
\end{aligned}
\tag{3.53}
$$

Observe that two terms in the bracket are both nonnegative, since $R = -||\boldsymbol{T}||^2 \leq 0$. Finally, we estimate the term B_2. Using the evolution inequality

$$
||\nabla \mathrm{Rm}||^2 \leq -\frac{1}{2}\blacksquare||\mathrm{Rm}||^2 + c||\mathrm{Rm}||^3 + c||\nabla^2\boldsymbol{T}||||\mathrm{Rm}||^{3/2} + c||\mathrm{Rm}||||\nabla\boldsymbol{T}||^2
$$

we obtain

$$
\begin{aligned}
B_2 =\ & \int ||\nabla\mathrm{Rm}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}dV \leq \int\Big[-\frac{1}{2}\blacksquare||\mathrm{Rm}||^2 + c||\mathrm{Rm}||^3 \\
& + c||\nabla^2\boldsymbol{T}||||\mathrm{Rm}||^{3/2} + c||\mathrm{Rm}||||\nabla\boldsymbol{T}||^2\Big]||\mathrm{Rm}||^{p-3}\eta^{2p}dV \\
\leq\ & -\frac{1}{2}\int\left(\blacksquare||\mathrm{Rm}||^2\right)||\mathrm{Rm}||^{p-3}\eta^{2p}dV + cA_1 \\
& + c\int||\nabla^2\boldsymbol{T}||||\mathrm{Rm}||^{p-3/2}\eta^{2p}dV + c\int||\nabla^2\boldsymbol{T}||^2||\mathrm{Rm}||^{p-2}\eta^{2p}dV.
\end{aligned}
\tag{3.54}
$$

For the first integral one has

$$
\begin{aligned}
-\frac{1}{2}\int\left(\blacksquare||\mathrm{Rm}||^2\right)||\mathrm{Rm}||^{p-3}\eta^{2p}dV =\ & \frac{1}{2}\int\left(\blacktriangle||\mathrm{Rm}||^2\right)||\mathrm{Rm}||^{p-3}\eta^{2p}dV \\
-\frac{1}{2}\int\left(\partial_t||\mathrm{Rm}||^2\right)||\mathrm{Rm}||^{p-3}\eta^{2p}dV =\ & -\frac{1}{2}\int\left(\partial_t||\mathrm{Rm}||^2\right)||\mathrm{Rm}||^{p-3}\eta^{2p}dV \\
-\frac{1}{2}\int\nabla||\mathrm{Rm}||^2\left[\left(\nabla||\mathrm{Rm}||^{p-3}\right)\eta^{2p} + ||\mathrm{Rm}||^{p-3}\left(\nabla\eta^{2p}\right)\right]dV &
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{p-3}{4}\int \left(\nabla||\mathrm{Rm}||^2\right)^2 ||\mathrm{Rm}||^{p-5}\eta^{2p}dV \\
&\quad + c\int ||\mathrm{Rm}||^{p-2}||\nabla\mathrm{Rm}||||\nabla\eta||\eta^{2p-1}dV - \frac{1}{2}\int \left(\partial_t||\mathrm{Rm}||^2\right)||\mathrm{Rm}||^{p-3}\eta^{2p}dV \\
&\le \frac{1}{50}B_2 + cA_4 - \frac{1}{2}\int \left(\partial_t||\mathrm{Rm}||^2\right)||\mathrm{Rm}||^{p-3}\eta^{2p}dV.
\end{aligned}
$$

Here we used the assumption that $p \ge 5$. On the other hand,

$$
\begin{aligned}
-\frac{1}{2}\int \left(\partial_t||\mathrm{Rm}||^2\right)||\mathrm{Rm}||^{p-3}\eta^{2p}dV &= -\frac{1}{2}\frac{d}{dt}\left[\int ||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right] \\
&\quad + \frac{1}{2}\int ||\mathrm{Rm}||^2\left(\partial_t||\mathrm{Rm}||^{p-3}\right)\eta^{2p}dV + \frac{1}{2}\int ||\mathrm{Rm}||^{p-1}\eta^{2p}\left(\partial_t dV\right) \\
&\le \frac{p-3}{4}\int ||\mathrm{Rm}||^{p-3}\left(\partial_t||\mathrm{Rm}||^2\right)\eta^{2p}dV + cA_1 - \frac{1}{2}\frac{d}{dt}\left[\int ||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right]
\end{aligned}
$$

so that

$$
-\frac{1}{2}\int \left(\partial_t||\mathrm{Rm}||^2\right)||\mathrm{Rm}||^{p-3}\eta^{2p}dV \le cA_1 - \frac{1}{p-1}\frac{d}{dt}\left[\int ||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right].
$$

Therefore

$$
\begin{aligned}
-\frac{1}{2}\int \left(\blacksquare||\mathrm{Rm}||^2\right)||\mathrm{Rm}||^{p-3}\eta^{2p}dV &\le \frac{1}{50}B_2 + cA_4 + cA_1 \\
&\quad - \frac{1}{p-1}\frac{d}{dt}\left[\int ||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right].
\end{aligned} \tag{3.55}
$$

To estimate the remainder two integrals, we recall from (2.35) that

$$
\nabla \boldsymbol{T} = \mathrm{Rm} * \varphi + \boldsymbol{T} * \boldsymbol{T} * \varphi \tag{3.56}
$$

and from (2.14) that

$$
\nabla\varphi = \boldsymbol{T} * \psi. \tag{3.57}
$$

From (3.56) we get

$$
||\nabla\boldsymbol{T}|| \le c||\mathrm{Rm}|| + c||\boldsymbol{T}||^2 \le c||\mathrm{Rm}||. \tag{3.58}
$$

In particular, the inequality (3.58) yields

$$
\int ||\nabla\boldsymbol{T}||^2||\mathrm{Rm}||^{p-2}\eta^{2p}dV \le c\int ||\mathrm{Rm}||^p\eta^{2p}dV \le cA_1. \tag{3.59}
$$

Taking the derivative of (3.56) and using (3.57) we obtain

$$
\nabla^2\boldsymbol{T} = \nabla\mathrm{Rm} * \varphi + \mathrm{Rm} * \boldsymbol{T} * \psi + \nabla\boldsymbol{T} * \boldsymbol{T} * \varphi + \boldsymbol{T} * \boldsymbol{T} * \boldsymbol{T} * \psi. \tag{3.60}
$$

The particular case $||\nabla^2\boldsymbol{T}|| \le c||\nabla\mathrm{Rm}|| + c||\mathrm{Rm}||||\boldsymbol{T}|| + c||\nabla\boldsymbol{T}||||\boldsymbol{T}|| + c||\boldsymbol{T}||^3$ leads to

$$
\begin{aligned}
c\int ||\nabla^2\boldsymbol{T}||||\mathrm{Rm}||^{p-3/2}\eta^{2p}dV &\le c\int \Big[||\nabla\mathrm{Rm}|| + ||\mathrm{Rm}||||\boldsymbol{T}|| + ||\nabla\boldsymbol{T}||||\boldsymbol{T}|| \\
&\quad + ||\boldsymbol{T}||^3\Big]||\mathrm{Rm}||^{p-3/2}\eta^{2p}dV \le c\int \left(||\nabla\mathrm{Rm}||||\mathrm{Rm}||^{p-3/2}\eta^p\right)\left(||\mathrm{Rm}||^{p/2}\eta^p\right)dV \\
&\quad + c\int ||\mathrm{Rm}||^p\eta^{2p}dV \le \frac{1}{50}B_2 + cA_1.
\end{aligned} \tag{3.61}
$$

Plugging (3.55), (3.59), and (3.61) into (3.54) we arrive at

$$B_2 \leq cA_4 + cA_1 - \frac{d}{dt}\left[\frac{1}{p-1}\int ||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right]. \tag{3.62}$$

Together with (3.53) and (3.62) we finally obtain

$$\begin{aligned}(A_1 + cKA_2)' \ &\leq \ cK(A_1 + cKA_2) + cKA_4 \\ &- \frac{d}{dt}\left[\frac{c}{K}\int ||\mathrm{Rm}||^{p-1}||\mathrm{Ric}||^2\eta^{2p}dV + c\int(-R)||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right].\end{aligned} \tag{3.63}$$

Equivalently,

Lemma 3.6 *If* $||\mathrm{Ric}|| \leq K$ *and* $p \geq 5$, *one has*

$$\begin{aligned}&\frac{d}{dt}\left[A_1 + cKA_2 + \frac{c}{K}\int ||\mathrm{Rm}||^{p-1}||\mathrm{Ric}||^2\eta^{2p}dV + c\int(-R)||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right] \\ &\leq \ cK(A_1 + cKA_2) + cKA_4.\end{aligned} \tag{3.64}$$

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