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Empty Monochromatic Simplices

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Abstract Let S be a k-colored (finite) set of n points in $\mathbb{R}^d$, $d \geq 3$, in general position, that is, no $(d+1)$ points of S lie in a common $(d-1)$-dimensional hyperplane. We count the number of empty monochromatic d-simplices determined by S, that is, simplices which have only points from one color class of S as vertices and no points of S in their interior. For $3 \leq k \leq d$ we provide a lower bound of $\Omega(n^{d-k+1+2^{-d}})$ and strengthen this to $\Omega(n^{d-2/3})$ for $k=2$.

On the way we provide various results on triangulations of point sets in $\mathbb{R}^d$. In particular, for any constant dimension $d \geq 3$, we prove that every set of n points (n sufficiently large), in general position in $\mathbb{R}^d$, admits a triangulation with at least $dn + \Omega(\log n)$ simplices.

Keywords Colored point sets · Empty monochromatic simplices · High dimensional triangulations · Simplicial complex

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1 Introduction

Let S be a finite set of n points in $\mathbb{R}^d$. Throughout this paper we assume that S is in general position, that is, no $(d+1)$ points of S lie in a common $(d-1)$-dimensional hyperplane. A more formal definition of "general position" can be found in Sect. 2.1. A subset S' of S is said to be empty if $\mathrm{Conv}(S') \cap S = S'$, where $\mathrm{Conv}(S')$ denotes the convex hull of S' (please see Sect. 2.1 for a detailed definition). A *k-coloring* of S is a partition of S into k nonempty sets called *color classes*. A subset of S is said to be *monochromatic* if all its elements belong to the same color class. A d-simplex is the d-dimensional version of a triangle.

The problem of determining the minimum number of empty triangles any set of n points in general position in the plane contains, has been widely studied [3, 9, 12, 19] and also the higher dimensional version of the problem has been considered [2]. In [12] it is noted that every set of n points in general position in $\mathbb{R}^d$ determines at least $\binom{n-1}{d} = \Omega(n^d)$ empty simplices. In [2] it is shown that in a random set of n points in $\mathbb{R}^d$—chosen uniformly at random on a convex, bounded set with nonempty interior—the expected number of empty simplices is at most $c_d\binom{n}{d} = O(n^d)$ (where c_d is a constant depending only on d).

The colored version of the problem has been introduced in [7] and was studied in [1], where $\Omega(n^{5/4})$ empty monochromatic triangles were shown to exist in every two colored set of n points in general position in the plane. This has later been improved to $\Omega(n^{4/3})$ in [15]. Further, arbitrarily large 3-colored sets without empty monochromatic triangles were shown to exist in the plane in [7].

In this paper we study the higher dimensional version of this colored variant. We generalize both, the dimension and the number of colors. Specifically, we consider the problem of counting the number of empty monochromatic d-simplices in a k-colored set of points in $\mathbb{R}^d$.

It is shown in [18] that every sufficiently large 4-colored set of points in general position in $\mathbb{R}^3$ contains an empty monochromatic tetrahedron. This is done by showing that any set of n points in general position in $\mathbb{R}^3$ can be triangulated with more than $3n$ tetrahedra.

The problem of triangulating a set of points with many simplices is intimately related to the problem of determining the minimum number of empty simplices in k-colored sets of points in $\mathbb{R}^d$. Remarkably this problem has received little attention. For the special case of $\mathbb{R}^3$, it even has been pronounced "the least significant" among the four extremal (maxmax, maxmin, minmax, minmin) problems in [10]. Consequently, only a trivial lower bound and an upper bound of $\frac{7}{15}n^2 + O(n)$ has been shown there. Nevertheless, in [5] sets of n points in $\mathbb{R}^d$ in general position are shown such that every triangulation of them has $O(n^{5/3})$ tetrahedra, for points in $\mathbb{R}^3$, and in general $O(n^{1/d+\lceil d/2\rceil\cdot(d-1)/d})$ simplices for points in $\mathbb{R}^d$. Furthermore, in [6] this minmax problem is stated as Open Problem 11 in the section "Extremal Number of Special Subconfigurations".

In this direction we give the first, although not asymptotically improving, nontrivial lower bound and show that for $d \geq 3$ every set of n points in general position in $\mathbb{R}^d$ admits a triangulation of at least $dn + \Omega(\log n)$ simplices, for n sufficiently large and d constant.

Table 1 Number of empty monochromatic d-simplices in k-colored sets of n (sufficiently large) points in $\mathbb{R}^d$

	$d=2$	$d\geq 3$
$k=2$	$\Omega(n^{4/3})$ ([15] and Theorem 33)	$\Omega(n^{d-2/3})$ (Theorem 33)
$3\leq k\leq d$	–	$\Omega(n^{d-k+1+2^{-d}})$ (Theorem 29)
$k=d+1$	none ([7])	at least linear[a] (Corollary 25)
$k\geq d+2$	none ([7])	unknown

[a]The linear lower bound for $d=3$ and $k=4$ has been proved already in [18]

The paper is organized as follows: in Sect. 2, known results on simplicial complexes and triangulations are reviewed; in Sect. 3, new results on simplicial complexes and triangulations are presented; using these results in Sect. 4, high-dimensional versions of the Order and Discrepancy Lemmas used in [1] are shown; in Sect. 5, the lemmas of Sect. 4 are put together to prove various results on the minimum number of empty monochromatic simplices in sets of points in $\mathbb{R}^d$. Our results are summarized in Table 1.

To provide a better general view on the paper, and especially to visualize the interrelation between the many lemmas, we present a "roadmap" through the paper in Fig. 1. The lemmas (and theorems and corollaries) are shown in boxes, given with their number, if applicable a special name, and the necessary preconditions. Main results have a bold frame. The lemmas are grouped to reflect their topical and section correlation. An arrow from a Lemma A to a Lemma B depicts, that the proof of Lemma B uses the result of Lemma A. Theorem 35 (stated and proven in the Conclusions section) is not depicted in Fig. 1, as there is no interrelation with other lemmas.

2 Preliminaries

In this section, following the notation of Matoušek [13], we state the definitions and known results regarding simplicial complexes and triangulations, that will be needed throughout the paper. Note that in this paper we consider the number, d, of dimensions and also the number, k, of different colors as constants. This means, that d and k do not depend on the size, n, of the considered finite set of points. But of course the required minimum size of the point set might depend on d and k.

2.1 Simplicial Complexes

Let X be a finite set of points in $\mathbb{R}^d$. The *convex hull* of X, denoted by $\mathrm{Conv}(X)$, is the intersection of all convex sets containing X. Alternatively, it may be defined as the set of points that can be written as a *convex combination* of elements of X:

$$\mathrm{Conv}(X)=\left\{\sum_{i=1}^{|X|}\alpha_i x_i \;\middle|\; x_i\in X,\alpha_i\in\mathbb{R},\alpha_i\geq 0,\sum_{i=1}^{|X|}\alpha_i=1\right\}.$$

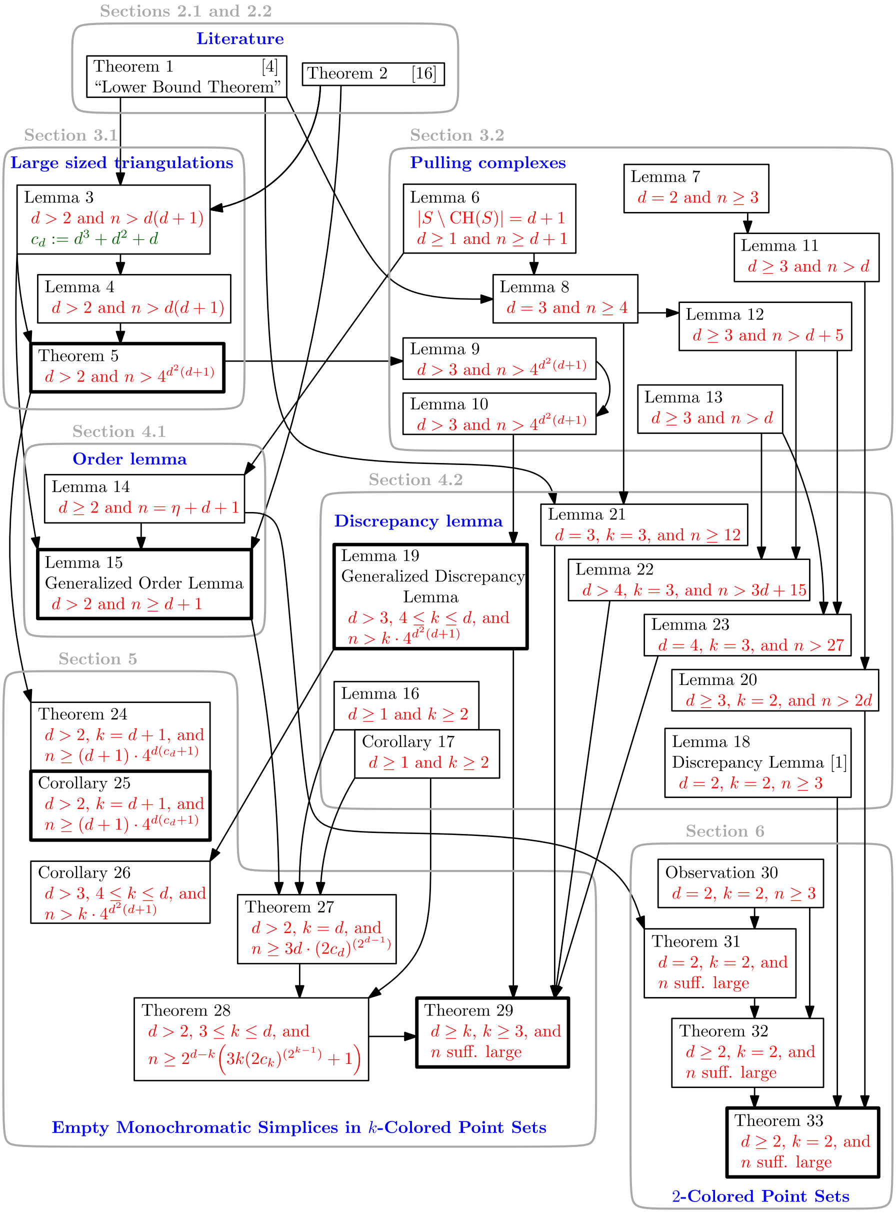

Fig. 1 Roadmap through the paper

We denote the boundary of $\mathrm{Conv}(X)$ with $\mathrm{CH}(X)$. A point of X is said to be a *convex hull point* if it lies in $\mathrm{CH}(X)$, otherwise it is called an *interior point*. A point set X is said to be in *convex position* if every point of X is a convex hull point.

Let $\mathbf{0}$ denote the d-dimensional zero vector. A set of points $\{x_1, \ldots, x_n\}$ in $\mathbb{R}^d$ is said to be *affinely dependent* if there exist real numbers $(\alpha_1, \ldots, \alpha_n)$, not all zero, such that $\sum_{i=1}^{n} \alpha_i x_i = \mathbf{0}$ and $\sum_{i=1}^{n} \alpha_i = 0$. Otherwise $\{x_1, \ldots, x_n\}$ is said to be *affinely independent*. A set of points X in $\mathbb{R}^d$ is in *general position* if each subset of X with at most $d+1$ elements is affinely independent.

A *simplex* σ is the convex hull of a finite affinely independent set A in $\mathbb{R}^d$. The elements of A are called the *vertices* of σ. If A consists of $m+1$ elements, we say that σ is of *dimension* $\dim \sigma := m$ or that σ is an m-simplex. The convex hull of any subset of vertices of a simplex σ is called a *face* of σ. A face of a simplex is again a simplex.

A *simplicial complex* $\mathcal{K}$ is a family of simplices satisfying the following properties:

- Each face of every simplex in $\mathcal{K}$ is also a simplex of $\mathcal{K}$.
- The intersection of two simplices $\sigma_1, \sigma_2 \in \mathcal{K}$ is either empty or a face of both, σ_1 and σ_2.

The *vertex set* of $\mathcal{K}$ is the union of the vertex sets of all simplices in $\mathcal{K}$. We say that $\mathcal{K}$ is of *dimension* m, if m is the highest dimension of any of its simplices. The *size* of a simplicial complex of dimension m is the number of its simplices of dimension m. The j-*skeleton* of $\mathcal{K}$ is the simplicial complex consisting of all simplices of $\mathcal{K}$ of dimension at most j. Hence the 0-skeleton is the vertex set of $\mathcal{K}$.

We now turn to finite sets of points in general position in $\mathbb{R}^d$. Let S be such a set of n elements. Note that since S is in general position we may regard $\mathrm{CH}(S)$ as a simplicial complex in a natural way. Such simplicial complexes are called *simplicial polytopes*. It is known that every simplicial polytope satisfies:

Theorem 1 ([4] Lower Bound Theorem) *For a simplicial polytope of dimension d let f_m be the number of its m-dimensional faces. Then*:

- $f_m \geq \binom{d}{m} f_0 - \binom{d+1}{m+1} m$ *for all* $1 \leq m \leq d-2$ *and*
- $f_{d-1} \geq (d-1) f_0 - (d+1)(d-2)$.

Note that in the Lower Bound Theorem, the word *dimension* refers to the dimension of the simplicial polytope as a polytope. Hence, a three dimensional simplicial polytope would be a two dimensional simplicial complex.

2.2 Triangulations

A *triangulation* $\mathcal{T}$ of S is a simplicial complex such that its vertex set is S and the union of all simplices of $\mathcal{T}$ is $\mathrm{Conv}(S)$. This definition generalizes the usual definition of triangulations of planar point sets. The *size* of a triangulation is the number of its d-simplices. The minimum size of any triangulation of S is known to be $n-d$. We explicitly mention this result for further use:

Theorem 2 [16] *Every triangulation of a set of n points in general position in $\mathbb{R}^d$ has size at least $n-d$.*

We will use the following operation of *inserting a point p into a triangulation $\mathcal{T}$* frequently: Let p be a point not in S but such that $S \cup \{p\}$ is also in general position,

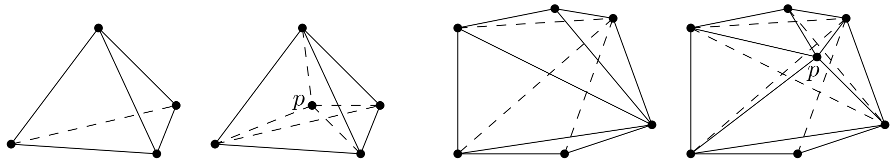

Fig. 2 Example in $\mathbb{R}^3$ for inserting a point p into a triangulation with (*left*) p inside the convex hull and (*right*) p outside the convex hull

and let $\mathcal{T}$ be a triangulation of S. If p lies in $\mathrm{Conv}(S)$ then p is contained in a unique d-simplex σ of $\mathcal{T}$. We remove σ from $\mathcal{T}$ and replace it with the $(d+1)$ d-simplices formed by taking the convex hull of p and each of the $(d+1)$ $(d-1)$-dimensional faces of σ. If, on the other hand, p lies outside $\mathrm{Conv}(S)$ then a set $\mathcal{F}$ of $(d-1)$-dimensional faces of $\mathrm{CH}(S)$ is visible from p. We get a set of d-simplices formed by taking the convex hull of p and each face of $\mathcal{F}$, and add these simplices to $\mathcal{T}$. In either case the resulting family of simplices is a triangulation of $S \cup \{p\}$ (see Fig. 2).

We distinguish two different types of triangulations of a set S of n points in general position in $\mathbb{R}^d$ by their construction: A *shelling triangulation* of S is constructed as follows. Choose any ordering $p_1, p_2, \dots, p_n$ of the elements of S and let $S_i = \{p_1, \dots, p_i\}$. Start by triangulating S_{d+1} with only one simplex. Afterwards, for every $i > d+1$ create the triangulation of S_i by inserting p_i into the triangulation of S_{i-1}. The final triangulation of this process, that of S_n, is a shelling triangulation. A *pulling triangulation* of S is constructed by choosing (if it exists) a point p of S, such that $S \setminus ((\mathrm{CH}(S) \cap S) \cup \{p\}) = \emptyset$. Then $S \setminus \{p\}$ is in convex position. Construct a d-simplex with p and each $(d-1)$-dimensional face of $\mathrm{CH}(S)$ that does not contain p.

3 Results on Triangulations and Simplicial Complexes

In this section we present some results on triangulations and simplicial complexes that will be needed later, but are also of independent interest. We begin by showing that every point set can be triangulated with a "large number" of simplices. We use the same strategy as in [18].

3.1 Large Sized Triangulations

First we prove a lower bound on the size for a triangulation of a convex set of points, by building a shelling triangulation for a special sequence of points.

Lemma 3 *Every set S of $n > d(d+1)$ points in convex and general position in $\mathbb{R}^d$ $(d > 2)$ has a triangulation of size at least $(d+1)n - c_d$, with $c_d = d^3 + d^2 + d$.*

Proof The 1-skeleton of $\mathrm{CH}(S)$ is a graph of n vertices and, by the Lower Bound Theorem (Theorem 1, for $m = 1$), of at least $dn - \frac{d(d+1)}{2}$ edges. Therefore, as long as $n > d(d+1)$ there will be a vertex of degree at least $2d$ in this graph.

Set $S_n := S$ and let G_n be the 1-skeleton (as a graph) of $\mathrm{CH}(S_n)$. In general once S_i is defined, let G_i be the 1-skeleton (as a graph) of $\mathrm{CH}(S_i)$. Let p_i be a vertex of degree at least $2d$ in G_i, with $n \geq i > d(d+1)$. We construct a shelling triangulation $\mathcal{T}_n$ of S_n, with size as claimed in the lemma.

Starting with S_n, iteratively remove a vertex p_i from S_i, i.e., $S_{i-1} = S_i \setminus \{p_i\}$. Observe that $|S_i| = i$. The iteration stops with $S_{i-1} = S_{d(d+1)}$ as $i > d(d+1)$. Construct an arbitrary shelling triangulation $\mathcal{T}_{d(d+1)}$ of $S_{d(d+1)}$. By Theorem 2, $\mathcal{T}_{d(d+1)}$ has size at least $d(d+1) - d = d^2$. Complete $\mathcal{T}_{d(d+1)}$ to a shelling triangulation $\mathcal{T}_n$ by inserting the points p_i in reversed order of their removal (i from $d(d+1)+1$ to n).

We prove that with each inserted point p_i at least $(d+1)$ d-simplices are added to the triangulation. Let ϱ_i be the degree of p_i in G_i and recall that $\varrho_i \geq 2d$. Consider the neighbors $q_1, \ldots, q_{\varrho_i}$ of p_i in G_i. Let Π be a $(d-1)$-dimensional hyperplane separating p_i and S_{i-1}, and let $q'_1, \ldots, q'_{\varrho_i}$ be the set of intersections of Π with the lines spanned by p_i and each of $q_1, \ldots, q_{\varrho_i}$.

Note that $q'_1, \ldots, q'_{\varrho_i}$ are a set of points in convex position in $\mathbb{R}^{d-1}$ and that the $(d-1)$-dimensional faces of $\mathrm{CH}(S_{i-1})$, which are visible to p_i, project to a triangulation of $q'_1, \ldots, q'_{\varrho_i}$ in Π. By Theorem 2, every triangulation of ϱ_i points in $\mathbb{R}^{d-1}$ has size at least $\varrho_i - (d-1) \geq d+1$. Thus, at least $(d+1)$ d-simplices are added when inserting p_i. Hence, the constructed shelling triangulation $\mathcal{T}_n$ has size at least $d^2 + (d+1)(n - d(d+1))$, which is the claimed bound of $(d+1)n - c_d$, with $c_d = d(d+1)^2 - d^2 = d^3 + d^2 + d$. □

Using this result it is easy to give a lower bound on the triangulation size for general point sets in dependence of a certain subset property.

Lemma 4 *Let S be a set of points in general position in $\mathbb{R}^d$ $(d > 2)$. Let P and Q be two disjoint sets, such that $S = P \cup Q$ and Q is in convex position. If $|Q| > d(d+1)$ then there exists a triangulation of S of size at least $(d+1)|Q| + |P| - c_d$, with c_d defined as in Lemma 3.*

Proof By Lemma 3, Q has a triangulation $\mathcal{T}$ of size at least $(d+1)|Q| - c_d$, if $|Q| > d(d+1)$. Inserting each point of P into $\mathcal{T}$ adds at least one d-simplex to $\mathcal{T}$ per point in P. This results in a triangulation of S with size at least $(d+1)|Q| + |P| - c_d$. □

Combining the previous two lemmas we prove a new non-trivial lower bound for the size of triangulations with an additive logarithmic term.

Theorem 5 *Every set S of $n > 4^{d^2(d+1)}$ points in general position in $\mathbb{R}^d$ $(d > 2)$, with h convex hull points, has a triangulation of size at least $dn + \max\{h, \frac{\log_2(n)}{2d}\} - c_d$, with c_d as defined in Lemma 3.*

Proof Let P be the set of convex hull points of S. We distinguish two cases:

- $|P| = h > \log_2(n)/(2d)$. By Lemma 3, there exists a triangulation of P of size at least $(d+1)h - c_d$, as $h > d(d+1)$. Insert the remaining $n - h$ points of $S \setminus P$ into this triangulation. Since these points are inside $\mathrm{Conv}(P)$, each of them contributes with d additional d-simplices to the final triangulation. Therefore, the resulting triangulation has size at least $dn + h - c_d > dn + \frac{\log_2(n)}{2d} - c_d$.

- $|P| = h \le \log_2(n)/(2d)$. By the Erdős–Szekeres Theorem (see [11]) and its best known upper bound (see [17]), S contains a subset Q of at least
$$|Q| > \frac{\log_2(n)}{2} > d(d+1)$$
points in convex position. Let $P' = P \setminus Q$. Apply Lemma 4 to obtain a triangulation $\mathcal{T}$ of $P' \cup Q$ of size at least $(d+1)|Q| + |P'| - c_d$. Insert the remaining points of $S \setminus (P' \cup Q)$ into $\mathcal{T}$. Since these inserted points are in the interior of $\mathrm{Conv}(P' \cup Q)$, each of them contributes with d additional d-simplices to the final triangulation. Therefore, this triangulation has size at least
$$\begin{aligned} &d\big(n - |Q| - \left|P'\right|\big) + (d+1)|Q| + \left|P'\right| - c_d \\ &\quad = dn + |Q| - (d-1)\left|P'\right| - c_d > dn + \frac{\log_2(n)}{2} - (d-1)\frac{\log_2(n)}{2d} - c_d \\ &\quad \ge dn + \frac{\log_2(n)}{2} - \frac{\log_2(n)}{2} + \frac{\log_2(n)}{2d} - c_d, \end{aligned}$$
which is $dn + \frac{\log_2(n)}{2d} - c_d$. □

Note that c_d in Lemma 3 can be improved to $\frac{d(d+1)^2}{2} + \frac{d(d+1)}{12} = \frac{d^3}{2} + \frac{13d^2}{12} + \frac{7d}{12}$. Instead of stopping the process at $S_{d(d+1)}$, we continue the iteration using a vertex of degree $2d-1$ for S_i with $d(d+1) \ge i > \frac{d(d+1)}{2}$, a vertex of degree $2d-2$ for S_i with $\frac{d(d+1)}{2} \ge i > \frac{d(d+1)}{3}$, and so on. This way, instead of a triangulation of size at least d^2, we can guarantee a triangulation $\mathcal{T}_{d(d+1)}$ of size at least
$$\begin{aligned} &\sum_{i=1}^{d}\left(\big(2d - i - (d-1)\big)\frac{d(d+1)}{i(i+1)}\right) \\ &\qquad \ge \frac{3}{4}d(d+1)^2 - d(d+1)\cdot\min\left\{\frac{d+1}{4} + \frac{1}{12}, \ln(d+1)\right\}, \end{aligned}$$
which results in the claimed improvement of c_d for $d \ge 3$. Thus, for $d = 3$ Theorem 5 can be improved to $3n + \max\{h, \frac{\log_2 n}{6}\} - 25$. Note that this corresponds to the bound from [10], that every set of n points in general position in $\mathbb{R}^3$, with h convex hull points, has a tetrahedralization of size at least $3(n-h) + 4h - 25$ for $h \ge 13$.

3.2 Pulling Complexes

Let S be a set of n points in general position in $\mathbb{R}^d$. In this section we present lemmas that allow us to construct d-simplicial complexes of large size on S, such that their d-simplices contain a pre-specified subset of S in their vertex set. We begin with a result for point sets, whose convex hull is a simplex.

Lemma 6 *Let S be a set of $n \ge d+1$ points in general position in $\mathbb{R}^d$ $(d \ge 1)$, such that* $\mathrm{Conv}(S)$ *is a d-simplex. For every convex hull point p of S, there exists a triangulation of S such that $(d-1)n - d^2 + 2$ of its d-simplices have p as a vertex.*

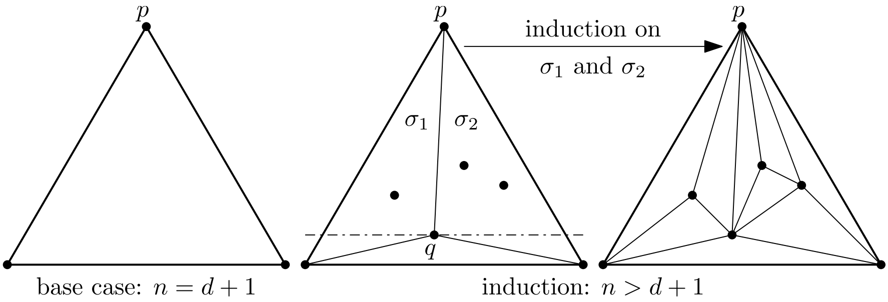

Fig. 3 Illustration of the proof of Lemma 6 for $n = 7$ and $d = 2$

Proof We use induction on n, see Fig. 3 for an illustration. Start with a triangulation $\mathcal{T}$ consisting only of the d-simplex $\mathrm{Conv}(S)$. If $n = (d + 1)$, $\mathcal{T}$ is a triangulation with $(d - 1)n - d^2 + 2 = (d - 1)(d + 1) - d^2 + 2 = 1$ empty simplex containing p as vertex.

Assume $n > d + 1$. Let q be the interior point of S closest to the only face of $\mathrm{Conv}(S)$ not incident to p. (If there exist more then one such closest points, then choose an arbitrary one of them as q.) Insert q into $\mathcal{T}$. This results in a triangulation of size $(d + 1)$ in which d of its d-simplices, $\sigma_1 \dots \sigma_d$, have p as a vertex. Note that the remaining d-simplex does not contain any point of S in its interior. We apply induction on $\sigma_1 \dots \sigma_d$. Let n_i $(1 \le i \le d)$ be the number of points of S interior to σ_i, $\sum_{i=1}^{d} n_i = n - (d + 1) - 1$. For each σ_i we obtain a triangulation such that $(d - 1)(n_i + (d + 1)) - d^2 + 2$ of its d-simplices have p as a vertex. The union of the triangulations of each σ_i is a triangulation of S, and

$$\sum_{i=1}^{d} \big((d-1)n_i + (d-1)(d+1) - d^2 + 2\big)$$

$$= (d-1)\sum_{i=1}^{d}(n_i) + d = (d-1)n - d^2 + 2$$

of its d-simplices have p as a vertex. □

The next three lemmas give, for every point of a general point set in $\mathbb{R}^d$, a lower bound on the number of interior disjoint d-simplices incident to p, for the cases $d = 2$, $d = 3$, and $d > 3$, respectively.

Lemma 7 *Let S be a set of $n \ge 3$ points in general position in $\mathbb{R}^2$. For every point p of S there exists a 2-dimensional simplicial complex of size at least $(n - 2)$ and such that all of its triangles have p as a vertex.*

Proof Do a cyclic ordering around p of the points of $S \setminus \{p\}$. Construct a 2-dimensional simplicial complex by forming a triangle with p and every two consecutive elements determining an angle less than π. This simplicial complex has at least $n - 2$ triangles and they all contain p as a vertex. □

Lemma 8 *Let S be a set of $n \geq 4$ points in general position in $\mathbb{R}^3$. For every point p of S there exists a triangulation of S such that at least*:

- $2n - 6$ *of its* 3*-simplices have p as a vertex*, *if p is an interior point of S.*
- $2n - \varrho(p) - 4$ *of its* 3*-simplices contain p as a vertex*, *if p is a convex hull point of S and $\varrho(p)$ is its degree in the* 1*-skeleton of* $\mathrm{CH}(S)$.

Proof Let S' be the set of convex hull points of S and $n' = |S'|$. Construct a pulling triangulation $\mathcal{T}'$ w.r.t. p of $S' \cup \{p\}$. By definition all 3-simplices of $\mathcal{T}'$ contain p as a vertex. For every 3-simplex σ of $\mathcal{T}'$, let η be the number of points of S interior to σ. By applying Lemma 6 we can triangulate σ, such that $2(\eta + 4) - 7 = 2\eta + 1$ of its 3-simplices have p as a vertex. Repeat this for every 3-simplex of $\mathcal{T}'$, to obtain a triangulation $\mathcal{T}$ of S.

By Theorem 1, $\mathrm{CH}(S)$ has (at least) $2n' - 4$ faces. For $d = 3$ this lower bound is tight.

- If p is an interior point of S, $\mathcal{T}'$ contains a 3-simplex for every face of $\mathrm{CH}(S)$. Therefore, summing over all these faces we get
$$\sum(2\eta + 1) = 2\sum(\eta) + 2n' - 4 = 2\bigl(n - n' - 1\bigr) + 2n' - 4 = 2n - 6$$
of the 3-simplices in $\mathcal{T}$ have p as a vertex.
- If p is a convex hull point of S, $\mathcal{T}'$ contains a 3-simplex for every face of $\mathrm{CH}(S)$ not having p as a vertex. This is equal to $2n' - 4 - \varrho(p)$, where $\varrho(p)$ is the degree of p in the 1-skeleton of $\mathrm{CH}(S)$. Therefore,
$$\begin{aligned}\sum(2\eta + 1) &= 2\sum(\eta) + 2n' - 4 - \varrho(p) = 2\bigl(n - n'\bigr) + 2n' - 4 - \varrho(p)\\ &= 2n - \varrho(p) - 4\end{aligned}$$
of the 3-simplices in $\mathcal{T}$ have p as a vertex. □

Lemma 9 *Let S be a set of $n > 4^{d^2(d+1)}$ points in general position in $\mathbb{R}^d$ $(d > 3)$. For every point p of S*, *there exists a d-dimensional simplicial complex $\mathcal{K}$ with vertex set S*, *such that $\mathcal{K}$ has size strictly larger than $(d - 1)n + \frac{\log_2 n}{2(d-1)} - 2c_{d-1}$ and all its d-simplices have p as a vertex*, *with c_d defined as in Lemma 3.*

Proof For every point $q \in S$ distinct from p let r_q be the infinite ray with origin p and passing through q. Let Π be a halving $(d - 1)$-dimensional hyperplane of S passing through p, not containing any other point of S. Further, let Π_1 and Π_2 be two $(d - 1)$-dimensional hyperplanes parallel to Π containing $\mathrm{Conv}(S)$ between them and not parallel to any of the rays r_q.

Project from p every point in $S \setminus \{p\}$ to Π_1 or Π_2, in the following way. Every ray r_q intersects either Π_1 or Π_2 in a point q'. Take q' to be the projection of q from p. Let S'_1 and S'_2 be these projected points in Π_1 and Π_2, respectively. Both, S'_1 and S'_2, are sets of points in general position in $\mathbb{R}^{d-1}$, with $|S'_1| = n_1 = \lfloor\frac{n-1}{2}\rfloor$ and $|S'_2| = n_2 = \lceil\frac{n-1}{2}\rceil$, where both, n_1 and n_2, are strictly larger than $4^{(d-1)^2 d}$.

By Theorem 5, there exist triangulations $\mathcal{T}_1$ of S'_1 and $\mathcal{T}_2$ of S'_2 of size at least $(d - 1)n_1 + \frac{\log_2(n_1)}{2(d-1)} - c_{d-1}$ and $(d - 1)n_2 + \frac{\log_2(n_2)}{2(d-1)} - c_{d-1}$, respectively. Consider the simplicial complexes $\mathcal{K}_1$ and $\mathcal{K}_2$ that arise from replacing every point q' in a simplex of $\mathcal{T}_1$ or $\mathcal{T}_2$ with its preimage q in $S \setminus \{p\}$. The $(d-1)$-simplices of $\mathcal{K}_1$ and $\mathcal{K}_2$ are all visible from p. Hence, we obtain a simplicial complex $\mathcal{K}$ of dimension d, by taking the convex hull of p and each $(d-1)$-simplex of $\mathcal{K}_1$ and $\mathcal{K}_2$. Obviously, all d-simplices of $\mathcal{K}$ contain p as a vertex. The size of $\mathcal{K}$ is at least

$$\begin{aligned}
&(d-1)(n_1+n_2) + \frac{\log_2(n_1)+\log_2(n_2)}{2(d-1)} - 2c_{d-1}\\
&\quad = (d-1)n - (d-1) + \frac{\log_2(n_1 n_2)}{2(d-1)} - 2c_{d-1}\\
&\quad \geq (d-1)n + \frac{\log_2(\frac{n(n-2)}{4})}{2(d-1)} - 2c_{d-1} - (d-1)\\
&\quad = (d-1)n + \frac{\log_2(n)}{2(d-1)} - 2c_{d-1} + \frac{\log_2(n-2) - \log_2(4)}{2(d-1)} - (d-1)\\
&\quad > (d-1)n + \frac{\log_2(n)}{2(d-1)} - 2c_{d-1} + \frac{2d^2(d+1)-1-2}{2(d-1)} - (d-1)\\
&\quad > (d-1)n + \frac{\log_2(n)}{2(d-1)} - 2c_{d-1} + (d-1)^2.
\end{aligned}$$

This is strictly larger than $(d-1)n + \frac{\log_2(n)}{2(d-1)} - 2c_{d-1}$. □

We now consider not only one point, but subsets X of point sets in $\mathbb{R}^d$ $(d > 3)$. The next three lemmas, applicable for $1 \leq |X| \leq d-3$, $|X| = d-1$, and $|X| = d-2$, respectively, provide lower bounds on the number of interior disjoint d-simplices which all share the points in X. Note that the second lemma in the row, Lemma 11, is true for $d \geq 3$.

Lemma 10 *Let S be a set of $n > 4^{d^2(d+1)}$ points in general position in $\mathbb{R}^d$ $(d > 3)$. For every set $X \subset S$ of r points $(1 \leq r \leq d-3)$, there exists a d-dimensional simplicial complex $\mathcal{K}$ with vertex set S, such that $\mathcal{K}$ has size strictly larger than $(d-r)n + \frac{\log_2 n}{2(d-r)} - 2c_{d-1}$ and all its d-simplices have X in their vertex set, with c_d defined as in Lemma 3.*

Proof The case $r = 1$ is shown in Lemma 9. Thus assume that $r > 1$. Let Π be the $(r-1)$-dimensional hyperplane containing X and let Π' be a $(d-(r-1))$-dimensional hyperplane orthogonal to Π. Project S orthogonally to Π', and let S' be the resulting image. The set X is projected to a single point p_X in Π'. Obviously $|S'| = n - r + 1 > 4^{(d-r+1)^2(d-r+2)}$. Apply Lemma 9 to S', and obtain a $(d-r+1)$-dimensional simplicial complex $\mathcal{K}'$ with vertex set S' of size at least

$$\begin{aligned}
&(d-r)(n-r+1) + \frac{\log_2(n-r+1)}{2(d-r)} - 2c_{d-r}\\
&\quad = (d-r)n + \frac{\log_2 n}{2(d-r)} - 2c_{d-r} - (d-r)(r-1) + \frac{\log_2(1-\frac{r-1}{n})}{2(d-r)}\\
&\quad > (d-r)n + \frac{\log_2 n}{2(d-r)} - 2c_{d-1},
\end{aligned}$$

such that all the $(d-r+1)$-simplices of $\mathcal{K}'$ have p_X as a vertex.

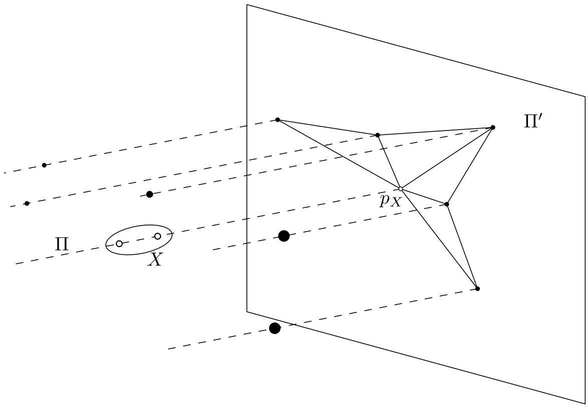

Fig. 4 Illustration of the proof of Lemma 11 for $n = 7$ and $d = 3$

To get $\mathcal{K}$ from $\mathcal{K}'$, lift each simplex of $\mathcal{K}'$ to the convex hull of the preimage of its vertex set. Thus $\mathcal{K}$ is a d-dimensional simplicial complex with vertex set S and size larger than $(d-r)n + \frac{\log_2 n}{2(d-r)} - 2c_{d-1}$. As all $(d-r+1)$-simplices of $\mathcal{K}'$ have p_X as a vertex, each d-simplex of $\mathcal{K}$ has X as a vertex subset. □

Lemma 11 *Let S be a set of $n > d$ points in general position in $\mathbb{R}^d$ $(d \geq 3)$. For every set $X \subset S$ of $d-1$ points, there exists a d-dimensional simplicial complex $\mathcal{K}$ with vertex set S, such that $\mathcal{K}$ has size at least $n-d$, and all d-simplices of $\mathcal{K}$ have X in their vertex set.*

Proof The proof is similar to that of Lemma 10, with the difference that we cannot apply Lemma 9.

Let Π be the $(d-2)$-dimensional hyperplane containing X and let Π' be a 2-dimensional hyperplane orthogonal to Π. Project S orthogonally to Π', and let S' be its image. The set X is projected to a single point p_X of Π' (see Fig. 4). Obviously $|S'| = n - d + 2 \geq 3$. Apply Lemma 7 to S', and obtain a 2-dimensional simplicial complex $\mathcal{K}'$ with vertex set S' of size at least $(n-d+2)-2 = n-d$, such that all triangles of $\mathcal{K}'$ have p_X as a vertex.

To get $\mathcal{K}$ from $\mathcal{K}'$, lift each triangle of $\mathcal{K}'$ to the convex hull of the preimage of its vertex set. Thus $\mathcal{K}$ is a d-dimensional simplicial complex with vertex set S and size $n-d$. Since all triangles of $\mathcal{K}'$ have p_X as a vertex, all d-simplices of $\mathcal{K}$ have X as a vertex subset. □

Note that Lemmas 10 and 11 leave a gap for $r = d-2$. In this case, the point set is projected to a 3-dimensional hyperplane, where the guaranteed bounds on incident 3-simplices vary significantly for extremal and interior points, see Lemma 8. Thus we make a weaker statement for this case, which will turn out to be sufficient anyhow.

Lemma 12 *Let S be a set of $n > d+5$ points in general position in $\mathbb{R}^d$ $(d > 3)$. Let $X \subset S$ be a subset of $d-2$ points. Denote with Π the $(d-3)$-dimensional hyperplane containing X and with Π' a 3-dimensional hyperplane orthogonal to Π.*

Project S orthogonally to Π', and let S' be the resulting image. The set X is projected to a single point p_X in Π'.

If p_X is an interior point of S', then there exists a d-dimensional simplicial complex $\mathcal{K}$ with vertex set S, such that $\mathcal{K}$ is of size at least $2n-2d-8$ and all d-simplices of $\mathcal{K}$ have X in their vertex set.

Proof Obviously $|S'| = n-d-1 > 4$. As p_X is assumed to be an interior point of S', apply Lemma 8 to S', and obtain a 3-dimensional simplicial complex $\mathcal{K}'$ with vertex set S' of size at least $2(n-d-1)-6 = 2n-2d-8$, such that all the 3-simplices of $\mathcal{K}'$ have p_X as a vertex.

To get $\mathcal{K}$ from $\mathcal{K}'$, lift each 3-simplex of $\mathcal{K}'$ to the convex hull of the preimage of its vertex set. Thus $\mathcal{K}$ is a d-dimensional simplicial complex with vertex set S and size at least $2n-2d-8$. As all 3-simplices of $\mathcal{K}'$ have p_X as a vertex, each d-simplex of $\mathcal{K}$ has X as a vertex subset. □

In the light of the previous lemma it is of interest to know the conditions for a subset X of S in $\mathbb{R}^d$ $(d > 3)$ to project to an interior point of S'. We make the following statement.

Lemma 13 *Let S be a set of $n > d$ points in general position in $\mathbb{R}^d$ $(d > 3)$ and let $X \subset S$ be a subset of $d-2$ points. With Π denote the $(d-3)$-dimensional hyperplane spanned by X and with Π' a 3-dimensional hyperplane orthogonal to Π. Project S orthogonally to Π' and denote with S' the resulting image of S and with p_X the image of X, respectively. Then p_X is an extremal point of S' if and only if* $\mathrm{Conv}(X)$ *is a $(d-3)$-dimensional facet of* $\mathrm{CH}(S)$.

Proof If $\mathrm{Conv}(X)$ is a $(d-3)$-dimensional facet of $\mathrm{CH}(S)$, then there exists a $(d-1)$-dimensional hyperplane Π_T "tangential" to $\mathrm{Conv}(S)$, containing only X and having all other points of S on one side. Thus, there exists a "tangential" plane $\Pi'_T = \Pi_T \cap \Pi'$ at p_X, such that all points of $S' \setminus \{p_X\}$ are on one side of Π'_T. Hence, p_X is extremal.

If $\mathrm{Conv}(X)$ is not a $(d-3)$-dimensional facet of $\mathrm{CH}(S)$, then all $(d-1)$-dimensional hyperplanes containing X have points of S on both sides, and therefore p_X is not extremal in S'. Assume the contrary: at least one $(d-1)$-dimensional hyperplane, Π_T, containing X exists, such that all points of $S \setminus X$ are on one side of Π_T. Then we could tilt Π_T keeping all of its contained points and consuming the ones it hits while tilting, until Π_T contains d points; i.e., until Π_T consumed two more points, q_1 and q_2. Still all points of S, except the ones contained in Π_T, are on one side of Π_T. Observe that a hyperplane spanned by d points (in a point set in general position) is a $(d-1)$-dimensional hyperplane. Hence, Π_T has become a supporting hyperplane of a $(d-1)$-dimensional facet, $\mathrm{Conv}(X \cup \{q_1, q_2\})$, of $\mathrm{CH}(S)$. As the convex hull of every subset of $(X \cup \{q_1, q_2\})$ is a facet of $\mathrm{CH}(S)$, this is a contradiction to the assumption that $\mathrm{Conv}(X)$ is not a $(d-3)$-dimensional facet of $\mathrm{CH}(S)$. □

4 Higher Dimensional Versions of the Order and Discrepancy Lemmas

We prove the higher dimensional versions of the Order and Discrepancy Lemmas from [1]. The proofs are essentially the same as in the planar case, with the difference that some facts we used in the plane are now provided by the lemmas in the previous sections.

Recall that in a partial order a *chain* is a set of pairwise comparable elements, whereas an *antichain* is a set of pairwise incomparable elements.

4.1 Order Lemma

Lemma 14 *Let S be a set of $\eta + d + 1$ points ($\eta \geq 0$) in general position in $\mathbb{R}^d$ ($d \geq 2$), such that* $\mathrm{Conv}(S)$ *is a d-simplex. Then there exists a triangulation of S, such that at least $(d-1)\eta + \eta^{(2^{(1-d)})} + 1$ of its d-simplices contain a convex hull point of S.*

Proof Let I be the set of the η interior points of S. Let $\mathcal{F} = \{F_1, \ldots, F_{d+1}\}$ be the set of the $(d-1)$-dimensional faces of $\mathrm{CH}(S)$. For each $F_i \in \mathcal{F}$ we define a partial order $\leq_{F_i}$ on I. We say that $p \leq_{F_i} q$ $(p, q \in I)$ if p is in the interior of the d-simplex $\mathrm{Conv}(F_i \cup \{q\})$. Our goal is to obtain a "long" chain C^* with respect to some $F^* \in \mathcal{F}$ such that $|C^*| \geq \eta^{(2^{(1-d)})}$.

By Dilworth's Theorem [8] w.r.t. $\leq_{F_{d+1}}$, there exists a chain or an antichain C_{d+1} in I of size at least $\sqrt{\eta} \geq \eta^{(2^{(1-d)})}$. If C_{d+1} is a chain then we obtain $C^* = C_{d+1}$, $|C^*| \geq \eta^{(2^{(1-d)})}$, and $F^* = F_{d+1}$. Otherwise, we iteratively apply Dilworth's Theorem w.r.t. $\leq_{F_i}$ to the points of the antichain C_{i+1}, i from d down to 3, to obtain a chain or antichain C_i of size at least $\sqrt{|C_{i+1}|} = \eta^{(2^{(i-d-2)})}$. As soon as C_i is a chain, terminate with $C^* = C_i$, $F^* = F_i$, and $|C^*| \geq \eta^{(2^{(1-d)})}$. Otherwise, the process ends with the antichain C_3 of size at least $\eta^{(2^{(1-d)})}$. Observe that an antichain with respect to all but two faces is a chain with respect to the remaining two faces. Hence, $C^* = C_3$, $F^* = F_2$, with $|C^*| \geq \eta^{(2^{(1-d)})}$.

Let $p_1 \leq_{F^*} \cdots \leq_{F^*} p_r$ $(r = |C^*|)$ be the points of C^*. Construct a triangulation $\mathcal{T}$ of S, starting with $\mathcal{T}$ consisting only of the d-simplex $\mathrm{Conv}(S)$. Then insert the points of C^* into $\mathcal{T}$ in the order $p_r, \ldots, p_1$. With each step one d-simplex is replaced by $(d+1)$ new ones. This results in an intermediate triangulation $\mathcal{T}$ of $((S \cap \mathrm{CH}(S)) \cup \{p_1 \ldots p_r\})$ consisting of $(dr + 1)$ many d-simplices, each of which having at least one point in $\mathrm{CH}(S)$ as a vertex.

Let σ_i, $1 \leq i \leq dr + 1$, be the d-simplices of $\mathcal{T}$, let η_i be the number of interior points of σ_i, and let p_i be a vertex of σ_i that is also in $\mathrm{CH}(S)$. By Lemma 6 there exists a triangulation of $S \cap \sigma_i$ such that $(d-1)(\eta_i + d + 1) - d^2 + 2 = (d-1)\eta_i + 1$ of its d-simplices have p_i as a vertex. Therefore, the remaining points can be inserted into $\mathcal{T}$, such that at least

$$\sum_{i=1}^{dr+1} \big((d-1)\eta_i + 1\big) = (d-1)(\eta - r) + (dr + 1) = (d-1)\eta + r + 1$$

of the d-simplices of $\mathcal{T}$ have at least one point in $\mathrm{CH}(S)$. Since $r \geq \eta^{(2^{(1-d)})}$, at least $(d-1)\eta + \eta^{(2^{(1-d)})} + 1$ many d-simplices have at least one point in $\mathrm{CH}(S)$. □

We are now able to prove the high-dimensional variation of the "Order Lemma":

Lemma 15 (Generalized Order Lemma) *Let S be a set of $n \geq d+1$ points in general position in $\mathbb{R}^d$ $(d > 2)$ with $h = |S \cap \mathrm{CH}(S)|$. Then there exists a triangulation of S, such that at least $(d-1)n + (n-h)^{(2^{(1-d)})} + 2h - c_d$ of its d-simplices have at least one point in* $\mathrm{CH}(S)$, *with c_d as defined in Lemma 3.*

Proof Let $S' = S \cap \mathrm{CH}(S)$ be the set of convex hull points of S. If $h > d(d+1)$, then by Lemma 3 there exists a triangulation of S' of size $\tau \geq (d+1)h - c_d$. If $h \leq d(d+1)$, then by Theorem 2 any triangulation of S' has size at least $\tau \geq h - d = (d+1)h - dh - d \geq (d+1)h - c_d$.

Let σ_i, $1 \leq i \leq \tau$, be the d-simplices of the triangulation of S', and let η_i be the number of interior points of σ_i. By Lemma 14 there exists a triangulation $\mathcal{T}_i$ of $S \cap \sigma_i$, such that at least $(d-1)\eta_i + \eta_i^{(2^{(1-d)})} + 1$ of the d-simplices of $\mathcal{T}_i$ have at least one point in $\mathrm{CH}(S \cap \sigma_i)$. In total we obtain a triangulation $\mathcal{T}$ of S, such that at least

$$\begin{aligned}\sum_{i=1}^{\tau}\big((d-1)\eta_i + \eta_i^{(2^{(1-d)})} + 1\big) &\geq (d-1)\sum_{i=1}^{\tau}(\eta_i) + \left(\sum_{i=1}^{\tau}\eta_i\right)^{(2^{(1-d)})} + \tau \\ &\geq (d-1)(n-h) + (n-h)^{(2^{(1-d)})} + (d+1)h - c_d \\ &= (d-1)n + (n-h)^{(2^{(1-d)})} + 2h - c_d\end{aligned}$$

of the d-simplices of $\mathcal{T}$ have at least one point in $\mathrm{CH}(S)$. □

4.2 Discrepancy Lemma

Let S be a k-colored set of n points in general position in $\mathbb{R}^d$ and let $S_1, S_2, \ldots, S_k$ be its color classes. Recall that we consider k and d to be constants w.r.t. n, i.e., k and d are independent of n. We define the *discrepancy* $\delta(S)$ of S to be the sum of differences between the sizes of its biggest chromatic class and the remaining classes. Let $S_{\max}$ be the chromatic class with the maximum number of elements. Then $\delta(S) = \sum(|S_{\max}| - |S_i|) = (k-1)|S_{\max}| - |S \setminus S_{\max}| = k|S_{\max}| - n$. Further, we denote with $S_{\min}$ the chromatic class with the least number of elements.

We start with two statements describing the interaction of $\delta(S)$, $S_{\max}$, and $S_{\min}$.

Lemma 16 *Let S be a k-colored set of n points in $\mathbb{R}^d$. Let $f_{(n,d,k)}$ be some function on k, d, and n. If $|S_{\min}| \leq \frac{n}{k} - (k-1) \cdot f_{(n,d,k)}$ then $|S_{\max}| \geq \frac{n}{k} + f_{(n,d,k)}$, and $\delta(S) \geq k \cdot f_{(n,d,k)}$.*

Proof From $|S_{\min}| \leq \frac{n}{k} - (k-1) \cdot f_{(n,d,k)}$ we get

$$|S \setminus S_{\min}| = n - |S_{\min}| \geq n - \frac{n}{k} + (k-1) \cdot f_{(n,d,k)} = (k-1) \cdot \left(\frac{n}{k} + f_{(n,d,k)}\right).$$

As there exist $(k-1)$ color classes besides $|S_{\min}|$, all not bigger than $|S_{\max}|$, we have $|S_{\max}| \geq \frac{n}{k} + f_{(n,d,k)}$. This leads to

$$\delta(S) = k|S_{\max}| - n \geq k\left(\frac{n}{k} + f_{(n,d,k)}\right) - n = k \cdot f_{(n,d,k)}. \qquad \square$$

The following corollary is a direct consequence of Lemma 16.

Corollary 17 *Let S be a k-colored set of n points in $\mathbb{R}^d$. Let $f_{(n,d,k)}$ be some function on k, d, and n. If $\delta(S) < k \cdot f_{(n,d,k)}$ then $|S_{\min}| > \frac{n}{k} - (k-1) \cdot f_{(n,d,k)}$.*

The previous two technical statements will be needed for the Theorems 27 and 28. For the sake of completeness we state the "original" Discrepancy Lemma for $d = k = 2$ from [1].

Lemma 18 (Discrepancy Lemma [1]) *Let S be a 2-colored set of $n \geq 3$ points in general position in $\mathbb{R}^2$, such that $\delta(S) \geq 2$. Then S determines at least $\frac{\delta(S)-2}{6}(n + \delta(S))$ empty monochromatic triangles.*

In the following we proof the high-dimensional variation of this "Discrepancy Lemma":

Lemma 19 (Generalized Discrepancy Lemma) *Let S be a k-colored set of $n > k \cdot 4^{d^2(d+1)}$ points in general position in $\mathbb{R}^d$, with $d \geq k > 3$. Then S determines $\Omega(n^{d-k+1} \cdot (\delta(S) + \log n))$ empty monochromatic d-simplices.*

Proof Consider a subset X of $d-k+1$ points of $S_{\max}$. From the requirements of the lemma we have $d > 3$, $|S_{\max}| \geq \lceil \frac{n}{k} \rceil > 4^{d^2(d+1)}$, and $1 \leq |X| \leq d-3$. Thus we may apply Lemma 10 to X which guarantees the existence of a d-simplicial complex $\mathcal{K}_X$ with vertex set $S_{\max}$, such that $\mathcal{K}_X$ has size strictly larger

$$\begin{aligned}
&\big(d - (d-k+1)\big)|S_{\max}| + \frac{\log_2 |S_{\max}|}{2(d-(d-k+1))} - 2c_{d-1} \\
&\quad = (k-1)|S_{\max}| + \frac{\log_2 |S_{\max}|}{2(k-1)} - 2c_{d-1}
\end{aligned}$$

and all d-simplices of $\mathcal{K}_X$ have X in their vertex set. Since every point of $S \setminus S_{\max}$ is in at most one d-simplex of $\mathcal{K}_X$, $\mathcal{K}_X$ contains at least $\delta(S) + \frac{\log_2 |S_{\max}|}{2(k-1)} - 2c_{d-1}$ empty monochromatic d-simplices.

We do this counting for each of the $\binom{|S_{\max}|}{d-k+1}$ subsets of $(d-k+1)$ points of $S_{\max}$, and overcount each empty monochromatic d-simplex at most $\binom{d+1}{d-k+1}$ times. Hence, in total we get $\frac{\binom{|S_{\max}|}{d-k+1}}{\binom{d+1}{d-k+1}} \cdot (\delta(S) + \frac{\log_2 |S_{\max}|}{2(k-1)} - 2c_{d-1})$ empty monochromatic d-simplices. As $|S_{\max}| \geq \lceil \frac{n}{k} \rceil$, and d, c_{d-1} (see Lemma 3), and k are constant w.r.t. n, we get $\Omega(n^{d-k+1} \cdot (\delta(S) + \log n))$ empty monochromatic d-simplices in S. $\square$

Observe, that this "Generalized Discrepancy Lemma" is not applicable for small values of k and d. With the 2-colored variant in $\mathbb{R}^2$ already provided in Lemma 18 ([1]), we generalize it to $\mathbb{R}^d$ in the next lemma.

Lemma 20 *Let S be a 2-colored set of $n > 2d$ points in general position in $\mathbb{R}^d$, with $d \geq 3$. Then S determines $\Omega(n^{d-1} \cdot \delta(S))$ empty monochromatic d-simplices.*

Proof Consider a subset X of $d-1$ points of $S_{\max}$. From the requirements of the lemma we have $d \geq 3$, $|S_{\max}| \geq \lceil \frac{n}{2} \rceil > d$, and $|X| = d-1$. Thus we may apply Lemma 11 to X which guarantees the existence of a d-simplicial complex $\mathcal{K}_X$ with vertex set $S_{\max}$, such that $\mathcal{K}_X$ has size at least $|S_{\max}| - d$ and all d-simplices of $\mathcal{K}_X$ have X in their vertex set. Since every point of $S \setminus S_{\max}$ is in at most one d-simplex of $\mathcal{K}_X$, $\mathcal{K}_X$ contains at least $\delta(S) - d$ empty monochromatic d-simplices.

We do this counting for each of the $\binom{|S_{\max}|}{d-1}$ subsets of $(d-1)$ points of $S_{\max}$, and overcount each empty monochromatic d-simplex at most $\binom{d+1}{d-1} = \binom{d+1}{2}$ times. Hence, in total we get $\frac{\binom{|S_{\max}|}{d-1}}{\binom{d+1}{2}} \cdot (\delta(S) - d)$ empty monochromatic d-simplices. As $|S_{\max}| \geq \lceil \frac{n}{2} \rceil$, and d is constant w.r.t. n, we get $\Omega(n^{d-1} \cdot \delta(S))$ empty monochromatic d-simplices in S. □

The still missing 3-colored case of the "Discrepancy Lemma" turns out to be quite difficult. In the remaining three lemmas of this section we will first prove the variant for $\mathbb{R}^3$, then give a general bound for $\mathbb{R}^d$ and $d > 4$, and lastly providing the missing case of $\mathbb{R}^4$.

Lemma 21 *Let S be a 3-colored set of $n \geq 12$ points in general position in $\mathbb{R}^3$. Then S determines at least $\frac{\delta(S)-10}{12} \cdot n + 3$ empty monochromatic 3-simplices.*

Proof Let p be a point of $S_{\max}$. From the requirements of the lemma we have $d = 3$ and $|S_{\max}| \geq \lceil \frac{n}{3} \rceil \geq 4$. Thus we may apply Lemma 8 to p which guarantees the existence of a 3-simplicial complex $\mathcal{K}_p$ with vertex set $S_{\max}$, such that all 3-simplices of $\mathcal{K}_p$ have p as a vertex, and $\mathcal{K}_p$ has size at least

- $2|S_{\max}| - 6$ if p is an interior point of $S_{\max}$ and
- $2|S_{\max}| - \varrho(p) - 4$ if p is a convex hull point of $S_{\max}$ and $\varrho(p)$ is the degree of p in the 1-skeleton of $\mathrm{CH}(S_{\max})$.

Since every point of $S \setminus S_{\max}$ is in at most one 3-simplex of $\mathcal{K}_p$, $\mathcal{K}_p$ contains at least $\delta(S) - 6$ empty monochromatic d-simplices if p is an interior point of $S_{\max}$, and $\delta(S) - \varrho(p) - 4$ empty monochromatic d-simplices if p is a convex hull point of $S_{\max}$.

We do this counting for each point in $S_{\max}$, and overcount each empty monochromatic 3-simplex at most 4 times. Denote with h the number of convex hull points of $S_{\max}$. We know from Theorem 1 that summing over all convex hull points of $S_{\max}$ we have $\sum \varrho(p) = 2 \cdot (3h - 6) = 6h - 12$. Hence, in total we get

$$\frac{1}{4} \cdot \Big(\big(\delta(S) - 6\big) \cdot \big(|S_{\max}| - h\big) + \big(\delta(S) - 4\big) \cdot h - \sum \varrho(p) \Big)$$
$$= \frac{1}{4} \cdot \big(\big(\delta(S) - 6\big) \cdot |S_{\max}| - 4h + 12 \big) \geq \frac{\delta(S) - 10}{4} \cdot |S_{\max}| + 3$$

empty monochromatic 3-simplices. As $|S_{\max}| \geq \lceil \frac{n}{3} \rceil$, we get at least $\frac{\delta(S)-10}{12} \cdot n + 3$ empty monochromatic 3-simplices in S. □

Lemma 22 *Let S be a 3-colored set of $n > 3d + 15$ points in general position in $\mathbb{R}^d$ $(d > 4)$. Then S determines $\Omega(n^{d-2} \cdot \delta(S))$ empty monochromatic d-simplices.*

Proof Consider a subset X of $d-2$ points of $S_{\max}$. Note that $|S_{\max}| \geq \lceil \frac{n}{3} \rceil$. Denote with Π the $(d-3)$-dimensional hyperplane containing X and with Π' a 3-dimensional hyperplane orthogonal to Π. Project $S_{\max}$ orthogonally to Π', and let $S'_{\max}$ be the resulting image. The set X is projected to a single point p_X in Π'.

By Lemma 13, p_X is an extremal point of $S'_{\max}$ only if $\mathrm{Conv}(X)$ is a $(d-3)$-dimensional facet of $\mathrm{CH}(S_{\max})$. By the upper bound theorem [14], the convex hull of a point set in $\mathbb{R}^d$ has size at most $\Theta(n^{\lfloor \frac{d}{2} \rfloor})$. Obviously, this bound applies to the number of all ξ-dimensional facets, $1 \leq \xi < d$, of $\mathrm{CH}(S_{\max})$, as d is constant, i.e., independent of $|S_{\max}|$.

On the other hand, the total number of different subsets of $d-2$ points of $S_{\max}$ is $\binom{|S_{\max}|}{d-2} \geq \binom{\frac{n}{3}}{d-2} = \Theta(n^{d-2})$. As $d-2 > \lfloor \frac{d}{2} \rfloor$ for $d > 4$, there exist

$$\Theta\left(n^{d-2}\right) - \Theta\left(n^{\lfloor \frac{d}{2} \rfloor}\right) = \Theta\left(n^{d-2}\right)$$

different subsets X, such that p_X is an interior point of $S'_{\max}$.

For each such subset X apply Lemma 12, as $|S_{\max}| \geq \lceil \frac{n}{3} \rceil > d + 5$ and $d > 3$. This guarantees for each X the existence of a d-simplicial complex $\mathcal{K}_X$ with vertex set $S_{\max}$, such that $\mathcal{K}_X$ has size at least $2|S_{\max}| - 2d - 8$ and all d-simplices of $\mathcal{K}_X$ have X in their vertex set. Since every point of $S \setminus S_{\max}$ is in at most one d-simplex of $\mathcal{K}_X$, $\mathcal{K}_X$ contains at least $\delta(S) - 2d - 8$ empty monochromatic d-simplices.

As we can do this counting for $\Theta(n^{d-2})$ different subsets, and overcount each empty monochromatic d-simplex at most $\binom{d+1}{d-2}$ times, we get at least

$$\Theta\left(n^{d-2}\right) \cdot \left(\delta(S) - 2d - 8\right)$$

empty monochromatic d-simplices in total. □

For $\mathbb{R}^4$ the simple asymptotic counting from the previous proof does not work. We have to take a more detailed look.

Lemma 23 *Let S be a 3-colored set of $n > 27$ points in general position in $\mathbb{R}^4$. Then S determines $\Omega(n^2 \cdot \delta(S))$ empty monochromatic 4-simplices.*

Proof Note that $|S_{\max}| \geq \lceil \frac{n}{3} \rceil$. Recall that the size of $\mathrm{CH}(S_{\max})$ is bounded by $O(|S_{\max}|^{\lfloor \frac{4}{2} \rfloor}) = O(|S_{\max}|^2)$. Thus, there are also at most quadratically many edges on $\mathrm{CH}(S_{\max})$. We distinguish two cases depending on the number of edges on $\mathrm{CH}(S_{\max})$.

(1) If less than quadratically many edges are on $\mathrm{CH}(S_{\max})$, then there exist $\Theta(|S_{\max}|^2)$ many pairs of points of $S_{\max}$ that are no 1-dimensional facet of $\mathrm{CH}(S_{\max})$. Consider a subset X of 2 points of $S_{\max}$, forming such a pair. Denote with Π the line containing X and with Π' a 3-dimensional hyperplane orthogonal to Π. Project $S_{\max}$ orthogonally to Π', and let $S'_{\max}$ be the resulting image. The set X is projected to a single point p_X in Π'. By Lemma 13, p_X is an interior point of $S'_{\max}$, as $\mathrm{Conv}(X)$ is not an edge of $\mathrm{CH}(S_{\max})$. Apply

Lemma 12 to X, as $|S_{\max}| \geq \lceil \frac{n}{3} \rceil > d+5$ and $d=4>3$. This guarantees for X the existence of a 4-simplicial complex $\mathcal{K}_X$ with vertex set $S_{\max}$, such that $\mathcal{K}_X$ has size at least $2|S_{\max}|-16$ and all 4-simplices of $\mathcal{K}_X$ have X in their vertex set. Since every point of $S \setminus S_{\max}$ is in at most one 4-simplex of $\mathcal{K}_X$, $\mathcal{K}_X$ contains at least $\delta(S)-16$ empty monochromatic 4-simplices. As we can do this counting for $\Theta(|S_{\max}|^2)=\Theta(n^2)$ different subsets, and overcount each empty monochromatic 4-simplex at most $\binom{5}{2}$ times, we get at least $\Theta(n^2)\cdot(\delta(S)-16)$ empty monochromatic 4-simplices in total.

(2) If there are $\Theta(|S_{\max}|^2)$ many edges on $\mathrm{CH}(S_{\max})$, then there are also $\Theta(|S_{\max}|^2)$ many tetrahedra on $\mathrm{CH}(S_{\max})$ (because the number of tetrahedra is at least a sixth of the number of edges), and obviously $|S_{\max} \cap \mathrm{CH}(S_{\max})| = \Theta(|S_{\max}|)$. For a point $p \in S_{\max}$ make a pulling triangulation $\mathcal{K}_p$ of $(S_{\max} \cap \mathrm{CH}(S_{\max})) \cup \{p\}$. Inserting the remaining points of $S_{\max}$ into $\mathcal{K}_p$ does not decrease the number of 4-simplices in $\mathcal{K}_p$, which have p as a vertex. Remove all 4-simplices from $\mathcal{K}_p$ that don't have p as a vertex. Then $\mathcal{K}_p$ is a 4-dimensional simplicial complex, such that every 4-simplex has p as a vertex and $\mathcal{K}_p$ is of size
   (a) $\Theta(|S_{\max}|^2)$, if p is an interior point of $S_{\max}$, or
   (b) $\Theta(|S_{\max}|^2)-\varrho(p)$, if p is an extremal point of $S_{\max}$, where $\varrho(p)$ is the number of tetrahedra in $\mathrm{CH}(S_{\max})$, having p as a vertex.

   For case (b) observe, that $\sum_{p\in(S_{\max}\cap \mathrm{CH}(S_{\max}))} \varrho(p) = 4\cdot\Theta(|S_{\max}|^2)$. Thus on average, at least $\Omega(|S_{\max}|)$ points of $S_{\max}$ have at most $O(|S_{\max}|)$ incident tetrahedra in $\mathrm{CH}(S_{\max})$. Hence, $\Theta(|S_{\max}|^2)-\varrho(p)=\Theta(|S_{\max}|^2)$ for $\Theta(|S_{\max}|)$ points in $S_{\max}$, which we can all choose for the point p.

   All 4-simplices of $\mathcal{K}_p$ are empty of points of $S_{\max}$ by construction. Since every point of $S \setminus S_{\max}$ is in at most one 4-simplex of $\mathcal{K}_p$, $\mathcal{K}_p$ contains at least $\Theta(|S_{\max}|^2) - 2|S_{\max}| + \delta(S) - 2d - 8 = \Theta(n^2)$ empty monochromatic 4-simplices. Note that $\delta(S)=O(n)$. As we can do this counting for $\Theta(|S_{\max}|)=\Theta(n)$ different points, and overcount each empty monochromatic 4-simplex at most 5 times, we get $\Omega(n^3) \geq \Omega(n^2\cdot\delta(S))$ empty monochromatic 4-simplices in total. □

With this last lemma in a line of five lemmas in total and including [1], we now have a "Discrepancy Lemma" type of statement for all k-colored point sets in $\mathbb{R}^d$, for every combination of $d \geq 2$ and $2 \leq k \leq d$.

5 Empty Monochromatic Simplices in k-Colored Point Sets

In this section we present our results on the minimum number of empty monochromatic d-simplices determined by any k-colored set of n points in general position in $\mathbb{R}^d$. Some first bounds follow directly from the results in the previous section.

Theorem 24 *Every $(d+1)$-colored set S of $n \geq (d+1)\cdot 4^{d(c_d+1)}$ points in general position in $\mathbb{R}^d$ $(d>2)$, c_d defined as in Lemma 3, determines an empty monochromatic d-simplex.*

Proof Let $S_{\max}$ be the largest chromatic class of S. From the requirements of the theorem we have $d > 2$ and $|S_{\max}| \geq \lceil \frac{n}{d+1} \rceil \geq 4^{d(c_d+1)} = 4^{d^4+d^3+d^2+d} > 4^{d^2(d+1)}$. By Theorem 5, $S_{\max}$ has a triangulation $\mathcal{T}$ of size at least $d|S_{\max}| + \frac{\log_2 |S_{\max}|}{2d} - c_d$. All the d-simplices of $\mathcal{T}$ are of the same color and empty of points of $S_{\max}$. There are at most $d|S_{\max}|$ points in S of the remaining colors, and each of these points is in at most one d-simplex of $\mathcal{T}$.

Therefore, at least $\frac{\log_2 |S_{\max}|}{2d} - c_d \geq \frac{2d(c_d+1)}{2d} - c_d = 1$ of the d-simplices of $\mathcal{T}$ are empty of points of S. □

Note that $d > 2$ is crucial here, as for $d = 2$ Devillers et al. [7] showed that there are arbitrarily large 3-colored sets which do not contain an empty monochromatic triangle.

As an immediate corollary of Theorem 24 we have:

Corollary 25 *Every $(d+1)$-colored set S of $n \geq (d+1) \cdot 4^{d(c_d+1)}$ points in general position in $\mathbb{R}^d$ $(d > 2)$, c_d defined as in Lemma 3, determines at least a linear number of empty monochromatic d-simplices.*

Proof By Theorem 24 there exists a constant $\mu_d \leq (d+1) \cdot 4^{d(c_d+1)}$ such that every subset of S of μ_d points determines at least one empty monochromatic d-simplex. Divide S (with parallel $(d-1)$-dimensional hyperplanes) into $\lfloor \frac{n}{\mu_d} \rfloor$ subsets of μ_d points each. Hence, in total there exist at least $\lfloor \frac{n}{\mu_d} \rfloor$ empty monochromatic d-simplices in S. □

The next result follows immediately from Lemma 19 and provides a first general lower bound.

Corollary 26 *Let S be a k-colored set of $n > k \cdot 4^{d^2(d+1)}$ points in general position in $\mathbb{R}^d$, with $d \geq k > 3$. Then S determines $\Omega(n^{d-k+1} \log n)$ empty monochromatic d-simplices.*

Proof This is a direct consequence of Lemma 19 since every colored set has discrepancy at least 0. □

We will further improve on this result in Theorem 29 below. The next theorem is central for this improvement and provides a relation between the number of empty monochromatic d-simplices of an arbitrary color in a d-colored point set $S \subset \mathbb{R}^d$, and convex subsets of S with high discrepancy.

Theorem 27 *Let S be a d-colored set of $n \geq 3d \cdot (2c_d)^{(2^{d-1})}$ points in general position in $\mathbb{R}^d$, $d > 2$ and c_d as defined in Lemma 3. For every $1 \leq j \leq d$, either there are $\Omega(n^{1+2^{-d}})$ empty monochromatic d-simplices of color j, or there is a convex set C in $\mathbb{R}^d$, such that $|S \cap C| = \Theta(n)$ and $\delta(S \cap C) = \Omega(n^{(2^{-d})})$.*

Proof The general idea for the proof is to iteratively peel convex layers of color j from the point set. For each layer we use the Generalized Order Lemma to obtain

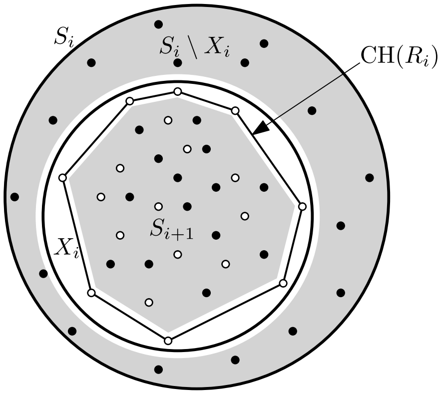

Fig. 5 2D-sketch to illustrate the nomenclature of the proof of Theorem 27. White points are points of chromatic class j, black points are points of the other color classes

roughly $n^{(2^{1-d})}$ empty monochromatic d-simplices of color j. If at any moment the discrepancy is large enough we terminate the process with the desired convex set C. Otherwise, the iteration stops after at most $\frac{1}{8}n^{(1-2^{-d})}$ steps.

Let S_i be the d-colored set of points in iteration step i. With $S_{i,l}$ we denote the chromatic classes of S_i, and with $S_{i,\max}/S_{i,\min}$ we denote the largest/smallest chromatic class of S_i, respectively. Note that a point of S_i can only be in one chromatic class, and that $\bigcup_{l=1}^{d} S_{i,l} = S_i$. The iteration starts with $S_1 = S$. For $i > 1$ smaller sets are constructed, such that $S_{i+1} \subset S_i$ and $S_{i+1,l} \subseteq S_{i,l}$. Let $\tilde{n} = \frac{n}{3d}$. As an invariant through all iterations we guarantee

$$\text{Invariant:} \quad |S_i| \geq (d+1)\tilde{n}.$$

The iteration stops either if a convex set C is found, with $|S \cap C| = \Theta(n)$ and $\delta(S \cap C) \geq \frac{\tilde{n}^{(2^{-d})}}{(d-1)}$, or after at most $\frac{1}{8}n^{(1-2^{-d})}$ steps.

Consider the ith step of the iteration. We will prove inequalities on the sizes of different subsets, their discrepancy, and the size of chromatic classes. With R_i we denote the jth chromatic class in step i, i.e., $S_{i,j}$ of S_i. Further, let h_i be the number of points in $R_i \cap \mathrm{CH}(R_i)$ and let $X_i = S_i \cap \mathrm{Conv}(R_i)$, such that the chromatic classes of X_i are $X_{i,l} = S_{i,l} \cap \mathrm{Conv}(R_i)$, with $X_{i,\max}$ and $X_{i,\min}$ being the largest and smallest chromatic class of X_i, respectively. See also Fig. 5 for an illustration of the different sets. The next iteration step $i+1$ will consist of $S_{i+1} = X_i \setminus (R_i \cap \mathrm{CH}(R_i))$ and $S_{i+1,l} = S_{i,l} \cap S_{i+1}$.

(1) $\delta(S_i) < \frac{\tilde{n}^{(2^{-d})}}{d-1}$.
If $\delta(S_i) \geq \frac{\tilde{n}^{(2^{-d})}}{d-1}$, then the iteration terminates with $C = \mathrm{Conv}(S_i)$, as $S \cap C = S_i$ and $|S_i| = \Theta(n)$ by the invariant.

(2) $|R_i| > \frac{|S_i|}{d} - \frac{\tilde{n}^{(2^{-d})}}{d} > \tilde{n}$.
By inequality (1), $\delta(S_i) < \frac{\tilde{n}^{(2^{-d})}}{d-1} = d \cdot \frac{\tilde{n}^{(2^{-d})}}{d(d-1)}$. Applying Corollary 17 we get $|S_{i,\min}| > \frac{|S_i|}{d} - (d-1) \cdot \frac{\tilde{n}^{(2^{-d})}}{d(d-1)} = \frac{|S_i|}{d} - \frac{\tilde{n}^{(2^{-d})}}{d} \geq \frac{(d+1)\tilde{n} - \tilde{n}^{(2^{-d})}}{d} > \tilde{n}$. Obviously $|R_i| \geq |S_{i,\min}|$, which proves the inequality.

(3) $\delta(X_i) < \frac{\tilde{n}^{(2^{-d})}}{d-1}$.

Obviously, $|X_i| \geq |R_i|$. Thus, by inequality (2), $|X_i| > \tilde{n} = \Theta(n)$. Hence, if $\delta(X_i) \geq \frac{\tilde{n}^{(2^{-d})}}{d-1}$, then the iteration terminates with $C = \mathrm{Conv}(X_i)$.

(4) $(d-1)|R_i| - |X_i \setminus R_i| > -\tilde{n}^{(2^{-d})}$.
Assume the contrary: $(d-1)|R_i| - |X_i \setminus R_i| \leq -\tilde{n}^{(2^{-d})}$, which can be rewritten to $d|R_i| \leq |X_i| - \tilde{n}^{(2^{-d})}$. From inequality (3) we know that $\delta(X_i) < \frac{\tilde{n}^{(2^{-d})}}{d-1}$, which implies by Corollary 17 that $|X_{i,\min}| > \frac{|X_i|}{d} - \frac{\tilde{n}^{(2^{-d})}}{d}$. As obviously $|R_i| \geq |X_{i,\min}|$, we get $|X_i| - \tilde{n}^{(2^{-d})} \geq d|R_i| > |X_i| - \tilde{n}^{(2^{-d})}$, which is a contradiction.

(5) $|S_i \setminus X_i| < 2\tilde{n}^{(2^{-d})}$.
Assume the contrary: $|S_i \setminus X_i| \geq 2\tilde{n}^{(2^{-d})}$. Using inequality (3) and the definition for the discrepancy we get

$$\frac{\tilde{n}^{(2^{-d})}}{d-1} > \delta(X_i) = (d-1)|X_{i,\max}| - |X_i \setminus X_{i,\max}| \geq (d-1)|R_i| - |X_i \setminus R_i|.$$

Further, we know that $|X_i \setminus R_i| = |S_i \setminus R_i| - |S_i \setminus X_i|$ and from inequality (2) we know $|R_i| > \frac{|S_i|}{d} - \frac{\tilde{n}^{(2^{-d})}}{d}$. Together with the assumption this leads to

$$\begin{aligned}\frac{\tilde{n}^{(2^{-d})}}{d-1} &> (d-1)|R_i| - |S_i \setminus R_i| + |S_i \setminus X_i| \\ &= d|R_i| - |S_i| + |S_i \setminus X_i| > |S_i| - \tilde{n}^{(2^{-d})} - |S_i| + 2\tilde{n}^{(2^{-d})} = \tilde{n}^{(2^{-d})},\end{aligned}$$

which is a contradiction.

(6) $\delta(S_{i+1}) < \frac{\tilde{n}^{(2^{-d})}}{d-1}$.
Note that $\delta(S_{i+1}) = \delta(X_i \setminus (R_i \cap \mathrm{CH}(R_i)))$ by the definition of the next iteration step. Although inequality (6) looks very similar to inequality (1), we have to prove it here, because we may not assume the invariant for the next step, yet. Using inequality (5) we can give the following bound:

$$|X_i| = |S_i| - |S_i \setminus X_i| > |S_i| - 2\tilde{n}^{(2^{-d})}.$$

From inequality (3) we get $(d-1)|X_{i,\max}| - |X_i \setminus X_{i,\max}| = \delta(X_i) < \frac{\tilde{n}^{(2^{-d})}}{d-1}$ and therefore $|R_i| \leq |X_{i,\max}| < \frac{|X_i| + \frac{\tilde{n}^{(2^{-d})}}{d-1}}{d}$. Combining these inequalities and using the invariant for $|S_i|$, we get

$$\begin{aligned}|S_{i+1}| &\geq |X_i| - |R_i| > |X_i| - \frac{|X_i| + \frac{\tilde{n}^{(2^{-d})}}{d-1}}{d} > \frac{d-1}{d}\bigl(|S_i| - 2\tilde{n}^{(2^{-d})}\bigr) - \frac{\tilde{n}^{(2^{-d})}}{d(d-1)} \\ &\geq \frac{(d-1)(d+1)}{d}\tilde{n} - \frac{2(d-1)^2+1}{d(d-1)}\tilde{n}^{(2^{-d})}.\end{aligned}$$

As $d > 2$ we may evaluate this relation to $|S_{i+1}| > \frac{8}{3}\tilde{n} - \frac{3}{2}\tilde{n}^{(2^{-d})} > \tilde{n} = \Theta(n)$. Hence, if $\delta(S_{i+1}) \geq \frac{\tilde{n}^{(2^{-d})}}{d-1}$, then the iteration terminates with $C = \mathrm{Conv}(S_{i+1})$.

(7) $h_i < 2\tilde{n}^{(2^{-d})}$.
As always, assume the contrary: $h_i \geq 2\tilde{n}^{(2^{-d})}$. We distinguish two cases on whether R_i is the largest chromatic class of X_i or not.

(a) If $R_i \neq X_{i,\max}$ then $S_{i+1,\max} = X_{i,\max}$ and

$$|S_{i+1} \setminus S_{i+1,\max}| = |X_i \setminus X_{i,\max}| - h_i.$$

Using inequality (6) and the definition for the discrepancy, we get

$$\begin{aligned}\frac{\tilde{n}^{(2^{-d})}}{d-1} > \delta(S_{i+1}) &= (d-1)|S_{i+1,\max}| - |S_{i+1} \setminus S_{i+1,\max}| \\ &= (d-1)|X_{i,\max}| - |X_i \setminus X_{i,\max}| + h_i = \delta(X_i) + h_i,\end{aligned}$$

which is a contradiction to the assumption, as $\delta(X_i) \geq 0$.

(b) If $R_i = X_{i,\max}$, recall that R_{i+1} denotes the jth color class of S_{i+1} and observe that $R_{i+1} = R_i \setminus (R_i \cap \mathrm{CH}(R_i))$. From inequality (3) and $R_i = X_{i,\max}$ we derive $\frac{\tilde{n}^{(2^{-d})}}{d-1} > \delta(X_i) = d|R_i| - |X_i| = d(|R_{i+1}| + h_i) - (|S_{i+1}| + h_i)$, and get $|R_{i+1}| < \frac{\tilde{n}^{(2^{-d})}}{d(d-1)} + \frac{|S_{i+1}|+h_i}{d} - h_i = \frac{|S_{i+1}|}{d} - (d-1) \cdot (\frac{h_i}{d} - \frac{\tilde{n}^{(2^{-d})}}{d(d-1)^2})$. As $|S_{i+1,\min}| \leq |R_{i+1}|$ we get from Lemma 16 that

$$\delta(S_{i+1}) \geq d \cdot \left(\frac{h_i}{d} - \frac{\tilde{n}^{(2^{-d})}}{d(d-1)^2}\right) = h_i - \frac{\tilde{n}^{(2^{-d})}}{(d-1)^2}.$$

Using inequality (6) and inserting the assumption for h_i, results in the contradiction $\frac{\tilde{n}^{(2^{-d})}}{d-1} > \delta(S_{i+1}) \geq \frac{\tilde{n}^{(2^{-d})}}{d-1} \cdot (2(d-1) - \frac{1}{d-1})$, as $d > 2$.

Using these inequalities we can provide a lower bound on the number of empty monochromatic d-simplices of color j per step and hence, in total, and prove the invariant on $|S_i|$. From inequality (2) we know that $|R_i| > \tilde{n} = \frac{n}{3d} \geq d+1$. Thus we may apply the Generalized Order Lemma (Lemma 15) to R_i, which guarantees at least $(d-1)|R_i| + (|R_i| - h_i)^{(2^{1-d})} + 2h_i - c_d$ interior disjoint d-simplices of color j with at least one point in $\mathrm{CH}(R_i)$ each. Only points of $(X_i \setminus R_i)$ can be in these d-simplices, and each of these $|X_i \setminus R_i|$ points lies inside at most one d-simplex. Therefore, there exist at least

$$(d-1)|R_i| - |X_i \setminus R_i| + \big(|R_i| - h_i\big)^{(2^{1-d})} + 2h_i - c_d =: \tau_i$$

empty monochromatic d-simplices of color j, each of them having at least one point in $\mathrm{CH}(R_i)$. Using the inequalities (4), (2), (7), and $h_i \geq 0$, we get

$$\tau_i > -\tilde{n}^{(2^{-d})} + \big(\tilde{n} - 2\tilde{n}^{(2^{-d})}\big)^{(2^{1-d})} + 0 - c_d \geq \frac{\tilde{n}^{(2^{1-d})}}{10},$$

where the last inequality holds for $\tilde{n} \geq (2c_d)^{(2^{d-1})}$.

Recall that the next iteration step $i+1$ considers $S_{i+1} = X_i \setminus (R_i \cap \mathrm{CH}(R_i))$ and $S_{i+1,l} = S_{i,l} \cap S_{i+1}$, for $1 \leq l \leq d$. Note that all empty monochromatic d-simplices of color j from step i have at least one vertex in $\mathrm{CH}(R_i)$. As the points of $\mathrm{CH}(R_i)$ are not in S_{i+1}, we do not overcount.

The iteration either terminates with a convex set C, such that $|S \cap C| = \Theta(n)$ and $\delta(S \cap C) \geq \frac{\tilde{n}^{(2^{-d})}}{d-1} = \Omega(n^{(2^{-d})})$, or it ends after $\frac{1}{8}n^{(1-2^{-d})}$ steps. With at least $\frac{\tilde{n}^{(2^{1-d})}}{10}$ empty monochromatic d-simplices of color j per step we get

$$\frac{\tilde{n}^{(2^{1-d})}}{10} \cdot \frac{1}{8} n^{(1-2^{-d})} = \frac{1}{80} \cdot \left(\frac{n}{3d}\right)^{(2^{1-d})} \cdot n^{(1-2^{-d})} = \Omega\big(n^{(1+2^{-d})}\big)$$

such simplices in total.

It remains to prove the invariant $|S_{i+1}| \geq (d+1)\tilde{n}$. After each step we have $S_{i+1} = X_i \setminus (X_i \cap \mathrm{CH}(R_i))$ and thus $|S_{i+1}| = |S_i| - |S_i \setminus X_i| - h_i$. With inequalities (5) and (7) we get $|S_{i+1}| > |S_i| - 2\tilde{n}^{(2^{-d})} - 2\tilde{n}^{(2^{-d})} = |S_i| - 4\tilde{n}^{(2^{-d})}$. Therefore, starting with $S_1 = S$, there are at least

$$\begin{aligned} n - 4\tilde{n}^{(2^{-d})} \cdot \frac{1}{8} n^{(1-2^{-d})} &= 3d\tilde{n} - \frac{1}{2} \cdot \frac{\tilde{n}^{(2^{-d})} \cdot 3d\tilde{n}}{(3d\tilde{n})^{(2^{-d})}} \\ &= \tilde{n}\left(3d - \frac{3d}{2 \cdot (3d)^{(2^{-d})}}\right) \\ &\geq \frac{3d}{2}\tilde{n} > (d+1)\tilde{n} \end{aligned}$$

points left after $\frac{1}{8} n^{(1-2^{-d})}$ steps, as $d > 2$. □

We generalize the last result to k-colored point sets, for $3 \leq k \leq d$.

Theorem 28 *Let S be a k-colored set of $n \geq 2^{d-k}(3k \cdot (2c_d)^{(2^{k-1})} + 1)$ points in general position in $\mathbb{R}^d$, $d > 2$ and c_d defined as in Lemma 3. For every $3 \leq k \leq d$ and every $1 \leq j \leq k$, either there are $\Omega(n^{d-k+1+2^{-d}})$ empty monochromatic d-simplices of color j, or there is a convex set C in $\mathbb{R}^d$, such that $|S \cap C| = \Theta(n)$ and $\delta(S \cap C) = \Omega(n^{(2^{-d})})$.*

Proof For fixed k we prove the theorem by induction on the dimension, and use Theorem 27 as an induction base for $d = k > 2$. Consider the induction step $(d-1) \longrightarrow d$, for $d > k$. Denote with S_j the jth, and with $S_{\min}$ the smallest chromatic class of S. If $\delta(S) \geq \frac{n^{(2^{-d})}}{k-1}$ then $C = \mathrm{Conv}(S)$ is the desired convex set, with $|S \cap C| = \Theta(n)$ and $\delta(S \cap C) = \Omega(n^{(2^{-d})})$. Thus assume that $\delta(S) < \frac{n^{(2^{-d})}}{k-1} = k \cdot \frac{n^{(2^{-d})}}{k(k-1)}$. From Corollary 17 we know that $|S_j| \geq |S_{\min}| > \frac{|S|}{k} - (k-1) \cdot \frac{n^{(2^{-d})}}{k(k-1)} = \frac{n - n^{(2^{-d})}}{k} \geq \frac{n}{2k} = \Theta(n)$.

Let $p \in S_j$ be a point of color j. For every point $q \in S \setminus \{p\}$ let r_q be the infinite ray with origin p and passing through q. Let Π' and Π'' be two $(d-1)$-dimensional hyperplanes containing $\mathrm{Conv}(S)$ between them and not parallel to any of the rays r_q. See Fig. 6 for a sketch. Project from p every point in $S \setminus \{p\}$ to Π' or Π'', in the following way. Every ray r_q intersects either Π' or Π'' in a point q' or q'', respectively. Take q' or q'' to be the projection of q from p. Let S' and S'' be the sets of these projected points in Π' and Π'', respectively. The bigger set, assume w.l.o.g. S' in Π', is a set of at least $\frac{n-1}{2}$ points in general position in $\mathbb{R}^{d-1}$.

Apply the induction hypothesis to S' and get either (a) $\Omega(n^{d-1-k+1+2^{-d}})$ empty monochromatic $(d-1)$-simplices of color j, or (b) a convex set C in $\mathbb{R}^{d-1}$, such that $|S' \cap C| = \Theta(n)$ and $\delta(S' \cap C) = \Omega(n^{(2^{-d+1})})$.

For case (b) observe, that the preimage of the point set of a convex set in Π' is the point set of a convex set in $\mathbb{R}^d$. Hence, C is a convex set in $\mathbb{R}^d$, such that $|S \cap C| = \Theta(n)$ and $\delta(S \cap C) = \Omega(n^{(2^{-d+1})})$, which trivially implies $\delta(S \cap C) = \Omega(n^{(2^{-d})})$.

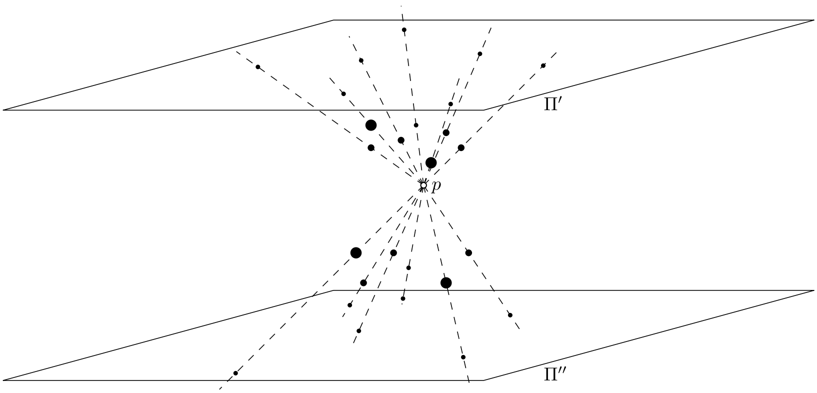

Fig. 6 Illustration of the projection, R^3 to two 2-dimensional hyperplanes in the sketch, in the proof of Theorem 28

For case (a) note that, if X is the vertex set of an empty monochromatic $(d-1)$-simplex of color j in Π', then $\text{Conv}(X \cup p)$ is an empty monochromatic d-simplex of color j in $\mathbb{R}^d$. Repeat the projection and the induction for each point $p \in S_j$ and assume that this always results in case (a) (because the proof is completed if case (b) happens once). This results in a total of $\frac{|S_j|}{d+1} \cdot \Omega(n^{d-k+2^{-d}}) = \Omega(n^{d-k+1+2^{-d}})$ empty monochromatic d-simplices of color j, as each d-simplex gets overcounted at most $(d+1)$ times. □

Combining the last theorem with the "Generalized Discrepancy Lemma" (Lemma 19) and its different versions for the 3-colored case (Lemmas 21 to 23), we can prove one of our main results.

Theorem 29 *Any k-colored set S of n points in general position in $\mathbb{R}^d$, $d \geq k \geq 3$, determines $\Omega(n^{d-k+1+2^{-d}})$ empty monochromatic d-simplices.*

Proof By Theorem 28 either there exist $\Omega(n^{d-k+1+2^{-d}})$ empty monochromatic d-simplices, or there exists a convex set C in $\mathbb{R}^d$, such that $|S \cap C| = \Theta(n)$ and $\delta(S \cap C) = \Omega(n^{(2^{-d})})$. In the latter case, there exist $\Omega(n^{d-k+1+2^{-d}})$ empty monochromatic d-simplices by applying Lemma 19 (for $d \geq k > 3$), Lemma 21 (for $d = k = 3$), Lemma 23 (for $d = 4$ and $k = 3$), or Lemma 22 (for $d > 4$ and $k = 3$) to the point set $(S \cap C)$. □

6 Empty Monochromatic Simplices in Two Colored Point Sets

For the sake of simplicity, we call the two color classes of a bichromatic point set S "red" and "blue", and denote these point sets with R and B, respectively. Observe, that the discrepancy $\delta(S) = (k-1)|S_{\max}| - |S \setminus S_{\max}|$ simplifies to $\delta(S) = ||R| - |B||$ for the bichromatic case $k = 2$. This is the same notion of discrepancy as used in [1] and [15].

Note further, that assuming an upper bound for the discrepancy,

$$\delta(S) = \big||R| - |B|\big| < f_n,$$

for a bicolored set of n points, leads to lower and upper bounds for the cardinality of both color classes in a simple way. The inequality reformulates to $|R| - |B| < f_n$ and $|R| - |B| > -f_n$. Using $|R| = |S| - |B|$ and $|B| = |S| - |R|$ we make the following simple observation, which will be used frequently later on.

Observation 30 *Let S be a bicolored set of n points in $\mathbb{R}^2$, partitioned into a red point set R and a blue point set B. Let f_n be some function on n. If $\delta(S) < f_n$, then $|B| - f_n < |R| < |B| + f_n$ and $|R| - f_n < |B| < |R| + f_n$, and $\frac{n-f_n}{2} < |R| < \frac{n+f_n}{2}$ and $\frac{n-f_n}{2} < |B| < \frac{n+f_n}{2}$.*

Using the Order and Discrepancy Lemmas from [1], Pach and Tóth [15] prove that the number of empty monochromatic triangles in bichromatic point sets in the plane is $\Omega(n^{4/3})$. We adapt this proof technique to our notation and explicitly state, in Theorem 31, an intermediate result showing the central trade off between many empty monochromatic triangles and large convex sets. This result will be generalized to higher dimensions in Theorem 32.

Theorem 31 *Let S be a bicolored set of n points in general position in $\mathbb{R}^2$, partitioned into a red point set R and a blue point set B. Then either there exist $\Omega(n^{4/3})$ empty red triangles, or there exists a convex set C in $\mathbb{R}^2$, such that $|S \cap C| = \Theta(n)$ and $\delta(S \cap C) = ||C \cap R| - |C \cap B|| = \Omega(\sqrt[3]{n})$.*

Proof Following the lines of [15], we call a point $p \in S$ *rich* if at least $\frac{\sqrt[3]{n}}{3}$ empty monochromatic triangles in S have p as a vertex. The general idea for the proof is to iteratively remove a rich red point from the point set. We show that it is possible to find either $\frac{n}{5}$ rich red points or a convex set C with the desired properties.

If there exists some convex set C in $\mathbb{R}^2$, such that $|S \cap C| = \Theta(n)$ and $\delta(S \cap C) = \Omega(\sqrt[3]{n})$, then the theorem is proven. Hence, assume its nonexistence. Let S_i be the bicolored set of points in iteration step i, and let R_i and B_i be its color classes. Further, let h_i be the number of convex hull points of R_i and let $X_i = S_i \cap \operatorname{Conv}(R_i)$. See also Fig. 5 for an illustration of the different sets. The iteration starts with $S_1 = S$. For $i > 1$ smaller sets S_{i+1} are constructed, by removing one rich red point from S_i. Considering the ith iteration ($1 \le i \le \frac{n}{5}$), we can state the following relations:

(1) $|S_i| = |S| - (i-1) > n - \frac{n}{5} + 1 = \Theta(n)$.

(2) $\delta(S_i) < \frac{\sqrt[3]{n}}{20}$.

By relation (1), $|S_i| = \Theta(n)$. Thus, if $\delta(S_i) \ge \frac{\sqrt[3]{n}}{20}$, then we can set $C = \operatorname{Conv}(S_i)$, implying $|S \cap C| = \Theta(n)$ and $\delta(S \cap C) = \Omega(\sqrt[3]{n})$, which we assumed not to exist.

(3) $|R_i| > \frac{2n}{5} - \frac{\sqrt[3]{n}}{40}$ and $|B_i| > \frac{2n}{5} - \frac{\sqrt[3]{n}}{40}$.

Using inequality (1) and (2), and Observation 30 we get

$$|R_i| > \frac{|S_i| - \frac{\sqrt[3]{n}}{20}}{2} > \frac{4n}{10} - \frac{\sqrt[3]{n}}{40} \quad \text{and} \quad |B_i| > \frac{|S_i| - \frac{\sqrt[3]{n}}{20}}{2} > \frac{4n}{10} - \frac{\sqrt[3]{n}}{40}.$$

(4) $\delta(X_i) < \frac{\sqrt[3]{n}}{20}$.
Obviously, $|X_i| \geq |R_i| = \Theta(n)$, by inequality (3). Thus, $\delta(X_i) \geq \frac{\sqrt[3]{n}}{20}$ again supplies us with some $C = \mathrm{Conv}\,(S_i)$, which we assumed not to exist.

(5) $|S_i \setminus X_i| < \frac{\sqrt[3]{n}}{10}$.
Note that $|S_i \setminus X_i| = |B_i \setminus X_i|$. Using inequality (4) and Observation 30 we get $|B_i \cap \mathrm{Conv}(X_i)| = |B_i| - |S_i \setminus X_i| = |X_i \setminus R_i| > |R_i| - \frac{\sqrt[3]{n}}{20}$. Using inequality (2) we get $|S_i \setminus X_i| < |B_i| - |R_i| + \frac{\sqrt[3]{n}}{20} \leq \delta(S_i) + \frac{\sqrt[3]{n}}{20} < 2\frac{\sqrt[3]{n}}{20}$.

(6) $h_i < \frac{\sqrt[3]{n}}{10}$.
Let $X_i' = X_i \setminus (X_i \cap \mathrm{CH}(R_i))$. Obviously, $|X_i'| \geq |B_i| - |S_i \setminus X_i|$, and consequently $|X_i'| > \frac{4n}{10} - \frac{\sqrt[3]{n}}{40} - \frac{\sqrt[3]{n}}{10} = \Theta(n)$ by inequality (3) and (5). Therefore, we can assume $\delta(X_i') = |(|R_i| - h_i) - |X_i \setminus R_i|| < \frac{\sqrt[3]{n}}{20}$, because the contrary would imply the existence of some $C = \mathrm{Conv}\,(X_i')$, which we assumed not to exist. From $|X_i \setminus R_i| - (|R_i| - h_i) < \frac{\sqrt[3]{n}}{20}$ and inequality (4) we get $h_i < |R_i| - |X_i \setminus R_i| + \frac{\sqrt[3]{n}}{20} = \delta(X_i) + \frac{\sqrt[3]{n}}{20} < \frac{\sqrt[3]{n}}{20} + \frac{\sqrt[3]{n}}{20}$.

Using these inequalities we can prove the existence of rich points. Let $p_1, \ldots, p_{h_i}$ be the convex hull points of R_i in counter clock-wise order. Triangulate $\mathrm{CH}(R_i)$ by adding the diagonals $p_1 p_j$, for $3 \leq j \leq (h_i - 1)$. In the resulting triangulation let $\triangle_j$, $2 \leq j \leq (h_i - 1)$, be the triangle $p_1 p_j p_{j+1}$. With $S(\triangle_j)$ denote the bicolored set of points interior to $\triangle_j$ and let $R(\triangle_j)$ and $B(\triangle_j)$ be its color classes.

(7) $\delta(S(\triangle_j)) = ||R(\triangle_j)| - |B(\triangle_j)|| < \frac{\sqrt[3]{n}}{10}$ for every $2 \leq j \leq (h_i - 1)$.
Assume the contrary: $\delta(S(\triangle_j)) \geq \frac{\sqrt[3]{n}}{10}$ for some $\triangle_j$, $2 \leq j \leq (h_i - 1)$. Consider the three regions $(\triangle_2 \cup \cdots \cup \triangle_{j-1})$, $\triangle_j$, and $(\triangle_{j+1} \cup \cdots \cup \triangle_{h_i - 1})$. At least one of these three regions contains at least

$$\frac{|X_i| - h_i}{3} = \frac{|S_i| - |S_i \setminus X_i| - h_i}{3} > \frac{1}{3}\left(n - \frac{n}{5} + 1 - \frac{\sqrt[3]{n}}{10} - \frac{\sqrt[3]{n}}{10}\right) > \frac{n}{5}$$

interior points, by inequality (1), (5), and (6).
If $|S(\triangle_j)| \geq \frac{n}{5} = \Theta(n)$, then we can set $C = \mathrm{Conv}\,(S(\triangle_j))$, which we assumed not to exist. Thus assume w.l.o.g. that region $(\triangle_2 \cup \cdots \cup \triangle_{j-1})$ has at least $\frac{n}{5}$ interior points, i.e., $|S(\triangle_2) \cup \cdots \cup S(\triangle_{j-1})| \geq \frac{n}{5} = \Theta(n)$. Note that also

$$\begin{aligned} &\left|S(\triangle_2) \cup \cdots \cup S(\triangle_{j-1}) \cup S(\triangle_j)\right| \\ &\quad = \left|S(\triangle_2) \cup \cdots \cup S(\triangle_{j-1})\right| + \left|S(\triangle_j)\right| \geq \frac{n}{5} = \Theta(n). \end{aligned}$$

Then either the points inside region $(\triangle_2 \cup \cdots \cup \triangle_{j-1})$ have high discrepancy, $\delta(S(\triangle_2) \cup \cdots \cup S(\triangle_{j-1})) \geq \frac{\sqrt[3]{n}}{20}$, and thus we can set $C = \mathrm{Conv}\,(S(\triangle_2) \cup \cdots \cup S(\triangle_{j-1}))$, or the discrepancy in region $(\triangle_2 \cup \cdots \cup \triangle_{j-1} \cup \triangle_j)$ is high,

$$\begin{aligned} &\delta\big(S(\triangle_2) \cup \cdots \cup S(\triangle_{j-1}) \cup S(\triangle_j)\big) \\ &\quad \geq \delta\big(S(\triangle_j)\big) - \delta\big(S(\triangle_2) \cup \cdots \cup S(\triangle_{j-1})\big) > \frac{\sqrt[3]{n}}{20} \end{aligned}$$

and thus we can set $C = \mathrm{Conv}(S(\triangle_2) \cup \cdots \cup S(\triangle_{j-1}) \cup S(\triangle_j))$. Both times the existence of a convex set C, with $|S \cap C| = \Theta(n)$ and $\delta(S \cap C) = \Omega(\sqrt[3]{n})$, is a contradiction to its assumed nonexistence and consequently a contradiction to the assumed existence of some $\triangle_j$, $2 \le j \le (h_i - 1)$, with $\delta(S(\triangle_j)) \ge \frac{\sqrt[3]{n}}{10}$.

By inequalities (3) and (6) we have

$$\sum_{j=2}^{h_i-1} \left(\left|R(\triangle_j)\right|\right) = |R_i| - h_i > \frac{4n}{10} - \frac{\sqrt[3]{n}}{40} - \frac{\sqrt[3]{n}}{10} > \frac{3n}{10}.$$

Hence, there exists a $\triangle_j$, such that $|R(\triangle_j)| > \frac{3n}{10(h_i-2)} > 3n^{2/3}$. Further, using inequality (7) and Observation 30 we have $|B(\triangle_j)| < |R(\triangle_j)| + \frac{\sqrt[3]{n}}{10}$. Applying Lemma 14 for $d = 2$ we know that there exist at least $|R(\triangle_j)| + \sqrt{|R(\triangle_j)|} + 1$ interior disjoint red triangles, each with a point in $\mathrm{CH}(\triangle_j)$. At least

$$\begin{aligned} \left|R(\triangle_j)\right| + \sqrt{\left|R(\triangle_j)\right|} + 1 - \left|B(\triangle_j)\right| &> \left|R(\triangle_j)\right| + \sqrt{\left|R(\triangle_j)\right|} + 1 - \left|R(\triangle_j)\right| - \frac{\sqrt[3]{n}}{10} \\ &> \sqrt{3n^{2/3}} - \frac{\sqrt[3]{n}}{10} > \sqrt[3]{n} \end{aligned}$$

of these triangles are empty of points, and at least a third of them has the same point p in $\mathrm{CH}(\triangle_j)$ and thus in $\mathrm{CH}(R_i)$. Hence, p is a rich point.

If $i < \frac{n}{5}$ then let $S_{i+1} = S_i \setminus \{p\}$, $i = i + 1$, and iterate. As all triangles counted so far have p as a vertex, and p does not belong to the point sets of future iterations, we do not overcount. The process either terminates with a convex set C, such that $|S \cap C| = \Theta(n)$ and $\delta(S \cap C) = \Omega(\sqrt[3]{n})$, or it ends after $\frac{n}{5}$ steps. For each rich point we can count at least $\frac{\sqrt[3]{n}}{3}$ empty red triangles. As we get $\frac{n}{5}$ rich points and do not overcount we get $\frac{n}{5} \cdot \frac{\sqrt[3]{n}}{3} = \Omega(n^{4/3})$ empty red triangles in total. □

Combining Theorem 31 with Lemma 18 proves the bound of $\Omega(n^{4/3})$ empty monochromatic triangles for the 2-colored case in the plane, already shown in [15]. However, Theorem 31 can be generalized to $\mathbb{R}^d$:

Theorem 32 *Let S be a bicolored set of n points in general position in $\mathbb{R}^d$ $(d \ge 2)$, partitioned into a red point set R and a blue point set B. Then either there exist $\Omega(n^{d-2/3})$ empty red d-simplices, or there exists a convex set C in $\mathbb{R}^d$, such that $|S \cap C| = \Theta(n)$ and $\delta(S \cap C) = \Omega(\sqrt[3]{n})$.*

Proof We prove the theorem by induction on the dimension d (recall that d is a constant, independent of n), and use Theorem 31 as an induction base for $d = 2$. Consider the induction step $(d-1) \longrightarrow d$, for $d > 2$. If $\delta(S) \ge \sqrt[3]{n}$ then $C = \mathrm{Conv}(S)$ is the desired convex set, with $|S \cap C| = \Theta(n)$ and $\delta(S \cap C) = \Omega(\sqrt[3]{n})$. Thus assume that $\delta(S) < \sqrt[3]{n}$. From Observation 30 we know that $|R| > \frac{n-\sqrt[3]{n}}{2} = \Theta(n)$.

Let $p \in R$ be a red point. For every point $q \in S \setminus \{p\}$ let r_q be the infinite ray with origin p and passing through q. Let Π' and Π'' be two $(d-1)$-dimensional hyperplanes containing $\mathrm{Conv}(S)$ between them and not parallel to any of the rays r_q.

See Fig. 6 (for the very similar proof of Theorem 28) for a sketch. Project from p every point in $S \setminus \{p\}$ to Π' or Π'', in the following way. Every ray r_q intersects either Π' or Π'' in a point q' or q'', respectively. Take q' or q'' to be the projection of q from p. Let S' and S'' be the sets of these projected points in Π' and Π'', respectively. The bigger set, assume w.l.o.g. S' in Π', is a set of at least $\frac{n-1}{2}$ points in general position in $\mathbb{R}^{d-1}$.

Apply the induction hypothesis to S' and get either (a) $\Omega(n^{d-1-2/3})$ empty red $(d-1)$-simplices, or (b) a convex set C in $\mathbb{R}^{d-1}$, such that $|S' \cap C| = \Theta(n)$ and $\delta(S' \cap C) = \Omega(\sqrt[3]{n})$.

For case (b) observe, that the preimage of a point set of a convex set in Π' is the point set of a convex set in $\mathbb{R}^d$. Hence, C is a convex set in $\mathbb{R}^d$, such that $|S \cap C| = \Theta(n)$ and $\delta(S \cap C) = \Omega(\sqrt[3]{n})$.

For case (a) note that, if X is the vertex set of an empty red $(d-1)$-simplex in Π', then $\mathrm{Conv}(X \cup p)$ is an empty red d-simplex in $\mathbb{R}^d$. Repeat the projection and the induction for each red point $p \in R$ and assume that this always results in case (a) (because the proof is completed if case (b) happens once). This results in a total of $\frac{|R|}{d+1} \cdot \Omega(n^{d-1-2/3}) = \Omega(n^{d-2/3})$ empty red d-simplices, as each d-simplex gets overcounted at most $(d+1)$ times. □

Combining Theorem 32 with the two variants of the "Discrepancy Lemma" for the bicolored case (Lemmas 18 and 20), allows us to generalize the bound on the number of empty monochromatic triangles for the bicolored case in the plane, to $\mathbb{R}^d$.

Theorem 33 *Any bicolored set S of n points in general position in $\mathbb{R}^d$, $d \geq 2$, determines $\Omega(n^{d-2/3})$ empty monochromatic d-simplices.*

Proof By Theorem 32 either there exist $\Omega(n^{d-2/3})$ empty monochromatic d-simplices, or there exists a convex set C in $\mathbb{R}^d$, such that $|S \cap C| = \Theta(n)$ and $\delta(S \cap C) = \Omega(\sqrt[3]{n})$.

In the former case the theorem is proven. In the latter case, if $d = 2$ then there exist $\Omega(n^{2-1+1/3}) = \Omega(n^{4/3})$ empty monochromatic triangles (2-simplices) by applying Lemma 18 to $(S \cap C)$, and if $d > 2$ then there exist $\Omega(n^{d-1+1/3}) = \Omega(n^{d-2/3})$ empty monochromatic d-simplices by applying Lemma 20 to $(S \cap C)$. □

7 Conclusions

In this paper we generalized known bounds on the number of empty monochromatic triangles and tetrahedra on colored point sets to higher dimensions. Our results are summarized in Table 1 (Sect. 1).

As main results, in Theorem 33, we proved that any bicolored point sets in $\mathbb{R}^d$ determines $\Omega(n^{d-2/3})$ empty monochromatic d-simplices. For $3 \leq k \leq d$, in Theorem 29, we proved that any k-colored point set in $\mathbb{R}^d$ determines $\Omega(n^{d-k+1-2^{-d}})$ empty monochromatic d-simplices. Further, we extended the linear lower bound for the number of empty monochromatic tetrahedra in 4-colored point sets in $\mathbb{R}^3$ to a linear lower bound for the number of empty monochromatic d-simplices in $(d+1)$-colored point sets in $\mathbb{R}^d$, Corollary 25.

Observe that in the "Generalized Discrepancy Lemma" (Lemma 19) in fact all empty monochromatic d-simplices of the provided bound are of the same color. This is also true for the Discrepancy Lemmas (Lemmas 18, 20, 21, 22, and 23) for small values of k and d, for which the "Generalized Discrepancy Lemma" is not applicable. As Corollary 26 is a direct consequence of Lemma 19, we can state that: Every k-colored set of $n > k \cdot 4^{d^2(d+1)}$ points in general position in $\mathbb{R}^d$, with $d \geq k > 3$, determines $\Omega(n^{d-k+1} \log n)$ empty monochromatic d-simplices, which are all of the same color.

In order to prove our lower bounds on the number of empty monochromatic d-simplices, we proved a result that is interesting on its own right. Theorem 5 shows that a simplicial complex with at least $dn + \max\{h, \frac{\log_2(n)}{2d}\} - c_d$, with $c_d = d^3 + d^2 + d$, d-simplices exists for any point set in $\mathbb{R}^d$ (points in general position).

Although still linear, this is a first non-trivial bound, of interest in view of the following open problem stated by Brass et al. [6]: What is the maximum number $R_d(n)$ such that every set of n points in general position in d-dimensional space has a triangulation consisting of at least $R_d(n)$ simplices? Moreover, Urrutia [18] posed the following open problem:

Problem 34 Is it true that for any point set in general position in $\mathbb{R}^3$ there exists a triangulation with super linear many 3-simplices?

A positive answer to this question implies that any k coloring of a set of points with n elements in general position, always contains an empty monochromatic simplex, k constant, and n sufficiently large.

Unfortunately, proving or disproving Problem 34 seems to be illusive and remains open. On the other hand, it is well known that any set of n points on the momentum curve (x, x^2, x^3) has a triangulation with a quadratic number of 3-simplices. Aside from this, we are not aware of many families of point sets in general position in $\mathbb{R}^3$ for which it is known that there exist triangulations with a quadratic number of 3-simplices.

We close our paper with the following result that somehow suggests that any point set with n elements in $\mathbb{R}^d$ is not *far* from a point set that generates a quadratic number of interior disjoint 3-simplices:

Theorem 35 *Any set X of n points in general position in $\mathbb{R}^3$ is contained in a set S with $2n$ points in general position in $\mathbb{R}^3$ such that S determines at least $\binom{n}{2}$ interior disjoint 3-simplices.*

Proof Let $v_1, \ldots, v_n$ be n different unit vectors, no two of which are parallel to each other, nor parallel to any segment determined by any two elements of X. For each point p_i, $1 \leq i \leq n$, in X let $q_i = p_i + \varepsilon \cdot v_i$ be a point of $S \setminus X$, where ε is a small enough constant. Let $\sigma_{i,j} = \mathrm{Conv}(\{p_i, q_i, p_j, q_j\})$ be a 3-simplex. As X is in general position, it is easy to see, that for all $(i, j) \neq (r, s)$ the 3-simplices $\sigma_{i,j}$ and $\sigma_{r,s}$ have disjoint interior. $\square$

Note that in many instances it will not be possible to complete the set of 3-simplices obtained in the proof above to a full triangulation of S. Nevertheless, this shows that the families of point sets admitting a quadratic number of interior disjoint empty 3-simplices may not have any special properties that would allow us to characterize them.

Clearly, the construction used in Theorem 35 can be generalized to higher dimensions. We conjecture:

Conjecture 1 *For each $d \geq 3$ and every constant k, there exists a constant $f(d, k)$ such that every set $S \in \mathbb{R}^d$ of more than $f(d, k)$ points with arbitrary k-coloring has a monochromatic empty d-simplex.*

Even more so, we believe that the answer to Problem 34 is "Yes" and that this actually extends to higher dimensions.

Conjecture 2 *For each $d \geq 3$ and every point set $S \in \mathbb{R}^d$ there exists a triangulation with super linear many d-simplices.*

Of course, proving this stronger Conjecture 2 would imply a proof for Conjecture 1: Construct a triangulation of super linear size on the biggest color class $R \subseteq S$. There exist only linear many differently colored points in $S \setminus R$ to fill the super linear many monochromatic d-simplices on R. Hence, there exists at least one monochromatic empty d-simplex.

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