



Method with horizontal fuzzy numbers for solving real fuzzy linear systems

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Abstract

The paper presents a method for solving real fuzzy systems of linear equations with the usage of horizontal fuzzy numbers (HFNs). Based on the multidimensional RDM interval arithmetic and a fuzzy number in a parametric form, the definition of a horizontal fuzzy number with nonlinear left and right borders was given. Additionally, the paper presents the properties of basic algebraic operations on these numbers. The usage of horizontal fuzzy numbers with linear and nonlinear borders was illustrated in the examples with $n \times n$ fuzzy linear systems. The obtained results are multidimensional and satisfy the fuzzy linear systems. Calculated solutions of fuzzy linear systems were compared with the results of other methods. The solution obtained with horizontal fuzzy numbers satisfies any equivalent form of fuzzy linear system, whereas the results of existing methods do not satisfy the equivalent forms of the system. The presented examples also show that the method with HFN delivers a full multidimensional solution (direct solution). The analyzed results of standard methods, which are only a part of span (indicator, indirect solution) of a full solution and/or include values that do not satisfy the fuzzy linear systems, are underestimated or overestimated. The proposed method gives a possibility to obtain a crisp solution together with a crisp system of equations; other methods do not possess these properties.

Keywords Fuzzy number · Horizontal fuzzy number · Fuzzy linear system · Fuzzy arithmetic · RDM arithmetic · Uncertainty theory · Artificial Intelligence · Granular computing

1 Introduction

Fuzzy sets and fuzzy arithmetic play an important role in the problems where variables are uncertain. It helps to describe real problems in the form of mathematical models and finds a solution of tasks with uncertain variables. Fuzzy arithmetic and fuzzy sets are used in the uncertainty theory (Bhiwani and Patre 2009), computing with words (Zadeh 2002), grey systems (Liu and Forest 2010), granular computing (Pedrycz et al. 2008).

The idea of fuzzy numbers and arithmetic on fuzzy numbers was first presented in (Chang and Zadeh 1972; Mizumoto and Tanaka 1976; Dubois and Prade 1978). Currently, the most popular and often used fuzzy numbers have vertical form and are used to solve, e.g., fuzzy equations (Bhi-

wani and Patre 2009; Buckley and Qu 1990; Buckley et al. 2002), fuzzy nonlinear equations with an uncertain variable (Abbasbandy and Asady 2004; Dymova 2010; Jafarian and Nia 2013). However, fuzzy systems of linear equations describe systems from the real world with parameters that are rather fuzzy than crisp numbers. These systems appear in different areas, for example in mathematics, engineering, physics and economics. The usage of fuzzy numbers for solving fuzzy system of linear equations is presented in, e.g., (Abbasbandy and Alavi 2005; Abbasbandy and Jafarian 2006; Behera and Chakraverty 2012; Dehghan et al. 2006; Friedman et al. 1998; Jafari et al. 2009; Yeh 2007; Lodwick and Dubois 2015); in these papers the triangular or trapezoidal fuzzy numbers are considered. In Boroujeni et al. (2015), the method for solving the fuzzy systems of dual form with triangular fuzzy numbers is presented.

The standard fuzzy arithmetic is based on vertical fuzzy sets (Aliiev et al. 2012; Dubois and Prade 1978; Hanss 2005; Klir 1997; Nguyen 2006; Rajati and Mendel 2012; Rajati et al. 2011; Zadeh 1996, 2002, 2007). Unfortunately, there were gaps because more complicated operations on the verti-

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cal representation of fuzzy numbers, e.g., a solution of fuzzy system of linear equations, provide the results that are not a full solution or are just a span of the solution (indirect result). Standard methods give the results which preclude obtaining the crisp solutions that satisfy the system of equations. Existing versions of fuzzy arithmetic assume that only vertical fuzzy number is the result of operations on fuzzy numbers. This assumption leads to many computational paradoxes observed in fuzzy arithmetic.

Why is the differentiation of direct and indirect solution important? Because the use of indirect results (e.g., span $[\underline{x}, \bar{x}]$), leads to imprecise, false or partly false solution. In Example 1, the differences between the results obtained by standard interval arithmetic (S-IA) and multidimensional RDM interval arithmetic were shown. The concept of the multidimensional Relative Distance Measure interval arithmetic (RDM-IA) was developed by A. Piegat (Piegat and Landowski 2013, 2017b; Landowski 2015, 2017).

Example 1 Solve interval equation $AX + B = AB$, where $A = [1, 2]$ and $B = [4, 6]$.

Standard interval arithmetic (S-IA)

$$\begin{aligned} X &= (AB - B)/A = ([1, 2][4, 6] - [4, 6])/[1, 2] \\ &= ([4, 12] - [4, 6])/[1, 2] = [-2, 8]/[1, 2] = [-2, 8]. \end{aligned}$$

Solution X does not satisfy the equation.

RDM interval arithmetic—multidimensional solution

Intervals A and B in RDM notation have a form: $A = 1 + \alpha_A$, $B = 4 + 2\alpha_B$, where $\alpha_A, \alpha_B \in [0, 1]$.

$$\begin{aligned} X &= (AB - B)/A = [(1 + \alpha_A)(4 + 2\alpha_B) \\ &\quad - (4 + 2\alpha_B)]/(1 + \alpha_A) = (4\alpha_A + 2\alpha_A\alpha_B)/(1 + \alpha_A), \\ &\quad \alpha_A, \alpha_B \in [0, 1]. \end{aligned}$$

RDM solution $X(\alpha_A, \alpha_B)$ satisfies the equation. It is true solution of the equation.

Deriving span from RDM solution

$$s(X) = [\min_{\alpha_A, \alpha_B} X(\alpha_A, \alpha_B), \max_{\alpha_A, \alpha_B} X(\alpha_A, \alpha_B)] = [0, 3]$$

Spans comparison

The S-IA gives span $s(X) = [-2, 8]$, span with the RDM-IA equals $s(X) = [0, 3]$.

Conclusions of Example 1:

1. The solution's span calculated according to S-IA is usually considerably greater than the span calculated with RDM-IA.
2. Condition of the correct span determining is possessed by the multidimensional RDM arithmetic.

3. The solution span does not have to be calculated stepwise, sequentially as in the case of S-IA. It should be calculated only on the basis of the full multidimensional RDM solution.

To fill the gaps, the article presents the method with the usage of horizontal fuzzy number (HFN) to find the solution of fuzzy system of linear equations.

The main contributions of the paper are as follows:

1. In the paper, the definition and the examples of the horizontal fuzzy number where the left and right borders can be nonlinear were presented. The previous papers (Piegat and Landowski 2015, 2017a, c) considered only triangular and trapezoidal horizontal fuzzy numbers.
2. The proofs of the properties of basic algebraic operations on horizontal fuzzy numbers are given.
3. The paper presents the multidimensional approach to the solution of the fuzzy linear system. The proposed method gives a possibility to solve the fuzzy linear systems with various types of fuzzy numbers, not only with triangular or trapezoidal fuzzy numbers.
4. The presented method gives the multidimensional solution which has a possibility of generation of every crisp solution of the fuzzy systems of linear equations and can also generate crisp system of linear equations from fuzzy linear system for the considered crisp solution.
5. In the article, it is shown that in multidimensional approach the solution of the fuzzy system of linear equations in a vertical fuzzy number form is only the secondary information (indicator); the direct solution is a multidimensional set.
6. The examples show that the methods in the cited papers do not give full solutions and/or generate results that do not satisfy the equivalent forms of the fuzzy linear systems. The proposed method gives full solutions which satisfy the fuzzy systems of linear equations and their equivalent forms.

The correctness of the results obtained in the examples is easy to check by substituting calculated values to a given problem. Many researchers do not check the correctness of the results in presented examples. This way of comparing of the obtained solutions by method with horizontal fuzzy numbers with results by other methods is called the testing point method (TP method). By using the TP method, the points in incorrect solution that does not satisfy the fuzzy linear system or points that are solutions but are not included in the obtained results can be found.

2 Horizontal fuzzy number

In the following section, the theoretical foundations of the horizontal fuzzy number and its arithmetic are presented.

In RDM interval arithmetic, a given value x from interval $X = [\underline{x}, \bar{x}]$ is described using RDM variable α_x , where $\alpha_x \in [0, 1]$, as shown in (1).

$$x = \underline{x} + \alpha_x(\bar{x} - \underline{x}) \tag{1}$$

Definition 1 (Landowski 2015) In notation RDM, the interval $X = [\underline{x}, \bar{x}]$ is described as a set of numbers given by (2).

$$X = \{x : x = \underline{x} + \alpha_x(\bar{x} - \underline{x}), \alpha_x \in [0, 1]\} \tag{2}$$

A fuzzy number in a parametric form is defined as intervals for every r -cut.

Definition 2 (Friedman et al. 1998) An arbitrary fuzzy number is an ordered pair of functions $(\underline{u}(r), \bar{u}(r))$, $0 \leq r \leq 1$, which satisfy the following requirements:

1. $\underline{u}(r)$ is a bounded left continuous nondecreasing function over $[0, 1]$.
2. $\bar{u}(r)$ is a bounded left continuous nonincreasing function over $[0, 1]$.
3. $\underline{u}(r) \leq \bar{u}(r)$, $0 \leq r \leq 1$.

Fuzzy number is a model of uncertain value of variable.

Based on RDM interval notation and parametric definition of fuzzy number, a horizontal fuzzy number with linear or/and nonlinear left and right borders can be defined as:

Definition 3 A horizontal fuzzy number U^r of an arbitrary fuzzy number $(\underline{u}(r), \bar{u}(r))$ is a set of numbers u^r for every $r, \alpha_u \in [0, 1]$ given by (3).

$$U^r = \{u^r \in \mathbb{R} : u^r = \underline{u}(r) + \alpha_u(\bar{u}(r) - \underline{u}(r)), r, \alpha_u \in [0, 1]\} \tag{3}$$

The variable α_u gives a possibility to obtain any value between left border $\underline{u}(r)$ and right border $\bar{u}(r)$ of a fuzzy number. For $\alpha_u = 0$ value of the left endpoint $\underline{u}(r)$ of a fuzzy number and for $\alpha_u = 1$ the right endpoint $\bar{u}(r)$ of a fuzzy number are obtained.

The four basic arithmetic operations $*$ \in $\{+, -, \cdot, /\}$ on horizontal fuzzy numbers $U^r = \{u^r : u^r = \underline{u}(r) + \alpha_u(\bar{u}(r) - \underline{u}(r)), r, \alpha_u \in [0, 1]\}$ and $V^r = \{v^r : v^r = \underline{v}(r) + \alpha_v(\bar{v}(r) - \underline{v}(r)), r, \alpha_v \in [0, 1]\}$ are defined as set (4), operation / occurs only if $0 \notin V^r$.

$$U^r * V^r = \{u^r * v^r : \begin{aligned} &u^r * v^r = [\underline{u}(r) + \alpha_u(\bar{u}(r) - \underline{u}(r))] \\ &* [\underline{v}(r) + \alpha_v(\bar{v}(r) - \underline{v}(r))], \\ &r, \alpha_u, \alpha_v \in [0, 1] \end{aligned} \tag{4}$$

The horizontal fuzzy number (HFN) $U^r = \{u^r \in \mathbb{R} : u^r = \underline{u}(r) + \alpha_u(\bar{u}(r) - \underline{u}(r)), r, \alpha_u \in [0, 1]\}$ can be written as $u(r, \alpha_u)$ defined by Eq. (5)

$$u(r, \alpha_u) = \underline{u}(r) + \alpha_u(\bar{u}(r) - \underline{u}(r)), \tag{5}$$

where $r, \alpha_u \in [0, 1]$.

Definition 4 (The direct solution) The direct solution of operation on n horizontal fuzzy numbers $u_1(r, \alpha_{u_1}), \dots, u_n(r, \alpha_{u_n})$, where $r, \alpha_{u_1}, \dots, \alpha_{u_n} \in [0, 1]$, is a set of numbers expressed in the form of multidimensional formula $Z^r = z(r, \alpha_{u_1}, \dots, \alpha_{u_n})$ with up to n HFN variables $\alpha_{u_1}, \dots, \alpha_{u_n} \in [0, 1]$.

Example 2 (Asady 2013) The fuzzy numbers A and B with nonlinear left and right borders, in parametric form $A = (-10 + 20r^{1/4}, 14 - 4r^3)$ and $B = (1 + \sqrt{r}, 30 - 27\sqrt{r})$, $r \in [0, 1]$, as horizontal fuzzy numbers are:

$$a(r, \alpha_a) = -10 + 20r^{1/4} + \alpha_a(24 - 4r^3 - 20r^{1/4}),$$

and

$$b(r, \alpha_b) = 1 + \sqrt{r} + \alpha_b(29 - 28\sqrt{r})$$

where $\alpha_a, \alpha_b, r \in [0, 1]$. Figures 1 and 2 show fuzzy numbers A and B in standard and horizontal forms.

Basic operations on horizontal fuzzy numbers X^r, Y^r and Z^r hold the algebraic properties.

Both horizontal fuzzy number addition and multiplication are commutative and associative. For three HFNs $X^r = x(r, \alpha_x), Y^r = y(r, \alpha_y)$ and $Z^r = z(r, \alpha_z)$, where $\alpha_x, \alpha_y, \alpha_z, r \in [0, 1]$, Eq. (6) hold.

$$\begin{aligned} X^r + Y^r &= Y^r + X^r \\ X^r \cdot Y^r &= Y^r \cdot X^r \\ X^r + (Y^r + Z^r) &= (X^r + Y^r) + Z^r \\ X^r \cdot (Y^r \cdot Z^r) &= (X^r \cdot Y^r) \cdot Z^r \end{aligned} \tag{6}$$

Proof $X^r + Y^r = x(r, \alpha_x) + y(r, \alpha_y) = y(r, \alpha_y) + x(r, \alpha_x) = Y^r + X^r$, $X^r \cdot Y^r = x(r, \alpha_x) \cdot y(r, \alpha_y) = y(r, \alpha_y) \cdot x(r, \alpha_x) = Y^r \cdot X^r$, $X^r + (Y^r + Z^r) = x(r, \alpha_x) + [y(r, \alpha_y) + z(r, \alpha_z)] = [x(r, \alpha_x) + y(r, \alpha_y)] + z(r, \alpha_z) = (X^r + Y^r) + Z^r$, $X^r \cdot (Y^r \cdot Z^r) = x(r, \alpha_x) \cdot [y(r, \alpha_y) \cdot z(r, \alpha_z)] = [x(r, \alpha_x) \cdot y(r, \alpha_y)] \cdot z(r, \alpha_z) = (X^r \cdot Y^r) \cdot Z^r$ \square

In horizontal fuzzy number arithmetic, there exist neutral elements of addition and multiplication. For addition the neutral element is a degenerate fuzzy number 0; for multiplication the neutral element is a degenerate fuzzy number 1. For any fuzzy horizontal number X^r hold Eq. (7).

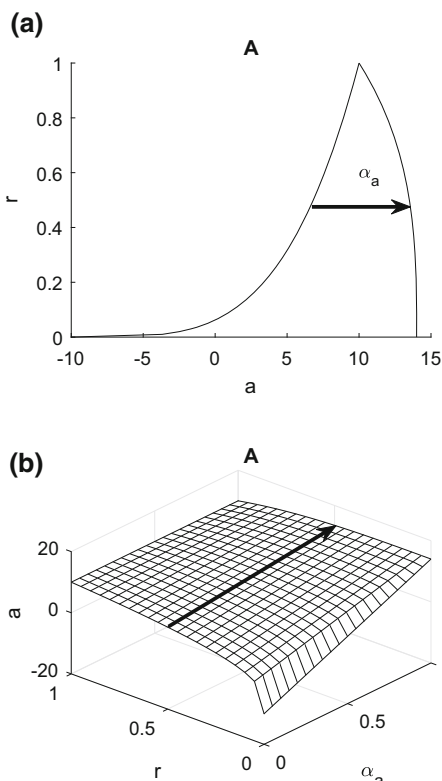


Fig. 1 Fuzzy number $A = (-10 + 20r^{1/4}, 14 - 4r^3)$ in standard (a) and horizontal (b) form

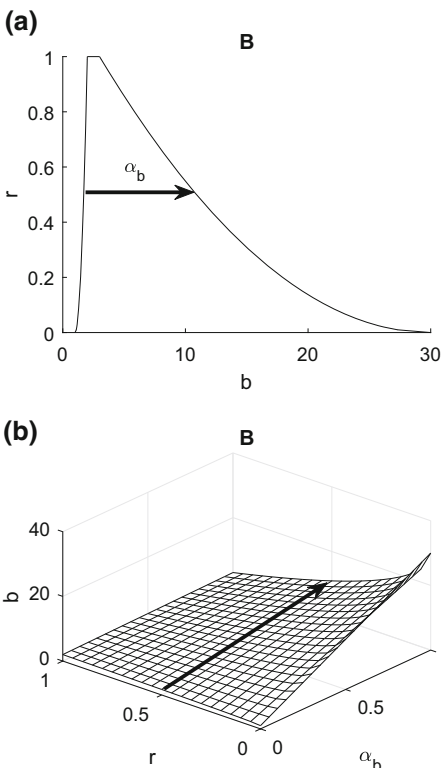


Fig. 2 Fuzzy number $B = (1 + \sqrt{r}, 30 - 27\sqrt{r})$ in standard (a) and horizontal (b) form

$$X^r + 0 = 0 + X^r = X^r$$

$$X^r \cdot 1 = 1 \cdot X^r = X^r \tag{7}$$

The horizontal fuzzy number $X^r = x(r, \alpha_x) = \underline{x}(r) + \alpha_x(\bar{x}(r) - \underline{x}(r))$, where $r, \alpha_x \in [0, 1]$, in HFN arithmetic has an additive inverse element (8).

$$-X^r = -x(r, \alpha_x) = -\underline{x}(r) - \alpha_x(\bar{x}(r) - \underline{x}(r)) \tag{8}$$

Proof $X^r + (-X^r) = [x(r, \alpha_x)] + [-x(r, \alpha_x)] = [\underline{x}(r) + \alpha_x(\bar{x}(r) - \underline{x}(r))] + [-\underline{x}(r) - \alpha_x(\bar{x}(r) - \underline{x}(r))] = 0 \quad \square$

In HFN arithmetic, there exists a multiplicative inverse element, so for $X^r = x(r, \alpha_x) = \underline{x}(r) + \alpha_x(\bar{x}(r) - \underline{x}(r))$, where $r, \alpha_x \in [0, 1]$, $0 \notin X^r$, the multiplicative inverse element is (9).

$$1/X^r = 1/[x(r, \alpha_x)] = 1/[\underline{x}(r) + \alpha_x(\bar{x}(r) - \underline{x}(r))] \tag{9}$$

Proof $X^r \cdot (1/X^r) = x(r, \alpha_x) \cdot [1/(x(r, \alpha_x))] = [\underline{x}(r) + \alpha_x(\bar{x}(r) - \underline{x}(r))]/[\underline{x}(r) + \alpha_x(\bar{x}(r) - \underline{x}(r))] = 1 \quad \square$

In HFN arithmetic for every HFNs $X^r = x(r, \alpha_x)$, $Y^r = y(r, \alpha_y)$ and $Z^r = z(r, \alpha_z)$, where $\alpha_x, \alpha_y, \alpha_z, r \in [0, 1]$, the distributive law holds, Eq. (10).

$$X^r (Y^r + Z^r) = X^r Y^r + X^r Z^r \tag{10}$$

Proof $X^r (Y^r + Z^r) = x(r, \alpha_x) \cdot [y(r, \alpha_y) + z(r, \alpha_z)] = x(r, \alpha_x) \cdot y(r, \alpha_y) + x(r, \alpha_x) \cdot z(r, \alpha_z) = X^r Y^r + X^r Z^r \quad \square$

The cancelation law for addition (11) holds for horizontal fuzzy numbers.

$$X^r + Z^r = Y^r + Z^r \Rightarrow X^r = Y^r \tag{11}$$

Proof For any horizontal fuzzy numbers $X^r = x(r, \alpha_x)$, $Y^r = y(r, \alpha_y)$ and $Z^r = z(r, \alpha_z)$, where $\alpha_x, \alpha_y, \alpha_z, r \in [0, 1]$.

$$X^r + Z^r = Y^r + Z^r, x(r, \alpha_x) + z(r, \alpha_z) = y(r, \alpha_y) + z(r, \alpha_z).$$

Adding to the both sides additive inverse element $-Z^r = -z(r, \alpha_z)$ there is:

$$x(r, \alpha_x) + z(r, \alpha_z) + [-z(r, \alpha_z)] = y(r, \alpha_y) + z(r, \alpha_z) + [-z(r, \alpha_z)], x(r, \alpha_x) = y(r, \alpha_y), X^r = Y^r$$

\square

Also, for horizontal fuzzy numbers the multiplicative cancellation law (12) holds if $0 \notin Z^r$.

$$X^r \cdot Z^r = Y^r \cdot Z^r \Rightarrow X^r = Y^r \tag{12}$$

Proof For any horizontal fuzzy numbers $X^r = x(r, \alpha_x)$, $Y^r = y(r, \alpha_y)$ and $Z^r = z(r, \alpha_z)$, where $\alpha_x, \alpha_y, \alpha_z, r \in [0, 1], 0 \notin Z^r$.

$$X^r \cdot Z^r = Y^r \cdot Z^r,$$

$$x(r, \alpha_x) \cdot z(r, \alpha_z) = y(r, \alpha_y) \cdot z(r, \alpha_z).$$

Multiplying the obtained equation by multiplicative inverse element $1/Z^r = 1/[z(r, \alpha_z)]$ there is:

$$\begin{aligned} x(r, \alpha_x) \cdot z(r, \alpha_z) \cdot \{1/[z(r, \alpha_z)]\} \\ = y(r, \alpha_y) \cdot z(r, \alpha_z) \cdot \{1/[z(r, \alpha_z)]\}, \\ x(r, \alpha_x) = y(r, \alpha_y), X^r = Y^r \end{aligned}$$

□

The direct solution obtained with horizontal fuzzy numbers is presented by formula (granule of information), and also solution can be shown by result indicators (indirect solutions) like span, cardinality distribution or the center of gravity.

Definition 5 (span, secondary solution) For the direct solution $Z^r = z(r, \alpha_{u_1}, \dots, \alpha_{u_n})$ of the basic arithmetic operations on n horizontal fuzzy numbers span is a fuzzy number given by (13).

$$s(Z^r) = \left(\min_{\alpha_{u_1}, \dots, \alpha_{u_n}} z(r, \alpha_{u_1}, \dots, \alpha_{u_n}), \max_{\alpha_{u_1}, \dots, \alpha_{u_n}} z(r, \alpha_{u_1}, \dots, \alpha_{u_n}) \right) \tag{13}$$

The special cases of fuzzy numbers are triangular $U = (a, b, c)$ and trapezoidal $V = (a, b, c, d)$ fuzzy numbers. Their parametric forms are $(\underline{u}(r), \bar{u}(r)) = (a + (b - a)r, c + (b - c)r)$ and $(\underline{v}(r), \bar{v}(r)) = (a + (b - a)r, d + (c - d)r)$, respectively, where $r \in [0, 1]$. The triangular and trapezoidal horizontal fuzzy numbers are presented in Piegat and Landowski (2015, 2017a) as:

$$u(r, \alpha_u) = [a + (b - a)r] + [(c - a) - r(c - a)]\alpha_u$$

and

$$v(r, \alpha_v) = [a + (b - a)r] + [(d - a) - r(d - a + b - c)]\alpha_v,$$

where $r, \alpha_u, \alpha_v \in [0, 1]$.

3 Solution of fuzzy linear systems with horizontal fuzzy numbers

Definition 6 (Abbasbandy and Alavi 2005; Friedman et al. 1998) (Fuzzy Linear System) The $n \times n$ linear system (14)

$$\begin{aligned} a_{11}x_1 + \dots + a_{1n}x_n &= y_1 \\ a_{21}x_1 + \dots + a_{2n}x_n &= y_2 \\ &\vdots \\ a_{n1}x_1 + \dots + a_{nn}x_n &= y_n \end{aligned} \tag{14}$$

where the coefficients matrix $A = (a_{ij}), 1 \leq i, j \leq n$ is a $n \times n$ matrix and each $y_i, 1 \leq i \leq n$, is a fuzzy number in parametric form which is called a fuzzy linear system in parametric form (FLS).

Steps to find the solution and the span of solution (indicator of solution) by the method with horizontal fuzzy numbers are as follows:

- Step 1: Write the given fuzzy numbers in algebraic system of equations in the HFN notation using HFN variables.
- Step 2: Calculate direct solution using algebraic operations.
- Step 3: To obtain span (indirect solution, indicator, secondary solution) of the solution find border values for HFN variable of the solution.
- Step 4: For border values find minimum and maximum that are span (secondary solution, indicator) of the direct solution.

4 Examples of solving fuzzy linear systems with horizontal fuzzy numbers

Example 3 (Friedman et al. 1998; Behera and Chakraverty 2012) Find a solution of 2×2 fuzzy system of linear equation (15)

$$\begin{aligned} x_1 - x_2 &= (r, 2 - r), \\ x_1 + 3x_2 &= (4 + r, 7 - 2r). \end{aligned} \tag{15}$$

In HFN notation, the fuzzy system of Eq. (15) has a form of (16),

$$\begin{aligned} x_1 - x_2 &= r + \alpha_1(2 - 2r), \\ x_1 + 3x_2 &= 4 + r + \alpha_2(3 - 3r), \end{aligned} \tag{16}$$

where $r, \alpha_1, \alpha_2 \in [0, 1]$.

The direct solution of fuzzy linear system (15) in HFN form is given by (17),

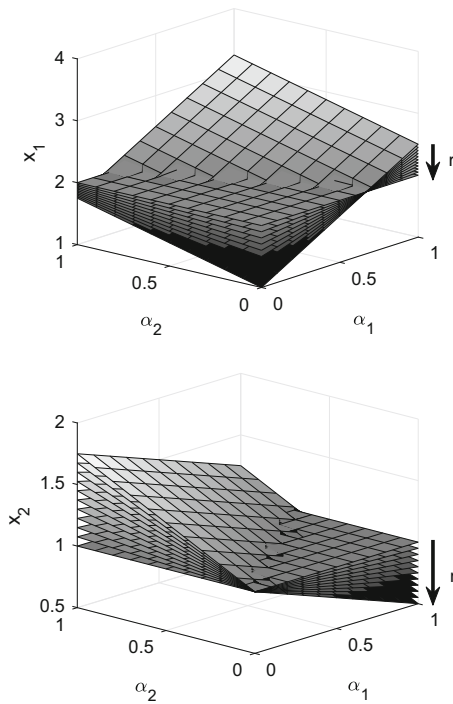


Fig. 3 Three-dimensional horizontal fuzzy number solution of fuzzy system of linear equations, where $r \in [0 : 0.1 : 1]$

$$\begin{aligned} x_1(r, \alpha_1, \alpha_2) &= 1 + r + 1.5\alpha_1(1 - r) + 0.75\alpha_2(1 - r) \\ x_2(r, \alpha_1, \alpha_2) &= 1 - 0.5\alpha_1(1 - r) + 0.75\alpha_2(1 - r) \end{aligned} \quad (17)$$

where $r, \alpha_1, \alpha_2 \in [0, 1]$.

Figure 3 shows solution (17) of fuzzy system of linear equations in three-dimensional space.

Substituting the obtained solution (17) to fuzzy system of Eq. (15) shows that the solution is correct, there is:

$$\begin{aligned} x_1 - x_2 &= 1 + r + 1.5\alpha_1(1 - r) + 0.75\alpha_2(1 - r) - [1 - 0.5\alpha_1(1 - r) + 0.75\alpha_2(1 - r)] = r + \alpha_1(2 - 2r) = (r, 2 - r), \\ x_1 + 3x_2 &= 1 + r + 1.5\alpha_1(1 - r) + 0.75\alpha_2(1 - r) + 3[1 - 0.5\alpha_1(1 - r) + 0.75\alpha_2(1 - r)] = 4 + r + \alpha_2(3 - 3r) = (4 + r, 7 - 2r). \end{aligned}$$

The direct solution of given fuzzy linear system (15) is presented by formula (17); every pair $(x_1(r, \alpha_1, \alpha_2), x_2(r, \alpha_1, \alpha_2))$ of solution depends on HFN variable $\alpha_1, \alpha_2 \in [0, 1]$ and $r \in [0, 1]$. For 2×2 fuzzy interval systems, point (crisp) solutions are coupled and cannot be considered separately.

The representation of the solution can be a span, but span is only the result indicator. Any two values from the span are not a pair of crisp solution. The choice of the crisp solution is possible for unique HFN variable $\alpha_1, \alpha_2 \in [0, 1]$. For example, from formula (17) one of the crisp point solutions is a pair of numbers $(x_1(0.5, 0, 0), x_2(0.5, 0, 0)) = (1.5, 1)$, $r = 0.5$.

Table 1 Border values of solution with horizontal fuzzy numbers for HFN variable $\alpha_1, \alpha_2 \in [0, 1]$ and $r \in [0, 1]$

α_1	0	0	1	1
α_2	0	1	0	1
$x_1(r, \alpha_1, \alpha_2)$	$1 + r$	$1.75 - 0.25r$	$2.5 - 0.5r$	$3.25 - 1.25r$
	min x_1		max x_1	
$x_2(r, \alpha_1, \alpha_2)$	1	$1.75 - 0.75r$	$0.5 + 0.5r$	$1.25 - 1.25r$
	max x_2		min x_2	

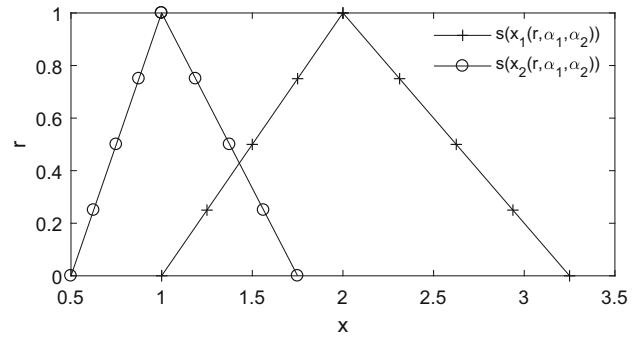


Fig. 4 Span of horizontal fuzzy number solution of fuzzy system of linear equations

To find the span of the solution, border values should be calculated. Table 1 presents border values solution (17) for variables α_1 and α_2 .

Considering the spans of (17), the solution $x_1(r, \alpha_1, \alpha_2)$ has a minimum value for $\alpha_1 = \alpha_2 = 0$ and maximum for $\alpha_1 = \alpha_2 = 1$, solution $x_2(r, \alpha_1, \alpha_2)$ has a minimum for $\alpha_1 = 1$ and $\alpha_2 = 0$ and maximum for $\alpha_1 = 0$ and $\alpha_2 = 1$. The spans of (17) in parametric form are presented by (18) and in graphical form in Fig. 4,

$$\begin{aligned} s(x_1(r, \alpha_1, \alpha_2)) &= (1 + r, 3.25 - 1.25r) \\ s(x_2(r, \alpha_1, \alpha_2)) &= (0.5 + 0.5r, 1.75 - 0.75r) \end{aligned} \quad (18)$$

where $r \in [0, 1]$.

The result for fuzzy system of linear equations (15) obtained by Friedman et al. (1998) and Behera and Chakraverty (2012) equals (19),

$$\begin{aligned} x_1 &= (1.375 + 0.625r, 2.875 - 0.875r) \\ x_2 &= (0.875 + 0.125r, 1.375 - 0.375r) \end{aligned} \quad (19)$$

where $r \in [0, 1]$.

The results (19) satisfy the fuzzy system of linear equations only in the form of (15), but do not satisfy equivalent fuzzy linear system, for example in the form of (20).

$$\begin{aligned} x_1 &= (r, 2 - r) + x_2, \\ x_1 &= (4 + r, 7 - 2r) - 3x_2. \end{aligned} \quad (20)$$

Substituting x_1 and x_2 from (19) to the right side of the equations of the fuzzy linear system in the form of (20) is:

$$\begin{aligned} (r, 2 - r) + x_2 &= (r, 2 - r) + (0.875 + 0.125r, \\ &1.375 - 0.375r) = (0.875 + 1.125r, 3.375 - 1.375r) \\ &\neq (1.375 + 0.625r, 2.875 - 0.875r) = x_1, \\ (4 + r, 7 - 2r) - 3x_2 &= (4 + r, 7 - 2r) \\ &- 3(0.875 + 0.125r, 1.375 - 0.375r) \\ &= (-0.125 - 0.125r, 4.375 - 2.375r) \\ &\neq (1.375 + 0.625r, 2.875 - 0.875r) = x_1. \end{aligned}$$

Similarly it can be shown that results (19) do not satisfy other forms of fuzzy linear system (15).

The solution obtained with horizontal fuzzy number (17) satisfies any equivalent form of fuzzy linear system (15) and the universality conditions.

Results (19) are just spans (indicators, indirect solutions). From (19), it is difficult to find a pair of (x_1, x_2) that satisfies the system of Eq. (15).

Spans of solutions (17) for $r = 0$ are $s(x_1^{r=0}) = (1, 3.25)$, $s(x_2^{r=0}) = (0.5, 1.75)$ and the supports in (19) equal $x_1^{r=0} = (1.375, 2.875)$, $x_2^{r=0} = (0.875, 1.375)$. As it can be seen, the spans for $r = 0$ in the solution obtained using horizontal fuzzy number have wider width than result (19). Result (19) is not a full span of solution of system (15); it can be checked by testing point method. For example, from (17) for $r = 0$, $\alpha_1 = 0.9$ and $\alpha_2 = 0.2$ values $x_1 = 2.5$ and $x_2 = 0.7$ are one of the solutions of (15), but $x_2 = 0.7$ does not belong to "solution" (19).

Example 4 (Friedman et al. 1998; Behera and Chakraverty 2012) Find a solution of 3×3 fuzzy system of linear equations (21),

$$\begin{aligned} x_1 + x_2 - x_3 &= (r, 2 - r), \\ x_1 - 2x_2 + x_3 &= (2 + r, 3), \\ 2x_1 + x_2 + 3x_3 &= (-2, -1 - r), \end{aligned} \tag{21}$$

where $r \in [0, 1]$.

The fuzzy system (21) using fuzzy horizontal number has a form of (22),

$$\begin{aligned} x_1 + x_2 - x_3 &= r + \alpha_1(2 - 2r), \\ x_1 - 2x_2 + x_3 &= 2 + r + \alpha_2(1 - r), \\ 2x_1 + x_2 + 3x_3 &= -2 + \alpha_3(1 - r), \end{aligned} \tag{22}$$

where $r, \alpha_1, \alpha_2, \alpha_3 \in [0, 1]$.

The direct solution with horizontal fuzzy numbers is given by (23),

$$\begin{aligned} x_1(r, \alpha_1, \alpha_2, \alpha_3) &= [-6 - 11r - 14\alpha_1(1 - r) \\ &- 4\alpha_2(1 - r) - \alpha_3(1 - r)]/(-13), \end{aligned}$$

$$\begin{aligned} x_2(r, \alpha_1, \alpha_2, \alpha_3) &= [14 + 4r - 2\alpha_1(1 - r) \\ &+ 5\alpha_2(1 - r) - 2\alpha_3(1 - r)]/(-13), \\ x_3(r, \alpha_1, \alpha_2, \alpha_3) &= [8 + 6r + 10\alpha_1(1 - r) \\ &+ \alpha_2(1 - r) - 3\alpha_3(1 - r)]/(-13), \end{aligned} \tag{23}$$

where $r, \alpha_1, \alpha_2, \alpha_3 \in [0, 1]$.

By equations in (23), every three numbers x_1, x_2 and x_3 that satisfy the fuzzy system of linear equations (21) can be found.

It can be proved that solution (23) is corrected by substituting (23) to the fuzzy system of equations (21). For the first equation starting from the left side:

$$\begin{aligned} x_1 + x_2 - x_3 &= [-6 - 11r - 14\alpha_1(1 - r) - 4\alpha_2(1 - r) \\ &- \alpha_3(1 - r)]/(-13) + [14 + 4r - 2\alpha_1(1 - r) \\ &+ 5\alpha_2(1 - r) - 2\alpha_3(1 - r)]/(-13) \\ &- [8 + 6r + 10\alpha_1(1 - r) + \alpha_2(1 - r) \\ &- 3\alpha_3(1 - r)]/(-13) = r + \alpha_1(2 - 2r) = (r, 2 - r), \end{aligned}$$

the right side is obtained.

Similarly, it can be shown that solution (23) satisfies the second and third equations, so the solution obtained using horizontal fuzzy numbers satisfies the fuzzy system of linear equations (21) and also its equivalent form.

The solutions of fuzzy linear systems are dependent. The point solution of three values that satisfy the fuzzy system of linear equations (21) can be calculated from the solution with horizontal fuzzy numbers (23). For example, for $r = 0$, $\alpha_1 = \alpha_3 = 0$ and $\alpha_2 = 1$ from solution (23) the point solution is $(x_1, x_2, x_3) = (10/13, -19/13, -9/13)$.

For $r \in [0, 1]$ the minimum and maximum values can be calculated. Table 2 presents border values of solution (23) for variables α_1, α_2 and α_3 .

The span (indicator, indirect solution) of solution (23) for $r \in [0, 1]$ and RDM variables from Table 2 is presented by (24).

$$\begin{aligned} s(x_1(r, \alpha_1, \alpha_2, \alpha_3)) &= (6/13 + 11r/13, 25/13 - 8r/13) \\ s(x_2(r, \alpha_1, \alpha_2, \alpha_3)) &= (-19/13 + r/13, -10/13 - 8r/13) \\ s(x_3(r, \alpha_1, \alpha_2, \alpha_3)) &= (-19/13 + 5r/13, -5/13 - 9r/13) \end{aligned} \tag{24}$$

Figure 5 shows graphical form of spans (24) of solution with horizontal fuzzy numbers.

The solution of the fuzzy system (21) obtained in (Friedman et al. 1998; Behera and Chakraverty 2012) according to M. Friedman et al. is not fuzzy numbers and cannot be

Table 2 Border values of solution with horizontal fuzzy numbers for variables $\alpha_1, \alpha_2, \alpha_3 \in [0, 1]$ and $r \in [0, 1]$

α_1	0	0
α_2	0	0
α_3	0	1
$x_1(r, \alpha_1, \alpha_2, \alpha_3)$	$6/13 + 11r/13$	$7/13 + 10r/13$
min x_1		
$x_2(r, \alpha_1, \alpha_2, \alpha_3)$	$-14/13 - 4r/13$	$-12/13 - 6r/13$
$x_3(r, \alpha_1, \alpha_2, \alpha_3)$	$-8/13 - 6r/13$	$-5/13 - 9r/13$
max x_3		
α_1	0	1
α_2	1	0
α_3	0	0
$x_1(r, \alpha_1, \alpha_2, \alpha_3)$	$10/13 + 7r/13$	$20/13 - 3r/13$
$x_2(r, \alpha_1, \alpha_2, \alpha_3)$	$-19/13 + r/13$	$-12/13 - 6r/13$
min x_2		
$x_3(r, \alpha_1, \alpha_2, \alpha_3)$	$-9/13 - 5r/13$	$-18/13 + 4r/13$
α_1	0	1
α_2	1	0
α_3	1	1
$x_1(r, \alpha_1, \alpha_2, \alpha_3)$	$11/13 + 6r/13$	$21/13 - 4r/13$
$x_2(r, \alpha_1, \alpha_2, \alpha_3)$	$-17/13 - r/13$	$-10/13 - 8r/13$
max x_2		
$x_3(r, \alpha_1, \alpha_2, \alpha_3)$	$-6/13 - 8r/13$	$-15/13 + r/13$
α_1	1	1
α_2	1	1
α_3	0	1
$x_1(r, \alpha_1, \alpha_2, \alpha_3)$	$24/13 - 7r/13$	$25/13 - 8r/13$
max x_1		
$x_2(r, \alpha_1, \alpha_2, \alpha_3)$	$-17/13 - r/13$	$-15/13 - 3r/13$
$x_3(r, \alpha_1, \alpha_2, \alpha_3)$	$-19/13 + 5r/13$	$-16/13 + 2r/13$
min x_3		

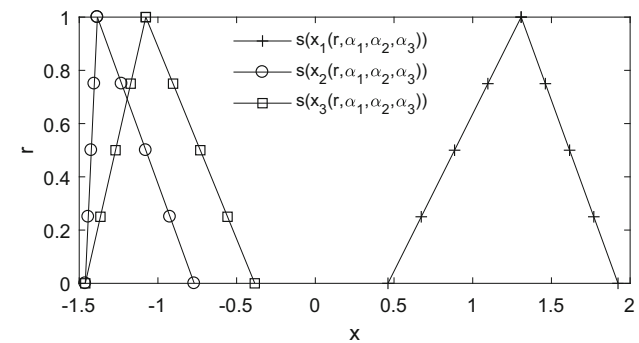


Fig. 5 Spans of horizontal fuzzy number solutions of 3×3 fuzzy system of linear equations

used in practice because $\bar{x}_2 \leq x_2$ and $\bar{x}_3 \leq x_3$. Lodwick and Dubois (2015) wrote that improper intervals are not accepted. Changing the borders in x_2 and x_3 , the indicator of “solution”

Table 3 Spans for $r = 0$ of the solution obtained using horizontal fuzzy numbers and of the “weak solution” by methods presented in (Friedman et al. 1998; Behera and Chakraverty 2012)

Solution	Span of the solution using HFN	The “weak solution”
x_1	$(6/13, 25/13)$	$(-30/13, 61/13)$
x_2	$(-19/13, -10/13)$	$(-21/13, -8/13)$
x_3	$(-19/13, -5/13)$	$(-38/13, 14/13)$

is obtained. M. Friedman et al. called these results as a “weak solution” in a form of (25), but it can only be called the “weak indicator of solution”.

$$\begin{aligned}
 x_1 &= (-30/13 + 47r/13, 61/13 - 44r/13) \\
 x_2 &= (-21/13 + 3r/13, -8/13 - 10r/13) \\
 x_3 &= (-38/13 + 24r/13, 14/13 - 28r/13) \tag{25}
 \end{aligned}$$

Results (25) from (Friedman et al. 1998; Behera and Chakraverty 2012) do not satisfy fuzzy linear system (21). Substituting x_1, x_2 and x_3 from (25) to the left side of fuzzy system of linear equations (21), there is:

$$\begin{aligned}
 x_1 + x_2 - x_3 &= (-30/13 + 47r/13, 61/13 - 44r/13) \\
 &\quad + (-21/13 + 3r/13, -8/13 - 10r/13) \\
 &\quad - (-38/13 + 24r/13, 14/13 - 28r/13) \\
 &= (-65/13 + 78r/13, 91/13 - 78r/13) \neq (r, 2 - r), \\
 x_1 - 2x_2 + x_3 &= (-30/13 + 47r/13, 61/13 - 44r/13) \\
 &\quad - 2(-21/13 + 3r/13, -8/13 - 10r/13) \\
 &\quad + (-38/13 + 24r/13, 14/13 - 28r/13) \\
 &= (-52/13 + 91r/13, 117/13 - 75r/13) \neq (2 + r, 3), \\
 2x_1 + x_2 + 3x_3 &= 2(-30/13 + 47r/13, 61/13 - 44r/13) \\
 &\quad + (-21/13 + 3r/13, -8/13 - 10r/13) \\
 &\quad + 3(-38/13 + 24r/13, 14/13 - 28r/13) \\
 &= (-195/13 + 169r/13, 156/13 - 182r/13) \\
 &\neq (-2, -1 - r).
 \end{aligned}$$

Also results (25) do not satisfy any fuzzy linear system equivalent to (21).

To compare results (23) and (25), spans (indirect solutions) for $r = 0$ are presented in Table 3.

Obtained fuzzy numbers in (Friedman et al. 1998; Behera and Chakraverty 2012) include values that are not a solution, and it is difficult to indicate the three points of solution of fuzzy linear system (21). For example, let us consider crisp solution from the obtained results in (Friedman et al. 1998; Behera and Chakraverty 2012). For $r = 0$ values $x_1 = -2, x_2 = -1.2$ and $x_3 = 1$ belong to the solution presented in Table 3 and in (25), but these values do not satisfy system (21):

Table 4 Chemical compositions of feeds (SalooKolayi et al. 2011)

Feed	E (Kcal/kg)	Pr (g/kg)	F (g/kg)	NDF (g/kg)
Alfalfa	1190	192	25	416
Barely grain	1860	124	22	208
Fat supplement	5020	0	845	0
Sugar beet molasses	1840	85	2	1
Soybean meal	2130	499	16	149
Sunflower meal	1380	284	14	403
Wheat bran	1610	173	43	425

Feed	NFC (g/kg)	Ca (g/kg)	P (g/kg)
Alfalfa	257	14.7	2.8
Barely grain	617	0.6	3.9
Fat supplement	0	120	0
Sugar beet molasses	798	1.5	0.3
Soybean meal	270	4	7.1
Sunflower meal	222	4.8	10
Wheat bran	296	1.3	11.8

$$\begin{aligned}
 & -2 + (-1.2) - 1 = -4.2 \notin (0, 2); \\
 & -2 - 2 \cdot (-1.2) + 1 = 1.4 \notin (2, 3); \\
 & 2 \cdot (-2) + (-1.2) + 3 \cdot 1 = -2.2 \notin (-2, -1).
 \end{aligned}$$

Spans in the form of intervals of HFN solution in (24) and for $r = 0$ in Table 3 are just the representations (indicators) of the full solution.

Values (x_1, x_2, x_3) that satisfy the fuzzy linear system (21) should be calculated from (23). Solutions $x_1(r, \alpha_1, \alpha_2, \alpha_3)$, $x_2(r, \alpha_1, \alpha_2, \alpha_3)$ and $x_3(r, \alpha_1, \alpha_2, \alpha_3)$ are dependent. Three numbers of the solution that satisfy system (21) can be obtained for unique HFN variables $\alpha_1, \alpha_2, \alpha_3 \in [0, 1]$ and $r \in [0, 1]$. For example, let us consider the crisp solution with HFN: values $x_1 = 1.5, x_2 = -1$ and $x_3 = -1.3$ calculated from the obtained solution (23) for $r = 0, \alpha_1 = 0.9, \alpha_2 = 0.2$ and $\alpha_3 = 0.1$ satisfy system (21):

$$\begin{aligned}
 & 1.5 + (-1) - (-1.3) = 1.8 \in (0, 2); \\
 & 1.5 - 2 \cdot (-1) + (-1.3) = 2.2 \in (2, 3); \\
 & 2 \cdot (1.5) + (-1) + 3 \cdot (-1.3) = -1.9 \in (-2, -1).
 \end{aligned}$$

The data in example 5 were taken from SalooKolayi et al. (2011), where authors solved linear fuzzy model to plan the feeding of the animals.

Example 5 (SalooKolayi et al. 2011) Find the plan of a diet for animals by seven feeds. The chemical compositions of feeds are given in Table 4, where: E—energy, Pr—protein, F—fat, NDF—neutral detergent fiber, NFC—non-fibrous carbohydrate, Ca—calcium and P—phosphor.

Table 5 Fuzzy values of nutrient requirements (SalooKolayi et al. 2011)

Nutrient	Fuzzy value of requirement
E (Kcal)	$(1480 + 40r, 1700 - 100r)$
Pr (g)	$(150 + 10r, 190 - 20r)$
F (g)	$(25 + 10r, 90 - 20r)$
NDF (g)	$(280 + 40r, 420 - 40r)$
NFC (g)	$(320 + 60r, 440 - 40r)$
Ca (g)	$(9.5 + 0.5r, 10.5 - 0.5r)$
P (g)	$(4.75 + 0.25r, 5.25 - 0.25r)$

The fuzzy values of nutrient requirements of feeds are presented in Table 5, $r \in [0, 1]$.

To find the feeding plan the 7×7 fuzzy system of linear equations (26) has to be solved,

$$\begin{aligned}
 & 1190x_1 + 1860x_2 + 5020x_3 + 1840x_4 + 2130x_5 + 1380x_6 \\
 & \quad + 1610x_7 = (1480 + 40r, 1700 - 100r) \\
 & 192x_1 + 124x_2 + 85x_4 + 499x_5 + 284x_6 + 173x_7 \\
 & \quad = (150 + 10r, 190 - 20r) \\
 & 25x_1 + 22x_2 + 845x_3 + 2x_4 + 16x_5 + 14x_6 + 43x_7 \\
 & \quad = (25 + 10r, 90 - 20r) \\
 & 416x_1 + 208x_2 + 149x_5 + 403x_6 + 425x_7 \\
 & \quad = (280 + 40r, 420 - 40r) \\
 & 257x_1 + 617x_2 + 798x_4 + 270x_5 + 222x_6 + 296x_7 \\
 & \quad = (320 + 60r, 440 - 40r) \\
 & 14.7x_1 + 0.6x_2 + 120x_3 + 1.5x_4 + 4x_5 + 4.8x_6 + 1.3x_7 \\
 & \quad = (9.5 + 0.5r, 10.5 - 0.5r) \\
 & 2.8x_1 + 3.9x_2 + 0.3x_4 + 7.1x_5 + 10x_6 + 11.8x_7 \\
 & \quad = (4.75 + 0.25r, 5.25 - 0.25r) \tag{26}
 \end{aligned}$$

where $r \in [0, 1]$.

The fuzzy system (26) in the form of HFN notation, where $\alpha_1, \dots, \alpha_7, r \in [0, 1]$, is presented by formula (27).

$$\begin{aligned}
 & 1190x_1 + 1860x_2 + 5020x_3 + 1840x_4 + 2130x_5 + 1380x_6 \\
 & \quad + 1610x_7 = 1480 + 40r + (220 - 140r)\alpha_1 \\
 & 192x_1 + 124x_2 + 85x_4 + 499x_5 + 284x_6 + 173x_7 \\
 & \quad = 150 + 10r + (40 - 30r)\alpha_2 \\
 & 25x_1 + 22x_2 + 845x_3 + 2x_4 + 16x_5 + 14x_6 + 43x_7 \\
 & \quad = 25 + 10r + (65 - 30r)\alpha_3 \\
 & 416x_1 + 208x_2 + 149x_5 + 403x_6 + 425x_7 \\
 & \quad = 280 + 40r + (140 - 80r)\alpha_4 \\
 & 257x_1 + 617x_2 + 798x_4 + 270x_5 + 222x_6 + 296x_7 \\
 & \quad = 320 + 60r + (120 - 100r)\alpha_5
 \end{aligned}$$

$$\begin{aligned}
 &14.7x_1 + 0.6x_2 + 120x_3 + 1.5x_4 + 4x_5 + 4.8x_6 + 1.3x_7 \\
 &= 9.5 + 0.5r + (1 - r)\alpha_6 \\
 &2.8x_1 + 3.9x_2 + 0.3x_4 + 7.1x_5 + 10x_6 + 11.8x_7 \\
 &= 4.75 + 0.25r + (0.5 - 0.5r)\alpha_7
 \end{aligned}
 \tag{27}$$

Solving fuzzy system of equations (27), the multidimensional solutions x_i ($i \in \{1, \dots, 7\}$) of fuzzy system of linear equations (26) expressed in the form of HFN notation using variable $\alpha_i \in [0, 1]$ and $r \in [0, 1]$ were obtained. Direct solutions are presented by formula (28).

$$\begin{aligned}
 x_1(r, \alpha_1, \dots, \alpha_7) &= -0.8642r + 1.5729 \\
 &- 0.0043\alpha_1(140r - 220) \\
 &+ 0.0128\alpha_2(30r - 40) + 0.0436\alpha_3(30r - 65) \\
 &+ 0.0004\alpha_4(80r - 140) + 0.0086\alpha_5(100r - 120) \\
 &- 0.1292\alpha_6(r - 1) + 0.0195\alpha_7(0.5r - 0.5) \\
 x_2(r, \alpha_1, \dots, \alpha_7) &= 1.9747r - 2.6044 \\
 &+ 0.0076\alpha_1(140r - 220) \\
 &- 0.0271\alpha_2(30r - 40) - 0.0982\alpha_3(30r - 65) \\
 &- 0.0099\alpha_4(80r - 140) - 0.0153\alpha_5(100r - 120) \\
 &+ 0.3722\alpha_6(r - 1) + 0.4134\alpha_7(0.5r - 0.5) \\
 x_3(r, \alpha_1, \dots, \alpha_7) &= 0.0617r - 0.0612 \\
 &+ 0.0003\alpha_1(140r - 220) \\
 &- 0.0008\alpha_2(30r - 40) - 0.0034\alpha_3(30r - 65) \\
 &- 0.0001\alpha_4(80r - 140) - 0.0005\alpha_5(100r - 120) \\
 &+ 0.0040\alpha_6(r - 1) + 0.0006\alpha_7(0.5r - 0.5) \\
 x_4(r, \alpha_1, \dots, \alpha_7) &= -0.8729r + 1.4080 \\
 &- 0.0033\alpha_1(140r - 220) \\
 &+ 0.0128\alpha_2(30r - 40) + 0.0480\alpha_3(30r - 65) \\
 &+ 0.0060\alpha_4(80r - 140) + 0.0053\alpha_5(100r - 120) \\
 &- 0.2014\alpha_6(r - 1) - 0.2452\alpha_7(0.5r - 0.5) \\
 x_5(r, \alpha_1, \dots, \alpha_7) &= -0.5966r + 0.7205 \\
 &- 0.0029\alpha_1(140r - 220) \\
 &+ 0.0060\alpha_2(30r - 40) + 0.0234\alpha_3(30r - 65) \\
 &+ 0.0018\alpha_4(80r - 140) + 0.0061\alpha_5(100r - 120) \\
 &- 0.0436\alpha_6(r - 1) + 0.0083\alpha_7(0.5r - 0.5) \\
 x_6(r, \alpha_1, \dots, \alpha_7) &= 2.2621r - 2.7735 \\
 &+ 0.0112\alpha_1(140r - 220) \\
 &- 0.0330\alpha_2(30r - 40) - 0.0956\alpha_3(30r - 65) \\
 &- 0.0045\alpha_4(80r - 140) - 0.0225\alpha_5(100r - 120) \\
 &+ 0.2043\alpha_6(r - 1) + 0.0043\alpha_7(0.5r - 0.5) \\
 x_7(r, \alpha_1, \dots, \alpha_7) &= -1.9622r + 2.7712 \\
 &- 0.0092\alpha_1(140r - 220) \\
 &+ 0.0299\alpha_2(30r - 40) + 0.0878\alpha_3(30r - 65)
 \end{aligned}$$

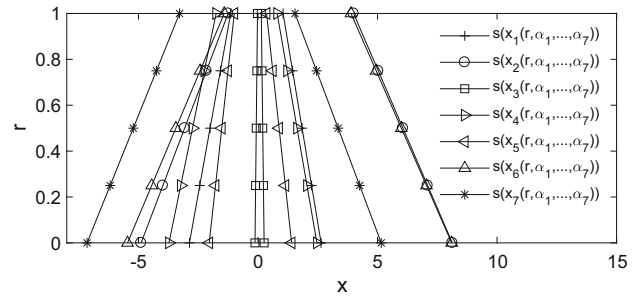


Fig. 6 Spans of horizontal fuzzy numbers solution of fuzzy system of linear equations 7×7

$$\begin{aligned}
 &+ 0.0057\alpha_4(80r - 140) + 0.0183\alpha_5(100r - 120) \\
 &- 0.2341\alpha_6(r - 1) - 0.2284\alpha_7(0.5r - 0.5)
 \end{aligned}
 \tag{28}$$

The spans (indirect solutions) of solutions (28) of fuzzy system of equations (26) are presented in parametric form by (29) and graphically in Fig. 6,

$$\begin{aligned}
 s(x_1(r, \alpha_1, \dots, \alpha_7)) &= (-2.87 + 1.73r, 2.64 - 1.59r) \\
 s(x_2(r, \alpha_1, \dots, \alpha_7)) &= (-4.89 + 3.64r, 8.13 - 4.13r) \\
 s(x_3(r, \alpha_1, \dots, \alpha_7)) &= (-0.13 + 0.1r, 0.26 - 0.12r) \\
 s(x_4(r, \alpha_1, \dots, \alpha_7)) &= (-3.72 + 1.97r, 2.46 - 1.66r) \\
 s(x_5(r, \alpha_1, \dots, \alpha_7)) &= (-2.04 + 1.05r, 1.44 - 1.05r) \\
 s(x_6(r, \alpha_1, \dots, \alpha_7)) &= (-5.46 + 4.05r, 8.1 - 4.21r) \\
 s(x_7(r, \alpha_1, \dots, \alpha_7)) &= (-7.15 + 3.87r, 5.16 - 3.61r)
 \end{aligned}
 \tag{29}$$

where $r \in [0, 1]$.

In this case, where the solution is a plan of diet for animals obtained values should be greater than or equal to zero. So the values that satisfy this condition should be picked from the solution. With HFN method, the crisp solution where values are greater than or equal to zero can be obtained. One of this crisp solution equals $(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (0.29, 0.22, 0.05, 0.05, 0, 0.13, 0.15)$; it can be obtained for $r = 0, \alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0.5, \alpha_4 = 0, \alpha_5 = 0, \alpha_6 = 1$ and $\alpha_7 = 0$. Because of the knowledge of the values r and α_i , the system of linear equations for this crisp solution can be obtained.

Example 6 Let us consider the fuzzy system of linear equations (30) where the border values are not linear,

$$\begin{aligned}
 x_1 - x_2 &= (1 + r^2, 4 - r^3) \\
 3x_1 - 2x_2 &= (3 + 2r^2, 6 - \sqrt{r})
 \end{aligned}
 \tag{30}$$

where $r \in [0, 1]$.

The fuzzy numbers $M = (1 + r^2, 4 - r^3)$ and $N = (3 + 2r^2, 6 - \sqrt{r})$, using variables $\alpha_1 \in [0, 1]$ and $\alpha_2 \in [0, 1]$, in

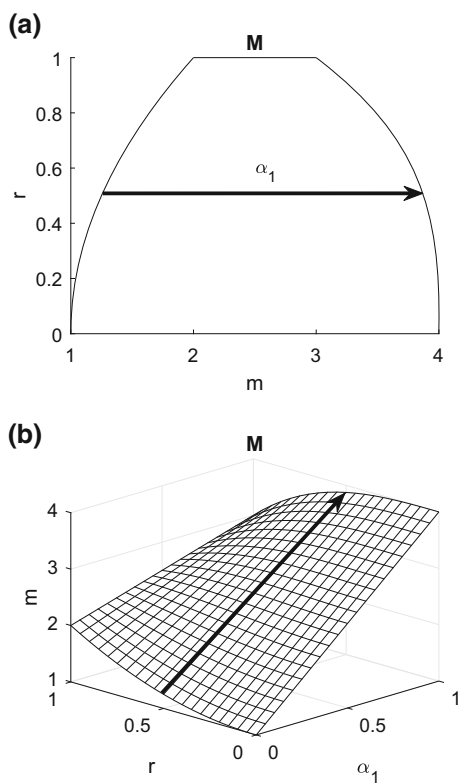


Fig. 7 Fuzzy number $M = (1 + r^2, 4 - r^3)$ in standard (a) and horizontal (b) form

HFN notation have a form $m(r, \alpha_1) = 1 + r^2 + \alpha_1(3 - r^2 - r^3)$ and $n(r, \alpha_2) = 3 + 2r^2 + \alpha_2(3 - \sqrt{r} - 2r^2)$, respectively.

The graphical presentation of fuzzy numbers M and N in standard and horizontal forms is shown in Figs. 7 and 8.

The fuzzy system of linear equations (30) with horizontal fuzzy numbers $m(r, \alpha_1)$ and $n(r, \alpha_2)$ is presented by (31),

$$\begin{aligned} x_1 - x_2 &= 1 + r^2 + \alpha_1(3 - r^2 - r^3) \\ 3x_1 - 2x_2 &= 3 + 2r^2 + \alpha_2(3 - \sqrt{r} - 2r^2) \end{aligned} \tag{31}$$

where $\alpha_1, \alpha_2, r \in [0, 1]$.

The direct solution of fuzzy system of linear equations (31) is given by (32),

$$\begin{aligned} x_1(r, \alpha_1, \alpha_2) &= 1 - 2\alpha_1(3 - r^2 - r^3) \\ &\quad + \alpha_2(3 - \sqrt{r} - 2r^2) \\ x_2(r, \alpha_1, \alpha_2) &= -r^2 - 3\alpha_1(3 - r^2 - r^3) \\ &\quad + \alpha_2(3 - \sqrt{r} - 2r^2) \end{aligned} \tag{32}$$

where $\alpha_1, \alpha_2, r \in [0, 1]$.

Figure 9 shows solutions x_1 and x_2 in 3-dimensional space for $r \in [0 : 0.1 : 1]$ and $\alpha_1, \alpha_2 \in [0, 1]$.

The obtained solutions (32) satisfy the fuzzy system of linear equations (30). Substituting solutions (32) to the left side of (30), the correctness of the solution can be checked:

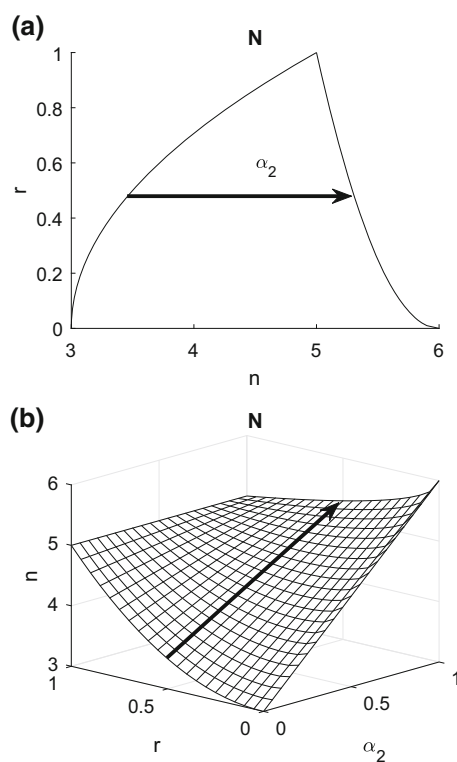


Fig. 8 Fuzzy number $N = (3 + 2r^2, 6 - \sqrt{r})$ in standard (a) and horizontal (b) form

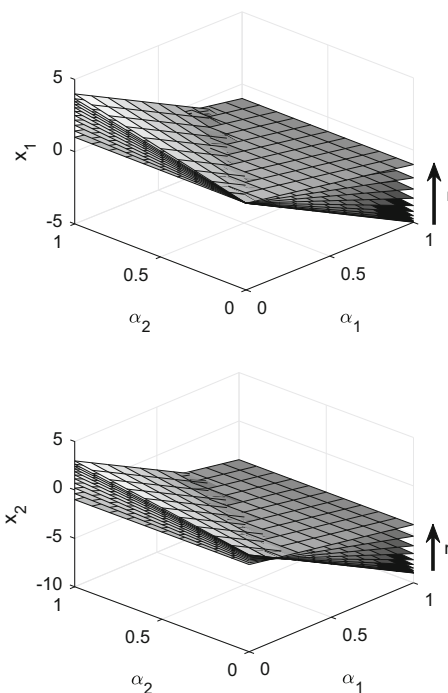


Fig. 9 Three-dimensional horizontal fuzzy number solution of fuzzy system of linear equations, where $r \in [0 : 0.1 : 1]$

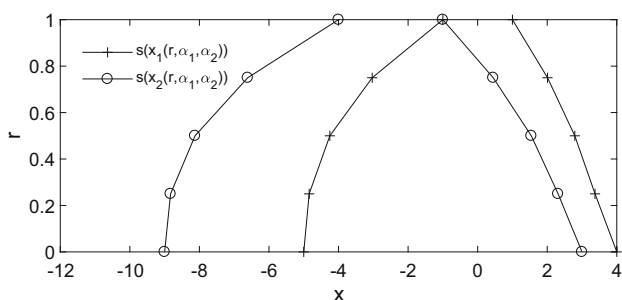


Fig. 10 Span of horizontal fuzzy number solution of fuzzy system of linear equations

$$\begin{aligned}
 x_1 - x_2 &= [1 - 2\alpha_1(3 - r^2 - r^3) + \alpha_2(3 - \sqrt{r} - 2r^2)] \\
 &\quad - [-r^2 - 3\alpha_1(3 - r^2 - r^3) \\
 &\quad + \alpha_2(3 - \sqrt{r} - 2r^2)] = 1 + r^2 \\
 &\quad + \alpha_1(3 - r^2 - r^3) = (1 + r^2, 4 - r^3), \\
 3x_1 - 2x_2 &= 3[1 - 2\alpha_1(3 - r^2 - r^3) + \alpha_2(3 - \sqrt{r} - 2r^2)] \\
 &\quad - 2[-r^2 - 3\alpha_1(3 - r^2 - r^3) \\
 &\quad + \alpha_2(3 - \sqrt{r} - 2r^2)] = 3 + 2r^2 \\
 &\quad + \alpha_2(3 - \sqrt{r} - 2r^2) = (3 + 2r^2, 6 - \sqrt{r}).
 \end{aligned}$$

The spans (indirect solutions) of the solution present (33) and are shown in graphical form in Fig. 10,

$$\begin{aligned}
 s(x_1(r, \alpha_1, \alpha_2)) &= (-5 + 2r^2 + 2r^3, 4 - \sqrt{r} - 2r^2) \\
 s(x_2(r, \alpha_1, \alpha_2)) &= (-9 + 2r^2 + 3r^3, 3 - \sqrt{r} - 3r^2) \quad (33)
 \end{aligned}$$

where $r \in [0, 1]$.

Using solution (32) of fuzzy system of linear equations, any crisp solution with appropriate crisp system of linear equations can be generated. For example for $r = 1$, $\alpha_1 = 0$ and $\alpha_2 = 1$ the crisp solution equals $x_1 = 1$ and $x_2 = -1$. From HFN notation of fuzzy system of linear equations (31), the crisp system of linear equations for the considered crisp solution can be obtained: $x_1 - x_2 = 2$, $3x_1 - 2x_2 = 5$.

5 Conclusions

The paper presents a method with horizontal fuzzy number for solving fuzzy systems of linear equations. The definition and the examples of horizontal fuzzy number where border values are nonlinear were given. Also it was proved that operations on horizontal fuzzy numbers hold basic algebraic properties. The solved numerical examples show that the solution is multidimensional and satisfies fuzzy linear systems. The solutions calculated with horizontal fuzzy numbers are associated, which means that every crisp value of a solution corresponds with other crisp values of solutions.

The solution obtained with the proposed method with HFN variables gives a possibility to calculate a crisp solution and a crisp system of linear equations which corresponds with this solution. This possibility is impossible with the methods that use vertical fuzzy numbers, because in these cases the results do not bring any information about crisp solution. Also it was proved that the obtained multidimensional solution satisfies the fuzzy system of linear equations. Moreover, in the analyzed examples it was shown that the obtained results as fuzzy numbers in (Friedman et al. 1998; Behera and Chakraverty 2012) are not full solutions and are underestimated or overestimated.

Existing methods for solving fuzzy linear systems assume that the direct solution is a vertical fuzzy number. This assumption leads to paradoxes. In the article, it was shown that the result in a vertical fuzzy number form is only the secondary information (indicator, indirect solution) of the solution being a multidimensional set. The method with horizontal fuzzy numbers gives a correct full multidimensional solution.

To show the applicability of the proposed method also the practical fuzzy system of linear equations and system with a fuzzy number where borders are nonlinear were solved. The proposed method solves the fuzzy linear system given in different equivalent forms. The methods with vertical fuzzy numbers for equivalent forms of fuzzy linear systems give different results.

Compliance with ethical standards

Conflict of interest The author declares that there is no conflict of interests regarding the publication of this paper.

Ethical approval This article does not contain any studies with human participants or animals performed by the author.

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