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Proof of the Katchalski–Lewis Transversal Conjecture for $T(3)$-Families of Congruent Discs*

Aladár Heppes

Rényi Institute of Mathematics, Hungarian Academy of Sciences,
Pf. 127, H-1364 Budapest, Hungary
aheppes@renyi.hu

Abstract. A family of disjoint closed congruent discs is said to have property $T(3)$ if to every triple of discs there exists a common line transversal. Katchalski and Lewis [10] proved the existence of a constant m_{disc} such that to every family of disjoint closed congruent discs with property $T(3)$ a straight line can be found meeting all but at most m_{disc} of the members of the family. They conjectured that this is true even with $m_{\text{disc}} = 2$. On one hand Bezdek [1] proved $m_{\text{disc}} \geq 2$ in 1991 and on the other hand Kaiser [9] showed $m_{\text{disc}} \leq 12$ in a recent paper. The present work is devoted to proving this conjecture showing that $m_{\text{disc}} \leq 2$.

1. Introduction

Definitions. Throughout this paper the term *disc* is used for a solid circle of diameter 1 and the term *unit disc* for a solid circle of radius 1. $D(X)$ denotes the closed disc and $U(X)$ the closed unit disc about X. A *line transversal* to a family of compact convex domains is a straight line having a non-empty intersection with every member of the family. If a family has a line transversal we also say that this family has property T. A family $\mathcal{F}$ of at least k compact convex domains is a $T(k)$*-family*, if every k-membered subset of it has property T.

Very likely Vincensini [13] was the first who made efforts to find Helly type results for families of domains using transversals. From his followers we refer here only to Grünbaum [5] who formulated the conjecture below concerning families of pairwise disjoint translates of a compact convex domain:

Conjecture G. *In a family $\mathcal{F}$ of pairwise disjoint translates of a compact convex set S in the plane $T(5) \Rightarrow T$.*

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Danzer [2] proved this conjecture for the special case when S is a disc as early as in 1957 and Grünbaum [5] proved it for parallelograms the following year, while for the proof of the general case, given by Tverberg [11], we had to wait until 1989.

These theorems are sharp in the sense that 5 cannot be replaced by 4. Although $T(4)$ does not imply T, even $T(3)$ has interesting consequences. Katchalski and Lewis [10] proved that in a family of property $T(3)$ there exists a transversal to *almost all* members of the family, more exactly, the number of the "exceptional" members is limited from above by a universal (shape-independent) constant m.

Definition. We say that $\mathcal{F}$ has property "$T - m$" if there exists a subfamily $\mathcal{G} \subset \mathcal{F}$, $|\mathcal{F} \setminus \mathcal{G}| \leq m$ such that $\mathcal{G}$ has a line transversal.

Theorem K-L [10]. *There is a constant m such that to every family $\mathcal{F}$ of pairwise disjoint translates of a compact convex set S in the plane $T(3) \Rightarrow T - m$.*

Katchalski and Lewis [10] proved that $m \leq 192\pi$, and conjectured that $m = 2$. Improvements on this inequality can be found in [12], and the current best value is 22 [8].

Families of special domains have also been considered. (Notice that problems of this nature are equivalent for affine images.) For families of squares Holmsen [7], [8] proved

Theorem H. *If $\mathcal{F}$ is a $T(3)$-family of disjoint translates of a closed square then $m_{\text{square}} = 4$ is the smallest value for which $T(3) \Rightarrow T - m_{\text{square}}$ holds.*

Thus, for squares (and parallelograms) the problem is fully settled. Holmsen's result also disproves the expected shape-independent upper bound $m \leq 2$ for the number of exceptional translates, at least the general validity of this inequality.

Families of discs have also been studied. By a construction of Bezdek [1] we have the following lower bound:

Theorem B. *To every $n > 5$ there exists a $T(3)$-family $\mathcal{F}$ of n disjoint closed congruent discs such that no straight line of the plane meets more than $n - 2$ members of the family, i.e. $T(3)$ does not imply $T - 1$.*

Thus we have $m_{\text{disc}} \geq 2$. Recently the upper bound $m = m_{\text{disc}} \leq 12$ was published by Kaiser [9]:

Theorem K. *If $\mathcal{F}$ is a $T(3)$-family of disjoint closed congruent discs then $T(3) \Rightarrow T - 12$.*

In the present paper our goal is to prove the following:

Theorem 1. *If $\mathcal{F}$ is a finite $T(3)$-family of $n > 5$ disjoint closed congruent discs then $T(3) \Rightarrow T - 2$.*

Remark. Theorems 1 and B imply that $m_{\text{disc}} = 2$ is the smallest value for which $T(3) \Rightarrow T - m_{\text{disc}}$. This result and Theorem H together show that while the smallest possible number m does not depend on the size of the (finite) family it does depend on the *shape* of the translates. Since in a sense the discs are the roundest and the squares the least round convex figures of central symmetry it can be expected that for every "shape-dependent m_s" the inequality

$$2 \leq m_s \leq 4$$

generally holds. It is an additional open question whether there exist compact convex figures for which $m = 3$ is best.

For more information on transversal problems the reader is advised to consult the survey papers by Eckhoff [3], Goodman et al. [4] and Wenger [14].

2. Preparation

Definition. A *transversal strip* of a family of discs is a closed (parallel) strip intersecting all members of the family and the *transversal width* of a family is the infimum of the width of its transversal strips. For each finite family of discs a transversal strip of *minimal* width also exists.

Definitions. The *sheaf* belonging to the two disjoint discs $D(X)$ and $D(Y)$ is the union of all straight lines intersecting both discs. This is a simply connected unbounded domain the boundary of which consists of parts of the four lines tangent to both discs. It is denoted by $\Sigma(X, Y)$ (Fig. 2.1(a)). Clearly, a third disc has a common transversal with $D(X)$ and $D(Y)$ iff it is intersecting the sheaf $\Sigma(X, Y)$. The outer parallel domain of radius $\frac{1}{2}$ of sheaf $\Sigma(X, Y)$, which is the locus of the *centers* of the discs intersecting $\Sigma(X, Y)$, is called the *center sheaf* belonging to X and Y and is denoted by $\Sigma^{c}(X, Y)$. Two of the (at most) six lines generating the boundary of the center sheaf $\Sigma^{c}(X, Y)$ are non-separating tangent lines to both unit discs $U(X)$ and $U(Y)$ and the four other lines are tangent to one of the two unit discs and pass through the center of the other one (see Fig. 2.1(b)).

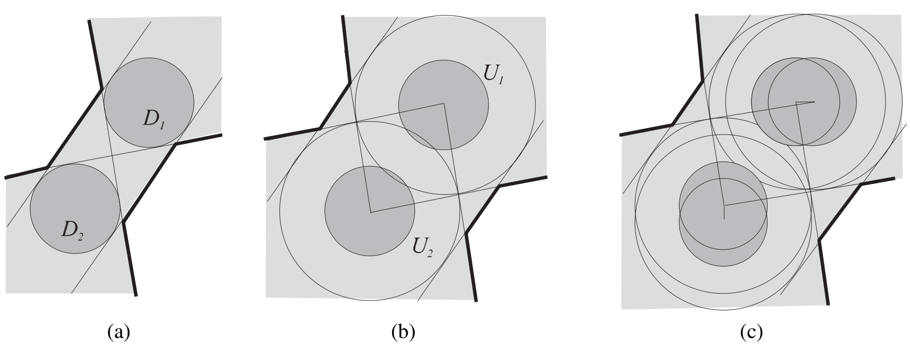

Fig. 2.1. (a) Sheaf, (b) center sheaf, and (c) generalized center sheaf.

Definition. The *generalized center sheaf* belonging to two orthogonal segments also play an important role in some parts of the proof (Fig. 2.1(c)). This is the union of all center sheaves belonging to the pairs of discs, one of them centered in the first segment and the other centered in the second one. $\Sigma^c(\overline{X_1X_2}, \overline{Y_1Y_2})$ denotes the generalized center sheaf belonging to the segments $X \in \overline{X_1X_2}$ and $Y \in \overline{Y_1Y_2}$, where $\overline{X_1X_2}$ and $\overline{Y_1Y_2}$ denote the segments on the x- and y-axis connecting X_1 and X_2 and Y_1 and Y_2, respectively. The generalized center sheaf is a domain bounded by parts of at most six straight lines—like in the case of the common center sheaf, however, the generalized center sheaf is usually not symmetric.

Definition. We say that two closed domains are *strictly* (*weakly*) *separated* if there exist a straight line such that the domains (the interiors of the domains) lie in different open half-planes defined by the line. Symbol $\lambda(XY)$ is used for the line connecting points X and Y and $\lambda^\ell(X, Y)$, $\lambda^r(X, Y)$, for the common non-horizontal support lines of sets X and Y on the left and on the right, respectively.

Throughout the paper σ denotes a (not necessarily unique) narrowest transversal strip of $\mathcal{F}$ and w denotes the width of σ. We assume that σ is horizontal. Strip σ^+ is the outer parallel domain of radius $\frac{1}{2}$ of strip σ. This is the narrowest strip covering all centers of $\mathcal{F}$ and its width is $1 + w$.

As Tverberg points out the nature of the problem allows us to assume that the discs are in "general position". For the inflation and perturbation technique he uses see [11]).

It will be assumed that

(i) no three discs have a common tangent line,
(ii) no three centers form the vertices of a right-angled triangle,
(iii) no pair of discs have a common tangent line at angle β or $-\beta$ to the Cartesian x-axis, where $\beta = \arccos(1/(2.34)) = 1.129236\ldots$.

As σ^+ is a narrowest strip, (ii) implies that there exist three centers on the boundary of the strip such that two centers, B and C, are lying on one of the boundary lines and are strictly separated from each other by the vertical line through the third center, A, lying on the other boundary. These are the *basic centers* of the family. In a properly chosen coordinate system these points are $A(0, a)$, $B(b, 0)$ and $C(c, 0)$, $a < -1$, $b > 0$ and $c < 0$. The strip bounded by the x-axis and line $y = -w$ will be denoted by σ^*.

Remarks. By these assumptions the investigation of several limiting configurations can be skipped. Assumption (i) has, e.g. the convenient consequence that any weakly separated set of discs is also strictly separated.

The proof of Theorem 1 is constructive. Since it follows different lines for families of small and large transversal width we discuss these cases separately. In every case a candidate transversal line is defined and it is proved that this particular line intersects all but at most two discs of the family. The method followed below can be easily extended to families of slightly overlapping discs or to shapes slightly different from a disc.

Definition. The discs not met by the candidate transversal line are called *exceptional discs* and their centers *exceptional centers*.

We cite without proof the following well-known fact:

Lemma 2.1. *Three pairwise disjoint closed discs have no line transversal iff each disc can be strictly separated from the union of the other discs.*

3. Proof of Theorem 1 for $w \geq w_0 = 0.17$

The transversal width w of $\mathcal{F}$ cannot be arbitrary large. By a recent result of the author [6] we have:

Lemma 3.1. *Let $\mathcal{F}$ be a $T(3)$-family of disjoint closed discs of diameter* 1. *Then the transversal width of $\mathcal{F}$ is $w < 0.65$.*

By Lemma 3.1 it is enough to show in this section that whenever $w \in [0.17, 0.65]$ holds the number n_{ex} of the exceptional centers is at most 2.

3.1. *No Exceptional Center Exists in Strip σ^* between B and C*

Assume

$$b \leq |c|. \tag{3.0}$$

The disjointness hypothesis on the discs implies $b - c \geq 1$ and $b \leq -c$ implies $c \leq -0.5$. The horizontal line $y = a + 0.5$, the upper tangent of the disc centered at A, will be our candidate transversal line for families with "large" transversal width, i.e. for $w \geq 0.17$.

Let

$$p = p(w) = 1/\sqrt{4 - 1/(1+w)^2}. \tag{3.1}$$

$P(p, 0)$ and $P^-(-p, 0)$ are two points such that the line parallel to $\lambda(AP)$ and passing through the origin is tangent to $D(P)$, $D(P^-)$ and $D(A)$ (Fig. 3.1). It can be assumed that

$$b \leq p$$

since $b > p$ would imply $c < -p$ and $(b > p) \wedge (c < -p)$ would imply that $D(A)$, $D(B)$ and $D(C)$ have no transversal in common.

Clearly, p is a decreasing function of w which attains its maximum in each interval for the smallest value of w. Thus

$$p \leq p_0 = p_0 = 0.5530\ldots < 0.554 \tag{3.2}$$

holds in the entire interval $w \in [0.17, 0.65]$.

For given A and B let $C^*(c^*, 0)$ denote the leftmost point of the center sheaf $\Sigma^{\mathrm{c}}(A, B)$ on the x-axis. This point is determined by line $\lambda^\ell(U(B), A)$ since for $c < -p$ the point of intersection of $\lambda^\ell(U(B), A)$ and $\lambda^\ell(U(A), U(B))$ is below the x-axis (Fig. 3.1).

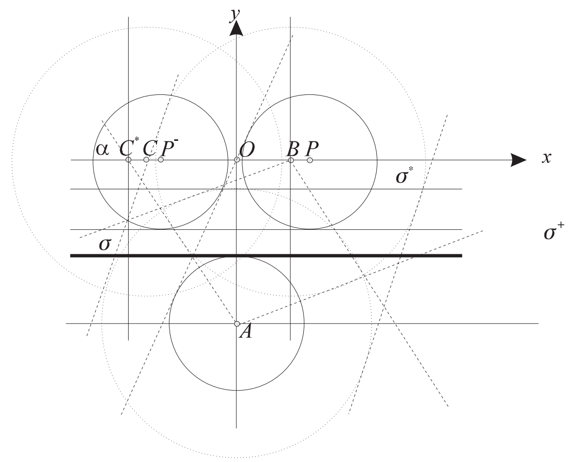

Fig. 3.1. The three discs defining the narrowest transversal strip and the candidate (partial) transversal line.

It holds

Proposition 3.2. *For given $w \geq w_0$ the length $f(w, b) = b - c^*$ of segment $\overline{BC^*}$ is a decreasing function of $b \in [0, p]$ and, for given $b \geq 0$, $f(w, b)$ is a decreasing function of w.*

The proof of Proposition 3.2 is left to the reader.

This proposition yields the maximum

$$f(w) = \max_{b \in [0,p]} f(w, b) = f(w, 0) = \frac{1}{\sqrt{1 - 1/(1 + w)^2}}. \tag{3.3}$$

As the distance of any two centers is at least 1 the length of the projection on the x-axis of a section connecting two centers lying in σ^* is at least

$$s(w) = \sqrt{1 - w^2}. \tag{3.4}$$

Proposition 3.3. *In the interval $w \in [0.17, 0.65]$ it holds that*

$$f(w) < 2s(w). \tag{3.5}$$

Proof. Note that the inequality holds for the endpoints of the interval [0.17, 0.65]. In this interval we also have $s''(w) < 0$ and $f''(w) > 0$. The result follows. □

Proposition 3.3 implies

Corollary 3.4. *For $w \in [0.17, 0.65]$ there is no center in σ^* between the vertical lines $x = c$ and $x = b$.*

3.2. *No Exceptional Center Exists in the Rest of Strip σ^**

To handle the problem the original intervals $a \in [-1.65, -1.17]$ and $b \in [0, 0.56]$ will be cut into a few smaller pieces. It is assumed in the following that $a \in [a_2, a_1]$ and $b \in [b_1, b_2]$ hold, where $[a_2, a_1]$ is one of the four subintervals

$$[-1.65, -1.52], [-1.52, -1.355], [-1.355, -1.24], [-1.24, -1.17] \tag{3.6}$$

and $[b_1, b_2]$ is—independently—one of the three subintervals

$$[0, 0.044], [0.044, 0.2], [0.2, 0.56]. \tag{3.7}$$

Let $[c_2, c_1]$ denote the feasibility interval for c while $a \in [a_2, a_1]$ and $b \in [b_1, b_2]$. The endpoints of the mentioned subintervals will be denoted by A_2, A_1, B_1, B_2, C_2 and C_1, respectively. This subdivision defines (numerically) 12 analogous subproblems.

We consider (for an arbitrary subproblem) the intersection Q of three domains: that of two generalized center sheaves $\Sigma^c(\overline{A_1A_2}, \overline{B_1B_2})$ and $\Sigma^c(\overline{A_1A_2}, \overline{C_1C_2})$ and strip σ^* belonging to the largest transversal width $w = |a_2 + 1|$ of the considered a-subinterval. Clearly, domain

$$Q = \sigma^* \cap \Sigma^c(\overline{A_1A_2}, \overline{B_1B_2}) \cap \Sigma^c(\overline{A_1A_2}, \overline{C_1C_2}) \tag{3.8}$$

is covering all exceptional centers.

Next we establish

Proposition 3.5. *Let M be a point in Q. Then $|\overline{MB}| < 1$ if M is to the right of line $x = b_1$ and $|\overline{MC}| < 1$ if M is to the left of line $x = c_1$.*

Proof. Denote by Q_ℓ the part of polygon Q to the left of the vertical line $x = c_1$ and by Q_r the part of Q to the right of the vertical line $x = b_1$. (In Fig. 3.2 domains Q_ℓ and Q_r are shaded.) It is simple to check (for each parameter box) that both parts have diameter smaller than 1. (The maximum of the diameters of the 24 polygons—two for each parameter box—is < 0.9996.) $\square$

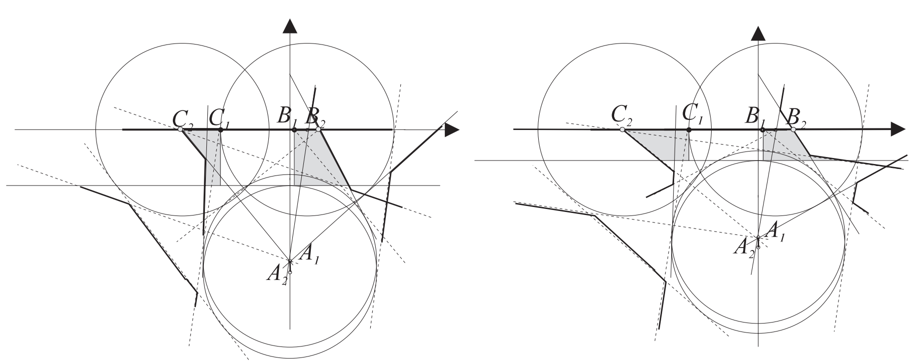

Fig. 3.2. Sketch for case $a \in [-1.65, -1.52], b \in [0.044, 0.2]$ and for case $a \in [-1.355, -1.24], b \in [0.044, 0.2]$.

Corollary 3.6. *If* $0.17 \le w \le 0.65$ *then there is no center in* σ^* *to the left of the vertical line* $x = c$ *and there is no center in* σ^* *to the right of the vertical line* $x = b$.

Corollaries 3.4 and 3.6 imply Theorem 1 for $w \ge w_0 = 0.17$. □

4. Proof of Theorem 1 for $0 < w \le w_0 = 0.17$

First a method, using Kaiser's idea [9] in proving $m_{\text{disc}} \le 12$, will be applied and a candidate transversal line defined. It will be proved that this particular line avoids at most two of the discs. The possibility of an assumed counterexample to this claim will be gradually excluded.

4.1. *The Candidate Transversal Line*

Let $A(0, a)$, $a < -1$, $B(b, 0)$, $b > 0$ and $C(c, 0)$, $c < 0$ denote the three basic centers determining the narrowest transversal strip, as defined in Section 2. (Notice that unlike in the proof for families of large transversal width (in Section 3) inequality $b \le |c|$ is not assumed this time.)

Let $0 < w \le 0.17$, and let λ^ℓ and λ^{r} be two lines received through rotating the x-axis by angles β and $-\beta$, respectively, where $\beta = \arccos(1/2.34)$—as before. Let H be the convex hull of the centers and denote by $A_i(x_i, y_i)$ the vertices on its lower half indexed from left to right so that $A(0, a)$ will be A_0. Then we have $y_0 = -(1 + w)$ and $y_i \ge -(1 + w)$ for $i \ne 0$. To each such center A_i two lines λ_i^ℓ parallel to λ^ℓ and λ_i^{r} parallel to λ^{r} are associated. These lines are tangent to the unit disc $U(A_i)$ from above and meet at point $A_i'(x_i, y_i + 2.34)$. It follows from our assumption (iii) on the "general position" of the centers that there is no center on λ_i^ℓ or on λ_i^{r}. Similarly, denote by λ_i^{l-} the line tangent to $D(A_i)$ from above and parallel to λ_i^ℓ. Line λ_i^{r-} is defined analogously. (Lines λ_i^{l-} and λ_i^{r-} intersect on or above the x-axis.)

Consider a lower support line $\lambda(\alpha)$ of H, passing through vertices A_i and A_j, $j = i$ or $j = i + 1$, where α denotes the angle of the x-axis and $\lambda(\alpha)$ in this order. (By our assumption (i) there are at most two centers on a support line.) Line $\lambda^+(\alpha)$ is the upper tangent of disc $D(A_i)$ and $\lambda^{++}(\alpha)$ is the upper tangent of unit disc $U(A_i)$, both parallel to $\lambda(\alpha)$. With the help of these lines three polygonal domains will be defined. The intersection of the closed half-plane $y \le 0$ and the open half-plane consisting of the points which are strictly above $\lambda^{++}(\alpha)$ is cut into (at most) three parts by lines λ_i^ℓ and λ_j^{r} (Fig. 4.1).

The part strictly to the left of λ_i^ℓ (strictly to the right of λ_j^{r}) is called $L(\alpha)$ $(R(\alpha))$ and the third part (lying between or on λ_i^ℓ and λ_j^{r}) is called $Q(\alpha)$. Any of these parts may contain centers. Integer functions $n_{\text{left}}(\alpha)$ and $n_{\text{right}}(\alpha)$ are introduced to count the number of centers in $L(\alpha)$ and in $R(\alpha)$, respectively.

Next, following Kaiser's idea, we prove

Proposition 4.1. *For every* α *either* $L(\alpha)$ *or* $R(\alpha)$ *is free of centers, consequently either* $n_{\text{left}}(\alpha) = 0$ *or* $n_{\text{right}}(\alpha) = 0$.

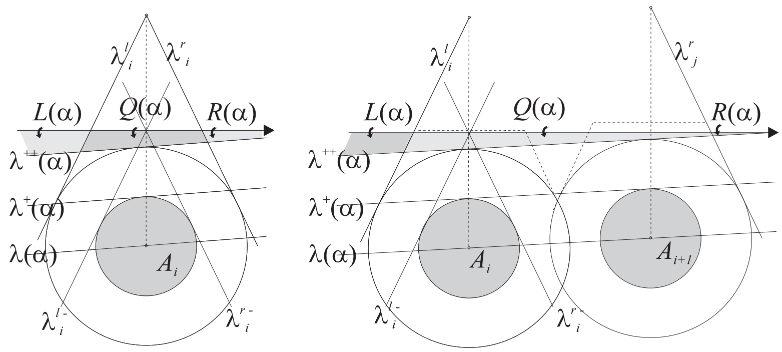

Fig. 4.1. The definition of $Q(\alpha)$ for $j = i$ and for $j = i + 1$.

Proof. Assume that there exist two centers G and G' contradicting the claim of the proposition. If $G \in L(\alpha)$ and $G' \in R(\alpha)$ then—for any $w \in (0, 0.17]$—$D(A_i)$ is weakly separated from $D(G) \cup D(G')$ by $\lambda^+(\alpha)$, $D(G)$ is weakly separated from $D(A_i) \cup D(G')$ by λ_i^{l-} and, finally, $D(G')$ is weakly separated from $D(A_i) \cup D(G)$ by λ_j^{r-}. Evidently, a sufficiently small upward translation of $\lambda^+(\alpha)$ provides strict separation of $D(A_i)$ from the other two discs. By our assumption (iii) no disc apart from $D(A_i)$ is touching λ_i^{l-}, thus a sufficiently small translation of λ_i^{l-} to the left results in a line strictly separating $D(G)$ from the other two discs. The case of $D(G')$ is analogous. Then, by Lemma 2.1, the three discs $D(A_i)$, $D(G)$ and $D(G')$ cannot have property $T(3)$. Contradiction. □

Clearly, $n_{\text{left}}(-\beta) = 0$ and $n_{\text{left}}(\alpha)$ is a left-continuous increasing function and, analogously, $n_{\text{right}}(\beta) = 0$ and $n_{\text{right}}(\alpha)$ is a right-continuous decreasing function in terms of α. Each function vanishes in a non-empty closed interval. By Proposition 4.1 the union of these two closed intervals covers $[-\beta, \beta]$. These facts have the following important corollary.

Corollary 4.2. *There exists $\alpha^* \in [-\beta, \beta]$ such that*

$$n_{\text{left}}(\alpha^*) = n_{\text{right}}(\alpha^*) = 0, \tag{4.1}$$

and all exceptional centers lie in domain $Q(\alpha^*)$.

From now on $\lambda^+(\alpha^*)$ will be the candidate transversal line and the number of the exceptional discs (the discs of the family not met by this line) will be denoted again by n_{ex}. It will be shown that $n_{\text{ex}} \le 2$, i.e. $Q(\alpha^*)$ contains at most two centers.

Observe that α^* need not be unique. The support lines for which (4.1) holds are called *balancing support lines*. The set of angles of the balancing support lines is the non-empty intersection of two closed intervals, either a whole closed interval or just a single value. Without loss of the generality we can assume that $\alpha^* \ge 0$.

4.2. *Basic Center A and the Candidate Transversal Line Are in Close Relation*

Our claim is

Proposition 4.3. *There exists a balancing support line through A.*

Proof. Select a balancing support line. Contrary to the claim of the proposition suppose that none of the support lines through $A \equiv A_0$ is balancing. Then the angle α^* of the selected balancing support line must be greater than $\alpha_{0,1}$, the angle of the x-axis and $\lambda(A_0A_1)$. This balancing support line is tangent to H at A_i for some $i = i^* \geq 1$. As $n_{\text{left}}(\alpha^*) = 0$—by the monotonicity property—$n_{\text{left}}(\alpha_{0,1}) = 0$ also holds. We can assume that

$$n_{\text{right}}(\alpha_{0,1}) \geq 1 \tag{4.2}$$

as $n_{\text{right}}(\alpha_{0,1}) = 0$ would imply that $\lambda(\alpha_{0,1})$ is a balancing support line.

Let α_1 be the angle of the upper tangent of $U(A_0)$ going through the point of intersection of λ_0^{r} and line $y = 0.17 - w$ (Fig. 4.2). (Observe that α_1 does not depend on w.) We have

$$\alpha_1 = \arccos\left(\frac{1}{\sqrt{(1+w_0)^2 + p_0^2}}\right) - \arcsin\left(\frac{p}{\sqrt{(1+w_0)^2 + p_0^2}}\right) = 0.24\cdots < 0.25. \tag{4.3}$$

Angle $\alpha_{0,1}$—the lower bound of the angle of the candidate line—cannot be larger than α_1 as in this case domain $R(\alpha_{0,1})$ would be empty (the three defining half-planes have no point in common) and this would contradict our assumption (4.2) (see Fig. 4.2(a)). Hence $\alpha_{0,1} \leq \alpha_1 < 0.25$. From $\alpha_{0,1} \leq 0.25$ it follows, however, that $x_1 > \cos(0.25) > 0.96$. Easily, the origin O is weakly separated from $U(A_1)$ by λ_1^{ℓ} if $\alpha_{0,1} \leq 0.25$ (Fig. 4.2(b)) and is strongly separated by a line sufficiently close to λ_1^{ℓ}, thus $L(\alpha_{0,1})$ contains the negative half of the x-axis. Consequently, $L(\alpha_{0,1})$ contains the basic center C as well, therefore $n_{\text{left}}(\alpha_{0,1}) \geq 1$, contradicting our assumption that $n_{\text{left}}(\alpha_{0,1}) = 0$. This concludes the proof of the proposition. □

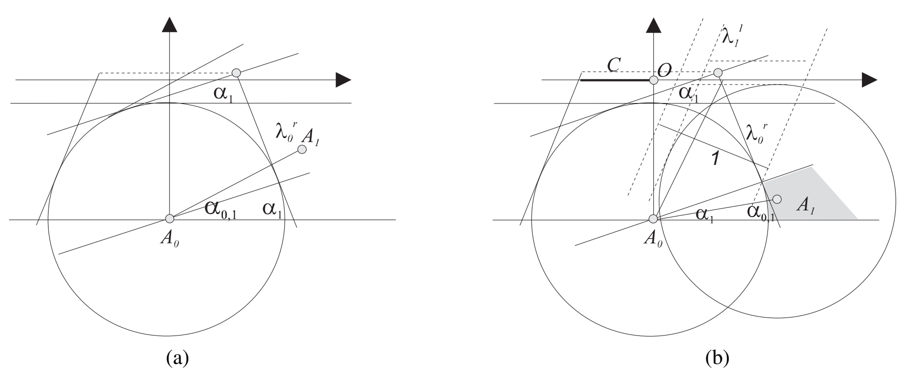

Fig. 4.2. (a) $\alpha_{0,1} < \alpha_1$ and (b) $\alpha_{0,1} > \alpha_1$.

4.3. *The Candidate Transversal Line (in the Assumed Counterexample) Is Tangent to Two Discs*

The case that $i = 0$, $j = 0$, when line $\lambda(\alpha^*)$ is passing through the single vertex A_0, can be excluded as the inequality $n_{\text{ex}} \le 2$ follows immediately from

Lemma 4.4. *Domain $Q(\alpha)$ contains at most two points of mutual distance* 1 *if $i = j = 0$ and $0 \le \alpha \le \beta$.*

Proof. Let $w = w_0 = 0.17$, $P_1(-p_0, 0)$ and $P_2(p_0, 0)$ are points of the x-axis lying on λ_0^ℓ and λ_0^{r}; $P_3(p_3, -0.17)$ and $P_4(0, -0.17)$ are points of line $y = -0.17$ lying on λ_0^{r} and on the y-axis, respectively; and finally, $P_5(p_5, q_5)$ is the point of tangency of $U(A)$ and λ_0^ℓ, where

$$\begin{aligned} p_3 &= p_0(1 + 2w_0)/(1 + w_0) = 0.6334\ldots, \\ p_5 &= \qquad -1/2p_0 \qquad\;\; = -0.9040\ldots, \\ q_5 &= \;(1 + p_5/p_0)(1 + w_0) \;= -0.7426\ldots. \end{aligned}$$

Clearly, $Q(\alpha)$ is part of pentagon $P_1P_2P_3P_4P_5$ for any admitted α, however, line $x = -0.3$ cuts this pentagon into two parts, each of diameter smaller than 1 (Fig. 4.3).

If $w < 0.17$ then $Q(\alpha)$ is even smaller and can be covered by a proper translate of the $Q(\alpha)$-domain belonging to the same α and to $w = 0.17$. Thus the claim of the lemma holds in this case as well. □

4.4. *Bounds on the Position of Basic Center C*

By Lemma 4.4 $i = 0$, $j = 1$ must hold in the assumed counterexample, i.e. $\alpha^* = \alpha_{0,1}$ and our candidate transversal line is $\lambda^+(\alpha_{0,1})$, the upper tangent of $D(A_0) \cup D(A_1)$. Two parts of $Q(\alpha_{0,1})$, the domain containing all exceptional centers, will be defined: $Q(\alpha_{0,1})^{\text{left}}$ and $Q(\alpha_{0,1})^{\text{right}}$ are the parts to the left of λ_0^{r} and to the right of λ_1^ℓ, respectively.

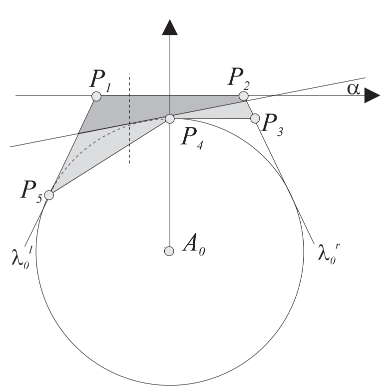

Fig. 4.3. $Q(\alpha)$ contains at most two centers

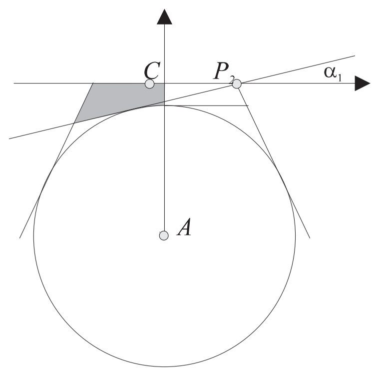

Fig. 4.4. $Q(\alpha)$ cannot have more than one center in $x < 0$ if $\alpha \leq 0.25$.

Proposition 4.5. *If* $\alpha_{0,1} \geq 0.25$ *then* $n_{\mathrm{ex}} \leq 2$.

Proof. Clearly, for $\alpha_{0,1} \geq 0.25$ $(\geq \alpha_1)$ line $\lambda^{++}(\alpha_{0,1})$ has no point in the domain below the x-axis and to the right of λ_0^r hence $Q(\alpha_{0,1}) \equiv Q(\alpha_{0,1})^{\mathrm{left}}$ and by Lemma 4.4 it contains at most two centers as claimed. □

Consequently, it will be assumed that $\alpha^* < 0.25$. First we show

Proposition 4.6. *C is the single exceptional center in the $x \leq 0$ half-plane.*

Proof. It is simple to check (see Fig. 4.4) that the diameter of the $x \leq 0$ part of $Q(\alpha_{0,1})$ is smaller than 1 for any $\alpha_{0,1} \leq 0.25$ and any $w \leq 0.17$. □

Denote the assumed two further exceptional centers of the family by $C'(c', d')$ and $C''(c'', d''), 0 < c' < c'', -0.17 \leq d' \leq 0, -0.17 \leq d'' \leq 0$. The disjointness hypothesis for the discs and Proposition 4.6 imply that even inequalities

$$c'' - c' \geq s_{\min}, \tag{4.4}$$

$$c' - c \geq s_{\min} \tag{4.5}$$

hold, where $s_{\min} = \min_{w \in [0, w_0]} s(w) = \sqrt{1 - w_0^2} > 0.985$.

Proposition 4.7. *We have*

$$c > -0.273, \qquad c' > 0.712 \quad \text{and} \quad c'' > 1.697. \tag{4.6}$$

Proof. C'', lying in $\Sigma^{\mathrm{c}}(A_0, C)$, must be—by definition—on or to the left of at least one of the three lines $\lambda^{\mathrm{r}}(A_0, U(C))$, $\lambda^{\mathrm{r}}(U(A_0), U(C))$ and $\lambda^{\mathrm{r}}(U(A_0), C)$. Clearly, $c'' > 1$ implies that C'' is to the right of the first two lines ($\lambda^{\mathrm{r}}(U(A_0), U(C))$ and $\lambda^{\mathrm{r}}(U(A_0), C)$), therefore C'' must be on or to the left of $\lambda^{\mathrm{r}}(A_0, U(C))$.

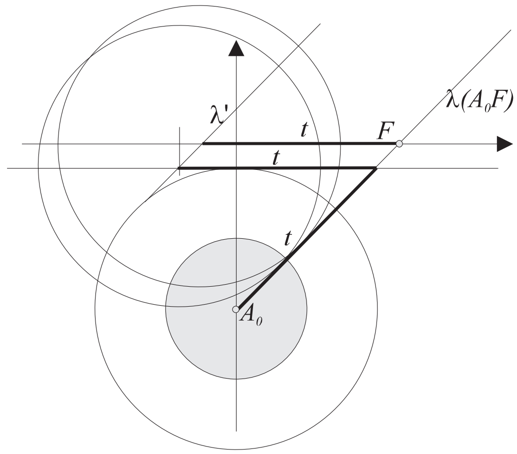

Fig. 4.5. Lower bound for C.

Consider point $F((1+w)\sqrt{t^2-1}, 0)$, $t > 1$, and line λ' parallel to $\lambda(A_0F)$ and tangent to $U(A_0)$ from above (Fig. 4.5). The horizontal distance of F and λ' is t. Since $\lambda(A_0F)$ is $\lambda^{r}(A_0, U(C))$ if C is on λ' then C'' cannot be to the right of $\lambda(A_0F)$ and therefore $c'' - c \le t$. To enable $c'' - c \ge 2s_{\min} = 1.97$—as (4.4) and (4.5) require—let $t \ge 1.97$. The leftmost point of λ' is attained in the $[0, 0.17]$ w-interval for $w = 0$ therefore $c \ge \sqrt{t^2-1} - t \ge \sqrt{1.97^2-1} - 1.97 = -0.272\ldots > -0.273$, $c' > 0.985 - 0.273 = 0.712$ and $c'' > 1.97 - 0.273 = 1.697$ as stated. □

On one hand the lower boundary of $Q(\alpha_{0,1})$, line $\lambda^{++}(\alpha_{0,1})$, runs below C'' and on the other hand C and C'' are in the horizontal strip σ^* of width ≤ 0.17. Hence inequality $\arctan(w/c'') < \arctan(0.17/1.697) < 0.1$ and Proposition 4.7 has immediate consequences for two angles:

Corollary 4.8. *The angle of the candidate transversal line is* $0 \le \alpha_{0,1} < 0.1$ *and for angle* γ *of the* x*-axis and* $\lambda(CC'')$ *it holds that* $-0.1 < \gamma \le 0$.

4.5. *Bounds on the Position of the Further Exceptional Centers*

To get the contradiction a lower bound will be derived for x_1—the x-coordinate for A_1—then upper bounds for $c'' - x_1$ and $x_1 - c'$.

Let

$$g(t) = \sqrt{4t^2-1} - t \quad \text{and} \quad h(t) = t - g(t). \tag{4.7}$$

$g(t)$ is one-half of the distance of two vertices on the same boundary component of a center sheaf of two discs if the distance of the defining centers is $2t \ge 2/\sqrt{3}$ (Figs. 4.6 and 4.7). Easily, $g(t)$ is increasing and $h(t)$ is decreasing in terms of t.

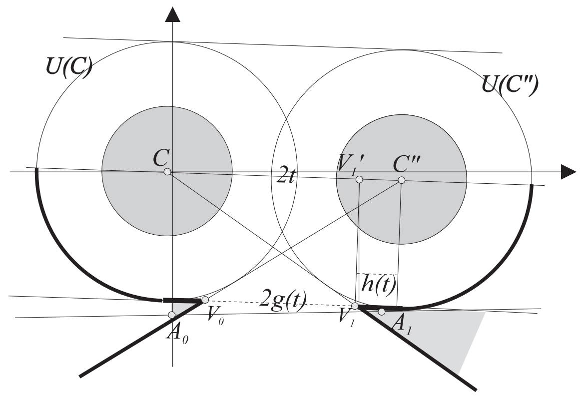

Fig. 4.6. The A_i-gap.

Let V_0 and V_1 denote the points of intersection of the lower tangent of $U(C)$ and $U(C'')$ with $\lambda^{\mathrm{r}}(U(C), C'')$ and $\lambda^{\ell}(C, U(C''))$, respectively, and let V_1' be the orthogonal projection of V_1 on $\lambda(CC'')$ (Fig. 4.6).

The following proposition holds.

Proposition 4.9. *We have*

$$|\overline{A_0 A_1}| > 1.42 \quad \textit{and} \quad x_1 > 1.41. \tag{4.8}$$

Proof. Let $2t$ denote the distance of C and C'' and consider the center sheaf $\Sigma^{\mathrm{c}}(C, C'')$. On one hand A_0 and A_1 must be in this center sheaf and on the other hand $U(C)$ and $U(C'')$ lie above $\lambda(A_0 A_1)$ therefore A_0 and A_1 cannot lie in the convex hull of $U(C) \cup U(C'')$. Thus the *reduced center sheaf*, which is feasible for A_0 and A_1, breaks into two connected components.

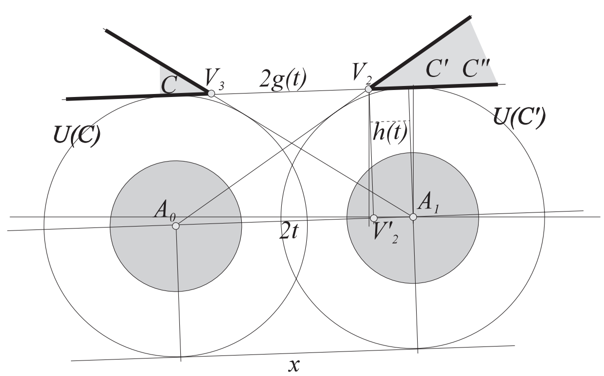

Fig. 4.7. The C-gap.

Since $|\overline{A_0C}| < |\overline{A_0C''}|$, center A_0 is in the same component as vertex V_0. We claim that A_0 and A_1 are lying in different components of the reduced center sheaf. This follows from the fact that (by Corollary 4.8) $x_1 > \cos(0.1) > 0.995$ while the x-coordinate of V_0, the rightmost point of the component, is at most $h(|\overline{CC''}| < 0.28$, thus the triangular intersection of the component and the $x \geq 0$ half-plane cannot contain a second center (A_1) to the right of A_0. By now applying the estimate based on the gap function we have $|\overline{A_0A_1}|) \geq 2g(|\overline{CC''}|/2) \geq 2g(s_{\min}) > 1.42$. Using the bounds of Corollary 4.8 on the angles we have $x_1 \geq 2g(s_{\min})\cos(0.1) > 1.41$, as claimed. □

Proposition 4.10. *It holds that*

$$c'' - x_1 < 0.38. \tag{4.9}$$

Proof. As, by (4.4) and (4.5), $|\overline{CC''}| \geq c'' - c \geq 1.97$ it holds that $|\overline{V_1'C''}| \leq h(0.985) = 0.272\ldots < 0.28$. By Corollary 4.8 the angle between $\overline{V_1V_1'}$ and the y-axis is smaller than 0.1 thus the difference of the x-coordinates of V_1 and V_1' is smaller than 0.1 and the claim of the proposition follows. □

Proposition 4.11. *It holds that*

$$x_1 - c' < 0.52. \tag{4.10}$$

Proof. Let V_2 and V_3 denote the points of intersection of the upper tangent of $U(A_0)$ and $U(A_1)$ with $\lambda^{\ell}(A_0, U(A_1))$ and $\lambda^{\mathrm{r}}(U(A_0), A_1)$, respectively. As C, C' and C'' are exceptional centers they must be above the upper tangent of $U(A_0)$ and $U(A_1)$ and within $\Sigma^{\mathrm{c}}(A_0, A_1)$ at the same time (see Fig. 4.7).

Since $|\overline{A_0A_1}| > 1.42$ the x-coordinate of V_3, and that of any center in the component containing vertex V_3, cannot be larger than $h(0.71) = 0.411\ldots < 0.42$. This upper bound and the lower bound $c > -0.273$ given in Proposition 4.7 do not admit a second exceptional center in this component. Consequently, C' and C'' must be in the other component of $\Sigma^{\mathrm{c}}(A_0, A_1)$, together with V_2.

Let V_2' be the orthogonal projection of V_2 on $\lambda(A_0A_1)$. $\Sigma^{\mathrm{c}}(A_0, A_1)$ and function $h(t)$ can be used to give a lower bound for c'. Let $t = |\overline{A_0A_1}|/2$. Coordinate c' cannot be much smaller than x_1 since C' cannot be to the left of V_2. By Proposition 4.9 $|\overline{A_0A_1}| > 1.42$. Then $h(|\overline{A_0A_1}|/2) < h(0.71) = 0.411\ldots < 0.42$, hence $|\overline{V_2'A_1}| < 0.42$. The difference of the x-coordinates of V_2 and V_2' is smaller than 0.1 since segment $\overline{V_2V_2'}$ is almost vertical (see Corollary 4.8). This yields the claim $|x_1 - c'| < 0.52$. □

Summing the bounds (4.9) and (4.10) given in Propositions 4.10 and 4.11 we get the upper bound

$$c'' - c' < 0.90,$$

while by (4.4) we have

$$c'' - c' > 0.985.$$

Contradiction. This concludes the proof of the theorem. □

Remark. By careful checking of the proofs one can observe that none of the estimates concerning the positions of the centers are sharp. This fact allows us to conclude that Theorem 1 keeps holding if the congruent discs of the family are not assumed to be mutually disjoint but a slight overlapping is admitted. Similarly, the same can be said about families of discs of slightly different size.

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