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Colourful and Fractional (p, q)-theorems

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Abstract Let $p \geq q \geq d+1$ be positive integers and let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. Assume that the elements of $\mathcal{F}$ are coloured with p colours. A p-element subset of $\mathcal{F}$ is heterochromatic if it contains exactly one element of each colour. The family $\mathcal{F}$ has the heterochromatic (p, q)-property if in every heterochromatic p-element subset there are at least q elements that have a point in common. We show that, under the heterochromatic (p, q)-condition, some colour class can be pierced

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by a finite set whose size we estimate from above in terms of d, p, and q. This is a colourful version of the famous (p, q)-theorem. (We prove a colourful variant of the fractional Helly theorem along the way.) A fractional version of the same problem is when the (p, q)-condition holds for all but an α fraction of the p-tuples in $\mathcal{F}$. We show that, in the case that $d = 1$, all but a β fraction of the elements of $\mathcal{F}$ can be pierced by $p - q + 1$ points. Here β depends on α and p, q, and $\beta \to 0$ as α goes to zero.

Keywords Colourful and fractional theorems · Gallai and Helly type results · The (p, q)-problem

1 Introduction

Helly's theorem states that if $\mathcal{F}$ is a finite family of convex sets in $\mathbb{R}^d$ such that every at most $(d + 1)$-element subfamily of $\mathcal{F}$ has nonempty intersection, then the whole family $\mathcal{F}$ has nonempty intersection. The condition can be relaxed leading to the so-called (p, q)-condition of Hadwiger and Debrunner [7] and the conclusion varies accordingly: Assuming $p \geq q \geq d + 1$, the family $\mathcal{F}$ has the (p, q)-property if among every p elements of $\mathcal{F}$ there are q with nonempty intersection. For example, in Helly's theorem the family of convex sets satisfies the $(d + 1, d + 1)$-condition in $\mathbb{R}^d$.

A set of points with the property that every element of $\mathcal{F}$ contains at least one of the points is said to *pierce* $\mathcal{F}$. The minimum number of points that can pierce $\mathcal{F}$ is called the *piercing number* of $\mathcal{F}$, and is denoted by $\tau(\mathcal{F})$.

Hadwiger and Debrunner [7] asked in 1957 if the (p, q)-condition implies that $\tau(\mathcal{F})$ is bounded as a function of d, p, and q. They proved this in [7] under the condition that $(d - 1)p < d(q - 1)$ in stronger from saying that $\tau(\mathcal{F}) \leq p - q + 1$. Note that the $(d - 1)p < d(q - 1)$ condition is always satisfied when $d = 1$. The general case had remained open for 35 years and was finally solved by Alon and Kleitman [1] by an ingenious and very powerful method.

Theorem 1 (Alon and Kleitman [1]) *Let p, q, d be positive integers with $p \geq q \geq d + 1$. Then there exists a number $m(p, q, d)$ such that $\tau(\mathcal{F}) \leq m(p, q, d)$ for every finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ satisfying the (p, q)-condition.*

We remark here that the necessity of the condition that $p \geq q \geq d + 1$ is shown by the example when $\mathcal{F}$ is a family of hyperplanes in general position. Note also that the (p, q)-property implies the $(p, q - 1)$-property. So the most important case of the (p, q)-problem occurs when $q = d + 1$.

In this paper we consider a colourful version of the (p, q)-problem. Let $\mathcal{F}_1, \ldots, \mathcal{F}_p$ be finite families of convex sets in $\mathbb{R}^d$. Their union is denoted by $\mathcal{F}$. One can think of $\mathcal{F}_i$ as containing the elements of $\mathcal{F}$ coloured by colour i. A *heterochromatic p-tuple* of $\mathcal{F}$ is just a collection of p sets $C_1, \ldots, C_p$ where $C_i \in \mathcal{F}_i$ for every $i \in [p] = \{1, \ldots, p\}$. Lovász [9] found a colourful version of Helly's theorem in 1974, its proof appeared first in Bárány [2] in 1982. The coloured version says the following.

Theorem 2 (Lovász [9] and Bárány [2]) *Let $\mathcal{F}_1, \dots, \mathcal{F}_{d+1}$ be finite families of convex sets* (*colour classes*) *in $\mathbb{R}^d$ with $\mathcal{F} = \cup_{j=1}^{d+1} \mathcal{F}_j$. If every heterochromatic* $(d+1)$*-tuple of $\mathcal{F}$ has a point in common, then there exists a family $\mathcal{F}_i$ whose elements have a point in common.*

The assumption of the colourful Helly theorem can be weakened in a way similar to that of the (p, q)-problem. The family $\mathcal{F}$ satisfies the *heterochromatic* (p, q)-condition, to be denoted by $(p, q)_H$, if every heterochromatic p-tuple of $\mathcal{F}$ contains an intersecting q-tuple.

We will use the Alon–Kleitman method to show the following.

Theorem 3 *Let p, q, d be positive integers with $p \geq q \geq d + 1$. Then there exists a number $M(p, q, d)$ such that the following holds. Given finite families $\mathcal{F}_1, \dots, \mathcal{F}_p$ of convex sets in $\mathbb{R}^d$ satisfying the $(p, q)_H$-property, there are $q - d$ indices $i \in [p]$ for which $\tau(\mathcal{F}_i) \leq M(p, q, d)$.*

The necessity of the condition $p \geq q \geq d + 1$ is shown by the example when all the $\mathcal{F}_i$ consist of hyperplanes in general position. One cannot hope for more than $q - d$ classes with bounded piercing number: this is shown by $q - d$ colour classes consisting of many copies of $\mathbb{R}^d$ and each of the remaining classes consisting of many hyperplanes in general position.

The (p, q)-property ($(p, q)_H$-property) can be weakened by requiring that all but an α fraction of the p-tuples (or heterochromatic p-tuples) of $\mathcal{F}$ satisfy the (p, q)-property ($(p, q)_H$-property). What can one hope for under this *fractional* (p, q)-condition? Perhaps $\mathcal{F}$ contains a subfamily $\mathcal{G}$ of size $\gamma|\mathcal{F}|$ with $\tau(\mathcal{G})$ bounded where γ depends only on α, d, p, q. It would be desirable to have $\gamma \to 1$ when $\alpha \to 0$. We will make a first step in this direction, focusing on the main case $q = d + 1$:

Theorem 4 *Let $\alpha > 0$ and let p, d be positive integers with $p \geq d + 1$. Then there exists a real number $\gamma(\alpha, p, d) > 0$ such that the following holds. Given finite families $\mathcal{F}_1, \dots, \mathcal{F}_p$ of convex sets in $\mathbb{R}^d$ satisfying the $(p, d + 1)_H$-condition for all but an α fraction of heterochromatic p-tuples of $\mathcal{F}$, some family $\mathcal{F}_i$ contains an intersecting subfamily of size $\gamma|\mathcal{F}_i|$.*

In the second half of the paper we will consider the same questions in dimension one, that is, when the convex sets in $\mathcal{F}$ are intervals in $\mathbb{R}$. In this case we prove precise results on the piercing number.

Theorem 5 *Let $p \geq q \geq 2$ be integers and $\mathcal{F}$ a finite family of intervals in $\mathbb{R}$ coloured with p colours. If $\mathcal{F}$ has the $(p, q)_H$-property, then there exists a colour class $\mathcal{F}_i \subset \mathcal{F}$ with the property that $\tau(\mathcal{F}_i) \leq \lfloor \frac{p-1}{q-1} \rfloor$. The bound is best possible in the sense that there is a family $\mathcal{F}$ satisfying the conditions for which $\tau(\mathcal{F}_i) \geq \lfloor \frac{p-1}{q-1} \rfloor$ for all $i \in [p]$.*

Further, for coloured intervals in $\mathbb{R}$ the fractional $(p, q)_H$-property implies the desired conclusion discussed above. Namely, we prove the following result which is a colourful and fractional version of the classical (p, q)-theorem of Hadwiger and Debrunner for finite families of intervals in the real line.

Theorem 6 *Let $p \geq q \geq 2$ be integers, set $\alpha_0 = \frac{1}{2}(p-q+3)^{-1/(p-q+2)}$ and let $\alpha \in [0, \alpha_0)$. Then there is a number $\beta = \beta(p,q,\alpha) \in [0,1)$ and an integer $n_0 = n_0(p,q,\alpha)$ such that the following holds. Let $\mathcal{F}$ be a finite and coloured family of intervals in $\mathbb{R}$ with colour classes $\mathcal{F}_1, \ldots, \mathcal{F}_p$ where each $|\mathcal{F}_i| \geq n_0$. If $\mathcal{F}$ satisfies the $(p,q)_H$-property with the exception of at most $\alpha \prod_{j=1}^{p} |\mathcal{F}_j|$ heterochromatic p-tuples, then there exists a colour class $\mathcal{F}_i \subset \mathcal{F}$ such that the elements of $\mathcal{F}_i$ can be pierced by at most $p-q+1$ points with the exception of at most $\beta|\mathcal{F}_i|$ intervals. Furthermore, $\beta = O(\alpha^{1/(p-q+2)})$.*

We will give an example showing that the dependence $\beta = O(\alpha^{1/(p-q+2)})$ is best possible. In Sect. 7 we state an extension of Theorem 6 where, under the same conditions, some colour class $\mathcal{F}_i$ is pierced by k points except for a small fraction of the intervals in $\mathcal{F}_i$. Here k is any integer from $\{\lfloor \frac{p-1}{q-1} \rfloor, \ldots, p-q+1\}$. The proof is given is Sect. 8.

Here comes the uncoloured (and fractional) version of Theorem 6. It follows from Theorem 6 quite easily.

Theorem 7 *Let $p \geq q \geq 2$ be positive integers, and let $\mathcal{F}$ be a finite family of n intervals in $\mathbb{R}$, and $\alpha \in [0,1)$. Then there exists a number $\beta = \beta(p,q,\alpha) \in [0,1)$ with the property that if the family $\mathcal{F}$ has the (p,q)-property with the exception of at most $\alpha\binom{n}{p}$ p-tuples, then the elements of $\mathcal{F}$ can pierced by $p-q+1$ points with the possible exception of at most βn elements. Furthermore $\beta = O(\alpha^{1/p})$.*

As a consequence of Theorems 6 and 7, when $q = 2$, we obtain the following result that shows how the monochromatic world, for intervals on the line, has influence on the behaviour of the heterochromatic world.

Corollary 1 *For every integer $p \geq 2$ and every $\alpha > 0$, there is $\beta = \beta(p,\alpha) > 0$ such that the following holds. Suppose that $\mathcal{F}$ is a finite family of intervals in $\mathbb{R}$ coloured with p colours. If for every colour i, the fraction of (monochromatic) p-tuples in $\mathcal{F}_i$ that are pairwise disjoint is bigger than α, then the fraction of heterochromatic p-tuples of $\mathcal{F}$ that are pairwise disjoint is larger than β.*

For an overview of this field and for further information we refer to the textbook by Matoušek [10] and the survey papers by Danzer, Grünbaum, and Klee [3], and Eckhoff [4, 5].

2 Preparations

In the above theorems the family $\mathcal{F}$ consists of general convex sets. However, we can assume that every $C \in \mathcal{F}$ is a polytope by the following standard argument. Let $\mathcal{G}$ be a subfamily of $\mathcal{F}$ with $\bigcap \mathcal{G} \neq \emptyset$, and let $z(\mathcal{G})$ be an arbitrary fixed point in $\bigcap \mathcal{G}$. The set Z consisting of the points $z(\mathcal{G})$ for all $\mathcal{G} \subset \mathcal{F}$ with $\bigcap \mathcal{G} \neq \emptyset$ is finite. Consider now a set $K \in \mathcal{F}$ and define $P(K)$ as the convex hull of all points $z(\mathcal{G}) \in Z$ with $K \in \mathcal{G}$.

Then $P(K)$ is a polytope, and the family $\mathcal{F}^* = \{P(K) : K \in \mathcal{F}\}$ has exactly the same intersection properties and same piercing number as $\mathcal{F}$ but consists of polytopes only.

As we have seen, the (p, q)-property implies the $(p, q-1)$-property. So the base case concerns the $(p, d+1)$-property. We will mainly work with this case when $d > 1$.

We will need a colourful version of the fractional Helly theorem. The original fractional Helly is due to Katchalski and Liu [8] and says the following.

Theorem 8 (Katchalski and Liu [8]) *Assume $\alpha \in (0, 1]$ and $\mathcal{F}$ is a family of n convex sets in $\mathbb{R}^d$. If at least $\alpha\binom{n}{d+1}$ of the $(d+1)$-tuples of $\mathcal{F}$ are intersecting, then $\mathcal{F}$ contains an intersecting subfamily of size $\frac{\alpha}{d+1}n$.*

The proof of Theorem 1 is based on the Alon–Kleitman lemma that will be stated next. We need the following definition. Given a finite family $\mathcal{G}$ of convex sets in $\mathbb{R}^d$, let $Z \subset \mathbb{R}^d$ be a finite set that contains one point from every nonempty intersection of elements of $\mathcal{G}$ (as described above). Now the *fractional packing number*, $\nu^*(\mathcal{G})$, of $\mathcal{G}$ is defined as

$$\nu^*(\mathcal{G}) = \max \sum_{K \in \mathcal{G}} x(K),$$

where the $x(K)$ are real variables subject to

$$\sum_{z \in K \in \mathcal{G}} x(K) \le 1 \ (\forall z \in Z), \text{ and } \quad x(K) \ge 0 \ (\forall K \in \mathcal{G}).$$

In other words, the real variables $x(K)$ assign weights between 0 and 1 to members of $\mathcal{G}$ in such a way that the sum of weights does not exceed 1 at any point of $\mathbb{R}^d$. Since the sum of $x(K)$ is the same at any point of the intersection of a subset of $\mathcal{G}$, the fractional packing number ν^* does not depend on the choice of Z.

Here comes the Alon–Kleitman lemma [1].

Lemma 1 *Let $\mathcal{G}$ be a finite family of convex sets in $\mathbb{R}^d$. Then $\tau(\mathcal{G})$ is bounded by a function of d and $\nu^*(\mathcal{G})$.*

When $\mathcal{G}$ is a finite family of convex sets in $\mathbb{R}^d$, a *blown-up copy of* $\mathcal{G}$, $\mathcal{G}^b$, is simply the same as $\mathcal{G}$ with some sets repeated (possibly deleted). The size of $\mathcal{G}^b$, $|\mathcal{G}^b|$ is the number of sets in it counted with multiplicities. The following lemma, also from [1], gives a simple and direct way to check whether $\nu^*(\mathcal{G}) \le \gamma$ for some $\gamma > 0$.

Lemma 2 *Let $\mathcal{G}$ be a finite family of convex sets in $\mathbb{R}^d$ and $\gamma > 0$. Then $\nu^*(\mathcal{G}) \le \gamma$ iff every blown-up copy of $\mathcal{G}$, say $\mathcal{G}^b$, contains an intersecting subfamily of size at least $\gamma^{-1}|\mathcal{G}^b|$.*

It will often be convenient to use the language of hypergraphs. A finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$, which is partitioned into p colour classes $\mathcal{F}_1, \ldots, \mathcal{F}_p$, gives rise to a p-partite hypergraph $\mathcal{H}$ with partition classes $\mathcal{F}_1, \ldots, \mathcal{F}_p$. The vertices of $\mathcal{H}$ are the convex sets $C \in \mathcal{F}$, its edges are of the form $e = (C_1, \ldots, C_p)$, where $C_1, \ldots, C_p$

is a heterochromatic p-tuple of $\mathcal{F}$ satisfying certain conditions. For instance $e \in \mathcal{H}$ if the heterochromatic p-tuple $C_1, \dots, C_p$ contains an intersecting q-tuple. We mention further that a blown-up copy $\mathcal{F}^b$ of the family $\mathcal{F}$ gives rise to a blown-up copy $\mathcal{H}^b$ of the corresponding hypergraph $\mathcal{H}$: the partition classes are simply $\mathcal{F}_i^b$ and $e = (C_1, \dots, C_p)$ is an edge in $\mathcal{H}^b$ iff it is an edge in $\mathcal{H}$.

3 Proof of Theorem 3

The proof uses the colourful version of the fractional Helly theorem.

Lemma 3 *Let $\mathcal{F}_1, \dots, \mathcal{F}_{d+1}$ be finite families of convex sets (colour classes) in $\mathbb{R}^d$, write $\mathcal{F}$ for their union and assume that $\alpha \in (0,1)$. If an α fraction of heterochromatic $(d+1)$-tuples of $\mathcal{F}$ are intersecting, then some $\mathcal{F}_i$ contains an intersecting subfamily of size $\frac{\alpha}{d+1}|\mathcal{F}_i|$.*

Proof The following is the standard method. Let $\mathcal{H}$ be the $(d+1)$-partite hypergraph with class i identified with $\mathcal{F}_i$ and edges $e \in \mathcal{H}$ corresponding to intersecting heterochromatic $(d+1)$-tuples of $\mathcal{F}$. Thus e is simply $(C_1, \dots, C_{d+1})$ with $C_i \in \mathcal{F}_i$ and $\bigcap_1^{d+1} C_i \neq \emptyset$. Set $C(e) = \bigcap_1^{d+1} C_i$. Define a *partial edge* as $f = (C_1, \dots, C_{i-1}, C_{i+1}, \dots, C_{d+1})$ if the intersection, $C(f)$, of these d convex sets is nonempty. Assume as we may that all $C \in \mathcal{F}$ are polytopes. Then all $C(e)$ and $C(f)$ are polytopes as well, and we can choose a vector $a \in \mathbb{R}^d$ so that the minimum of the scalar product ax over all x in $C(e)$ and the minimum over all x in $C(f)$ is reached at unique points $x(e)$ and $x(f)$.

To the best of our knowledge, the following claim was proved first by Wegner in [11]. For the sake of completeness, we present a short and simple proof here.

Claim 1 *For every $e \in \mathcal{H}$ there is a partial edge $f \subset e$ with $x(e) = x(f)$.*

Proof Let $H = \{x \in \mathbb{R}^d : ax < ax(e)\}$, this is an open halfspace and the definition of $x(e)$ implies that

$$H \cap C(e) = H \cap C_1 \cap \dots \cap C_{d+1} = \emptyset.$$

So these $d+2$ convex sets have empty intersection. By Helly's theorem some $d+1$ of them have empty intersection. This $(d+1)$-tuple cannot be $C_1, \dots, C_{d+1}$ so it is $H, C_1, \dots, C_{i-1}, C_{i+1}, \dots, C_{d+1}$ for some i. This means that $\bigcap_{j \neq i} C_j$ is disjoint from H. But it contains $x(e)$ so $x(f) = x(e)$ with $f = (C_1, \dots, C_{i-1}, C_{i+1}, \dots, C_{d+1})$. □

Now let $N_i = |\mathcal{F}_i|$ for all i and let $N = N_1 \dots N_{d+1}$. Write $\mathcal{H}_i$ for the d-partite hypergraph whose edges are the partial edges f missing class i. Clearly, $|\mathcal{H}_i| \leq N/N_i$. For $f \in \mathcal{H}_i$ let $\mathcal{F}_i(f) = \{C \in \mathcal{F}_i : x(f) \in C\}$. Note that $\mathcal{F}_i(f)$ is an intersecting subfamily of $\mathcal{F}_i$. We define α_i by

$$\alpha_i N_i = \max_{f \in \mathcal{H}_i} |\mathcal{F}_i(f)|.$$

We finish the proof by double counting the pairs (e, f) with $e \in \mathcal{H}$, $f \subset e$, $f \in \mathcal{H}_i$ for some i, and $x(e) = x(f)$. Claim 1 says that the number of such pairs is at least $\alpha N_1 \dots N_{d+1} = \alpha N$. Hence

$$\begin{aligned}\alpha N &\le \text{number of such pairs } (e, f)\\ &= \sum_{i=1}^{d+1} \sum_{f\in\mathcal{H}_i} \text{number of } e \in \mathcal{H} \text{ with } (e, f) \text{ being such a pair}\\ &\le \sum_{i=1}^{d+1} \sum_{f\in\mathcal{H}_i} \left|\left\{C \in \mathcal{F}_i : x(f) \in C\right\}\right| \le \sum_{i=1}^{d+1} \sum_{f\in\mathcal{H}_i} \alpha_i N_i\\ &\le \sum_{i=1}^{d+1} \alpha_i N_i \frac{N}{N_i} = \sum_{i=1}^{d+1} \alpha_i N.\end{aligned}$$

This implies that $\alpha \le \sum_1^{d+1} \alpha_i$ and so $\alpha_i \ge \frac{\alpha}{d+1}$ for some i. □

Proof of Theorem 3 We are going to use the Alon–Kleitman lemma (Lemma 1). We set $\gamma = (d+1)\binom{p}{d+1}$ and want to show first that $\nu^*(\mathcal{F}_i) \le \gamma$ for some $i \in [p]$. So we have to prove, by using Lemma 2, that in every blown-up copy $\mathcal{F}^b$ of $\mathcal{F}$ some $\mathcal{F}_i^b$ contains an intersecting subfamily of size $\gamma^{-1}|\mathcal{F}_i^b|$.

We are going to use the complete p-partite hypergraph $\mathcal{H}$ associated with the family $\mathcal{F}$, and its blown-up copy $\mathcal{H}^b$. When $e = (C_1, \dots, C_p)$ is an edge of $\mathcal{H}^b$ (or what is the same, of $\mathcal{H}$) and J is a subset of $[p]$, we write $e(J)$ for the *partial edge* $(C_j : j \in J)$. For $I \in \binom{[p]}{d+1}$ define the $(d+1)$-partite hypergraph $\mathcal{H}^b(I)$ whose classes are $\mathcal{F}_i^b$, $i \in I$, and $f = (C_i : i \in I)$ is an edge of $\mathcal{H}^b(I)$ if $\bigcap_{i\in I} C_i \ne \emptyset$.

Claim 2 *Some $\mathcal{H}_i^b$ has at least $\delta|\mathcal{H}_i^b|$ edges where*

$$\delta = \binom{p}{d+1}^{-1}.$$

This follows from double counting the pairs (e, f) with $e \in \mathcal{H}^b$ and $f = e(I) \in \mathcal{H}^b(I)$. Set $|\mathcal{F}_i^b| = N_i$ (repeated sets counted with their multiplicity) and define $N = N_1 \dots N_p$. The $(p, d+1)_H$-condition implies that for every $e \in \mathcal{H}^b$ there is an $I \in \binom{[p]}{d+1}$ such that $e(I) \in \mathcal{H}^b(I)$. This gives the first inequality below:

$$\begin{aligned}N &\le \text{number of such pairs } (e, f)\\ &= \sum_{\text{all } I} \sum_{f\in\mathcal{H}^b(I)} \left|\left\{e \in \mathcal{H}^b : f = e(I)\right\}\right|\\ &\le \sum_{\text{all } I} \sum_{f\in\mathcal{H}^b(I)} \prod_{j\notin I} N_j\end{aligned}$$

$$= N \sum_{\text{all } I} \frac{1}{\prod_{i \in I} N_i} |\mathcal{H}^b(I)|.$$

This implies that some $\mathcal{H}^b(I)$ indeed has at least $\delta|\mathcal{H}^b(I)|$ edges. □

This finishes the proof quite quickly. The edge density in some $\mathcal{H}^b(I)$ is at least δ. By the coloured fractional Helly theorem (Lemma 3), some $\mathcal{F}_i^b$ with $i \in I$ has an intersecting subfamily of size $\delta/(d+1)|\mathcal{F}_i^b|$. Consequently, by Lemma 2, $\nu^*(\mathcal{F}_i) \leq (\delta/(d+1))^{-1} = \gamma$.

This was the proof for the base case $q = d+1$. For the general case of Theorem 3 we need to find $q-d$ families $\mathcal{F}_i$ with bounded piercing number. This is quite easy: We find the first one, say $\mathcal{F}_1$, with the previous proof. Then the family $\mathcal{F} \setminus \mathcal{F}_1$ is $p-1$ coloured, and satisfies the $(p-1, q-1)$ condition. The previous proof gives another family, say $\mathcal{F}_2$ with bounded τ. We repeat this process $q-d$ times and get $q-d$ families with bounded piercing number. □

4 Proof of Theorem 4

The proof is simple and short. Let $\mathcal{H}$ be the p-partite hypergraph whose classes are $\mathcal{F}_1, \ldots, \mathcal{F}_p$ and where $e = (C_1, \ldots, C_p)$ is an edge if the p-tuple $C_1, \ldots, C_p$ contains an intersecting $(d+1)$-tuple. Set $N_i = |\mathcal{F}_i|$ and $N = N_1 \cdots N_p$ as before. Also, for $I \in \binom{[p]}{d+1}$ let $\mathcal{H}(I)$ be the $(d+1)$-partite hypergraph with classes $\mathcal{F}_i, i \in I$ and where $f = (C_i : i \in I)$ is an edge if $\bigcap_{i \in I} C_i \neq \emptyset$. Apply the previous double counting to the hypergraph $\mathcal{H}$ (instead of $\mathcal{H}^b$). The $(p, d+1)_H$-condition with α fraction exceptions guarantees that $\mathcal{H}$ has $(1-\alpha)N$ edges. The rest of the double counting is the same and we conclude that some $\mathcal{H}(I)$ has at least $(1-\alpha)\delta \prod_{i \in I} N_i$ edges with the same δ as before. The colourful fractional Helly theorem implies that some $\mathcal{F}_i$ (with $i \in I$) has an intersecting subfamily of size $(1-\alpha)\delta/(d+1)|\mathcal{F}_i|$. □

5 Coloured Families of Intervals in $\mathbb{R}$

Let p be a positive integer, and let $\mathcal{F}$ be a finite family of intervals in $\mathbb{R}$, coloured with p colours. The intervals with colour i form the subfamily $\mathcal{F}_i$. We may assume (after applying the standard method from Sect. 2) that all intervals in $\mathcal{F}$ are closed. Clearly, there is a $\delta > 0$ such that any two disjoint intervals in $\mathcal{F}$ are at least at distance δ from each other. Now replace each interval $I \in \mathcal{F}$ by an open interval I^* containing I and contained in a $\delta/3$ neighbourhood of I. This gives rise to a new family $\mathcal{F}^*$. It is evident that this can be done in such a way that no two intervals in $\mathcal{F}^*$ have a common endpoint. It is also clear that $\mathcal{F}^*$ has the same intersection pattern and the same values for $\tau(\mathcal{F}^*)$ and $\tau(\mathcal{F}_i^*)$ as $\mathcal{F}$. From now on we assume that $\mathcal{F}$ consists of bounded open intervals no two of which have a common endpoint.

The following lemma, in a slightly different setting, was proved by Gyárfás and Lehel in [6]. For the sake of completeness, we present the short and simple proof.

Lemma 4 (Gyárfás and Lehel [6]) *Assume that $\mathcal{F}$ is a finite family of intervals in $\mathbb{R}$, coloured with p colours such that each colour class contains at least p pairwise disjoint intervals. Then there exists a pairwise disjoint heterochromatic p-tuple in $\mathcal{F}$.*

The *proof* goes by induction on p. The case $p = 1$ is obvious. For the induction step $p - 1 \to p$, $(p \geq 2)$ let a be the leftmost right endpoint of all intervals in $\mathcal{F}$. We assume, without loss of generality, that a is the right endpoint of some interval I_1 from the first colour class $\mathcal{F}_1$. Delete all intervals from $\mathcal{F} \setminus \mathcal{F}_1$ that contain a. The resulting family $\mathcal{F}'$ of intervals is coloured with $p - 1$ colours, and each colour class $\mathcal{F}'_j$ contains at least $p - 1$ disjoint intervals as only intervals containing the point a have been deleted from $\mathcal{F}_i$. The induction hypothesis guarantees the existence of disjoint intervals $I_j \in \mathcal{F}'_j \subset \mathcal{F}_j$, $j \in \{2, \ldots, p\}$. All of these $p - 1$ intervals are to the right of a, and so $I_1, I_2, \ldots, I_p$ is a heterochromatic p-tuple consisting of disjoint intervals. □

We need the following lemma.

Lemma 5 *Let $p \geq q \geq 2$ be integers and $\mathcal{F}$ a finite family of intervals in $\mathbb{R}$ coloured with p colours. If $\mathcal{F}$ has the $(p, q)_H$-property, then there is a colour class $\mathcal{F}_i$ such that $\tau(\mathcal{F}_i) \leq p - q + 1$.*

Note that for $p = 2$, Lemma 5 is the colourful Helly theorem (Theorem 2) in one dimension.

The *proof* is indirect, elementary and constructive. We describe the argument in detail because the construction will be used later to improve the upper bound on $\tau(\mathcal{F}_i)$.

Assume, on the contrary, that $\tau(\mathcal{F}_i) \geq p - q + 2$ for each $i = 1, \ldots, p$. We will find a heterochromatic p-tuple in $\mathcal{F}$ in which no q elements intersect, and thus reach a contradiction.

The indirect assumption implies that each colour class $\mathcal{F}_i$ must contain at least $p - q + 2$ pairwise disjoint intervals. Lemma 4 yields the existence of a pairwise disjoint heterochromatic $(p - q + 2)$-tuple of intervals $\{I_1, \ldots, I_{p-q+2}\}$ with $I_j \in \mathcal{F}_j$ for $j = 1, \ldots, p - q + 2$.

Select one arbitrary interval $I_k \in \mathcal{F}_k$ from each one of the remaining colour classes $k = p - q + 3, \ldots, p$. Clearly, the set of intervals $\{I_1, \ldots, I_p\}$ is a heterochromatic p-tuple with the property that any q-element subset of it must contain two disjoint intervals from the set $\{I_1, \ldots, I_{p-q+2}\}$ and thus cannot be intersecting. □

Note that in the case $q = 2$, the upper bound in Lemma 5 is best possible. This fact is shown by the following example.

Example 1 Let $p \geq q = 2$ be positive integers. For every $i \in [p]$ the family $\mathcal{F}_i$ consists of the same $p - 1$ pairwise disjoint intervals $I_1, \ldots, I_{p-1}$. So $\mathcal{F}$ consists of p copies of each I_j. The pigeonhole principle shows that $\mathcal{F}$ has the $(p, 2)_H$-property. At the same time, $\tau(\mathcal{F}_i) = p - 1$ for each colour class.

6 Proof of Theorem 5

Lemma 5 implies that $\tau(\mathcal{F}_i) \le p - q - 1$ for at least one colour class. It is easy to see (we omit the details) that

$$\left\lfloor \frac{p-1}{q-1} \right\rfloor = \max\left\{ m \in \mathbb{N} \mid q \le \left\lceil \frac{p}{m} \right\rceil \right\}. \tag{1}$$

Set

$$m := \min\{\tau(\mathcal{F}_i) : i = 1, \dots, p\}.$$

This implies that there are at least m pairwise disjoint intervals in each colour class $\mathcal{F}_i \subset \mathcal{F}$. According to Lemma 5, $1 \le m \le p - q + 1$. Let

$$p = km + r, \quad \text{where } k, r \in \mathbb{N} \text{ and } 0 \le r < m.$$

For each $0 \le l \le k - 1$, Lemma 4 yields the existence of m pairwise disjoint intervals $\{I_{lm+1}, \dots, I_{(l+1)m}\}$ of mutually different colours with $I_{lm+j} \in \mathcal{F}_{lm+j}$ for $j = 1, \dots, m$.

If $r > 0$, then, again by Lemma 4, there exist r pairwise disjoint intervals $\{I_{km+1}, \dots, I_p\}$ of mutually different colours, one from each of the remaining r colour classes $\mathcal{F}_{km+1}, \dots, \mathcal{F}_p$. The set $\{I_1, \dots, I_p\}$ just constructed is a pairwise disjoint heterochromatic p-tuple of intervals, which consists of $\lceil p/m \rceil$ groups and each group contains m disjoint intervals (all of them of distinct colours) except the last group which contains r disjoint intervals.

If $q > \lceil p/m \rceil$, then the pigeonhole principle guarantees that any q-element subset of $\{I_1, \dots, I_p\}$ contains two intervals from the same group and so they are disjoint. This contradicts the hypothesis of the theorem, implying that $q \le \lceil p/m \rceil$. Formula (1) then shows that indeed $m \le \lfloor \frac{p-1}{q-1} \rfloor$. □

The following example shows that upper bound in Theorem 5 is best possible.

Example 2 Let $p \ge q \ge 2$ be positive integers and let $m = \lfloor \frac{p-1}{q-1} \rfloor$. Let the family $\mathcal{F}$ consist of m pairwise disjoint intervals $I_1, I_2, \dots, I_m$, each taken with multiplicity p, and let the colour classes be $\mathcal{F}_i := \{I_1, \dots, I_m\}$, for all $i = 1, \dots, p$.

It is clear that $\mathcal{F}$ satisfies the $(p, q)_H$-property because any heterochromatic p-tuple of intervals must contain at least q copies of one of the intervals $I_1, \dots, I_m$, again by the pigeonhole principle. Further, $\tau(\mathcal{F}_i) = \lfloor \frac{p-1}{q-1} \rfloor$ for all $i = 1, \dots, p$.

Remark 1 There is no similar theorem in the uncoloured case: the (p, q)-condition implies $\tau(\mathcal{F}) \le p - q + 1$ (by the Hadwiger–Debrunner results [7]) and this bound is best possible, as shown by $p - q + 1$ disjoint intervals, one of them taken with arbitrary (large) multiplicity, and the others with multiplicity one. This means that, not surprisingly, the $(p, q)_H$-condition on p repeated copies of $\mathcal{F}$ is stronger than the (p, q)-condition on $\mathcal{F}$.

Remark 2 Under the hypotheses of Theorem 5, there exists a colour class, say $\mathcal{F}_1 \subset \mathcal{F}$, with $\tau(\mathcal{F}_1) \le \lfloor \frac{p-1}{q-1} \rfloor$. Then the subfamily $\mathcal{F} \setminus \mathcal{F}_1$ satisfies the $(p-1, q-1)_H$ property and Theorem 5 guarantees the existence of a colour class, say $\mathcal{F}_2 \subset \mathcal{F} \setminus \mathcal{F}_1$, with $\tau(\mathcal{F}_2) \le \lfloor \frac{p-2}{q-2} \rfloor$. Repeating this argument $q-2$ times, we obtain $q-2$ colour classes, say $\mathcal{F}_k$, $k = 1, \dots, q-2$, with $\tau(\mathcal{F}_k) \le \lfloor \frac{p-k}{q-k} \rfloor$.

Let $p \ge 3$. Assume that the family $\mathcal{F}$ is coloured with p colours and has the $(p, p-1)_H$-property. Applying the above argument to $\mathcal{F}$, we obtain that $p-3$ of the colour classes of $\mathcal{F}$ have piercing number one and one colour class has piercing number at most two.

7 An Extension of Theorem 6 and a Construction

Theorem 5 says that, under the $(p, q)_H$-condition, some colour class of the family $\mathcal{F}$ of intervals can be pierced by $\lfloor \frac{p-1}{q-1} \rfloor$ points. Thus, it is not surprising that Theorem 6 can be generalised so that all intervals of some colour class are pierced by k points, where $k \in \{\lfloor \frac{p-1}{q-1} \rfloor, \dots, p-q+1\}$:

Theorem 9 *Let $p \ge q \ge 2$ be integers, k another integer with $\lfloor \frac{p-1}{q-1} \rfloor \le k \le p-q+1$, $h = q-1+\lfloor (q-p-1)/k \rfloor$, and $\alpha \in [0, \alpha_0)$ where $\alpha_0 = \frac{1}{2}(k+2)^{-1/(p-h)}$. Then there is a number $\beta = \beta(p, q, k, \alpha) \in [0, 1)$ and an integer $n_0 = n_0(p, q, k, \alpha)$ such that the following holds. Let $\mathcal{F}$ be a finite and coloured family of intervals in $\mathbb{R}$ with colour classes $\mathcal{F}_1, \dots, \mathcal{F}_p$ where each $|\mathcal{F}_i| \ge n_0$. If $\mathcal{F}$ satisfies the $(p, q)_H$-property with the exception of at most $\alpha \prod_{j=1}^{p} |\mathcal{F}_j|$ heterochromatic p-tuples, then there exists a colour class $\mathcal{F}_i \subset \mathcal{F}$ such that the elements of $\mathcal{F}_i$ can be pierced by at most k points with the exception of at most $\beta |\mathcal{F}_i|$ intervals. Furthermore, $\beta = O(\alpha^{1/(p-h)})$.*

Note that this is exactly Theorem 6 when $k = p-q+1$ and $h = q-2$. We mention further that, as one can easily see, the h defined above is the largest integer l satisfying $\lfloor \frac{p-l}{q-l} \rfloor \le k$.

In the next section we shall prove Theorems 9 and 6 simultaneously. The proof will use the following construction. Assume that $\mathcal{G}$ is a finite family of bounded open intervals in $\mathbb{R}$ with no two intervals having the same endpoint. Suppose that a is the right endpoint of some interval from $\mathcal{G}$. We construct a subfamily $\mathcal{G}(a)$ of $\mathcal{G}$ as follows. Denote by $T(a)$ the collection of all intervals $I \in \mathcal{G}$ lying to the left of a and by $\mathcal{G}(a)$ the collection of all intervals to the right of a.

Now let $\mathcal{G} = \{I_1, \dots, I_n\}$, each I_i is open and no two intervals have a common endpoint. Define $t := \lceil \gamma n \rceil$ where $\gamma > 0$ is a parameter.

The right endpoints of the I_js form an increasing sequence of n distinct numbers. Let a_1 be its tth element, in other words, a_1 is the tth smallest right endpoint of the intervals in $\mathcal{G}$. Then $T_1 = T(a_1)$ consists of exactly t intervals and every interval in $\mathcal{G}^1 = \mathcal{G}(a_1)$ is to the right of a_1.

Assume that the families $\mathcal{G}^j \subset \mathcal{G}^{j-1} \subset \dots \subset \mathcal{G}$ have already been constructed. Assuming that $|\mathcal{G}^j| \ge t$, let a_{j+1} the tth smallest right endpoint of the intervals in $\mathcal{G}^j$. Then $T_{j+1} = T(a_{j+1})$ consists of exactly t intervals, and we set $\mathcal{G}^{j+1} = \mathcal{G}^j(a_{j+1})$. We can continue this construction as long as $|\mathcal{G}^j| \ge t$.

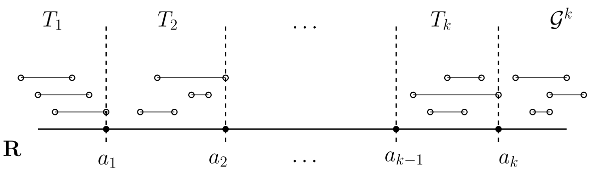

Fig. 1 Construction of T_i and $\mathcal{G}^k$

Fact *The points $a_1, \ldots, a_k$ pierce all but $kt + |\mathcal{G}^k|$ intervals from $\mathcal{G}$, cf. Fig. 1.*

8 Proof of Theorems 9 and 6

We assume again that all intervals in $\mathcal{F}$ are open and no two of them have a common endpoint. Let $n_i = |\mathcal{F}_i|$, $t_i = \lceil \gamma n_i \rceil$ where $\gamma = (2\alpha)^{1/(p-h)}$, and define $\beta = (k+2)\gamma$. Note that $\beta < 1$ follows because $\alpha < \alpha_0$.

For each colour class $\mathcal{F}_i \subset \mathcal{F}$, $i \in [p]$ we apply the above construction giving points $a_1^i, \ldots, a_j^i$ and sets $T_1^i, \ldots T_j^i$, and call the class *short* if the construction cannot be continued up to $j = k$. We note that we are done if some $\mathcal{F}_i$ is short; the Fact from Sect. 7 shows that points $a_1^i, \ldots, a_j^i$ pierce all but at most $jt_i + |\mathcal{F}_i^j| < (j+1)t_i < (k+1)\lceil \gamma n_i \rceil < \beta n_i$ intervals from $\mathcal{F}_i$. Here the last inequality follows from the choice of β and $n_i \geq n_0$ and $\alpha < \alpha_0$.

So we assume that there are no short colour classes, that is, a_k^i exists for all i. Let T^i denote the set of intervals in $\mathcal{F}_i$ that are to the right of a_k^i. It follows that $|T_j^i| = t_i$ for $j = 1, \ldots, k$ and any two intervals from two different sets among $T_1^i, \ldots, T_k^i, T^i$ are disjoint.

We are going to show that $|T^i| < t_i$ for some i. This will finish the proof since then $\mathcal{F}_i$ is pierced by the points $a_1^i, \ldots, a_k^i$ except for at most $kt_i + |T^i| < (k+1)t_i = (k+1)\lceil \gamma n_i \rceil < \beta n_i$ intervals where, again, the last inequality follows the same way as above. So assume, on the contrary, that $|T_i| \geq t_i$ for all i.

For $i \in [p-h]$ we define a family of intervals $\mathcal{G}_i$ by setting

$$\mathcal{G}_i := \left\{ \left(-\infty, a_1^i\right), \left(a_1^i, a_2^i\right), \ldots, \left(a_k^i, \infty\right) \right\},$$

their union, $\mathcal{G}$, is a family of intervals coloured with $p-h$ colours.

Claim 3 *For each $i \in [p-h]$ there is an interval $I_{j(i)} \in \mathcal{G}_i$ such that no $q-h$ of the $I_{j(i)}$s intersect.*

Proof If $k = p - q + 1$, then $h = q - 2$, and Lemma 4 guarantees the existence of a pairwise disjoint heterochromatic $(k+1)$-tuple in $\mathcal{G}$. If $k < p - q + 1$, then no $\mathcal{G}_i$ can be pierced by k points, and so by Theorem 5, $\mathcal{G}$ does not have the $(p-h, q-h)_H$-property. (This is where we use the choice of h.) Consequently, there are intervals $I_{j(i)} \in \mathcal{G}_i$ for each $i \in [p-h]$ such that no $q-h$ of the $I_{j(i)}$s intersect. □

Define S_i as the set of intervals from $\mathcal{F}_i$ that are contained in $I_{j(i)}$, so S_i coincides with some T_j^i or T^i. Consequently, $|S_i| \geq t_i$ for all i.

We count those heterochromatic p-tuples that contain one interval from every S_i, $i \in [p-h]$. Such a p-tuple cannot contain an intersecting q-tuple. Their number is at least

$$\prod_{i=1}^{p-h} |S_i| \prod_{j=p-h+1}^{p} |\mathcal{F}_j| \geq \prod_{i=1}^{p-h} t_i \prod_{j=p-h+1}^{p} n_j \geq \gamma^{p-h} \prod_{i=1}^{p} n_i = 2\alpha \prod_{i=1}^{p} n_i,$$

a contradiction, as $\mathcal{F}$ contains at most $\alpha \prod_1^p n_i$ heterochromatic p-tuples with no intersecting q-tuple. □

Remark 3 This proof gives a little more, namely the following. Under the conditions of the theorem there are at least $h+1$ colour classes $\mathcal{F}_i$ that can be pierced by k points except for βn_i intervals. The argument is easy: assume there are l short colour classes. We are done if $l \geq h+1$. Suppose then that $l \leq h$. There are $p-l \geq p-h$ non-short colour classes and any $p-h$ of them can be used in the above proof to give another non-short colour class with the required piercing property. We can repeat the argument getting further and further non-short colour classes until we have a total of $h+1$ colour classes, each pierced by a set of size at most k except for a β fraction of the intervals in the class.

The following example shows that the order of magnitude of β in Theorem 9 is optimal.

Example 3 Let $p \geq q \geq 2$ be positive integers, define k and h as above, let $0 < \beta < 1/(p-h+1)$ be a real number to be specified later, and set $\delta = (k+1)\beta$. Fix pairwise disjoint intervals $I_1, \dots, I_{k+1}$ and a big interval I containing their union. The family $\mathcal{F}_i$ is the same for all $i \in [p]$: it contains each of $I_1, \dots, I_{k+1}$ with multiplicity βn, and the interval I with multiplicity $(1-\delta)n$. Hence such an $\mathcal{F}_i$ is pierced by k points except for βn intervals.

Suppose that a given heterochromatic p-tuple P of $\mathcal{F}$ is *bad* in the sense that it does not contain an intersecting q-tuple. Say, the p-tuple contains exactly l copies of I and s_j copies of I_j, $j \in [k+1]$. We check that $l \leq h$. This is trivial if $k = p-q+1$ since then $h = q-2$ and $l > h$ would imply $l \geq q-1$. Thus P would contain an intersecting p-tuple. If $k < p-q+1$ and $l > h$, then $s_j \leq q-1-l$ for all j, and the definition of h would give

$$p = s_1 + \cdots + s_{k+1} + l \leq (k+1)(q-1-l) + l = k(q-1-l) + q-1 < p,$$

a contradiction.

We call the sequence $s_1, \dots, s_{k+1}, l$ the *profile* of P. The number of possible profiles of bad p-tuples with l copies of I is an integer $f(p,q,l)$, independent of n. Set $f(p,q) = \sum_0^h f(p,q,l)$.

The number of bad p-tuples with a fixed profile $s_1, \dots, s_{k+1}, l$ is

$$\big((1-\delta)n\big)^l (\beta n)^{s_1} (\beta n)^{s_2} \cdots (\beta n)^{s_{k+1}} = (1-\delta)^l \beta^{p-l} n^p.$$

As $\beta < 1/(p-h+1)$ the total number of bad p-tuples is

$$\sum_{l=0}^{h} f(p,q,l)(1-\delta)^l \beta^{p-l} n^p$$
$$\leq \sum_{l=0}^{h} f(p,q,h)(1-\delta)^h \beta^{p-h} n^p$$
$$= f(p,q)\big(1-(k+1)\beta\big)^h \beta^{p-h} n^p = \alpha n^p,$$

when we define β by requiring $f(p,q)(1-(k+1)\beta)^h \beta^{p-h} = \alpha$. It is easy to see that for α small enough there is a unique solution β in the interval $(0, 1/(p-h+1))$ and $\beta = \Omega(\alpha^{1/(p-h)})$. The order of magnitude $\beta = O(\alpha^{1/(p-h)})$ in Theorem 9 is indeed best possible.

9 Proof of Theorem 7

Set $|\mathcal{F}| = n$, $t = \lceil \gamma n \rceil$ where $\gamma = (q-1)^{(p-1)/p} \alpha^{1/p}$, and $k = p - q + 1$. We apply the construction of Sect. 7 to $\mathcal{F}$. If it stops before reaching a_k, then we are done the same way as before. So assume the construction produces points $a_1, \dots, a_k$ and families of intervals $T_1, \dots, T_k, T$ from $\mathcal{F}$. Then $|T_i| = t$ for all i and we are done, again, if $|T| < t$. So assume, for a contradiction, that $|T| \geq t$.

Next we derive a lower bound on the number of p-tuples in $\mathcal{F}$ that contain no intersecting q-tuple. We only consider the following specific types of p-tuples: all intervals are from $T_1 \cup \cdots T_k \cup T$ with at least one interval and at most $q-1$ intervals from every set $T_1, \dots, T_k$ and T. We will call such a p-tuple *bad*. Every q-tuple from a bad p-tuple contains intervals from at least two of the sets $T_1, \dots, T_k, T$ and thus its intersection is empty. Therefore a bad p-tuple does not have the q-intersection property.

A bad p-tuple has, say, s_i intervals from T_i for $i = 1, \dots, k$, and l intervals from T. Then $p = s_1 + \cdots + s_k + l$ and $s_1, \dots, s_k$ and l are integers from $[q-1]$. Call the sequence $s_1, \dots, s_k, l$ the *profile* of the given p-tuple, and let $g(p,q,l)$ be the number of profiles of bad p-tuples with $|T| = l$. The number of bad p-tuples with given profile $s_1, \dots, s_k, l$ is

$$\binom{|T|}{l} \prod_{i=1}^{k} \binom{t}{s_i} \geq \left(\frac{|T|}{l}\right)^l \prod_{i=1}^{k} \left(\frac{t}{s_i}\right)^{s_i} > \left(\frac{|T|}{q-1}\right)^l \prod_{i=1}^{k} \left(\frac{t}{q-1}\right)^{s_i}$$
$$= \left(\frac{|T|}{q-1}\right)^l \left(\frac{t}{q-1}\right)^{p-l}.$$

Let N denote the total number of bad p-tuples. As $g(p,q,l) \geq 1$,

$$N > \sum_{l=1}^{q-1} g(p,q,l) \left(\frac{|T|}{q-1}\right)^l \left(\frac{t}{q-1}\right)^{p-l} \geq \frac{1}{(q-1)^p} \sum_{l=1}^{q-1} |T|^l t^{p-l},$$

which is a non-decreasing function of $|T|$. As $|T| \geq t$, we have

$$N > (q-1)\frac{1}{(q-1)^p}t^p \geq \frac{1}{(q-1)^{p-1}}\left((q-1)^{\frac{p-1}{p}}\alpha^{\frac{1}{p}}\right)^p n^p = \alpha n^p > \alpha\binom{n}{p}.$$

This contradicts the assumption of Theorem 7, and so $|T| < t$ must be true. Further, $a_1, \dots, a_k$ pierce all but at most $(k+1)t$ intervals from $\mathcal{F}$ and so $\beta = O(\alpha^{1/p})$. □

Under the conditions of Theorem 7 one can give a better bound, namely, $\beta = O(\alpha^{1/(p-q+2)})$ provided $n > p^p/\alpha$. To prove this one should take each set in $\mathcal{F}$ with multiplicity p giving colour classes $\mathcal{F}_1, \dots, \mathcal{F}_p$ and apply Theorem 6 to this new family. We omit the details. We mention that the monochromatic version of Example 3 shows that this β is of order $\alpha^{1/(p-q+2)}$ when α is small and $n > p^p/\alpha$.

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