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The Alexandroff Dimension of Digital Quotients of Euclidean Spaces

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Abstract. Alexandroff T_0-spaces have been studied as topological models of the supports of digital images and as discrete models of continuous spaces in theoretical physics. Recently, research has been focused on the dimension of such spaces. Here we study the small inductive dimension of the digital space $X(\mathcal{W})$ constructed in [15] as a minimal open quotient of a fenestration $\mathcal{W}$ of $\mathbb{R}^n$. There are fenestrations of $\mathbb{R}^n$ giving rise to digital spaces of Alexandroff dimension different from n, but we prove that if $\mathcal{W}$ is a fenestration, each of whose elements is a bounded convex subset of $\mathbb{R}^n$, then the Alexandroff dimension of the digital space $X(\mathcal{W})$ is equal to n.

1. Introduction

In digital image processing and computer graphics, it is necessary to describe topological properties of n-dimensional digital images, hence the search for models of the supports of such images. In order to make it possible to process images by computer, a real-valued function defined on $\mathbb{R}^n$ called an "n-dimensional continuous image" is digitized to obtain a function defined on a discrete subspace D of $\mathbb{R}^n$ with integer values called an "n-dimensional digital image". In practice, D is usually the set of all points of $\mathbb{R}^n$ with integer coordinates, $D = \mathbb{Z}^n$. Any algorithm for finding objects in the image, and describing their "forms", requires that topological properties, such as connectivity, be introduced in the set D. Considering D as a topological subspace of $\mathbb{R}^n$ does not produce interesting results since this subspace has discrete topology.

Until a few years ago, connectivity concepts in D were based mainly on graph-theoretic rather than topological notions. The theory of neighbourhood graphs on $\mathbb{Z}^2$ and $\mathbb{Z}^3$, developed by Rosenfeld and Kak [20] and others, provided a theoretical foundation for important processing methods such as boundary and surface detection, and thinning. Applying graph theory, combinatorics, and using some ideas from topology, this model was generalized to a theory of neighbourhood structures by Klette and Voss for describing images in two and three dimensions, and later to a theory of incidence structures by Voss (for n-dimensional images), see [23]. This latter model has been used for designing an efficient surface-following algorithm, and discrete functions (modelling images) have been defined on these structures. On the other hand, using combinatorial topology and homotopy theory, topological structures were constructed in [11] on neighbourhood graphs for representing two- and three-dimensional images.

Another idea was the identification of D with the basic (open) cells of a combinatorial structure called a cellular complex by Kovalevsky [13]; this model has been applied to a number of algorithms in image processing, for example, for surface detection [14].

During the past few years topological models have been used to describe the structure of the digital image defined on D. The basic idea of these models is, given some topological space Y and a discrete set $D \subseteq Y$, to construct a (non-discrete) topology on D. Based on the Khalimsky topology τ on the integers, given by the subbase $\{\{2n-1, 2n, 2n+1\}\colon\ n \in \mathbb{Z}\}$, *digital n-space* was defined as a product of n copies of $(\mathbb{Z}, \tau)$, and Jordan curve (surface) theorems were proved for digital 2-space and digital 3-space, see, for example, [9], [10] and [12], providing a new theoretical foundation of boundary and surface-following algorithms.

A more general construction of a digital space was proposed by Kronheimer [15]: Starting with a collection $\mathcal{W}$ (called a *fenestration*) of pairwise disjoint regular open subsets of $\mathbb{R}^n$ whose union is dense in $\mathbb{R}^n$, a quotient space $X(\mathcal{W})$ of the Euclidean space is obtained by extending $\mathcal{W}$ to a partition $X(\mathcal{W})$ of $\mathbb{R}^n$, with the quotient topology, and which has the property that the projection map of $\mathbb{R}^n$ onto $X(\mathcal{W})$ is open. For any fixed fenestration $\mathcal{W}$, $X(\mathcal{W})$ turns out to be unique up to homeomorphism if a certain minimality condition is also imposed, then we call $X(\mathcal{W})$ the *digital space* constructed from $\mathcal{W}$. Digital n-space turns out to be the digital space constructed from a particular fenestration of $\mathbb{R}^n$. Kronheimer has shown that for a locally finite fenestration $\mathcal{W}$ of $\mathbb{R}^n$, the digital space $X(\mathcal{W})$, whenever it is semiregular (that is, if the space has a base of regular open sets), is a locally finite (and hence Alexandroff) T_0-space. A topological space is called an *Alexandroff space* if every point has a minimal neighbourhood, or, equivalently, if any intersection of open sets is open. We note that the cellular complex of Kovalevsky mentioned above as a model of the supports of digital images, is a digital space constructed from a particular fenestration and hence is an Alexandroff T_0-space; in fact, it is homeomorphic to digital n-space.

Other approaches to digital topology, which lead to locally finite spaces, are the model based on complexes in [16] and the model of molecular spaces developed by Ivashchenko [8]. This paper applies digital topological models in theoretical physics as well.

A problem on which research has been focused recently, is that of the dimension of a digital space. A digital image which is obtained by the "discretization" of an image defined on $\mathbb{R}^n$, should be modelled by a digital space of dimension n (in some sense to be defined). In previous papers [24], [25] we studied the small inductive dimension (*ind*) of an Alexandroff space (there called the *Alexandroff dimension*). The same concept was independently studied by Evako et al. in [6]. In general, the digital space constructed from a locally finite fenestration can have Alexandroff dimension different from n (see Examples 2.7 and 2.8). However, we prove that if $\mathcal{W}$ is a locally finite fenestration, each of whose elements is a bounded convex subset of $\mathbb{R}^n$, then each element of $\mathcal{W}$ is the interior of a polyhedron, and then the Alexandroff dimension of the digital space $X(\mathcal{W})$ is equal to n. This result gives a topological foundation of the concept of an "n-dimensional digital image" widely used in computer graphics and image processing.

The paper is organized as follows: In Section 2 we give all the necessary formal definitions and some examples. Section 3 presents topological properties of particular convex subsets of $\mathbb{R}^n$, which in Section 4 are applied to study locally finite fenestrations of $\mathbb{R}^n$ and to prove the main result of this paper (Theorem 4.10).

2. Dimension, Fenestrations and Digital Spaces

In [15] Kronheimer, generalizing the construction of the Khalimsky line, proposed the construction of a digital space $X(\mathcal{W})$ as a quotient space of an arbitrary topological space Y, starting with a collection $\mathcal{W}$ of disjoint open subsets of Y whose union is dense and then extending this family to a partition of Y which is topologized with the quotient topology. The collection $\mathcal{W}$ is identified with an open discrete dense subspace D of Y. The motivation for this is to construct a "digital" space mirroring properties of Y. We will apply this technique to the space $\mathbb{R}^n$ (with the Euclidean topology). Throughout the paper the terms closure (cl), interior (int) and frontier (fr), with no subindices refer to the space $\mathbb{R}^n$.

As mentioned in the Introduction, a space (X, τ) is said to be an *Alexandroff space* if each point of X has a minimal neighbourhood or equivalently if τ is closed under arbitrary intersections. A topology τ on a set X determines a preorder $\leq_\tau$ on X as follows:

$$x \leq_\tau y \quad \Leftrightarrow \quad x \in \mathrm{cl}_\tau(\{y\})$$

and it is easy to see that if (X, τ) is a T_0-space, then $\leq_\tau$ is a partial order, usually called the *specialization order* of τ. It is not hard to see that a function between Alexandroff spaces is continuous if and only if it preserves the specialization order.

Conversely, each partial order $\leq$ on a set X determines a unique T_0 Alexandroff topology $\tau_\leq$ on X whose minimal open sets are of the form

$$\{y\colon\ y \geq x\} \qquad (x \in X).$$

These are classical results of Alexandroff [1]; however, even more is true (see [5]): Let $\mathcal{P}$ denote the category of partially ordered sets and order-preserving functions and $\mathcal{A}_0$ the category of Alexandroff T_0-spaces and continuous maps.

Theorem 2.1. *The categories $\mathcal{A}_0$ and $\mathcal{P}$ are isomorphic; moreover, the functors F and G defined by*

$$F((X, \tau)) = (X, \leq_\tau) \quad \textit{and} \quad G((X, \leq)) = (X, \tau_\leq),$$

and which preserve maps, are (mutually inverse) isomorphisms.

For the definition of the *small inductive dimension* of a topological space X we refer the reader to [3] or [17] (but note that in [3] spaces are required to be regular). The *partial order dimension* of a poset $(X, \leq)$ is defined as

$$\sup\{|C| - 1\colon\ C \text{ is a chain of distinct elements in } (X, \leq)\}$$

which may be a non-negative integer or ∞.

The following result was proved in [24]:

Proposition 2.2. *If (X, τ) is an Alexandroff T_0-space and $\leq_\tau$ its specialization order, then the small inductive dimension of (X, τ) is equal to the partial order dimension of $(X, \leq_\tau)$.*

In light of Proposition 2.2, in future we will not distinguish between the partial order dimension of $(X, \leq)$ and the small inductive dimension of the corresponding Alexandroff T_0-space $(X, \tau_{\leq})$. Both will be referred to as the *Alexandroff dimension* of X and written $\mathrm{Adim}(X)$.

Definition 2.3. A *fenestration* of $\mathbb{R}^n$ is a collection of pairwise disjoint non-empty proper regular open subsets of $\mathbb{R}^n$, whose union is dense in $\mathbb{R}^n$. If X is a partition of $\mathbb{R}^n$ that contains a fenestration $\mathcal{W}$ and is such that the projection map from $\mathbb{R}^n$ to X (with the quotient topology) is open, then X is called a $\mathcal{W}$*-grid* of $\mathbb{R}^n$.

That $\mathcal{W}$-grids exist can easily be checked; indeed, $\mathcal{W} \bigcup \{\{x\}\colon\ x \in \mathbb{R}^n \setminus \bigcup \mathcal{W}\}$ is one such.

We are interested here in locally finite fenestrations $\mathcal{W}$ of $\mathbb{R}^n$, that is, those fenestrations in which each point $x \in \mathbb{R}^n$ has an open neighbourhood which meets only a finite number of elements of $\mathcal{W}$. It is well known that if $\mathcal{W}$ is locally finite, then for each $\mathcal{V} \subseteq \mathcal{W}$, the family $\overline{\mathcal{V}} = \{\mathrm{cl}(V)\colon\ V \in \mathcal{V}\}$ is also locally finite and $\mathrm{cl}(\bigcup \mathcal{V}) = \bigcup \overline{\mathcal{V}}$ (see Theorems 1.1.11 and 1.1.13 of [3]).

We note in passing that the idea of a fenestration of $\mathbb{R}^n$ has appeared previously in the literature of discrete geometry, although, in general, families of closed sets are considered. For example, in [18] Quaisser defined a *division* of $\mathbb{R}^n$ as a closed covering $\mathcal{G}$ of $\mathbb{R}^n$ with the property that the interiors of any two distinct elements A, B (called *tiles*) of $\mathcal{G}$ do not intersect. If $\mathcal{G}$ is a locally finite division, and each element of $\mathcal{G}$ is homeomorphic to a closed disc, then $\mathcal{G}$ is called a *mosaic* (again see [18]). A countable division is called a *tiling* of $\mathbb{R}^n$ in [22] and [7]. The remarks following Definition 2.3 clearly imply that if $\mathcal{W}$ is a locally finite fenestration of $\mathbb{R}^n$, then $\overline{\mathcal{W}}$ is a tiling of $\mathbb{R}^n$.

Discrete geometry (see [18], [7] and [22]) deals mainly with divisions of $\mathbb{R}^n$ (and in [18] of non-Euclidean spaces) in relation to their existence and their classification. For the study of the existence problem, particular properties (such as convexity, geometrical regularity or uniform boundedness, etc.) are imposed on the tiles or on the division itself (for example, that the intersection of any two tiles is connected). The classification of divisions may be realized in many different ways; for example, two mosaics M, M' are said to be topologically equivalent if there is a homeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$ such that $M' = \{f(T) : T \in M\}$. Other classification schemes are based on properties of the symmetry group of the division, or on combinatorial properties of the boundaries of the tiles. The aim of our paper, in contrast, is to construct a digital space (in some canonical way) from any given tiling of $\mathbb{R}^n$ whose tiles are convex n-dimensional polyhedra, with the property that the Alexandroff dimension of this space is equal to n.

Given a fenestration $\mathcal{W}$ of $\mathbb{R}^n$, in general there exist many $\mathcal{W}$-grids, however, the following property defined in [15] distinguishes a special $\mathcal{W}$-grid.

Definition 2.4. A $\mathcal{W}$-grid X of $\mathbb{R}^n$ is called *minimal* if any continuous open map of X onto some $\mathcal{W}$-grid, which is injective on $\mathcal{W}$, is a homeomorphism.

It was shown in Theorem 4.4 of [15] that for any fenestration $\mathcal{W}$, and for any $\mathcal{W}$-grid X, there exists a continuous open map $f_{\min}$ of X onto a minimal $\mathcal{W}$-grid $X_{\min}$, which is injective on $\mathcal{W}$ and satisfies $f_{\min} \circ \pi = \pi_{\min}$, where $\pi, \pi_{\min}$ are the quotient maps from $\mathbb{R}^n$ onto X and $X_{\min}$, respectively. This minimal $\mathcal{W}$-grid is then clearly unique up to homeomorphism.

Definition 2.5. For any locally finite fenestration $\mathcal{W}$ on $\mathbb{R}^n$, the minimal $\mathcal{W}$-grid is called the *digital space constructed from* $\mathcal{W}$.

Example 2.6. The *standard fenestration* of $\mathbb{R}^n$ is

$$\mathcal{W} = \{(m_1, m_1 + 1) \times (m_2, m_2 + 1) \times \cdots \times (m_n, m_n + 1),\ m_1, m_2, \ldots, m_n \in \mathbb{Z}\}.$$

The digital space X constructed from this fenestration is homeomorphic to the n-dimensional Khalimsky space (digital n-space) and $\mathrm{Adim}(X) = n$ (see [10] and [15]).

However, there exist fenestrations of $\mathbb{R}^2$, whose minimal $\mathcal{W}$-grids have Alexandroff dimensions different from 2, as we see in the following examples.

Example 2.7. Let $\mathcal{W}$ be the fenestration of $\mathbb{R}^2$ given by

$$\mathcal{W} = \{W_n\colon\ n \in \mathbb{N}\},$$

where $W_1 = D_1$, $W_n = D_n \backslash \mathrm{cl}(D_{n-1})$ for each integer $n \geq 2$ and where D_r is the open disc centred in $(0, 0)$ with radius r. $\mathcal{W}$ is locally finite, and the minimal $\mathcal{W}$-grid is given by $X = \mathcal{W} \bigcup \{F_n\colon\ n \in \mathbb{N}\}$, where $F_n = \mathrm{fr}(D_n)$, for each $n \geq 1$. The digital space X constructed from this fenestration is homeomorphic to an infinite connected subspace of the Khalimsky line [10] and hence $\mathrm{Adim}(X) = 1$.

Example 2.8. Let $W_1 = \{(x, y) \in \mathbb{R}^2\colon\ x < 0, y > 0\}$, $W_2 = \{(x, y) \in \mathbb{R}^2\colon\ x < 0, y < 0\}$, $W_3 = \{(x, y) \in \mathbb{R}^2\colon\ x > 0, y > \sin(1/x)/x\}$, $W_4 = \{(x, y) \in \mathbb{R}^2\colon\ x > 0, y < \sin(1/x)/x\}$. $\mathcal{W} = \{W_1, W_2, W_3, W_4\}$ is a locally finite fenestration of $\mathbb{R}^2$, and the minimal $\mathcal{W}$-grid is given by $X = \mathcal{W} \bigcup \{x_1, x_2, x_3, x_4, x_5\}$, where x_1 is the strictly positive part of the y-axis, x_2 the strictly negative part of the y-axis, x_3 the strictly negative part of the x-axis, x_4 the graph of the function $\sin(1/x)/x$ for $x > 0$, and $x_5 = \{(0, 0)\}$. Then $x_5 < x_1 < x_4 < W_4$ and it is easy to check that there is no longer specialization chain of distinct elements in X. Therefore, $\mathrm{Adim}(X) = 3$.

3. Cones, Convex Sets and Polyhedra

Recall that a subset $H \subseteq \mathbb{R}^n$ is an *affine subspace* if there is some $a \in H$ such that $H - a = \{h - a\colon h \in H\}$ is a (vector) subspace of $\mathbb{R}^n$. The *affine dimension* of H is then defined to be the (vector space) dimension of $H - a$. Furthermore, since H (with the relative Euclidean topology) is homeomorphic to $\mathbb{R}^m$ for some $m \leq n$, the vector space dimension of H coincides with the topological dimension functions *ind*, *Ind* and *dim* (see for example Theorems 7.3.3 and 7.3.19 of [3]) and, for simplicity, we denote this common value by $\dim(H)$.

For $a, b \in \mathbb{R}^n$, we employ the notations $[a, b] = \{(1-r)a + rb;\ r \in \mathbb{R}, 0 \leq r \leq 1\}$ (*closed segment*), $(a, b) = [a, b]\backslash\{a, b\}$ (*open segment*). Recall that a subset M of $\mathbb{R}^n$ is *convex* if $[a, b] \subseteq M$ for any $a, b \in M$. For any $M \subseteq \mathbb{R}^n$, the minimal convex set which contains M is $\operatorname{conv}(M) = \{\sum_{i=0}^{k} r_i a_i\colon a_i \in M, r_i \geq 0 \text{ for } i = 0, 1, \ldots, k, \sum_{i=0}^{k} r_i = 1\}$. The closure and the interior of any convex set are convex, and any open convex set is regular open, that is $\operatorname{int}(\operatorname{cl}(A)) = A$, and any closed convex set with non-empty interior is regular closed, that is $\operatorname{cl}(\operatorname{int}(A)) = A$ (see [7]). For any non-empty convex set $C \subseteq \mathbb{R}^n$, there is precisely one affine subspace H of $\mathbb{R}^n$ called the *carrier* of C which contains C and satisfies $\operatorname{int}_H(C) \neq \emptyset$. Among all affine subspaces containing C, this H has minimal affine dimension and we define the *dimension of the convex set* C as $\dim(C) = \dim(H)$. For more details, we refer the reader to [21] and [4].

Denote by $\mathcal{L}$ the vector space of all (continuous) linear mappings from the real vector space $\mathbb{R}^n$ into $\mathbb{R}$. A *hyperplane* of $\mathbb{R}^n$ is a maximal proper affine subspace, and it is well known that $H \subseteq \mathbb{R}^n$ is a hyperplane if and only if there exists $a \in \mathbb{R}$ and a surjective $f \in \mathcal{L}$ such that $H = f^{-1}(\{a\})$, this implies that H is closed. Then the sets $f^{-1}((-\infty, a])$ and $f^{-1}([a, \infty))$ are called *closed half-spaces* determined by H. It is easy to prove that if $f \in \mathcal{L}$, $f \neq 0$ and $a \in \mathbb{R}$, then $\operatorname{int}(f^{-1}([a, \infty))) = f^{-1}((a, \infty))$, and $\operatorname{int}(f^{-1}((-\infty, a])) = f^{-1}((-\infty, a))$. Thus the hyperplane $H = f^{-1}(\{a\})$ is the frontier of each of its closed half-spaces.

Finally recall that a *polyhedron* is a bounded subset of $\mathbb{R}^n$ which can be represented as a finite intersection of closed half-spaces. Thus a polyhedron is a convex compact subset of $\mathbb{R}^n$.

We also require the following known results.

Lemma 3.1 [19, Section 31.17]. *A closed convex n-dimensional set in $\mathbb{R}^n$ is a finite intersection of closed half-spaces of $\mathbb{R}^n$ if and only if its frontier is contained in a finite union of proper affine subspaces of $\mathbb{R}^n$.*

Lemma 3.2 [19, p. 311 and Section 31.18]. *An n-dimensional polyhedron P has at least one face of dimension k, for each $0 \leq k \leq n-1$, and $\operatorname{fr}(P)$ is the union of the $(n-1)$-dimensional faces of P.*

Lemma 3.3 [2, Part II.2 Section 7.1, Theorem 20]. *If $M \subseteq \mathbb{R}^n$, then the topological dimension (ind, Ind or dim) of M is n if and only if $\operatorname{int}(M) \neq \emptyset$.*

Corollary 3.4. *If H is a k-dimensional affine subspace of $\mathbb{R}^n$, and P is a polyhedron such that $P \subseteq H$, then $\dim(P) \leq k-1 \iff \operatorname{int}_H(P) = \emptyset$.*

If A and B are disjoint convex open sets in $\mathbb{R}^n$, then $\mathrm{cl}(A) \cap \mathrm{cl}(B)$ has empty interior and hence, by Lemma 3.3, is a convex subset of a k-dimensional affine subspace of $\mathbb{R}^n$, where $0 \le k \le n-1$.

Definition 3.5. If V is a convex subset of $\mathbb{R}^n$, $V \neq \emptyset$ and $p \in \mathbb{R}^n \backslash V$, then define *the cone over V with vertex p* by $C(V, p) = \mathrm{conv}(V \bigcup \{p\})$.

It is clear from this definition that $x \in C(V, p)$ if and only if there exists $v \in V$ such that $x \in [p, v]$.

The following lemmas concerning cones and convex sets may be known, but we include their proof for completeness.

Lemma 3.6. *Let V be convex and $p \in \mathbb{R}^n$; if there exist $r \in (0, 1)$ and $v \in V$ such that $p - r(v - p) \in C(V, p)$, then $p \in V$.*

Proof. Clearly, there is some $w \in V$ such that $p - r(v - p) = (1 - s)p + sw$; hence $p = (s/(s + r))w + (r/(s + r))v$, which by the convexity of V, implies that $p \in V$. ☐

Corollary 3.7. *If V is convex and $p \notin V$, then $p \notin \mathrm{int}(C(V, p))$.*

Proof. If $p \in \mathrm{int}(C(V, p))$, then there is $r \in (0, 1)$ such that for any $v \in V$, $p - r(v - p) \in C(V, p)$, which implies by Lemma 3.6 that $p \in V$. This is a contradiction. ☐

Lemma 3.8. *If V is an open convex set and $p \notin V$, then $\mathrm{int}(C(V, p)) = C(V, p)\backslash\{p\}$.*

Proof. One inclusion is clear by Corollary 3.7. For the converse, suppose that $x \in C(V, p)\backslash\{p\}$. Then there is some $v \in V$ such that $x \in (p, v]$. If $x = v$, then $x \in \mathrm{int}(C(V, p))$ since V is open, so suppose $x \neq v$. Then there exists $r \in (0, 1)$ such that $x = p + r(v - p) = (1 - r)p + rv \in (1 - r)p + rV$. However, this latter set is a subset of $C(V, p)$ since $C(V, p)$ is convex, and $(1 - r)p + rV$ is open because V is open and $x \mapsto (1 - r)p + rx$ is a homeomorphism if $r \neq 0$. In consequence $x \in \mathrm{int}(C(V, p))$. ☐

Lemma 3.9. *If V is a convex subset of $\mathbb{R}^n$ and $x \in \mathrm{int}(V)$, $y \in V$, then $(x, y) \subseteq \mathrm{int}(V)$.*

Proof. Let $z \in (x, y)$. Then there exists $r \in (0, 1)$ such that $z = x + r(y - x) = (1 - r)x + ry \in (1 - r)\,\mathrm{int}(V) + ry$ and, as in the proof of Lemma 3.8, this latter set is open. The convexity of V implies $(1 - r)\,\mathrm{int}(V) + ry \subseteq V$, consequently $z \in \mathrm{int}(V)$. ☐

4. Digital Spaces of Polyhedral Fenestrations

In this section all fenestrations are assumed to be locally finite and $U(x)$ always denotes an open disc centred at x.

Definition 4.1. A locally finite fenestration of $\mathbb{R}^n$ is called *polyhedral* if each of its elements is the interior of a polyhedron. For any polyhedral fenestration $\mathcal{W}$, and for $x \in \mathbb{R}^n$, we define

$$N_x = \{W \in \mathcal{W}\colon\ x \in \mathrm{cl}(W)\} \quad \text{and} \quad P_x = \bigcap \{\mathrm{cl}(W)\colon\ W \in N_x\}.$$

Note that N_x is a finite set, and hence P_x is a polyhedron whose carrier we denote by H_x.

The following simple result will be needed later.

Lemma 4.2. *If $\mathcal{W}$ is a polyhedral fenestration, then for any $x \in \mathbb{R}^n$:*

(i) $x \in \mathrm{int}(\bigcup\{\mathrm{cl}(W)\colon\ W \in N_x\})$.
(ii) *If U is open in $\mathbb{R}^n$ such that $U \subseteq \bigcup\{\mathrm{cl}(W)\colon\ W \in N_x\}$, then $U \cap \mathrm{cl}(W) = \emptyset$ for all $W \in \mathcal{W}\backslash N_x$.*

Proof. (i) Clearly, $x \in M = \bigcup\{\mathrm{cl}(W)\colon\ W \in N_x\} = \mathrm{cl}(\bigcup\{W\colon\ W \in N_x\})$. Suppose to the contrary that $x \notin \mathrm{int}(M)$, that is to say, $x \in \mathrm{cl}(\mathbb{R}^n\backslash M)$. Since $\mathcal{W}\backslash N_x$ is locally finite and $\bigcup(\mathcal{W}\backslash N_x)$ is dense in $\mathbb{R}^n\backslash M$, it follows that there is some $W* \in \mathcal{W}\backslash N_x$ such that $x \in \mathrm{cl}(W*)$, a contradiction.

(ii) Suppose to the contrary that $U \cap \mathrm{cl}(W*) \neq \emptyset$ for some $W* \in \mathcal{W}\backslash N_x$. Since U is open, $U \cap W* \neq \emptyset$, which is a contradiction since $\bigcup\{\mathrm{cl}(W)\colon\ W \in N_x\}$ and $W*$ are disjoint. □

The following technical lemma is the key to the proof of our main theorem.

Lemma 4.3. *For any polyhedral fenestration $\mathcal{W}$,*

$$\mathrm{int}_{H_x}(P_x) = \mathrm{int}\left(\bigcup\{\mathrm{cl}(W)\colon\ W \in N_x\}\right) \bigcap H_x.$$

Proof. We first consider the cases $\dim(P_x) = 0$ and $\dim(P_x) = n$: If $\dim(P_x) = 0$, then $P_x = H_x = \mathrm{int}_{H_x}(P_x) = \{x\}$, and $\mathrm{int}(\bigcup\{\mathrm{cl}(W)\colon\ W \in N_x\}) \cap H_x = \{x\}$, by Lemma 4.2(i). If $\dim(P_x) = n$, then $N_x = \{W\}$ for some $W \in \mathcal{W}$, for if N_x has two or more elements, then the remark following Corollary 3.4 implies that $\dim(P_x) \leq n - 1$. Hence $H_x = \mathbb{R}^n$ and $P_x = \mathrm{cl}(W)$, and so $\mathrm{int}_{H_x}(P_x) = \mathrm{int}(\bigcup\{\mathrm{cl}(W)\colon\ W \in N_x\}) \cap H_x = W$. Now suppose that $1 \leq \dim(P_x) \leq n - 1$.

To show that $\mathrm{int}(\bigcup\{\mathrm{cl}(W)\colon\ W \in N_x\}) \cap H_x \subseteq \mathrm{int}_{H_x}(P_x)$, let $z \in \mathrm{int}(\bigcup\{\mathrm{cl}(W)\colon\ W \in N_x\}) \cap H_x$. Then there exists an open disc $U(z)$ such that $U(z) \subseteq \bigcup\{\mathrm{cl}(W)\colon\ W \in N_x\}$. We claim that $U(z) \cap H_x \subseteq P_x$, which will complete the proof that $z \in \mathrm{int}_{H_x}(P_x)$. To prove our claim, let $t \in U(z) \cap H_x$ and suppose that there exists $W^* \in N_x$ such that $t \notin \mathrm{cl}(W^*)$. Clearly, $x \neq t$ and

$$t \in \mathrm{int}\left(\bigcup\{\mathrm{cl}(W)\colon\ W \in (N_x\backslash\{W^*\})\}\right)\backslash\mathrm{cl}(W^*),$$

and hence there exists an open disc $U(t)$ such that

$$U(t) \subseteq \left(\bigcup\{\mathrm{cl}(W)\colon\ W \in (N_x\backslash\{W^*\})\}\right)\backslash\mathrm{cl}(W^*).$$

Since H_x is the carrier of P_x, $\mathrm{int}_{H_x}(P_x) \neq \emptyset$, so let $p \in \mathrm{int}_{H_x}(P_x) \subseteq P_x \subseteq \mathrm{cl}(W^*)$; clearly, $p \notin U(t)$. Since $p \in \mathrm{int}_{H_x}(P_x)$ and $t \in H_x$, for sufficiently small $r \in (0, 1)$, $y = p + r(t - p) \in P_x$, but then $y \in (p, t)$ which by Lemma 3.9 is contained in $\mathrm{int}(C(U(t), p))$. However, $y \in P_x \subseteq \mathrm{cl}(W^*)$, and so there exists $s \in \mathrm{int}(C(U(t), p)) \cap W^*$, which implies by Lemma 3.8 that there exists $c \in U(t) \subseteq \bigcup\{\mathrm{cl}(W)\colon W \in (N_x \backslash \{W^*\})\}$ such that $s \in (p, c]$. Thus there is some $W' \in N_x$, $W' \neq W^*$ such that $c \in \mathrm{cl}(W')$. Since $p \in P_x \subseteq \mathrm{cl}(W')$ and $\mathrm{cl}(W')$ is convex, it follows that $[p, c] \subseteq \mathrm{cl}(W')$. This implies that $s \in \mathrm{cl}(W') \cap W^*$, giving a contradiction.

In order to prove that $\mathrm{int}_{H_x}(P_x) \subseteq \mathrm{int}(\bigcup\{\mathrm{cl}(W)\colon W \in N_x\}) \cap H_x$, let $z \in \mathrm{int}_{H_x}(P_x)$. By Lemma 4.2(i) we have $y \in \mathrm{int}(\bigcup\{\mathrm{cl}(W)\colon W \in N_y\})$ for each $y \in \mathbb{R}^n$ and hence it suffices to show that $N_z = N_x$. This is trivially true if $z = x$ and so we suppose $z \neq x$. Since $z \in P_x$, $N_x \subseteq N_z$ and so it remains to prove that $N_z \subseteq N_x$.

Since $x \in \mathrm{int}(\bigcup\{\mathrm{cl}(W)\colon W \in N_x\})$, $x \neq z$ and $z \in \mathrm{int}_{H_x}(P_x)$, we can choose disjoint open discs $U(x)$ and $U(z)$ such that

(i) $U(x) \subseteq \bigcup\{\mathrm{cl}(W)\colon W \in N_x\}$, and
(ii) $U(z) \cap H_x \subseteq P_x$.

Now, in order to prove $N_z \subseteq N_x$, suppose $W^* \in N_z \backslash N_x$ and let $C_1 = C(U(x), z)$. There are two cases to consider:

(a) If $\mathrm{int}(C_1) \cap W^* \neq \emptyset$, then let $s \in \mathrm{int}(C_1) \cap W^*$. By Lemma 3.8, there is $c \in U(x)$ such that $s \in (z, c]$. From (i), there is $W' \in N_x$ such that $c \in \mathrm{cl}(W')$, and since $z \in P_x \subseteq \mathrm{cl}(W')$, the convexity of $\mathrm{cl}(W')$ implies that $s \in \mathrm{cl}(W')$, that is, $s \in \mathrm{cl}(W') \cap W^*$, a contradiction.

(b) If, on the other hand, $\mathrm{int}(C_1) \cap W^* = \emptyset$, note that $z \notin W*$ for otherwise, $P_x = \mathrm{cl}(W*)$, contradicting the fact that $\dim(P_x) \leq n - 1$. Then since $z \notin U(x)$, by Lemma 3.6, we can choose $r \in (0, 1)$ sufficiently small so that $p = z - r(x - z) \in U(z)$ but $p \notin C_1$; we define $C_2 = C(U(x), p)$. Since $z \in H_x$ (an affine subspace) and $\dim(H_x) = \dim(P_x) \geq 1$, it follows that $p \in H_x$ and hence $p \in P_x$, by (ii). Now, by Lemma 3.9, observe that $z \in (p, x) \subseteq \mathrm{int}(C_2)$ and since $z \in \mathrm{cl}(W*)$, it follows that $\mathrm{int}(C_2) \cap W^* \neq \emptyset$. A contradiction can now be obtained exactly as in case (a), using p in place of z. ☐

Corollary 4.4. *For any polyhedral fenestration $\mathcal{W}$,*

$$\mathrm{fr}_{H_x}(P_x) \subseteq \mathrm{fr}\left(\bigcup\{\mathrm{cl}(W)\colon W \in N_x\}\right).$$

Proof. If $z \in \mathrm{fr}_{H_x}(P_x) = P_x \backslash \mathrm{int}_{H_x}(P_x)$, then clearly $z \in (\bigcup\{\mathrm{cl}(W)\colon W \in N_x\})$. However, if $z \in \mathrm{int}(\bigcup\{\mathrm{cl}(W)\colon W \in N_x\})$, since $z \in H_x$, it follows from Lemma 4.3 that $z \in \mathrm{int}_{H_x}(P_x)$, which is a contradiction. Consequently, $z \in \mathrm{fr}(\bigcup\{\mathrm{cl}(W)\colon W \in N_x\})$. ☐

Corollary 4.5. *If $\mathcal{W}$ is a polyhedral fenestration, then $z \in \mathrm{int}_{H_x}(P_x)$ if and only if $N_z = N_x$.*

Proof. The necessity has been shown in the proof of Lemma 4.3. For the sufficiency, suppose that $N_z = N_x$. Then $z \in P_z = P_x = \mathrm{int}_{H_x}(P_x) \bigcup \mathrm{fr}_{H_x}(P_x)$. However, if $z \in$

$\mathrm{fr}_{H_x}(P_x)$, then, by Corollary 4.4, $z \in \mathrm{fr}(\bigcup\{\mathrm{cl}(W)\colon\ W \in N_x\})$; since $\mathcal{W}$ is locally finite and $\bigcup\mathcal{W}$ is dense in $\mathbb{R}^n$, it is a consequence of the comments following Definition 2.3 that $z \in \bigcup\{\mathrm{cl}(W)\colon\ W \in \mathcal{W}\backslash N_x\}$. Consequently there exists $W^* \in \mathcal{W}\backslash N_x$ such that $z \in \mathrm{cl}(W^*)$, that is, $W^* \in N_z\backslash N_x$, a contradiction. □

Corollary 4.6. *If $\mathcal{W}$ is a polyhedral fenestration, then*

$$\bigcup\{\mathrm{int}_{H_y}(P_y)\colon\ y \in \mathbb{R}^n \ \textit{such that}\ N_y \subseteq N_x\} = \mathrm{int}\left(\bigcup\{\mathrm{cl}(W)\colon\ W \in N_x\}\right).$$

Proof. Suppose that $z \in \mathrm{int}_{H_y}(P_y)$ for some $y \in \mathbb{R}^n$ such that $N_y \subseteq N_x$. Corollary 4.5 implies that $N_z = N_y \subseteq N_x$, implying that $z \in \bigcup\{\mathrm{cl}(W)\colon\ W \in N_x\}$. However, if $z \in \mathrm{fr}(\bigcup\{\mathrm{cl}(W)\colon\ W \in N_x\})$, then, as in the proof of Corollary 4.5, there exists $W^* \in \mathcal{W}\backslash N_x$ such that $z \in \mathrm{cl}(W^*)$. However, then $W^* \in N_z \subseteq N_x$, which is a contradiction. Hence $z \in \mathrm{int}(\bigcup\{\mathrm{cl}(W)\colon\ W \in N_x\})$.

Now if $z \in \mathrm{int}(\bigcup\{\mathrm{cl}(W)\colon\ W \in N_x\})$, then, by Lemma 4.2(ii), there exists an open neighbourhood U of z such that $U \subseteq \bigcup\{\mathrm{cl}(W)\colon\ W \in N_x\}$ and $U \cap \mathrm{cl}(W) = \emptyset$ for all $W \in \mathcal{W}\backslash N_x$; this implies that $N_z \subseteq N_x$. Lemma 4.2(i) implies $z \in \mathrm{int}(\bigcup\{\mathrm{cl}(W)\colon\ W \in N_z\})$, and obviously $z \in P_z \subseteq H_z$. Hence by Lemma 4.3, $z \in \mathrm{int}_{H_z}(P_z)$ and so $z \in \bigcup\{\mathrm{int}_{H_y}(P_y)\colon\ y \in \mathbb{R}^n$ such that $N_y \subseteq N_x\}$. □

Corollary 4.7. *If $\mathcal{W}$ is a polyhedral fenestration, and $P_x \subset P_y$ (that is, $P_x \neq P_y$) for $x, y \in \mathbb{R}^n$, then $P_x \subseteq \mathrm{fr}_{H_y}(P_y)$ and $\dim(P_x) \le \dim(P_y) - 1$.*

Proof. Supposing $P_x \subset P_y$, note first that $N_y \subset N_x$ (that is, $N_y \neq N_x$). If $z \in P_x$, then $N_x \subseteq N_z$. However, if $z \in \mathrm{int}_{H_y}(P_y)$, then, by Corollary 4.5, $N_y = N_z \supseteq N_x$, that is, $N_y = N_x$, which is a contradiction. Hence $z \in \mathrm{fr}_{H_y}(P_y)$ and the result follows.

To prove the second assertion, note that $P_x \subseteq \mathrm{fr}_{H_y}(P_y)$ implies that $\mathrm{int}_{H_y}(P_x) = \emptyset$ and then Corollary 3.4 implies that $\dim(P_x) \le \dim(P_y) - 1$ since $\dim(P_y) = \dim(H_y)$. □

The easy proof of the following proposition, which depends on the fact that $\mathcal{W}$ is locally finite and $\mathrm{fr}(W)$ is compact for any $W \in \mathcal{W}$, is left to the reader.

Proposition 4.8. *If $\mathcal{W}$ is a locally finite fenestration of $\mathbb{R}^n$, such that each $W \in \mathcal{W}$ is a bounded convex subset of $\mathbb{R}^n$, then for any $W \in \mathcal{W}$, $\mathrm{fr}(W)$ intersects the frontiers of only finitely many elements of $\mathcal{W}$.*

The next result is part of the folklore, see [22], we include a proof here for completeness.

Proposition 4.9. *If $\mathcal{W}$ is a locally finite fenestration of $\mathbb{R}^n$ such that any $W \in \mathcal{W}$ is a bounded convex subset of $\mathbb{R}^n$, then $\mathcal{W}$ is a polyhedral fenestration.*

Proof. Let $W \in \mathcal{W}$; $\mathrm{cl}(W)$ is convex, n-dimensional (since W is open), bounded and hence compact. Applying Lemma 3.1, in order to prove that $\mathrm{cl}(W)$ is a polyhedron, it suffices to show that $\mathrm{fr}(W)$ is contained in a finite union of affine proper subspaces of $\mathbb{R}^n$. However, if $x \in \mathrm{fr}(W)$, then there exists $W' \in \mathcal{W}$, $W' \neq W$ such that $x \in \mathrm{cl}(W) \cap \mathrm{cl}(W') = \mathrm{fr}(W) \cap \mathrm{fr}(W')$. By the comments following Corollary 3.4, $\mathrm{cl}(W) \cap \mathrm{cl}(W')$ is contained in some proper affine subspace of $\mathbb{R}^n$ and, by Proposition 4.8, there exist only a finite number of such W'. This implies that $\mathrm{fr}(W)$ is contained in a finite union of proper affine subspaces, completing the proof. ☐

In our main theorem we use a construction due to Kronheimer. Given a fenestration $\mathcal{W}$ of $\mathbb{R}^n$, a decomposition of $\mathbb{R}^n$ which contains $\mathcal{W}$, denoted in Section 6 of [15] by $\Delta\times$, is constructed by identifying two points of $\mathbb{R}^n$ if and only if for every open neighbourhood of either, there exists an open neighbourhood of the other which intersects the same collection of elements of $\mathcal{W}$. Kronheimer proved that $\Delta\times$ is exactly the minimal $\mathcal{W}$-grid on $\mathbb{R}^n$ if and only if the natural projection map of $\mathbb{R}^n$ on $\Delta\times$ is open [15, Theorem 6.2], but this does not imply that this minimal grid is semiregular [15, Examples 6.6]. He also showed that whenever the minimal $\mathcal{W}$-grid Δ on $\mathbb{R}^n$ is semiregular, then $\Delta = \Delta\times$ [15, Theorem 6.5]; however, the semiregularity of $\Delta\times$ does not imply that it is the minimal $\mathcal{W}$-grid [15, Examples 6.6]. In the proof of our main theorem, we obtain a minimal semiregular $\mathcal{W}$-grid on $\mathbb{R}^n$.

Theorem 4.10. *If $\mathcal{W}$ is a locally finite fenestration of $\mathbb{R}^n$, such that any element W of $\mathcal{W}$ is a bounded convex subset of $\mathbb{R}^n$, then the digital space constructed from $\mathcal{W}$ is semiregular and has Alexandroff dimension equal to n.*

Proof. From Proposition 4.9, each $W \in \mathcal{W}$ is the interior of some polyhedron. We will construct a minimal $\mathcal{W}$-grid on $\mathbb{R}^n$ with Alexandroff dimension n and the result will follow from unicity (see the comments following Definition 2.4).

Using the terminology of Definition 4.1, we define the following equivalence relation on $\mathbb{R}^n$:

$$x \sim y \iff N_x = N_y, \qquad x, y \in \mathbb{R}^n.$$

Let $[x] = \{y \in \mathbb{R}^n\colon\ y \sim x\}$, and let

$$X = \mathbb{R}^n/\!\sim = \{[x], x \in \mathbb{R}^n\}.$$

Furthermore, let $\pi\colon \mathbb{R}^n \to X$ be defined by $\pi(x) = [x]$ and let τ be the quotient topology on X. Note that $N_x \subseteq N_y \iff P_y \subseteq P_x$, and Corollary 4.5 implies that

$$\pi(y) = [x] \iff N_x = N_y \iff P_x = P_y \iff$$
$$y \in \mathrm{int}_{H_x}(P_x) \iff x \in \mathrm{int}_{H_y}(P_y).$$

In consequence

$$\pi^{-1}([x]) = \mathrm{int}_{H_x}(P_x).$$

We claim that (X, τ) is semiregular and is the minimal $\mathcal{W}$-grid with $\mathrm{Adim}(X) = n$. Note that X is $\Delta\times$ of Section 6 of [15] and hence to prove that X is the minimal $\mathcal{W}$-grid, it suffices by Theorem 6.2 of [15] to show that X is a $\mathcal{W}$-grid, that is to say, that the map π is open.

We first characterize the minimal open neighbourhoods of the points of X.

For each $[x] \in X$, let

$$U_X([x]) = \{[y] \in X\colon\ y \in \mathbb{R}^n \text{ such that } N_y \subseteq N_x\}.$$

Clearly, $[x] \in U_X([x])$, and we will show that $U_X([x])$ is the minimal open neighbourhood of $[x]$. Now, $\pi^{-1}(U_X([x])) = \bigcup\{\pi^{-1}([y]) : y \in \mathbb{R}^n \text{ such that } N_y \subseteq N_x\} = \bigcup\{\mathrm{int}_{H_y}(P_y)\colon\ y \in \mathbb{R}^n \text{ such that } N_y \subseteq N_x\} = \mathrm{int}(\bigcup\{\mathrm{cl}(W)\colon\ W \in N_x\})$, by Corollary 4.6, implying that $U_X([x])$ is open in X.

Suppose that there is an open $V \subseteq X$ containing $[x]$ such that $V \subset U_X([x])$, and let $[z] \in U_X([x])\backslash V$. Then $N_z \subseteq N_x$, or, equivalently, $P_x \subseteq P_z$, implying $x \in P_z$. Since P_z is convex and closed with non-empty interior in its carrier H_z, it is regular closed in H_z, and because H_z is a closed subspace of $\mathbb{R}^n$, one obtains $x \in \mathrm{cl}_{H_z}(\mathrm{int}_{H_z}(P_z)) = \mathrm{cl}(\pi^{-1}([z]))$. This implies that the open neighbourhood $\pi^{-1}(V)$ of x intersects $\pi^{-1}([z])$, which is a contradiction. Thus X is an Alexandroff space and for each $x \in X$, $U_X([x])$ is its minimal open neighbourhood; moreover, this neighbourhood is finite since N_x is finite. Furthermore, it is now clear that X is a T_0-space, since if $U_X([y]) = U_X([x])$, then $N_x = N_y$ and hence $[x] = [y]$.

The rest of the proof is in three parts:

(i) $\mathrm{Adim}(X) = n$.

Let $\le$ denote the specialization order of the T_0 space X; that is to say, if $[x], [y] \in X$,

$$[x] \le [y] \iff [x] \in \mathrm{cl}_X([y]) \iff [y] \in U_X([x]) \iff N_y \subseteq N_x.$$

Now from Corollary 4.7

$$[x] < [y] \iff P_x \subset P_y \implies P_x \subseteq \mathrm{fr}_{H_y}(P_y) \text{ and } \dim(P_x) \le \dim(P_y) - 1.$$

Since the dimension of the closures of the elements of $\mathcal{W}$ cannot exceed n, we obtain $\mathrm{Adim}(X) \le n$.

In order to prove that $\mathrm{Adim}(X) \ge n$, we have only to construct a chain of $n+1$ distinct elements of X. By Lemma 3.2, any polyhedron of dimension k ($k \ge 1$) has faces of all dimensions $k-1, k-2, \ldots, 0$, and its frontier (in its carrier) is the union of all its $(k-1)$-dimensional faces.

Let $x \in W$ for some $W \in \mathcal{W}$. Then $P_x = \mathrm{cl}(W)$ and $\dim(P_x) = n$. Let F_{n-1} be an $(n-1)$-dimensional face of P_x, and H_{n-1} its carrier. By Proposition 4.8, F_{n-1} intersects only a finite number of frontiers of elements of $\mathcal{W}$. This implies that F_{n-1} is covered by a finite number of the polyhedra P_z, say $F_{n-1} \subseteq \bigcup\{P_{z_i}\colon\ 1 \le i \le k\}$ where $z_1, z_2, \ldots, z_k \in F_{n-1}$. If $\dim(P_{z_i}) \le n-2$, for each i, then, by Corollary 3.4, each P_{z_i} has empty interior in the carrier H_{n-1} of F_{n-1}. Hence, in H_{n-1} which is homeomorphic to $\mathbb{R}^{n-1}$, the closed set F_{n-1} is contained in the nowhere dense set $\bigcup\{P_{z_i}\colon\ 1 \le i \le k\}$ and again by Corollary 3.4 we have $\dim(F_{n-1}) \ne n-1$, which is a contradiction . Consequently, there is some $1 \le j \le k$ such that $\dim(P_{z_j}) = n-1$. Let $y_{n-1} = z_j$ and observe that $P_{y_{n-1}} \subseteq F_{n-1} \subseteq \mathrm{fr}_{H_x}(P_x)$, implying $[y_{n-1}] < [x]$.

It is evident that this process can be continued, until the selection of a 0-dimensional face F_0 of P_{y_1}. Then $F_0 = \{y_0\}$ and $[y_0] < [y_1]$, which completes the construction of the chain

$$[y_0] < [y_1] < \cdots < [y_{n-1}] < [x] \qquad \text{in } X.$$

This proves $\mathrm{Adim}(X) \geq n$, and concludes the proof of (i).

(ii) X is a $\mathcal{W}$-grid.

If $w \in W \in \mathcal{W}$, then $\pi[W] = [w]$ and so X contains the fenestration $\mathcal{W}$. It remains only to show that π is an open mapping.

Let M be an open subset of $\mathbb{R}^n$, and let $[x] \in \pi[M]$. To prove that $\pi[M]$ is open in X, it suffices to show that $U_X([x]) \subseteq \pi(M)$. To this end, let $[y] \in U_X([x])$. Since $[x] \in \pi[M]$, it follows that there exists $z \in M$ such that $\pi(z) = [x] \iff N_z = N_x \iff P_z = P_x$. Also, since $[y] \in U_X([x])$, $N_y \subseteq N_x \iff P_x \subseteq P_y$. However, P_y is a regular closed subset of H_y and $z \in P_z = P_x \subseteq P_y \subset H_y$, thus $z \in \mathrm{cl}_{H_y}(\mathrm{int}_{H_y}(P_y)) = \mathrm{cl}_{H_y}(\pi^{-1}([y]))$. Since $z \in M$ and M is open in $\mathbb{R}^n$, $M \cap H_y$ is an open neighbourhood of z in H_y, which must then intersect $\mathrm{int}_{H_y}(P_y)$. Hence there exists $s \in M \cap \mathrm{int}_{H_y}(P_y)$. However, $s \in \mathrm{int}_{H_y}(P_y)$ implies by Corollary 4.5 that $N_s = N_y$. Thus $\pi(s) = [y]$, and since $s \in M$ we obtain $[y] \in \pi(M)$, so proving (ii).

As we stated above, it now follows from Theorem 6.2 of [15] that X is the minimal $\mathcal{W}$-grid. Thus it is the digital space constructed from $\mathcal{W}$ and has Alexandroff dimension equal to n. However, we show even more, that:

(iii) X is semiregular.

It is sufficient to show that $U_X([x])$ is regular open for any $[x] \in X$, and since $U_X([x])$ is open, it remains only to prove that $\mathrm{int}_X(\mathrm{cl}_X(U_X([x]))) \subseteq U_X([x])$. To this end suppose that $[y] \in \mathrm{int}_X(\mathrm{cl}_X(U_X([x])))$; it suffices to show that $N_y \subseteq N_x$.

Clearly, $U_X([y]) \subseteq \mathrm{cl}_X(U_X([x]))$ and since π is an open mapping, it follows (see Problem 1.4.C of [3]) that $\pi-1[U_X([y])] \subseteq \mathrm{cl}(\pi^{-1}[U_X([x])])$. However, by Corollary 4.6 we have that for each $[z] \in X$,

$$\begin{aligned}\pi^{-1}(U_X([z])) &= \bigcup\{\mathrm{int}_{H_y}(P_y)\colon\ y \in \mathbb{R}^n \text{ such that } N_y \subseteq N_z\}\\ &= \mathrm{int}\left(\bigcup\{\mathrm{cl}(W)\colon\ W \in N_z\}\right),\end{aligned}$$

and hence we obtain

$$\mathrm{int}\left(\bigcup\{\mathrm{cl}(W)\colon\ W \in N_y\}\right) \subseteq \mathrm{cl}(\mathrm{int}\left(\bigcup\{\mathrm{cl}(W)\colon\ W \in N_x\})\right) = \bigcup\{\mathrm{cl}(W)\colon\ W \in N_x\}$$

since a finite union of regular closed sets is regular closed. Now by Lemma 4.2(i), $y \in \mathrm{int}(\bigcup\{\mathrm{cl}(W)\colon\ W \in N_y\}) \subseteq \mathrm{int}(\bigcup\{\mathrm{cl}(W)\colon\ W \in N_x\})$. By Lemma 4.2(ii), $\mathrm{int}(\bigcup\{\mathrm{cl}(W)\colon\ W \in N_x\}) \cap \mathrm{cl}(V) = \emptyset$ for all $V \in \mathcal{W}\backslash N_x$, which immediately implies that $N_y \subseteq N_x$, completing the proof of semiregularity of X. $\square$

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