The Euler Characteristic Is the Unique Locally Determined Numerical Invariant of Finite Simplicial Complexes which Assigns the Same Number to Every Cone

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Abstract. We show that a proof given by Levitt in [L] suffices to prove the stronger theorem stated in the title.

Let \( S \) denote the set of finite simplicial complexes, where we consider two complexes to be the same if they are combinatorially equivalent. Say a function

\[
\rho: S \rightarrow \mathbb{R}
\]

is \textit{locally determined} if there is a function

\[
d: S \rightarrow \mathbb{R}
\]

such that, for all finite simplicial complexes \( M \),

\[
\rho(M) = \sum_{v \text{ of } M} d(\text{link}(v))
\]

(\text{where link}(v) denotes the link of } v \text{ in } M, \text{ and we sometimes write link}(v, M) \text{ if there is possible confusion}). In this case, we write \( \rho = \rho_d \). We are primarily interested in those locally determined functions with topological significance. For example, a natural problem is to find all locally defined functions which only depend on the homotopy type of the underlying complex. This problem was posed, and completely answered, by

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Levitt. In [L] it is proved that

**Theorem A** [L]. The only locally determined real-valued homotopy invariants of finite simplicial complexes are multiples of the Euler characteristic.

It is easy to see that the Euler characteristic is, in fact, locally determined. Namely, define

$$e: S \to \mathbb{R}$$

by setting, for any finite simplicial complex $N$,

$$e(N) = 1 + \sum_i \frac{(-1)^{i+1}}{i+2} \text{(number of } i \text{ - simplices in } N).$$

Then, for any finite simplicial complex $M$,

$$\rho_e(M) = \chi(M),$$

where

$$\chi(M) = \sum_i (-1)^i \text{(number of } i \text{ - simplices in } M)$$

is the Euler characteristic of $M$.

Levitt also proves the following refinement of Theorem A.

**Theorem B** [L]. Let $\rho_d$ be any locally determined real-valued invariant of finite simplicial complexes which is a homotopy invariant, then there is a constant $c$ so that $\rho_d M = c \cdot e$.

Our goal in this paper is simply to point out that an argument used in [L] suffices to give a simple direct proof of the stronger theorem stated in the title. We quickly review some standard definitions. For any simplicial complex $N$, cone$N$ is the join of $N$ and a single vertex. We say a simplicial complex $M$ is a cone if $M = \text{cone}(N)$ for some $N$. In the proof of Theorem B in [L], Levitt essentially proves the following theorem.

**Theorem 1.** Let $\rho_d$ be any locally determined real-valued function on the set of finite simplicial complexes. Suppose there is a constant $c$ so that $\rho_d(M) = c$ whenever $M$ is a cone. Then $d = c \cdot e$.

Before proving this theorem we note that since all cones are homotopy equivalent, every locally determined homotopy invariant satisfies the hypotheses of Theorem 1. Thus Theorem 1 implies Theorem B (and hence Theorem A). It is interesting to observe that Theorem 1 implies that the converse is true. Namely, any locally determined invariant which gives the same answer for every cone is a homotopy invariant. Moreover, every cone is collapsible, and hence all cones are simple-homotopy equivalent (see [C] for definitions). Therefore, the above theorem implies the following strengthening of Levitt’s Theorem A.
Corollary 2. The only locally determined real-valued simple-homotopy invariants of finite simplicial complexes are multiples of the Euler characteristic.

We actually prove something somewhat more general. Namely, \( \rho_d \) need not be defined on all finite simplicial complexes. Say 

\[
T \subset S
\]

is star-closed if:

(i) \( T \) contains the simplicial complex consisting of a single vertex.
(ii) \( T \) is closed under taking stars. That is, if \( M \in T \) and \( v \) is a vertex in \( M \), then \( \text{star}(v, M) \in T \).

We will prove the following generalization of Theorem 1.

Theorem 3. Let \( T \subset S \) be any star-closed set, and let \( \rho_d \) be any locally determined real-valued function on \( T \) such that there is a constant \( c \) with \( \rho_d(M) = c \) for every cone \( M \in T \). Then \( d = c \cdot e \).

The set of combinatorial manifolds with (possibly empty) boundary is star-closed. Hence we have the corollary

Corollary 4. Let \( \rho_d \) be any locally determined function on the set of combinatorial manifolds with (possibly empty) boundary such that there is a constant \( c \) such that \( \rho_d(M) = c \) whenever \( M \) is a cone, then \( d = c \cdot e \).

This implies Theorem \( \text{A}' \) of [L], where there is the stronger hypothesis that \( \rho_d \) be a PL homeomorphism invariant.

Proof of Theorem 3. (The reader should note that we are following the proof of Levitt’s Theorem B in [L], and we include this proof merely for the sake of being self-contained.) Let \( \rho_d \) satisfy the hypotheses of the theorem. Without loss of generality, we may assume that, for every cone \( M \), \( \rho_d(M) = 1 \). (Namely, if \( c \neq 0 \) we can consider \( \rho_{\bar{d}} \) where \( \bar{d}' = (1/c)d \). If \( c = 0 \) we can consider \( \rho_{\bar{d}} \) where \( \bar{d}' = d + e \).) Our goal is to show that, for each finite simplicial complex \( M \in T \) and every vertex \( v \in M \), \( d(\text{link}(v)) = e(\text{link}(v)) \).

The proof will be by induction on the number \( \ell(v) \) (or \( \ell(v, M) \)) of simplices in the link of \( v \) in \( M \).

Suppose \( \ell(v) = 0 \). Let \( M' = \text{star}(v) \). Then \( M' \) consists only of the vertex \( v \), and by hypothesis \( M' \in T \). Since \( \text{star}(M') \) is a cone (on the empty set), we have

\[
1 = \rho_d(M') = d(\text{link}(v, M')).
\]

However, \( \text{link}(v, M') = \emptyset = \text{link}(v, M) \), so \( d(\text{link}(v, M)) = 1 \). On the other hand, \( 1 = e(\text{link}(v)) \). Thus, if \( \ell(v) = 0 \), \( d(\text{link}(v)) = e(\text{link}(v)) \).

Suppose \( d(\text{link}(v)) = e(\text{link}(v)) \) whenever \( v \in M \in T \) satisfies \( \ell(v) \leq k - 1 \). Let \( M \in T \) and let \( v \) be a vertex in \( M \) satisfying \( \ell(v, M) = k \). Let \( M' = \text{star}(v, M) \). Then
$M' \in T$ is a cone, so

$$1 = \rho_d(M') = \sum_{\text{vertices } w \text{ in } M'} d(\text{link}(w, M')).$$

(1)

It is easy to see that, for each vertex $w$ of $M'$, $\ell(w, M') \leq k$. Moreover, if $\ell(w, M') = k$, then $\text{link}(w, M')$ is combinatorially equivalent to $\text{link}(v, M') = \text{link}(v, M)$. Define

$$V_1 = \{\text{vertices } w \text{ in } M' \text{ with } \ell(w, M') < k\},$$

$$V_2 = \{\text{vertices } w \text{ in } M' \text{ with } \ell(w, M') = k\}$$

and let $n = \#V_2$. By induction, for all $w \in V_1$,

$$d(\text{link}(w, M')) = e(\text{link}(w, M')).$$

For all $w \in V_2$,

$$d(\text{link}(w, M')) = d(\text{link}(v, M')) = d(\text{link}(v, M)).$$

Substituting into formula (1), we have

$$1 = n \cdot d(\text{link}(v, M)) + \sum_{w \in V_1} e(\text{link}(w, M')).$$

On the other hand, we already know

$$1 = \chi(M') = \sum_{\text{vertices } w \text{ in } M'} e(\text{link}(w, M')) = n \cdot e(\text{link}(v, M)) + \sum_{w \in V_1} e(\text{link}(w, M')).$$

Hence we must have

$$d(\text{link}(v, M)) = e(\text{link}(v, M)),$$

as desired.

\[ \square \]

References


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