



# Commuting Involution Graphs for 4-Dimensional Projective Symplectic Groups

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## Abstract

For a group  $G$  and  $X$  a subset of  $G$  the commuting graph of  $G$  on  $X$ , denoted by  $\mathcal{C}(G, X)$ , is the graph whose vertex set is  $X$  with  $x, y \in X$  joined by an edge if  $x \neq y$  and  $x$  and  $y$  commute. If the elements in  $X$  are involutions, then  $\mathcal{C}(G, X)$  is called a commuting involution graph. This paper studies  $\mathcal{C}(G, X)$  when  $G$  is a 4-dimensional projective symplectic group over a finite field and  $X$  a  $G$ -conjugacy class of involutions, determining the diameters and structure of the discs of these graphs.

**Keywords** Involutions · Commuting involution graphs · Symplectic Groups

**Mathematics Subject Classification** 05C12 · 20E99

## 1 Introduction

For  $G$  a group and  $X$  a subset of  $G$ , the commuting graph of  $G$  on  $X$ ,  $\mathcal{C}(G, X)$ , is the graph whose vertex set is  $X$  with  $x, y \in X$  joined whenever  $x \neq y$  and  $xy = yx$ . In effect commuting graphs first appeared in the paper of Brauer and Fowler [14], famous for containing a proof that up to isomorphism only finitely many non-abelian simple groups can have a given centralizer of an involution. The commuting graphs considered in [14] had  $X = G \setminus \{1\}$ —such graphs have played an important role in recent work related to the Margulis–Platanov conjecture (see [41]). Various kinds of commuting graphs have been deployed in the study of finite groups, particularly the non-abelian simple groups. For example, the analysis and subsequent construction by Fischer [24] of the three simple Fischer groups used the commuting graph on the conjugacy class of 3-transpositions. While a computer-free uniqueness proof of the Lyons simple group by Aschbacher and Segev [6]

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employed a commuting graph where the vertices consisted of the 3-central subgroups of order 3. For  $G$  either a symmetric group, or more generally a finite Coxeter group, or a projective special linear group and  $X$  a certain conjugacy class of  $G$ , the structure of  $\mathcal{C}(G, X)$  has been investigated at length by Bundy [16], Bates, Bundy, Hart, Perkins and Rowley [9–13], Nawawi and Rowley [38], Jafari [31]. And for the double covers of symmetric groups and 3-dimensional unitary groups, see Aubad [8] and Everett [23] respectively. Also, commuting involution graphs for infinite Coxeter groups have been analysed in Perkins [40], Hart and Clarke [29]. A different flavour of graph (also called a commuting graph) has also been studied extensively. In this case, for a group  $G$ , the vertex set is  $G \setminus Z(G)$  with two distinct elements being joined if they commute. For a selection of work on various aspects of these graphs, consult Akbari, Mohammadian, Radjavi and Raja [3], Britnell and Gill [15], Cassell [19], Giudici and Kuzma [25], Giudici and Parker [26], Giudici and Pope [27], Iranmanesh and Jafarzadeh [30], Leshchenko and Zorya [34], Leshchenko [33], Mahmoudifar and Moghaddamfar [35], Mohammadian, Erfanian, Forrokhi and Wilkens [36], Morgan and Parker [37], Parker [39]. Recently there has been work on commuting graphs for rings (see, for example, [1, 2, 20, 21, 43]).

This paper investigates  $\mathcal{C}(G, X)$  when  $G$  is a finite 4-dimensional projective symplectic group and  $X$  is a  $G$ -conjugacy class of involutions. Such graphs are referred to as commuting involution graphs. From now on  $H$  will denote the symplectic group  $Sp(4, q)$ ,  $q = p^a$  and  $p$  a prime. Let  $V$  be the natural (symplectic)  $GF(q)H$ -module, and set  $G = H/Z(H)$ . So  $G \cong PSp(4, q)$  and  $G \cong H$  when  $p = 2$ . For  $t \in X$ , we define

$$\Delta_i(t) = \{x \in X \mid d(t, x) = i\}$$

where  $d$  is the standard distance metric on  $\mathcal{C}(G, X)$ . So  $\Delta_i(t)$  consists of all vertices in  $\mathcal{C}(G, X)$  distance  $i$  from  $t$ —we call  $\Delta_i(t)$  the  $i$ th disc of  $t$  (or just a disc of  $t$ ).

In the case when  $p = 2$ ,  $G$  has three conjugacy classes of involutions. Recalling that for an involution  $x$  of  $G$ ,  $V(x) = \{v \in V \mid (v, v^x) = 0\}$  these three classes  $X_1, X_2, X_3$  may be described thus (see [7])

$$\begin{aligned} X_1 &= \{x \in G \mid x^2 = 1, \dim C_V(x) = 3\}; \\ X_2 &= \{x \in G \mid x^2 = 1, \dim C_V(x) = 2, \dim V(x) = 3\}; \quad \text{and} \\ X_3 &= \{x \in G \mid x^2 = 1, \dim C_V(x) = 2, V(x) = V\}. \end{aligned}$$

Our four main theorems are as follows.

**Theorem 1.1** *Suppose that  $p = 2$  and  $i = 1, 3$ . Then  $\mathcal{C}(G, X_i)$  is connected of diameter 2 with the disc sizes being*

$$\begin{aligned} |\Delta_1(t)| &= q^3 - 2; \quad \text{and} \\ |\Delta_2(t)| &= q^3(q - 1). \end{aligned}$$

**Theorem 1.2** *Suppose that  $p = 2$ . Then  $\mathcal{C}(G, X_2)$  is connected of diameter 4, the disc sizes being*

$$\begin{aligned}
 |\Delta_1(t)| &= q^2(2q - 3); \\
 |\Delta_2(t)| &= 2q^2(q - 1)^2; \\
 |\Delta_3(t)| &= 2q^3(q - 1)^2; \text{ and} \\
 |\Delta_4(t)| &= q^4(q - 1)^2.
 \end{aligned}$$

Turning to the case when  $p$  is odd, we have that there are two  $G$ -involution conjugacy classes  $Y_1$  and  $Y_2$ . We shall let  $Y_1$  denote the  $G$ -conjugacy class whose elements are the images of an involution in  $H$ , and  $Y_2$  to denote the  $G$ -conjugacy class whose elements are the image of an element of  $H$  of order 4 which squares to the non-trivial element of  $Z(H)$ .

**Theorem 1.3** *If  $p$  is odd, then  $\mathcal{C}(G, Y_1)$  is connected of diameter 2 with disc sizes*

$$\begin{aligned}
 |\Delta_1(t)| &= \frac{1}{2}q(q^2 - 1); \text{ and} \\
 |\Delta_2(t)| &= \frac{1}{2}(q^4 - q^3 + q^2 + q - 2).
 \end{aligned}$$

**Theorem 1.4**

(i) If  $q \equiv -1 \pmod{4}$  then  $\mathcal{C}(G, Y_2)$  is connected of diameter 3. Furthermore,

$$\begin{aligned}
 |\Delta_1(t)| &= \frac{1}{2}q(q^2 + 2q - 1); \\
 |\Delta_2(t)| &= \frac{1}{16}(q + 1)(3q^5 - 2q^4 + 8q^3 - 30q^2 + 13q - 8); \text{ and} \\
 |\Delta_3(t)| &= \frac{1}{16}(q^2 - 1)(5q^4 - 9q^3 + 7q^2 - 3q + 8).
 \end{aligned}$$

(ii) If  $q \equiv 1 \pmod{4}$  then  $\mathcal{C}(G, Y_2)$  is connected of diameter 3. Furthermore,

$$\begin{aligned}
 |\Delta_1(t)| &= \frac{1}{2}q(q^2 + 1); \\
 |\Delta_2(t)| &= \frac{1}{16}(q - 1)(3q^5 - 6q^4 + 32q^3 - 10q^2 - 27q - 8); \text{ and} \\
 |\Delta_3(t)| &= \frac{1}{16}(q - 1)(5q^5 + 22q^4 - 8q^3 + 34q^2 + 51q + 24).
 \end{aligned}$$

Theorems 1.1 and 1.2 are established in Sect. 2. While in Sect. 3 we give a proof of Theorem 1.3. The structure and properties of  $\mathcal{C}(G, Y_2)$ , in Sect. 4, are a much tougher nut to crack than the other four cases. The reason for this is that for  $\mathcal{C}(G, X_i)$ , ( $i = 1, 2, 3$ ) and  $\mathcal{C}(G, Y_1)$  the graph can be studied effectively by working in  $H = Sp(4, q)$  and looking at certain configurations in the natural symplectic module  $V$  involving  $C_V(x)$  for various  $x \in X$  ( $X = X_i, i = 1, 2, 3$  or

$XZ(H)/Z(H) = Y_1$ ). The key point being that, in these four cases for  $x \in X$ ,  $C_V(x)$  is a non-trivial subspace of  $V$  whereas, for  $x$  of order 4 and squaring into  $Z(H)$ ,  $C_V(x)$  is trivial. If we change tack and look at  $G$  acting on the projective symplectic space things are not much better. When  $q \equiv -1 \pmod{4}$  elements of  $Y_2$  fix no projective points, while in the case  $q \equiv 1 \pmod{4}$  they fix  $2q + 2$  projective points. However, even in the latter case, the fixed projective points didn't appear to be of much assistance. It is the isomorphism  $PSp(4, q) \cong O(5, q)$  that comes to our rescue. If now  $V$  is the 5-dimensional orthogonal module and  $x \in Y_2$ , then  $\dim C_V(x) = 3$ . Even so, probing  $\mathcal{C}(G, Y_2)$  turns out to be a lengthy process. Fix  $t \in Y_2$ . Then by Lemma 4.3,  $Y_2 \subseteq \bigcup_{U \in \mathcal{U}_1} C_G(U)$  where  $\mathcal{U}_1$  is the set of all 1-subspaces of  $C_V(t)$  and as a result, by Lemma 4.4,  $\mathcal{C}(G, Y_2)$  may be viewed as the union of commuting involution graphs for various subgroups of  $G$ . Up to isomorphism there are three of these commuting involution graphs (called  $\mathcal{C}(G^-, Y^-)$ ,  $\mathcal{C}(G^+, Y^+)$  and  $\mathcal{C}(G^0, Y^0)$  in Sect. 4). After studying these three commuting involution graphs in Theorems 4.6, 4.8 and 4.14 it follows immediately (Theorem 4.15) that  $\mathcal{C}(G, Y_2)$  is connected and has diameter at most 3. Using the sizes of the discs in  $\mathcal{C}(G^-, Y^-)$ ,  $\mathcal{C}(G^+, Y^+)$  and  $\mathcal{C}(G^0, Y^0)$  we then complete the proof of Theorem 1.4. This “patching together” of the discs is quite complicated—for example we must confront such issues as  $t$  and  $x$  in  $Y_2$  being of distance 3 in each of the commuting involution subgraphs which contain both  $t$  and  $x$ , yet they have distance 2 in  $\mathcal{C}(G, Y_2)$  (see Lemmas 4.29–4.34).

Our group theoretic notation is standard as given, for example, in [5] or [28].

## 2 Structure of $\mathcal{C}(G, X_i), i = 1, 2, 3$

We begin looking at  $G_0 = Sp_{2n}(q)$  where  $n \geq 2$ ,  $q = p^a$  and  $p = 2$ . Let  $V_0$  denote the  $GF(q)G_0$ -symplectic module of dimension  $2n$  and let  $t_0$  be an involution in  $G_0$  for which  $\dim C_V(t_0) = 2n - 1$ . Put  $X_0 = t_0^{G_0}$ , the  $G_0$ -conjugacy class of  $t_0$ .

**Theorem 2.1**  $\mathcal{C}(G_0, X_0)$  is connected and has diameter 2.

**Proof** For  $x \in X_0$ ,

$$C_{G_0}(x) \leq \text{Stab}_{G_0}(C_{V_0}(x))$$

with  $\text{Stab}_{G_0}(C_{V_0}(x))$  having shape  $q^{2n-1}SL_{2n-2}(q)(q-1)$ . Set  $K_x = O^{2'}(\text{Stab}_{G_0}(C_{V_0}(x)))$ . Then  $K_x \sim q^{2n-1}SL_{2n-2}(q)$  and  $C_{G_0}(x) = K_x$ . Let  $x \in X_0 \setminus \{t_0\}$ . If  $C_{V_0}(t_0) = C_{V_0}(x)$ , then  $x \in K_{t_0}$  and so  $x \in \Delta_1(t_0)$ . Now suppose that  $C_{V_0}(t_0) \neq C_{V_0}(x)$ . Then  $\dim(C_{V_0}(t_0) \cap C_{V_0}(x)) = 2n - 2$ . Let  $U$  be a 1-dimensional subspace of  $C_{V_0}(t_0) \cap C_{V_0}(x)$ . Since  $[V_0, t_0]$  is a 1-space and  $G_0$  acts transitively on the 1-spaces of  $V_0$ , there exists  $y \in X_0$  such that  $[V_0, y] = U$ . So  $[V_0, y] \leq C_{V_0}(t_0) \cap C_{V_0}(x)$  and hence  $y$  leaves both  $C_{V_0}(t_0)$  and  $C_{V_0}(x)$  invariant. Thus  $y \in K_{t_0} \cap K_x = C_{G_0}(t_0) \cap C_{G_0}(x)$  and so  $d(t_0, x) \leq 2$  and we see that  $\mathcal{C}(G_0, X_0)$  is connected. Since  $\mathcal{C}(G_0, X_0)$  cannot have diameter 1 (as then  $\langle X_0 \rangle$  would be abelian), the theorem follows. □

The remainder of this section is devoted to establishing Theorems 1.1 and 1.2. So we have  $G = Sp(4, q)$  with  $q = p^a$  and  $p = 2$ . For  $V$ , the natural  $GF(q)$  module for  $G$ , we choose the symplectic basis  $\{v_1, v_2 | v_3, v_4\}$  with  $(v_1, v_4) = (v_2, v_3) = 1$ . Thus the matrix defining this form is

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and we may suppose that  $G = \{A \in GL(4, q) | A^T J A = J\}$ . We further define

$$S = \left\{ \left( \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & ad + b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| a, b, c, d \in GF(q) \right\},$$

$$Q_1 = \left\{ \left( \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| a, b, c \in GF(q) \right\} \text{ and } Q_2 = \left\{ \left( \begin{pmatrix} 1 & 0 & b & c \\ 0 & 1 & d & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| b, c, d \in GF(q) \right\}.$$

**Lemma 2.2**

- (i)  $S \in Syl_2 G$ .
- (ii)  $S = Q_1 Q_2$  with  $Q_1^\# \cup Q_2^\#$  consisting of all the involutions of  $S$ .

**Proof** It is straightforward to check that  $S$  is a subgroup of  $G$ . Since  $|G| = q^4(q^2 - 1)(q^4 - 1)$  and  $|S| = q^4$ , we have part (i). Part (ii) is an easy calculation. □

The following three involutions are elements of  $G$ .

$$t_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad t_2 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad t_3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

**Lemma 2.3**

- (i) For  $i = 1, 2, 3, t_i \in X_i$ .
- (ii)  $C_G(t_1) \sim q^3 SL(2, q)$  with  $O_2(C_G(t_1)) = Q_1$  of order  $q^3$ .
- (iii)  $C_G(t_2) = S$ .
- (iv)  $|X_1| = q^4 - 1$ .
- (v)  $|X_2| = (q^2 - 1)(q^4 - 1)$ .

**Proof**

- (i) Let  $v = (\alpha, \beta, \gamma, \delta) \in V$ . Then  $v^{t_1} = (\alpha, \beta, \gamma, \alpha + \delta)$ ,  $v^{t_2} = (\alpha, \beta, \alpha + \gamma, \alpha + \beta + \delta)$  and  $v^{t_3} = (\alpha, \alpha + \beta, \gamma, \gamma + \delta)$ . Hence  $[v, t_1] = (0, 0, 0, \alpha)$ ,  $[v, t_2] = (0, 0, \alpha, \alpha + \beta)$  and  $[v, t_3] = (0, \alpha, 0, \gamma)$ . Consequently  $\dim [V, t_1] = 1$  and  $\dim [V, t_2] = 2 = \dim [V, t_3]$ . Thus  $t_1 \in X_1$ . Now

$$(v, v^{t_2}) = \alpha(\alpha + \beta + \delta) + \beta(\alpha + \gamma) + \gamma\beta + \delta\alpha = \alpha^2 = 0$$

implies that  $\alpha = 0$  and so  $\dim V(t_3) = 3$ . Therefore  $t_2 \in X_2$ . Turning to  $t_3$  we have that

$$(v, v^{t_3}) = \alpha(\gamma + \delta) + \beta\gamma + \gamma(\alpha + \beta) + \delta\alpha = 0$$

implies that  $V(t_2) = V$ , as  $v$  is an arbitrary vector of  $V$ . Hence  $t_3 \in X_3$ , and we have (i).

- (ii) By direct calculation we see that

$$C_G(t_1) = \left\{ \left( \begin{array}{cccc} 1 & b & c & d \\ 0 & f & g & h \\ 0 & k & m & n \\ 0 & 0 & 0 & 1 \end{array} \right) \middle| \begin{array}{l} b, c, d, f, g, h, k, m, n \in GF(q) \\ gk + fm = 1 \\ b + hk + fn = 0 \\ c + mh + gn = 0 \end{array} \right\}.$$

Moreover

$$SL_2(q) \cong R = \left\{ \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & f & g & 0 \\ 0 & k & m & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) \middle| \begin{array}{l} f, g, k, m \in GF(q) \\ fg + km = 1 \end{array} \right\} \leq C_G(t_1)$$

with  $Q_1$  a normal elementary abelian subgroup of  $C_G(t_1)$  and  $|Q_1| = q^3$ . So  $C_G(t_1) = RQ_1$ . Thus (ii) holds.

- (iii) This is a routine calculation. From parts (ii) and (iii)  $|C_G(t_1)| = q^4(q^2 - 1)$  and  $|C_G(t_2)| = q^4$ . Combining this with  $|G| = q^4(q^2 - 1)(q^4 - 1)$  yields (iv) and (v). □

**Lemma 2.4**  $|C_G(t_1) \cap X_1| = q^3 - 1$ .

**Proof** Let  $s$  be an involution in  $S$ . Then, by Lemma 2.2(ii),  $s \in Q_1^\# \cup Q_2^\#$ . Let  $v = (\alpha, \beta, \gamma, \delta)$  be a vector in  $V$ . Assume for the moment that  $s \in Q_1$ . Then

$$s = \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where  $a, b, c \in GF(q)$ . So  $v^s = (\alpha, \alpha a + \beta, b\beta + \gamma, c\alpha + b\beta + a\gamma + \delta)$ . Suppose that at least one of  $a$  and  $b$  is non-zero. If  $v \in C_V(s)$ , then we have

$a\alpha = b\beta = c\alpha + b\beta + a\gamma = 0$ . If, say,  $a \neq 0$  then this gives  $\alpha = 0$  and  $b\beta + a\gamma = 0$ . Hence  $\gamma = \lambda\beta$  for some  $\lambda \in GF(q)$ . Thus  $\dim C_V(s) = 2$ , with the same conclusion if  $b \neq 0$ .

When  $a = b = 0$  we see that  $\dim C_V(s) = 3$ . Therefore we conclude that

$$|Q_1 \cap X_1| = q - 1. \tag{2.4.1}$$

Now we suppose  $s \in Q_2 \setminus Q_1$ . Then

$$s = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where  $a, b, c \in GF(q)$  and  $c \neq 0$ . Here  $v^s = (\alpha, \beta, a\alpha + c\beta + \gamma, b\alpha + a\beta + \delta)$  and so, if  $v \in C_V(s)$ ,  $a\alpha + c\beta = b\alpha + a\beta = 0$ . Suppose that  $a = 0$  and  $b \neq 0$ . Then  $c\beta = b\alpha = 0$  which yields  $\alpha = 0 = \beta$ . Hence  $\dim C_V(s) = 2$ . Likewise, when  $a \neq 0$  and  $b = 0$  we get  $\dim C_V(s) = 2$ . On the other hand,  $a = 0 = b$  gives  $\dim C_V(s) = 3$ .

Now consider the case when  $a \neq 0 \neq b$  and  $a^2 + bc = 0$ . From  $a\alpha + c\beta = 0$  we obtain  $\beta = a\alpha c^{-1}$  and so  $0 = b\alpha + a\beta = b\alpha + a^2c^{-1}\alpha = (b + a^2c^{-1})\alpha$ . Since  $a^2 + bc = 0$ , this equation holds for all  $\alpha \in GF(q)$  and consequently  $\dim C_V(s) = 3$ . Similar considerations show that  $\dim C_V(s) = 2$  when  $a \neq 0 \neq b$  and  $a^2 + bc \neq 0$ . So, to summarize, for  $s \in Q_2 \setminus Q_1$ ,  $s \in X_1$  when either  $a = 0 = b$  or  $a \neq 0 \neq b$  and  $a^2 + bc = 0$ . For the former, there are  $q - 1$  such involutions (as  $c \neq 0$ ). For the latter, there are  $q - 1$  choices for each of  $b$  and  $c$  and in each case  $a$  is uniquely determined (as  $GF(q)^\#$  is cyclic of odd order), so giving  $(q - 1)^2$  involutions. Therefore

$$|(X_1 \cap S) \setminus Q_1| = |X_1 \cap (Q_2 \setminus Q_1)| = q(q - 1). \tag{2.4.2}$$

Since any two distinct Sylow 2-subgroups of  $SL(2, q)$  have trivial intersection and  $SL(2, q)$  possesses  $q + 1$  Sylow 2-subgroups, Lemma 2.3(ii) together with (2.4.1) and (2.4.2) yields that

$$\begin{aligned} |C_G(t_1) \cap X_1| &= (q - 1) + q(q - 1)(q + 1) \\ &= (q - 1)(1 + q^2 + q) = q^3 - 1. \end{aligned}$$

This proves Lemma 2.4. □

*Proof of Theorem 1.1* As is well-known—see for example [18] —  $G$  has an outer automorphism arising from the Dynkin diagram of type  $C_2 = B_2$ . This outer automorphism interchanges the two involution conjugacy classes  $X_1$  and  $X_3$  and as a consequence  $\mathcal{C}(G, X_1)$  and  $\mathcal{C}(G, X_3)$  are isomorphic graphs. Thus we need only consider  $\mathcal{C}(G, X_1)$ . From Lemma 2.4, as  $\Delta_1(t) = (C_G(t_1) \cap X_1) \setminus \{t_1\}$ ,

$$|\Delta_1(t_1)| = (q^3 - 1) - 1 = q^3 - 2.$$

By Theorem 2.1,  $\mathcal{C}(G, X_1)$  has diameter 2. Hence, by Lemma 2.3(iv),

$$|\Delta_2(t_1)| = |X_1| - (q^3 - 1) = (q^4 - 1) - (q^3 - 1) = q^4 - q^3 = q^3(q - 1),$$

so proving Theorem 1.1.

Before moving on to prove Theorem 1.2 we need additional preparatory material. If  $W$  is a subspace of  $V$ , then  $W^\perp$  denotes the subspace of  $V$  defined by

$$W^\perp = \{v \in V \mid (v, w) = 0 \text{ for all } w \in W\}$$

and we recall that  $\dim W + \dim W^\perp = \dim V = 4$ .

By Lemma 2.3(i), (iii) we see that  $C_V(C_G(t_2)) = \{(0, 0, 0, \alpha) \mid \alpha \in GF(q)\}$  is 1-dimensional. For  $x \in X_2$  set  $U_1(x) = C_V(C_G(x))$  and  $U_2(x) = C_V(x)$ . So  $\dim U_1(x) = 1$  and  $\dim U_2(x) = 2$  (with the subscripts acting as a reminder). We denote the stabilizer in  $G$  of  $U_1(t_2)$ , respectively  $U_2(t_2)$ , by  $P_1$ , respectively  $P_2$ . Then  $P_i \sim q^3SL_2(q)(q - 1)$  for  $i = 1, 2$ . Also  $Q_i = O_2(P_i)$  with  $C_{P_i}(Q_i) = Q_i$  for  $i = 1, 2$ .

We start analyzing  $\mathcal{C}(G, X_2)$  by determining  $\Delta_1(t_2)$ . For  $x \in X_2$  we let  $Z_{C_G(x)}$  denote  $Z(C_G(x)) \cap X_2$ .

**Lemma 2.5**

$$X_2 = \bigcup_{R \in \text{Syl}_2G} Z_R.$$

**Proof** Clearly  $X_2 = \bigcup_{R \in \text{Syl}_2G} Z_R$  by Lemma 2.3(iii). If  $Z_R \cap Z_T = \emptyset$  for  $R, T \in \text{Syl}_2G$ , then we have some  $x \in Z(R) \cap Z(T) \cap X_2$  whence, using Lemma 2.3(iii),  $R = C_G(x) = T$ . So the lemma holds.  $\square$

**Lemma 2.6** *Let  $R, T \in \text{Syl}_2G$ . If there exists  $x \in Z_R$  and  $y \in Z_T$  such that  $[x, y] = 1$ , then  $[Z_R, Z_T] = 1$ .*

**Proof** Since  $xy = yx$ ,  $y \in C_G(x) = R$ . Hence  $Z(R) \leq C_G(y) = T$  and so  $[Z_R, Z_T] = 1$ .  $\square$

Let  $\Delta$  be the building for  $G$  and  $\mathcal{C}(\Delta)$  denote the chamber graph of  $\Delta$ . We may view the vertices (chambers) of  $\mathcal{C}(\Delta)$  as being  $\{N_G(R) \mid R \in \text{Syl}_2G\}$  with two distinct chambers  $N_G(R)$  and  $N_G(T)$  being adjacent whenever  $\langle N_G(R), N_G(T) \rangle \leq P_i^g$  for some  $g \in G$  and some  $i \in \{1, 2\}$ . We use  $d^c$  to denote the standard distance metric in  $\mathcal{C}(\Delta)$  and for a chamber  $c$  put  $\Delta_j^c(c) = \{d \in \mathcal{C}(\Delta) \mid d^c(c, d) = j\}$ . The structure of  $\mathcal{C}(\Delta)$  is well-known.

**Lemma 2.7**  *$\mathcal{C}(\Delta)$  has diameter 4 and  $|\Delta_1^c(c)| = 2q$ ;  $|\Delta_2^c(c)| = 2q^2$ ;  $|\Delta_3^c(c)| = 2q^3$ ; and  $|\Delta_4^c(c)| = q^4$ .*

**Proof** A straightforward calculation.  $\square$

We now introduce a graph  $\mathcal{Z}$  whose vertex set is  $V(\mathcal{Z}) = \{Z_R \mid R \in \text{Syl}_2G\}$  with  $Z_R, Z_T \in V(\mathcal{Z})$  joined if  $Z_R \neq Z_T$  and  $[Z_R, Z_T] = 1$ .



**Lemma 2.8** *The graphs  $\mathcal{Z}$  and  $\mathcal{C}(\Delta)$  are isomorphic.*

**Proof** Define  $\varphi : V(\mathcal{Z}) \rightarrow V(\mathcal{C}(\Delta))$  by  $\varphi : Z_R \mapsto N_G(R)$  ( $R \in \text{Syl}_2 G$ ). If  $\varphi(Z_R) = \varphi(Z_T)$  for  $R, T \in \text{Syl}_2 G$ , then  $N_G(R) = N_G(T)$  and so  $R = T$  and then  $Z_R = Z_T$ . Thus  $\varphi$  is a bijection between  $V(\mathcal{Z})$  and  $V(\mathcal{C}(\Delta))$ . Suppose  $N_G(R)$  and  $N_G(T)$  are distinct, adjacent chambers in  $\mathcal{C}(\Delta)$ . Without loss of generality we may assume  $T = S$ . Then  $N_G(R), N_G(S) \leq P_i$  for  $i \in \{1, 2\}$ . The structure of  $P_i$  then forces  $Z(R), Z(S) \leq Q_i$ . Since  $Q_i$  is abelian, we deduce that  $[Z_R, Z_S] = 1$ . So  $Z_R$  and  $Z_S$  are adjacent in  $\mathcal{Z}$ . Conversely, suppose  $Z_R$  and  $Z_S$  are adjacent in  $\mathcal{Z}$ . Then  $[Z_R, Z_S] = 1$  with, by Lemma 2.5,  $Z_R \cap Z_S = \emptyset$ . Hence  $Z_R \subseteq S$  and so by Lemma 2.2(ii),  $Z_R \subseteq Q_1 \cup Q_2$ . Now  $Q_1 \cap Q_2 \cap X_2 = Z_S$  and so we must have  $Z_R \subseteq Q_i$  for  $i \in \{1, 2\}$ . The structure of  $P_i$  now gives  $N_G(R) \leq P_i$  and therefore  $N_G(R)$  and  $N_G(S)$  are adjacent in  $\mathcal{C}(\Delta)$ , which proves the lemma.  $\square$

**Proof of Theorem 1.2** Since for all  $x_1, x_2 \in X_2$ ,  $[x_1, x_2] = 1$  if and only if  $[Z_{C_G(x_1)}, Z_{C_G(x_2)}] = 1$  by Lemma 2.5, then for  $i > 1$ ,  $d^{\mathcal{C}}(x_1, x_2) = i$  if and only if  $d^{\mathcal{Z}}(Z_{C_G(x_1)}, Z_{C_G(x_2)}) = i$  (where  $d^{\mathcal{Z}}$  denotes the distance in  $\mathcal{Z}$ ). Note that if  $d^{\mathcal{C}}(x_1, x_2) = 1$ , then either  $Z_{C_G(x_1)} = Z_{C_G(x_2)}$  or  $d^{\mathcal{Z}}(Z_{C_G(x_1)}, Z_{C_G(x_2)}) = 1$ . Since  $X_2$  is a disjoint union of the elements of  $\mathcal{Z}$ , then  $\mathcal{C}(G, X_2)$  is connected of diameter 4. Now

$$\Delta_1(t) = \bigcup_{\substack{R \in \text{Syl}_2 G \\ [Z_S, Z_R] = 1}} Z_R \quad \text{and} \quad \Delta_i(t) = \bigcup_{\substack{R \in \text{Syl}_2 G \\ d^{\mathcal{Z}}(Z_S, Z_R) = i}} Z_R, \quad i > 1$$

and so  $|\Delta_1(t)| = |Z_S| + 2q|Z_S| - 1$ . From  $|Z_S| = (q - 1)^2$  we get  $|\Delta_1(t)| = (q - 1)^2 + 2q(q - 1)^2 - 1 = q^2(2q - 3)$ . The remaining disc sizes are immediate from the structure of the chamber graph  $\mathcal{C}(\Delta)$ .  $\square$

### 3 Structure of $\mathcal{C}(G, Y_1)$

This section is devoted to the proof of Theorem 1.3. In order to investigate the disc structure of  $\mathcal{C}(G, Y_1)$  it is advantageous for us to work in  $H = Sp_4(q)$  (and so  $\bar{H} = H/Z(H) \cong G$ ). We assume that  $\{v_1, v_2, v_3, v_4\}$  is a hyperbolic basis for  $V$  with  $(v_2, v_1) = (v_4, v_3) = 1$ . Thus if  $J$  is the matrix defining this form then

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and  $J$  has two diagonal blocks  $J_0$  where  $J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ .

We may suppose that for  $t \in Y_1$ , we have  $\bar{s} = t$  where  $s = \left( \begin{array}{c|c} -I_2 & \\ \hline & I_2 \end{array} \right)$ . Put  $X = s^H$ . Then  $Y_1 = \{\bar{x} | x \in X\}$ . For  $x \in X$ , set  $N_x = N_H(\langle x, Z(H) \rangle)$ . Evidently, for

$\bar{x}_1, \bar{x}_2 \in Y_1$  (where  $x_1, x_2 \in X$ )  $\bar{x}_1$  and  $\bar{x}_2$  commute if and only if  $x_1 \in N_{x_2}$  (or equivalently  $x_2 \in N_{x_1}$ ). Now  $N_s$  consists of  $g \in H$  for which  $s^g = s$  or  $s^g = -s$ .

Letting  $g = \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$  where  $A, B, C$  and  $D$  are  $2 \times 2$  matrices over  $GF(q)$ , direct

calculation reveals that either  $B = C = 0$  or  $A = D = 0$ . Also, as  $g \in H$ , we must have  $A^T J_0 A = D^T J_0 D = J_0$  and therefore

$$N_s = \left\{ \left( \begin{array}{c|c} A & \\ \hline & B \end{array} \right), \left( \begin{array}{c|c} & A \\ \hline B & \end{array} \right) \mid A, B \in SL_2(q) \right\} \\ \sim (SL_2(q) \times SL_2(q)) : 2$$

**Lemma 3.1**

$$|\Delta_1(t)| = \frac{1}{2}q(q^2 - 1).$$

**Proof** Since  $X = s^H$  consists of all the involutions in  $H \setminus Z(H)$ , a quick calculation gives

$$X \cap N_s = \left\{ \left( \begin{array}{c|c} & A \\ \hline A^{-1} & \end{array} \right) \mid A \in SL_2(q) \right\} \cup \{s, -s\}.$$

Under the natural homomorphism to  $G$ , for  $x \in X$   $\bar{x} = \overline{-x}$ , and so  $|\Delta_1(t)| = \frac{1}{2}|SL_2(q)| = \frac{1}{2}q(q^2 - 1)$ . □

Put  $E = \langle v_3, v_4 \rangle$ . Then  $E^\perp = \langle v_1, v_2 \rangle$  and we note that  $C_V(s) = E$ . Furthermore we have that  $\text{Stab}_H(\{E, E^\perp\}) = N_s$ . Put  $\Sigma = \{\{F, F^\perp\} \mid F \text{ is a hyperbolic 2-subspace of } V\}$ . Now let  $\beta \in GF(q)$  and set  $U_\beta = \langle (1, 0, 1, 0), (0, \beta, 0, -\beta - 1) \rangle$ . Then  $U_\beta$  is a hyperbolic 2-subspace of  $V$  and so  $\{U_\beta, U_\beta^\perp\} \in \Sigma$ . The  $N_s$ -orbit of  $\{U_\beta, U_\beta^\perp\}$  will be denoted by  $\Sigma_\beta$ .

**Lemma 3.2** *Let  $F$  be a hyperbolic 2-subspace of  $V$  with  $F \neq E$  or  $E^\perp$ . Then  $\{F, F^\perp\} \in \Sigma_\beta$  for some  $\beta \in GF(q)$ . Moreover, for  $\beta \in GF(q)$ ,  $\Sigma_\beta = \Sigma_{-\beta-1}$ .*

**Proof** Since  $F \neq E$  or  $E^\perp$ , we may find  $w_1 \in F$  with  $w_1 = (\alpha_1, \beta_1, \gamma_1, \delta_1)$  and  $\{\alpha_1, \beta_1\} \neq \{0\} \neq \{\gamma_1, \delta_1\}$ . Now  $N_s$  contains two  $SL_2(q)$  subgroups for which  $\langle v_1, v_2 \rangle$  and  $\langle v_3, v_4 \rangle$  are natural  $GF(q)SL_2(q)$ -modules. Because  $SL_2(q)$  acts

transitively on the non-zero vectors of such modules, we may suppose  $w_1 = (1, 0, 1, 0)$ . Now choose  $w_2 \in F$  such that  $(w_1, w_2) = 1$  (and so  $\langle w_1, w_2 \rangle = F$ ). Then if  $w_2 = (\alpha, \beta, \gamma, \delta)$  we must have  $\beta + \delta = -1$  and so  $w_2 = (\alpha, \beta, \gamma, -\beta - 1)$ . The matrices in  $N_s$  fixing  $w_1$  are

$$C_{N_s}(w_1) = \left\{ \left( \begin{array}{cc|cc} 1 & & & \\ a_1 & 1 & & \\ \hline & & 1 & \\ & & a_2 & 1 \end{array} \right), \left( \begin{array}{cc|cc} & & 1 & \\ & & a_1 & 1 \\ \hline 1 & & & \\ a_2 & 1 & & \end{array} \right) \mid a_1, a_2 \in GF(q) \right\}.$$

Let

$$g = \left( \begin{array}{cc|cc} 1 & & & \\ a_1 & 1 & & \\ \hline & & 1 & \\ & & a_2 & 1 \end{array} \right)$$

where  $a_1, a_2 \in GF(q)$ . Then  $w_1^g = w_1$ .

We single out the cases  $\beta = 0$  and  $\beta = -1$  for special attention. If, say,  $\beta = 0$ , then  $w_2 = (\alpha, 0, \gamma, -1)$ . Hence  $w_2 - \alpha w_1 = (0, 0, \gamma - \alpha, -1)$  and  $F = \langle w_1, w_2 - \alpha w_1 \rangle$ . Since  $(0, 0, \gamma - \alpha, -1)g = (0, 0, (\gamma - \alpha) - a_2, -1)$  and choosing  $a_2 = -\gamma + \alpha$ , we obtain  $Fg = U_0$ . For  $\beta = -1$  a similar argument works (using  $w_2 - \gamma w_1$  instead of  $w_2 - \alpha w_1$ ). So we may assume that  $\beta \neq 0, -1$ . From

$$w_2 g = (\alpha, \beta, \gamma, -\beta - 1) = (\alpha + \beta a_1, \beta, \gamma + (-\beta - 1)a_2, -\beta - 1)$$

by a suitable choice of  $a_1$  and  $a_2$ , as  $\beta \neq 0, -1$ , we get  $w_2 g = (0, \beta, 0, -\beta - 1)$ , whence  $Fg = U_\beta$ . Thus we have shown  $\{F, F^\perp\} \in \Sigma_\beta$  for some  $\beta \in GF(q)$ . Finally, for  $\beta \in GF(q)$ ,  $\Sigma_\beta = \Sigma_{-\beta-1}$  follows from

$$(0, \beta, 0, -\beta - 1) \left( \begin{array}{cc|cc} & & 1 & \\ & & & 1 \\ \hline 1 & & & \\ & 1 & & \end{array} \right) = (0, -\beta - 1, 0, \beta).$$

□

Let  $\phi : GF(q) \setminus \{-1\} \rightarrow GF(q)$  be defined by

$$\phi(\lambda) = -\left(1 + (\lambda + 1)^{-2}(1 - \lambda^2)\right)^{-1} \quad (\lambda \in GF(q)).$$

There is a possibility that this is not well-defined should  $1 + (\lambda + 1)^{-2}(1 - \lambda^2) = 0$ . This would then give  $(\lambda + 1)^2 + (1 - \lambda^2) = 0$  from which we infer that  $\lambda = -1$ . So we conclude that  $\phi$  is well-defined.

**Lemma 3.3**  $\phi$  is injective.

**Proof** Suppose  $\phi(\lambda) = \phi(\mu)$  for  $\lambda, \mu \in GF(q) \setminus \{-1\}$  with  $\lambda \neq \mu$ . Hence

$$\left(1 + (\lambda + 1)^{-2}(1 - \lambda^2)\right)^{-1} = \left(1 + (\mu + 1)^{-2}(1 - \mu^2)\right)^{-1}.$$

Simplifying and using the fact that  $q$  is odd gives

$$\mu^2 + \mu - \mu\lambda^2 - \lambda^2 - \lambda + \lambda\mu^2 = 0,$$

and then

$$(\mu + \lambda)(\mu - \lambda) + (\mu - \lambda) + \lambda\mu(\mu - \lambda) = 0.$$

Hence  $(\mu - \lambda)(\mu + \lambda + 1 + \lambda\mu) = 0$ . Since  $\mu \neq \lambda$ , we get  $\mu + \lambda + 1 + \lambda\mu = 0$  from which we deduce that either  $\lambda = -1$  or  $\mu = -1$ , a contradiction. So the lemma holds.  $\square$

**Proof of Theorem 1.3** We first show that  $\text{Diam } \mathcal{C}(G, Y_1) = 2$ . So let  $x \in X$  be such that  $x \notin \{t\} \cup \Delta_1(t)$ . Now  $\{C_V(x), C_V(x)^\perp\} \in \Sigma$  as  $C_V(x) \neq E$  or  $E^\perp$  (otherwise  $x \in \{s, -s\}$  and then  $\bar{x} = t$ ). Hence  $\{C_V(x), C_V(x)^\perp\} \in \Sigma_\mu$  for some  $\mu \in GF(q)$  by

Lemma 3.2. Let  $y = \left(\begin{array}{c|c} & I_2 \\ \hline I_2 & \end{array}\right)$ .

Then  $\bar{y} \in \Delta_1(t)$ . Our aim is to choose an  $x_\lambda \in N_y \cap X$  (so  $\bar{x}_\lambda \in \Delta_1(\bar{y})$ ) for which  $\{C_V(x_\lambda), C_V(x_\lambda)^\perp\} \in \Sigma_\mu$ . Since  $\Sigma_\mu$  is an  $N_s$ -orbit, there exists  $h \in N_s$  such that  $\{C_V(x_\lambda), C_V(x_\lambda)^\perp\}^h = \{C_V(x), C_V(x)^\perp\}$ . As a consequence either  $x = x_\lambda^h$  or  $x_\lambda^{-1h}$  and therefore  $\bar{x} = \bar{x}_\lambda^{\bar{h}}$ , whence  $d(t, \bar{x}) \leq 2$ .  $\square$

We first look at the case when  $\mu = -2^{-1}$ . Then  $\mu = -\mu - 1$  and hence

$$U_{-2^{-1}} = \langle (1, 0, 1, 0), (0, 1, 0, 1) \rangle.$$

Observing that  $U_{-2^{-1}} = C_V(y)$ , we see that for  $\mu = -2^{-1}$ ,  $\bar{x} \in \Delta_1(y)$ , which we are not concerned with here. So we may assume  $\mu \neq -2^{-1}$ .

Let  $x_\lambda = \left(\begin{array}{c|c} \lambda I_2 & -B \\ \hline B & -\lambda I_2 \end{array}\right)$  where  $\lambda \in GF(q) \setminus \{0\}$  and such that  $B$  has zero trace

and determinant  $1 - \lambda^2$ . So  $x_\lambda \in X \cap N_y$ . We now move onto the case when  $\mu = 0$  (or equivalently  $\mu = -1$ ). Here we take  $\lambda = 1$  and  $B = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}$ , noting that  $B$  satisfies the conditions to ensure that  $\bar{x}_1 \in \Delta_1(\bar{y})$ . Let  $v = (\alpha, \beta, \gamma, \delta) \in V$ . Then  $v \in C_V(x_1)$  precisely when

$$\begin{aligned} 2\gamma + 2\delta &= 0; & -2\gamma - 2\delta &= 0; \\ -2\alpha - 2\beta - \gamma &= \gamma; & 2\alpha + 2\beta - \delta &= \delta; \end{aligned}$$

and thus the only conditions we get are  $\gamma = -\beta - \alpha$  and  $\alpha + \beta = \delta$ . Thus

$$C_V(x_1) = \{(\alpha, \beta, -\alpha - \beta, \alpha + \beta)\} \\ = \langle (1, 0, -1, 1), (0, 1, -1, 1) \rangle.$$

It is straightforward to check that  $\{C_V(x_1), C_V(x_1)^\perp\} \in \Sigma_0$ . Therefore we may also assume that  $\mu \neq 0, -1$ . Choosing  $B = \begin{pmatrix} \lambda & \lambda^{-1} \\ -\lambda & -\lambda \end{pmatrix}$  we see that the requisite conditions are satisfied. Take  $v = (\alpha, \beta, \gamma, \delta) \in V$  and calculating  $v^{x_2}$  gives the relations

$$(\lambda - 1)\alpha + \gamma\lambda - \delta\lambda = 0; \quad (\lambda - 1)\beta + \gamma\lambda^{-1} - \delta\lambda = 0; \\ -\lambda\alpha + \lambda\beta - (\lambda + 1)\gamma = 0; \quad -\lambda^{-1}\alpha + \lambda\beta - (\lambda + 1)\delta = 0;$$

which, after rearranging gives

$$\alpha = \lambda(\lambda - 1)^{-1}(\delta - \gamma); \quad \beta = \lambda(\lambda - 1)^{-1}\delta - \lambda^{-1}(\lambda - 1)^{-1}\gamma; \\ \gamma = \lambda(\lambda + 1)^{-1}(\beta - \alpha); \quad \delta = \lambda(\lambda + 1)^{-1}\alpha - \lambda^{-1}(\lambda + 1)^{-1}\alpha;$$

and note that the relations for  $\gamma$  and  $\delta$  are satisfied after substitution for  $\alpha$  and  $\beta$ . Hence

$$C_V(x_\lambda) = \left\{ \left( \alpha, \beta, \lambda(\lambda + 1)^{-1}(\beta - \alpha), \lambda(\lambda + 1)^{-1}\beta - \lambda^{-1}(\lambda + 1)^{-1}\alpha \right) \right\} \\ = \left\langle \left( 1, 0, -\lambda(\lambda + 1)^{-1}, -\lambda^{-1}(\lambda + 1)^{-1} \right), \left( 0, 1, \lambda(\lambda + 1)^{-1}, \lambda(\lambda + 1)^{-1} \right) \right\rangle. \tag{3.3.1}$$

We want to determine which  $N_s$ -orbit,  $\Sigma_\beta$ , that  $C_V(x_\lambda)$  lies in. Our representative,  $U_\beta$ , for  $\Sigma_\beta$  has  $w_1 = (1, 0, 1, 0)$  as one component of the hyperbolic pair, so we need an element of  $N_s$  to send the first generator in (3.3.1) to  $w_1$ . We need to find conditions on  $C, D \in SL_2(q)$  such that

$$(1, 0, -\lambda(\lambda + 1)^{-1}, -\lambda^{-1}(\lambda + 1)^{-1}) \begin{pmatrix} C & | \\ \hline & D \end{pmatrix} = (1, 0, 1, 0)$$

and so without loss of generality we can take  $C = I_2$ . This reduces to solving

$$\left( -\lambda(\lambda + 1)^{-1}, -\lambda^{-1}(\lambda + 1)^{-1} \right) \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} = (1, 0)$$

and after multiplying out, we get that  $d_3 = -(d_1 + 1)\lambda^2 - \lambda$  and  $d_4 = -d_2\lambda^2$ . Since  $D$  has determinant 1, we find that  $d_2 = \lambda^{-1}(\lambda + 1)^{-1}$  and so  $d_4 = -\lambda(\lambda + 1)^{-1}$ . Without loss of generality, by taking  $d_1 = 1$  we have that

$$D = \begin{pmatrix} 1 & \lambda^{-1}(\lambda + 1)^{-1} \\ -2\lambda^2 - \lambda & -\lambda(\lambda + 1)^{-1} \end{pmatrix}$$

and a quick check shows that the first generator in (3.3.1) is mapped to  $w_1$ . Using the same matrix, by multiplying on the right of the second generator in (3.3.1), we get

$$(0, 1, \lambda(\lambda + 1)^{-1}, \lambda(\lambda + 1)^{-1}) \left( \begin{array}{c|c} I_2 & \\ \hline & D \end{array} \right) = (*, 1, *, (\lambda + 1)^{-2}(1 - \lambda^2)) = u'$$

and  $\langle w_1, u' \rangle$  is a hyperbolic 2-subspace conjugate to some  $U_\beta$ . Recall that for a fixed  $\beta \in GF(q)$ ,  $N_s$  is transitive on  $\{(\alpha, \beta, \gamma, -\beta - 1) | \alpha, \gamma \in GF(q)\}$ . Hence, we need only find the hyperbolic pair representing such a conjugate of  $U_\beta$ , to determine  $\beta$ . This is found by requiring that some multiple of  $u'$  has inner product 1 with  $w_1$ , that is

$$\beta \cdot 1 = -1 - \beta((\lambda + 1)^{-2}(1 - \lambda^2))$$

for some  $\beta \in GF(q)$ . By expanding, we get that  $\beta = -\left(1 + (\lambda + 1)^{-2}(1 - \lambda^2)\right)^{-1}$  and so  $C_V(x_\lambda) \in \Sigma_\beta$ . By Lemma 3.3,  $\phi : \lambda \mapsto -\left(1 + (\lambda + 1)^{-2}(1 - \lambda^2)\right)^{-1}$  is an injective map from  $GF(q) \setminus \{-1\}$  into  $GF(q)$ . Since  $\mu \neq -2^{-1}$ ,  $\mu \neq -\mu - 1$  and therefore there exists  $\lambda \in GF(q) \setminus \{-1\}$  such that  $\phi(\lambda) = \mu$  or  $-\mu - 1$ . Bearing in mind that  $U_\mu = U_{-\mu-1}$  by Lemma 3.2, we conclude that  $\{C_V(x_\lambda), C_V(x_\lambda)^\perp\} \in \Sigma_\mu$ . Consequently we have proved that  $\text{Diam } \mathcal{C}(G, Y_1) = 2$ .

From  $|G| = \frac{q^4}{2}(q^2 - 1)(q^4 - 1)$  and  $|C_G(t)| = q^2(q^2 - 1)^2$  we see that  $|Y_1| = \frac{q^2}{2}(q^2 + 1)$ . Using Lemma 3.1 then gives

$$|\Delta_2(t)| = \frac{1}{2}(q^4 - q^3 + q^2 + q - 2),$$

which completes the proof of Theorem 1.3.

### 4 Structure of $\mathcal{C}(G, Y_2)$

In this section we present a proof of Theorem 1.4. The uncovering of the disc structures of  $\mathcal{C}(G, Y_2)$  will be a long haul. As discussed in Sect. 1, it will be advantageous for us to use the well known isomorphism that  $PSp(4, q) \cong O(5, q)$  (see Corollary 12.32 of [44]). So we take  $G = O(5, q)$  and from now on  $V$  will denote the 5-dimensional  $GF(q)$  orthogonal module for  $G$ . Thus the elements of  $G$  are  $5 \times 5$  orthogonal matrices with respect to the orthogonal form  $(, )$  which have spinor norm a square in  $GF(q)$ . We may assume that the Gram matrix with respect to  $(, )$  is

$$J = \left( \begin{array}{cc|ccc} 0 & 1 & & & \\ 1 & 0 & & & \\ \hline & & 0 & 0 & 1 \\ & & 0 & -2 & 0 \\ & & 1 & 0 & 0 \end{array} \right).$$

Let

$$t = \left( \begin{array}{c|ccc} I_2 & & & \\ \hline & 0 & 0 & 1 \\ & 0 & -1 & 0 \\ & 1 & 0 & 0 \end{array} \right).$$

Then  $t \in G$  and  $Y_2 = t^G$ . Let  $\delta = \pm 1$  where  $q \equiv \delta \pmod{4}$ .

**Lemma 4.1**

- (i)  $\dim(C_V(t)) = 3$ .
- (ii)  $C_V(t)^\perp = [V, t]$  is a 2-subspace of  $V$  of  $\delta$ -type.
- (iii)  $V = C_V(t) \perp C_V(t)^\perp$ .

**Proof** An easy calculation. □

Put  $L_t = C_G(t) \cap C_G([V, t])$ .

**Lemma 4.2**

- (i) Let  $x \in Y_2$ . Then  $t = x$  if and only if  $C_V(t) = C_V(x)$ .
- (ii)  $C_G(t) = \text{Stab}_G(C_V(t)) \sim (L_2(q) \times \frac{q-\delta}{2}).2^2$ .
- (iii)  $L_t$  acts faithfully on  $C_V(t)$  and  $L_t \cong L_2(q)$ .

**Proof**

- (i) Suppose  $C_V(x) = C_V(t)$ . Then, using Lemma 4.1 (ii),  $[V, x] = C_V(x)^\perp = C_V(t)^\perp = [V, t]$ . Hence by Lemma 4.1(iii),  $tx$  acts trivially on  $V$  and thus  $tx = 1$ . Therefore  $t = x$  and (i) holds.
- (ii) Plainly  $C_G(t) \leq \text{Stab}_G(C_V(t))$ , and if  $g \in \text{Stab}_G(C_V(t))$ , then  $C_V(t) = C_V(t)^g = C_V(t^g)$ . Hence, as  $t^g \in Y_2$ ,  $t = t^g$  by part (i). So  $g \in C_G(t)$  and thus  $C_G(t) = \text{Stab}_G(C_V(t))$ . That  $\text{Stab}_G(C_V(t)) \sim (L_2(q) \times \frac{q-\delta}{2}).2^2$  can be read off from Proposition 4.1.6 of [32].
- (iii) For any  $g \in C_G(t)$ , we have  $[V, t]^g = C_V(t)^{\perp g} = C_V(t^g)^\perp = C_V(t)^\perp = [V, t]$  and so  $C_G(t) \leq \text{Stab}_G[V, t]$ . If any element in  $L_t$  acts trivially on  $C_V(t)$ , then it would act trivially on  $V$  and thus be the identity. Hence  $L_t$  acts faithfully on  $C_V(t)$ . Let  $v \in C_V(t)$  and by Lemma 4.1(iii), we have  $[V, t] \leq \langle v \rangle^\perp$ . Hence

$\langle v \rangle^\perp = [V, t] \oplus W$  where  $W \leq C_V(t)$ . But since  $\dim(\langle v \rangle^\perp) = 4$ , we have  $\dim(W) = 2$  and so  $C_V(t) \not\leq \langle v \rangle^\perp$ . Therefore for all  $u \in C_V(t)$ ,  $(v, u) = 0$  if and only if  $v = 0$  and thus  $(\ , \ )$  is non-degenerate on restriction to  $C_V(t)$ . Hence we have  $L_t \hookrightarrow GO(C_V(t)) \sim GO_3(q)$  as  $L_t$  fixes  $[V, t]$  pointwise, by definition. Since  $L_t \leq G$  and acts as determinant 1 on  $[V, t]$ , then it must act as determinant 1 on  $C_V(t)$ . In addition, as  $L_t$  fixes  $[V, t]$  pointwise, when the elements of  $L_t$  are decomposed as products of reflections, the vectors reflected will lie in  $C_V(t)$ . Since the spinor norm of the elements of  $L_t$  are a square in  $GF(q)$  and the vectors reflected lie in  $C_V(t)$ , then the spinor norm doesn't change on restriction to  $C_V(t)$ . Hence,  $L_t \sim O_3(q) \sim L_2(q)$  proving (iii). □

Let  $\mathcal{U}_i$  denote the set of  $i$ -dimensional subspaces of  $C_V(t)$ ,  $i = 1, 2$ . In proving Theorem 1.4, our divide and conquer strategy is based on the following observation.

**Lemma 4.3**

$$Y_2 \subseteq \bigcup_{U \in \mathcal{U}_1 \cup \mathcal{U}_2} C_G(U).$$

**Proof** Let  $x \in Y_2 \setminus \{t\}$  and set  $U = C_V(t) \cap C_V(x)$ . By Lemmas 4.1(i) and 4.2(i),  $U \in \mathcal{U}_1 \cup \mathcal{U}_2$ . Since  $t, x \in C_G(U)$ , we have Lemma 4.3. □

The three cases we must chase down are presaged by our next result.

**Lemma 4.4**

- (i) Let  $U_0$  be an isotropic 1-subspace of  $C_V(t)$ . Then  $C_G(U_0) \sim q^3 : L_2(q)$ .
- (ii) Let  $U_\varepsilon$  be a 1-subspace of  $C_V(t)$ , such that  $U_\varepsilon^\perp \cap C_V(t)$  is a 2-space of  $\varepsilon$ -type ( $\varepsilon = \pm 1$ ). Then

$$C_G(U_\varepsilon) \sim \begin{cases} SL_2(q) \circ SL_2(q) & \delta = \varepsilon \\ L_2(q^2) & \delta = -\varepsilon. \end{cases}$$

**Proof** Let  $U_0$  be an isotropic 1-subspace of  $C_V(t)$ . From Proposition 4.1.20 of [32], we know that  $\text{Stab}_G(U_0) \sim C_0 : (C_1 \times C_2)\langle r \rangle$  where  $C_1$  acts as scalars on  $U_0$ ,  $r$  a reflection of  $U_0$  and  $C_0 \sim q^3$ ,  $C_2 \sim L_2(q)$  fixing  $U_0$  pointwise. Hence  $C_G(U_0) \sim q^3 : L_2(q)$ , so proving (i).

If  $\delta = 1$ , then  $[V, t]$  is a 2-subspace of  $V$  of  $+$ -type, and hence  $U_+^\perp = (U_+^\perp \cap C_V(t)) \perp [V, t]$  is a 4-subspace of  $+$ -type. Similarly,  $U_-^\perp = (U_-^\perp \cap C_V(t)) \perp [V, t]$  is a 4-space of  $-$ -type. If  $\delta = -1$ , then  $[V, t]$  is a 2-subspace of  $V$  of  $-$ -type, and the results for when  $\delta = 1$  interchange. Let  $W_+$  and  $W_-$  be 4-subspaces of  $V$  of  $+$  and  $-$ -type respectively, such that  $W_+^\perp$  and  $W_-^\perp$  are 1-subspaces of  $C_V(t)$ ,



observing that  $\text{Stab}_G(W_{\pm}) = \text{Stab}_G(W_{\pm}^{\perp})$ . From Proposition 4.1.6 of [32], we have

$$\begin{aligned} \text{Stab}_G(W_+) &\sim C_+ \langle s_+ \rangle \\ \text{Stab}_G(W_-) &\sim C_- \langle s_- \rangle \end{aligned}$$

where  $C_+ \sim SL_2(q) \circ SL_2(q)$  fixes  $W_+^{\perp}$  pointwise,  $C_- \sim L_2(q^2)$  fixes  $W_-^{\perp}$  pointwise and  $s_+, s_-$  are reflections of  $W_+^{\perp}$  and  $W_-^{\perp}$  respectively. This proves (ii) and hence the lemma.  $\square$

**Lemma 4.5**

- (i) Let  $U_0$  be a 2-subspace of  $C_V(t)$  such that  $U_0^{\perp} \cap C_V(t)$  is an isotropic 1-space. Then  $C_G(U_0) \sim q^2 : \frac{q-\delta}{2}$ .
- (ii) Let  $U_{\varepsilon}$  be a 2-subspace of  $C_V(t)$  of  $\varepsilon$ -type ( $\varepsilon = \pm 1$ ). Then  $C_G(U_{\varepsilon}) \cong L_2(q)$ .

**Proof** See Propositions 4.1.6 and 4.1.20 of [32].  $\square$

Define the following subsets of  $\mathcal{U}_i, i = 1, 2$ .

$$\begin{aligned} \mathcal{U}_1^+ &= \{U \in \mathcal{U}_1 \mid C_G(U) \cong SL_2(q) \circ SL_2(q)\} \\ \mathcal{U}_1^- &= \{U \in \mathcal{U}_1 \mid C_G(U) \cong L_2(q^2)\} \\ \mathcal{U}_1^0 &= \{U \in \mathcal{U}_1 \mid C_G(U) \sim q^3 : L_2(q)\} \\ \mathcal{U}_2^+ &= \{U \in \mathcal{U}_2 \mid U \text{ is of } +\text{-type}\} \\ \mathcal{U}_2^- &= \{U \in \mathcal{U}_2 \mid U \text{ is of } -\text{-type}\} \\ \mathcal{U}_2^0 &= \left\{ U \in \mathcal{U}_2 \mid C_G(U) \sim q^2 : \frac{q-\delta}{2} \right\}. \end{aligned}$$

In the notation of Lemma 4.4,  $\mathcal{U}_1^+$  is the case  $\delta = \varepsilon$  while  $\mathcal{U}_1^-$  is when  $\delta = -\varepsilon$ . Note by Lemmas 4.4 and 4.5 that  $\mathcal{U}_i = \mathcal{U}_i^0 \cup \mathcal{U}_i^+ \cup \mathcal{U}_i^-$ ,  $i = 1, 2$ . We now study  $C_G(U) \cap Y_2$  for  $U \in \mathcal{U}_1$ . By Lemma 4.4 there are three possibilities for the structure of  $C_G(U)$ . First we look at the case  $U \in \mathcal{U}_1^-$ , and set  $G^- = C_G(U)$ . Then  $G^- \cong L_2(q^2)$  by definition of  $\mathcal{U}_1^-$ . Define  $\Delta_i^-(t) = \{x \in G^- \cap Y_2 \mid d^-(t, x) = i\}$  where  $i \in \mathbb{N}$  and  $d^-$  is the distance metric on the commuting graph  $\mathcal{C}(G^-, G^- \cap Y_2)$ .

**Theorem 4.6** *If  $q \neq 3$  then  $\mathcal{C}(G^-, G^- \cap Y_2)$  is connected of diameter 3 with*

$$\begin{aligned} |\Delta_1^-(t)| &= \frac{1}{2}(q^2 - 1); \\ |\Delta_2^-(t)| &= \frac{1}{4}(q^2 - 1)(q^2 - 5); \quad \text{and} \\ |\Delta_3^-(t)| &= \frac{1}{4}(q^2 - 1)(q^2 + 7). \end{aligned}$$

**Proof** Since  $q^2 \equiv 1 \pmod{4}$  and  $q \neq 3$  implies  $q^2 > 13$ , this follows from Theorem 1.1(iii) of [11].  $\square$

We move on to analyze  $G^+ = C_G(U)$  where  $U \in \mathcal{U}_1^+$ . Hence, by definition of  $\mathcal{U}_1^+$ ,  $G^+ \cong L_1 \circ L_2$  where  $L_1 \cong SL_2(q) \cong L_2$  (with the central product identifying  $Z(L_1)$  and  $Z(L_2)$ ). Set  $Y^+ = G^+ \cap Y_2$ . We begin by describing  $Y^+$ .

**Lemma 4.7**

$$Y^+ = \{x_1x_2 \mid x_i \in L_i \text{ and } x_i \text{ has order } 4, i = 1, 2\}.$$

**Proof** Apart from the central involution  $z$  of  $G^+$ , all other involutions of  $G^+$  are of the form  $g_1g_2$  where  $g_i \in L_i$  ( $i = 1, 2$ ) has order 4. Since all involutions in  $L_i/Z(G^+)$  are conjugate, it quickly follows that  $\{g_1g_2 \mid g_i \in L_i \text{ and } g_i \text{ has order } 4, i = 1, 2\}$  is a  $G^+$ -conjugacy class. Now  $z$  acts as  $-1$  on  $U^\perp$  and thus  $\dim C_V(z) = 1$ . Therefore  $t \neq z$  whence, as  $t \in G^+$ , the lemma holds.  $\square$

Let  $d^+$  denote the distance metric on the commuting graph  $\mathcal{C}(G^+, Y^+)$  and, for  $i \in \mathbb{N}$ ,  $\Delta_i^+(t) = \{x \in Y^+ \mid d^+(t, x) = i\}$ .

**Theorem 4.8** Assume that  $q \notin \{3, 5, 9, 13\}$ . Then  $\mathcal{C}(G^+, Y^+)$  is connected of diameter 3 with

$$\begin{aligned} |\Delta_1^+(t)| &= \frac{1}{2}(q - \delta)^2 + 1; \\ |\Delta_2^+(t)| &= \frac{1}{8}(q - \delta)^3(q - 4 - \delta) + (q - \delta)(q - 2 - \delta); \text{ and} \\ |\Delta_3^+(t)| &= \frac{3}{8}q^4 + \frac{1}{2}(1 + 3\delta)q^3 - \frac{1}{4}(7 + 6\delta)q^2 + \frac{7}{2}(1 + \delta)q - \frac{1}{8}(29 + 20\delta). \end{aligned}$$

**Proof** Let  $\overline{G^+} = G^+/Z(G^+) (= \overline{L_1} \times \overline{L_2})$ . Note that for  $x_1x_2 \in Y^+$ ,  $x_1^{-1}x_2 = x_1x_2^{-1}$  and  $x_1x_2 = x_1^{-1}x_2^{-1}$  and so the inverse image of  $\overline{x_1x_2}$  contains two elements of  $Y^+$ . Let  $d^{(i)}$  denote the distance metric on the commuting graph of  $\overline{L_i}$  and  $\Delta_j^{(i)}(\overline{x_i})$  the  $j^{\text{th}}$  disc of  $\overline{x_i}$  in the commuting graph of  $\overline{L_i}$ . By Lemma 4.7,  $t = t_1t_2$  where, for  $i = 1, 2$ ,  $t_i \in L_i$  has order 4. Let  $x = x_1x_2 \in Y^+$  with  $x \neq t$ . Then  $tx = xt$  if and only if  $tx$  has order 2. So, bearing in mind that  $Y^+ \cup \{z\}$  (where  $\langle z \rangle = Z(G^+)$ ) are all the involutions of  $G^+$ , we have that  $tx = xt$  if and only if one of the following holds:  $x_1 = t_1, x_2 = t_2^{-1}; x_1 = t_1^{-1}, x_2 = t_2; \overline{x_1} \in \Delta_1^{(1)}(\overline{t_1})$  and  $\overline{x_2} \in \Delta_1^{(2)}(\overline{t_2})$ . Thus

$$\Delta_1^+(t) = \left\{x_1x_2 \mid \overline{x_i} \in \Delta_1^{(i)}(\overline{t_i}), i = 1, 2\right\} \cup \{t_1t_2^{-1}\}. \tag{4.8.1}$$

Hence, using [11],

$$|\Delta_1^+(t)| = 2\left(\frac{1}{2}(q - \delta)\right)^2 + 1 = \frac{1}{2}(q - \delta)^2 + 1. \tag{4.8.2}$$

Next we examine  $\Delta_2^+(t)$ . Let  $x \in Y^+$ . Assume that  $x = x_1t_2$  or  $x_1t_2^{-1}$  where  $\overline{x_1} \in \Delta_1^{(1)}(\overline{t_1})$ . Then  $x \in \Delta_1^+(t_1t_2^{-1})$  (recall  $t_1t_2^{-1} = t_1^{-1}t_2$ ) which implies, by (4.8.1),

that  $x \in \Delta_2^+(t)$ . If  $x = t_1x_2$  or  $t_1^{-1}x_2$  where  $\bar{x}_2 \in \Delta_1^{(2)}(\bar{t}_2)$ , we similarly get  $x \in \Delta_2^+(t)$ . Therefore

$$\left\{x_1x_2 \mid \bar{x}_1 \in \Delta_1^{(1)}(\bar{t}_1), \bar{x}_2 = \bar{t}_2\right\} \cup \left\{x_1x_2 \mid \bar{x}_2 \in \Delta_1^{(2)}(\bar{t}_2), \bar{x}_1 = \bar{t}_1\right\} \subseteq \Delta_2^+(t). \tag{4.8.3}$$

Now suppose  $x = x_1x_2$  where  $\bar{x}_1 \in \Delta_2^{(1)}(\bar{t}_1)$  and  $\bar{x}_2 \in \Delta_1^{(2)}(\bar{t}_2)$ . So there exists  $\bar{y}_1 \in \bar{L}_1$  such that  $(\bar{t}_1, \bar{y}_1, \bar{x}_1)$  is a path of length 2 in the commuting graph for  $\bar{L}_1$ . Then  $(t = t_1t_2, y_1x_2^{-1}, x_1x_2 = x)$  is a path of length 2 in  $\mathcal{C}(G^+, Y^+)$ . Thus, by (4.8.1),  $x \in \Delta_2^+(t)$ . If, on the other hand,  $\bar{x}_1 \in \Delta_1^{(1)}(\bar{t}_1)$  and  $\bar{x}_2 \in \Delta_2^{(2)}(\bar{t}_2)$  we obtain the same conclusion. Should we have  $\bar{x}_1 \in \Delta_2^{(1)}(\bar{t}_1)$  and  $\bar{x}_2 \in \Delta_2^{(2)}(\bar{t}_2)$ , similar arguments also give  $x \in \Delta_2^+(t)$ . So

$$\begin{aligned} &\left\{x_1x_2 \mid \bar{x}_1 \in \Delta_2^{(1)}(\bar{t}_1), \bar{x}_2 \in \Delta_1^{(2)}(\bar{t}_2)\right\} \cup \left\{x_1x_2 \mid \bar{x}_1 \in \Delta_1^{(1)}(\bar{t}_1), \bar{x}_2 \in \Delta_2^{(2)}(\bar{t}_2)\right\} \\ &\cup \left\{x_1x_2 \mid \bar{x}_1 \in \Delta_2^{(1)}(\bar{t}_1), \bar{x}_2 \in \Delta_2^{(2)}(\bar{t}_2)\right\} \subseteq \Delta_2^+(t). \end{aligned} \tag{4.8.4}$$

Since  $x = x_1x_2 \in \Delta_2^+(t)$  implies  $d^{(i)}(\bar{t}_i, \bar{x}_i) \leq 2$  for  $i = 1, 2$ ,  $\Delta_2^+(t)$  is the union of the two sets in (4.8.3) and (4.8.4). Thus, employing [11],

$$|\Delta_2^+(t)| = \frac{1}{8}(q - \delta)^3(q - 4 - \delta) + (q - \delta)(q - 2 - \delta). \tag{4.8.5}$$

Now, as  $q \notin \{3, 5, 9, 13\}$ , by [11] the commuting graph for  $\bar{L}_i$  is connected of diameter 3. Arguing as above we deduce that  $\mathcal{C}(G^+, Y^+)$  is also connected with diameter 3. Because  $|Y^+| = 2 \left| \bar{t}_1^{-1} \bar{t}_1 \right| \left| \bar{t}_2^{-1} \bar{t}_2 \right| = \frac{1}{2}q^2(q + \delta)^2$ , combining (4.8.2) and (4.8.5) we may determine  $|\Delta_3^+(t)|$  to be as stated, so completing the proof of Theorem 4.8.  $\square$

Finally we look at  $C_G(U)$  where  $U \in \mathcal{U}_1^0$ . This will prove to be trickier than the other two cases. Put  $G^0 = C_G(U)$ . So  $G^0 \sim q^3 : L_2(q)$ . We require an explicit description of  $G^0$  which we now give. Let  $Q = \{(\alpha, \beta, \gamma) \mid \alpha, \beta, \gamma \in GF(q)\}$  and

$$L = \left\{ \left( \begin{array}{ccc} a^2 & 2ab & b^2 \\ ac & ad + bc & bd \\ c^2 & 2cd & d^2 \end{array} \right) \mid \begin{array}{l} a, b, c, d \in GF(q) \\ ad - bc = 1 \end{array} \right\}.$$

with  $L$  acting on  $Q$  by right multiplication. Then  $Q \sim q^3$  and  $L \cong L_2(q)$ . Since  $Q$  is the 3-dimensional  $GF(q)L$ -module (see the description on page 15 of [4]),  $G^0 \cong Q \rtimes L$ . We will identify this semidirect product with  $G^0$ , writing  $G^0 = QL$ . Any  $g \in G^0$  has a unique expression  $g = g_Q g_L$  where  $g_Q \in Q$  and  $g_L \in L$  - in what follows we use such subscripts to describe this expression. Set  $Y^0 = G^0 \cap Y_2$ , let  $d^0$  denote the distance metric and  $\Delta_i^0(t)$  the  $i^{\text{th}}$  disc of the commuting graph  $\mathcal{C}(G^0, Y^0)$ . In determining the discs of  $\mathcal{C}(G^0, Y^0)$  we make use of the commuting involution

graph of  $L \cong L_2(q)$  (as given in [11]). So we shall use  $d^L$  to denote the distance metric on  $\mathcal{C}(L, L \cap Y^0)$  and for  $x \in L \cap Y^0$  and  $i \in \mathbb{N}$ ,  $\Delta_i^L(x) = \{y \in L \cap Y^0 \mid d^L(x, y) = i\}$ . It is straightforward to check that

$$L \cap Y^0 = \left\{ \left( \begin{array}{ccc} a^2 & 2ab & b^2 \\ ac & bc - a^2 & -ab \\ c^2 & -2ac & a^2 \end{array} \right) \middle| \begin{array}{l} a, b, c \in GF(q) \\ a^2 + b^2 = -1 \end{array} \right\}$$

and, as  $G^0$  has one conjugacy class of involutions,  $Y^0 = \{x_Q x_L \mid x_L \in L \cap Y^0 \text{ and } x_L \text{ inverts } x_Q\}$ . Without loss of generality, we take

$$t = t_L = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and, up until Theorem 4.14, we will assume that  $q \notin \{3, 5, 9, 13\}$ . Thus the diameter of  $\mathcal{C}(L, L \cap Y^0)$  is 3.

**Lemma 4.9**

- (i)  $Qt \cap Y^0 = \{(\alpha, \beta, -\alpha)t \mid \alpha, \beta \in GF(q)\}$  and  $|Qt \cap Y^0| = q^2$ .
- (ii)  $Qt \cap \Delta_1^0(t) = \emptyset$ .

**Proof** A straightforward calculation. □

**Lemma 4.10** *We have*

$$\Delta_1^0(t) = \left\{ x \mid x_Q = (\alpha, 0, \alpha), x_L = \begin{pmatrix} a^2 & 2ab & b^2 \\ ab & b^2 - a^2 & -ab \\ b^2 & -2ab & a^2 \end{pmatrix}, a^2 + b^2 = -1 \right\},$$

and  $|\Delta_1^0(t)| = \frac{1}{2}q(q - \delta)$ .

**Proof** Let  $x, y \in Y^0$ . If  $[x, y] = 1$  then clearly  $[x_L, y_L] = 1$ . From [11] we have

$$\Delta_1^L(t) = \left\{ \left( \begin{array}{ccc} a^2 & 2ab & b^2 \\ ab & b^2 - a^2 & -ab \\ b^2 & -2ab & a^2 \end{array} \right) \middle| a^2 + b^2 = -1 \right\}.$$

If  $x_Q = (\alpha, \beta, \gamma)$  and  $x_L \in \Delta_1^L(t)$  then  $[t, x] = 1$  implies  $\alpha = \gamma$  and  $\beta = 0$ . Moreover, every  $x = (\alpha, 0, \alpha)x_L$ , where  $x_L \in \Delta_1^L(t)$ , is in  $Y^0$ . Hence,  $\Delta_1^0(t)$  is as described above. By [11], for any involution  $x_L \in L$  we have  $|\Delta_1^L(x_L)| = \frac{1}{2}(q - \delta)$  and there are  $q$  possible values that  $\alpha$  can take for a fixed such  $x_L$ , proving the lemma. □

**Lemma 4.11** *Let  $x \in Y^0$  with  $x_L \in \Delta_1^L(t)$ . If  $x \notin \Delta_1^0(t)$ , then  $x \in \Delta_2^0(t)$ .*

**Proof** Suppose  $x \in Y^0$  where  $x_Q = (\alpha, \beta, \gamma)$  and

$$x_L = \begin{pmatrix} a^2 & 2ab & b^2 \\ ab & b^2 - a^2 & -ab \\ b^2 & -2ab & a^2 \end{pmatrix}.$$

Then  $x_L$  inverts  $x_Q$  if and only if

$$\begin{aligned} a^2\alpha + 2ab\beta + b^2\gamma &= -\alpha \\ ab\alpha + (b^2 - a^2)\beta - ab\gamma &= -\beta \\ b^2\alpha - 2ab\beta + a^2\gamma &= -\gamma. \end{aligned} \tag{4.11.1}$$

Suppose first that  $\delta = -1$ . Then, since  $-1$  is not square in  $GF(q)$ , we must have  $a, b \neq 0$ . Rearranging the first equation gives  $\alpha = 2ab^{-1}\beta + \gamma$  and (4.11.1) remains consistent. Note that when  $\beta = 0$ , we have  $\alpha = \gamma$  and so  $x \in \Delta_1^0(t)$ . So assume  $\beta \neq 0$ . Let  $y \in \Delta_1^0(t)$  where  $y_Q = (ab^{-1}\beta + \gamma, 0, ab^{-1}\beta + \gamma)$  and

$$y_L = \begin{pmatrix} b^2 & -2ab & a^2 \\ -ab & a^2 - b^2 & ab \\ a^2 & 2ab & b^2 \end{pmatrix}.$$

It is a routine calculation to show that  $[x, y] = 1$ , proving the lemma for  $\delta = -1$ . Now assume  $\delta = 1$ . If  $a, b \neq 0$  then the argument from the previous case still holds, so assume first that  $a = 0$ , and hence  $b$  is an element in  $GF(q)$  that squares to  $-1$ . Then (4.11.1) simplifies to  $\alpha = \gamma$ , and so  $x_Q = (\alpha, \beta, \alpha)$ . Let  $z \in \Delta_1^0(t)$  where  $z_Q = (\alpha, 0, \alpha)$  and

$$z_L = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

An easy calculation shows that  $[x, z] = 1$ . Similarly, assuming  $b = 0$  then  $a$  is an element of  $GF(q)$  squaring to  $-1$  and (4.11.1) simplifies to  $\beta = 0$ . Then  $x_Q = (\alpha, 0, \gamma)$  and if  $w \in \Delta_1^0(t)$  where  $w_Q = (2^{-1}(\alpha + \gamma), 0, 2^{-1}(\alpha + \gamma))$  and

$$w_L = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

then an easy check shows that  $[x, w] = 1$ , proving the lemma for  $\delta = 1$ . □

**Lemma 4.12** *We have  $Qt \cap Y^0 \subseteq \{t\} \cup \Delta_2^0(t) \cup \Delta_3^0(t)$ . Moreover,*

$$\begin{aligned} |Qt \cap \Delta_2^0(t)| &= \frac{1}{2}(q^2 - (1 + \delta)q + \delta); \text{ and} \\ |Qt \cap \Delta_3^0(t)| &= \frac{1}{2}(q^2 + (1 + \delta)q - (2 + \delta)). \end{aligned}$$

**Proof** If  $x \in Qt \cap Y^0$  and  $x \neq t$  then  $x_Q = (\alpha, \beta, -\alpha)$  and  $x \notin \Delta_1^0(t)$  by Lemma 4.9. Let  $y \in \Delta_1^0(t)$  where  $y_Q = (\gamma, 0, \gamma)$  and

$$y_L = \begin{pmatrix} a^2 & 2ab & b^2 \\ ab & b^2 - a^2 & -ab \\ b^2 & -2ab & a^2 \end{pmatrix}$$

with  $a^2 + b^2 = -1$ . Then  $[x, y] = 1$  if and only if  $-a^2\alpha = ab\beta$  and  $-b^2\beta = abx$ .

Assume first that  $\delta = -1$ . Since  $-1$  is not square in  $GF(q)$ , we have  $a, b \neq 0$  and so  $\alpha = -a^{-1}b\beta$ . Hence if  $y \in Qt$  is such that  $y_Q = (-a^{-1}b\beta, \beta, a^{-1}b\beta)$ , then  $y \in \Delta_2^0(t)$ . By looking at  $\Delta_1^t(t)$ , we see there are  $q + 1$  ordered pairs  $(a, b)$  that satisfy  $a^2 + b^2 = -1$ . However, if  $(a, b) \neq (c, d)$  where  $a^2 + b^2 = c^2 + d^2 = -1$  and  $a^{-1}b = c^{-1}d$ , then an easy calculation shows that  $(c, d) = (-a, -b)$ . Hence there are  $\frac{1}{2}(q + 1)$  distinct values of  $a^{-1}b$  satisfying the relevant conditions. If  $\beta = 0$ , then  $x = t$  and if  $\beta \neq 0$  there are  $\frac{1}{2}(q^2 - 1)$  elements in  $Qt \cap \Delta_2^0(t)$ .

Assume now that  $\delta = 1$ . If  $a, b \neq 0$  then the arguments of the previous case still hold, with the exception that there are now  $q - 1$  ordered pairs  $(a, b)$  that satisfy  $a^2 + b^2 = -1$ . However, as  $a, b \neq 0$  we exclude the pairs  $(\pm i, 0)$  and  $(0, \pm i)$  where  $i$  is an element of  $GF(q)$  squaring to  $-1$ . Hence there are  $q - 5$  ordered pairs  $(a, b)$  satisfying  $a^2 + b^2 = -1$ ,  $a, b \neq 0$  and thus  $\frac{1}{2}(q - 5)$  distinct values of  $a^{-1}b$ . Hence there are  $\frac{1}{2}(q - 5)(q - 1)$  elements  $z \in Qt \cap \Delta_2^0(t)$  such that  $z_Q = (-a^{-1}b\beta, \beta, a^{-1}b\beta)$  where  $\beta \neq 0$  (note that if  $\beta = 0$ , then  $z = t$ ). Suppose  $a = 0$ , then  $b \neq 0$  and so  $\beta = 0$ . Hence  $x_Q = (\alpha, 0, -\alpha)$  and all such  $x$  lie in  $\Delta_2^0(t)$  if  $\alpha \neq 0$ . Similarly, if  $b = 0$  then  $a \neq 0$  and  $x_Q = (0, \beta, 0)$  where  $\beta \neq 0$  and all such  $x$  lie in  $\Delta_2^0(t)$ . Therefore,  $|Qt \cap \Delta_2^0(t)| = \frac{1}{2}(q - 5)(q - 1) + 2(q - 1) = \frac{1}{2}(q - 1)^2$  as required.

Hence it suffices to show that the remaining involutions all lie in  $\Delta_3^0(t)$ . Let  $w \in Qt$  be such that  $w_Q = (\gamma, \varepsilon, -\gamma)$ . Choose  $s \in Y^0$  such that  $s_Q = (ab\varepsilon - b^2\gamma, ab\gamma - a^2\varepsilon, b^2\gamma - ab\varepsilon)$  with  $ab\gamma \neq a^2\varepsilon$  and

$$s_L = \begin{pmatrix} b^2 & -2ab & a^2 \\ -ab & a^2 - b^2 & ab \\ a^2 & 2ab & b^2 \end{pmatrix},$$

with  $a^2 + b^2 = -1$ . It is an easy check to show that  $s \in \Delta_2^0(t)$ , and moreover  $[w, s] = 1$ . This accounts for the remaining involutions in  $Qt$ , thus proving the lemma. □

**Lemma 4.13** *Suppose  $x \in Y^0$  with  $x_L \in \Delta_2^t(t)$ . Then  $x \in \Delta_2^0(t)$ .*

**Proof** It can be shown (see Remark 2.3 of [11], noting the result holds for any odd  $q$ ) that for a fixed  $a, b \in GF(q)$  such that  $a^2 + b^2 = -1$ ,

$$C_L \left( \begin{pmatrix} a^2 & 2ab & b^2 \\ ab & b^2 - a^2 & -ab \\ b^2 & -2ab & a^2 \end{pmatrix} \right) = \left\{ \begin{pmatrix} c^2 & 2cd & d^2 \\ ce & de - c^2 & -cd \\ e^2 & -2ce & c^2 \end{pmatrix} \mid \begin{matrix} c^2 + de = -1 \\ b(e + d) = -2ac \end{matrix} \right\}.$$

Let  $y \in Y^0$  be such that  $y_Q = (\alpha, \beta, \gamma)$  and

$$y_L = \begin{pmatrix} c^2 & 2cd & d^2 \\ ce & de - c^2 & -cd \\ e^2 & -2ce & c^2 \end{pmatrix} \in \Delta_2^L(t).$$

So there exists  $a, b \in GF(q)$  such that  $a^2 + b^2 = -1$  and  $b(e + d) = -2ac$  with  $d \neq e$ . Since  $y_L$  inverts  $y_Q$ , we have

$$\begin{aligned} c^2\alpha + 2cd\beta + d^2\gamma &= -\alpha \\ c\epsilon\alpha + (de - c^2)\beta - cd\gamma &= -\beta \\ e^2\alpha - 2ce\beta + c^2\gamma &= -\gamma. \end{aligned} \tag{4.13.1}$$

Assume first that  $\delta = -1$ . Since  $-1$  is not square in  $GF(q)$ , then  $d, e \neq 0$  and any  $a, b \in GF(q)$  such that  $b(d + e) = -2ac$  and  $a^2 + b^2 = -1$  must also be non-zero. Moreover, if  $c = 0$  then  $d = -e^{-1}$  and  $b(d - d^{-1}) = 0$  implying that  $d = -1$ . But then  $y_L = t \notin \Delta_2^L(t)$ , so  $c \neq 0$ . The system (4.13.1) now simplifies to  $\alpha = 2ce^{-1}\beta + de^{-1}\gamma$ . Let  $x \in \Delta_1^0(t)$  be such that  $x_Q = (\epsilon, 0, \epsilon)$  and

$$x_L = \begin{pmatrix} a^2 & 2ab & b^2 \\ ab & b^2 - a^2 & -ab \\ b^2 & -2ab & a^2 \end{pmatrix}$$

where  $\epsilon = -abc^{-1}e^{-1}(\gamma + (d - e)^{-1}(2c + a^{-1}be - ab^{-1}e - (ab)^{-1}e)\beta)$ . Using the PolynomialAlgebra command in MAGMA [17] we verify that  $[x, y] = 1$  and so  $y \in \Delta_2^0(t)$ .

Assume now that  $\delta = 1$ . Let  $a, b \in GF(q)$  be such that  $a^2 + b^2 = -1$  and  $b(d + e) = -2ac$ . Suppose  $c, d, e \neq 0$  and  $d \neq -e$ . Then  $b(d + e) = -2ac \neq 0$  and so  $a, b \neq 0$ . The argument for the case when  $\delta = -1$  then holds. Suppose then  $c, d, e \neq 0$  and  $d = -e$ . Then  $b(d + e) = -2ac = 0$  and since  $c \neq 0$  we must have  $a = 0$  and  $b^2 = -1$ . The system (4.13.1) then becomes  $\alpha = 2ce^{-1}\beta - \gamma$ . If  $x \in \Delta_1^0(t)$  is such that  $x_Q = (-c^{-1}e^{-1}\beta, 0, -c^{-1}e^{-1}\beta)$  and

$$x_L = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

then a routine check shows that  $[x, y] = 1$ .

Now assume  $c \neq 0$  and  $d = 0$ . Since  $y_L \in \Delta_2^L(t)$ , we must have  $e \neq 0$  and so

$c^2 = -1$ . The system (4.13.1) becomes  $\alpha = 2ce^{-1}\beta$  and using MAGMA [17] we deduce that if  $x \in \Delta_1^0(t)$  where  $x_Q = (\varepsilon, 0, \varepsilon)$ ,

$$x_L = \begin{pmatrix} a^2 & 2ab & b^2 \\ ab & b^2 - a^2 & -ab \\ b^2 & -2ab & a^2 \end{pmatrix}$$

and  $\varepsilon = (ce^{-1}(1 - a^2) - ab)\beta - 2^{-1}b^2\gamma$ , then  $[x, y] = 1$ . Similarly, if  $c \neq 0$  and  $e = 0$ , then  $d \neq 0$  and  $c^2 = -1$ . The system (4.13.1) becomes  $\beta = 2^{-1}cd\gamma$  and [17] will verify that if  $x \in \Delta_1^0(t)$  where  $x_Q = (\varepsilon, 0, \varepsilon)$ ,

$$x_L = \begin{pmatrix} a^2 & 2ab & b^2 \\ ab & b^2 - a^2 & -ab \\ b^2 & -2ab & a^2 \end{pmatrix}$$

and  $\varepsilon = 2^{-1}(\gamma - b^2\alpha + abcd\gamma - a^2\gamma)$ , then  $[x, y] = 1$ .

Finally, if  $c = 0$  then  $d = -e^{-1}$  and so  $a^2 = -1$  and  $b = 0$  satisfies the relevant conditions. Note that if  $d = \pm 1$  then  $y_L = t$ , so we may assume  $d \neq \pm 1$ . The system (4.13.1) becomes  $\alpha = d^2\gamma$ , so if  $x \in \Delta_1^0(t)$  where  $x_Q = (2d^2\gamma(1 - d^2)^{-1}, 0, 2d^2\gamma(1 - d^2)^{-1})$  and

$$x_L = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

then a routine check again shows that  $[x, y] = 1$ . Therefore, for all  $y \in Y^0$  such that  $y_L \in \Delta_2^L(t)$ , there exists  $x \in \Delta_1^L(t)$  such that  $[x, y] = 1$ , so proving the lemma.  $\square$

**Theorem 4.14** *If  $q \notin \{3, 5, 9, 13\}$ , then  $\mathcal{C}(G^0, Y^0)$  is connected of diameter 3, with disc sizes*

$$\begin{aligned} |\Delta_1^0(t)| &= \frac{1}{2}q(q - \delta); \\ |\Delta_2^0(t)| &= \frac{1}{4}(q^4 - (2\delta + 2)q^3 + (1 + 2\delta)q^2 - 2q + 2\delta); \text{ and} \\ |\Delta_3^0(t)| &= \frac{1}{4}(q^4 + 2(1 + 2\delta)q^3 - (3 + 2\delta)q^2 + 2(1 + \delta)q - 2(2 + \delta)). \end{aligned}$$

**Proof** It is known that  $\mathcal{C}(L, L \cap Y^0)$  has diameter 3. Hence, for any  $h_i \in \Delta_i^L(t)$ , there exists  $h_{i\pm 1} \in \Delta_{i\pm 1}^L(t)$  that commutes with  $h_i$ ,  $i = 1, 2$ . Therefore for any  $x \in Y^0$  where  $x_L \in \Delta_i^L(t)$ , there exists  $y \in Y^0$  with  $y_L \in \Delta_{i\pm 1}^L(t)$  and such that  $[x, y] = 1$ . Since any  $z \in Y^0$  where  $z_L \in \Delta_3^L(t)$  must commute with some  $w \in Y^0$  with  $w_L \in \Delta_2^L(t)$  (which lies in  $\Delta_2^0(t)$  by Lemma 4.13),  $z \in \Delta_3^0(t)$ . This finally covers all possible involutions in  $Y^0$  and so the diameter of  $\mathcal{C}(G^0, Y^0)$  is 3. Now for each



$x_L \in L \cap Y^0$ ,  $|Qx_L \cap Y^0| = q^2$  by Lemma 4.9, and therefore there are  $\frac{1}{2}q^2(q - \delta)$  involutions  $y \in Y^0$  such that  $y_L \in \Delta_1^L(t)$ . From Lemma 4.10,  $|\Delta_1^0(t)| = \frac{1}{2}q(q - \delta)$ . Therefore

$$\left| \bigcup_{x_L \in \Delta_1^L(t)} Qx_L \cap \Delta_2^0(t) \right| = \frac{1}{2}q^2(q - \delta) - \frac{1}{2}q(q - \delta) = \frac{1}{2}q(q - 1)(q - \delta).$$

There are  $q^2|\Delta_2^L(t)|$  involutions  $z \in Y^0$  such that  $z_L \in \Delta_2^L(t)$ , which is known to be  $\frac{1}{4}q^2(q - \delta)(q - 4 - \delta)$  (see [11]). Also, by Lemma 4.12,  $|Qt \cap \Delta_2^0(t)| = \frac{1}{2}(q^2 - (1 + \delta)q - \delta)$ . Hence

$$\begin{aligned} |\Delta_2^0(t)| &= |Qt \cap \Delta_2^0(t)| + \left| \bigcup_{x_L \in \Delta_1^L(t)} Qx_L \cap \Delta_2^0(t) \right| + q^2|\Delta_2^L(t)| \\ &= \frac{1}{4}(q^4 - (2\delta + 2)q^3 + (1 + 2\delta)q^2 - 2q + 2\delta). \end{aligned}$$

Finally, there are  $|Y^0| = q^2|L \cap Y^0| = \frac{1}{2}q^3(q + \delta)$  involutions in  $G^0$  and therefore

$$\begin{aligned} |\Delta_3^0(t)| &= |Y^0| - |\Delta_2^0(t)| - |\Delta_1^0(t)| - 1 \\ &= \frac{1}{4}(q^4 + 2(1 + 2\delta)q^3 - (3 + 2\delta)q^2 + 2(1 + \delta)q - 2(2 + \delta)) \end{aligned}$$

which proves Theorem 4.14. □

**Theorem 4.15**  $\mathcal{C}(G, Y_2)$  is connected of diameter at most 3.

**Proof** For  $q \leq 13$ , this is easily checked using MAGMA [17], so assume  $q > 13$ . Combining Lemma 4.3 with Theorems 4.6, 4.8 and 4.14 yields the theorem. □

We now focus on finding the disc sizes of  $\mathcal{C}(G, Y_2)$ . First, we need the following four lemmas.

**Lemma 4.16** The sets  $\mathcal{U}_1^+$ ,  $\mathcal{U}_1^-$  and  $\mathcal{U}_1^0$  are single  $C_G(t)$ -orbits. Moreover,

$$\begin{aligned} |\mathcal{U}_1^0| &= q + 1; \\ |\mathcal{U}_1^+| &= \frac{1}{2}q(q + \delta); \text{ and} \\ |\mathcal{U}_1^-| &= \frac{1}{2}q(q - \delta). \end{aligned}$$

**Proof** Since  $C_G(t)$  acts orthogonally on  $C_V(t)$ , the first statement is immediate. Recall the Gram matrix  $J$  for  $V$  with respect to  $(\ , \ )$  and the basis  $\{v_i\}$ . Observe that  $C_V(t) = \{(\alpha, \beta, \gamma, 0, \gamma) \mid \alpha, \beta, \gamma \in GF(q)\}$  and so a basis for  $C_V(t)$  is  $\{v_1, v_2, v_3 + v_5\}$ . Let  $v = (\alpha, \beta, \gamma, 0, \gamma)$  be a non-zero vector in  $C_V(t)$  and so  $(v, v) = 2\alpha\beta + 2\gamma^2$ .

Suppose  $v$  is isotropic, so  $C_G(\langle v \rangle) \sim q^3 : L_2(q)$  and  $(v, v) = 2\alpha\beta + 2\gamma^2 = 0$ . If

$\gamma = 0$ , then  $\alpha\beta = 0$  and so either  $\alpha = 0$  or  $\beta = 0$  (but not both as  $v \neq 0$ ). Hence there are  $2(q - 1)$  such vectors with  $\gamma = 0$ . If  $\gamma \neq 0$ , then  $\alpha = -\beta^{-1}\gamma^2$  and there are  $(q - 1)^2$  such vectors satisfying this. Hence there are  $2(q - 1) + (q - 1)^2 = (q - 1)(q + 1)$  non-zero isotropic vectors contained in  $C_V(t)$  and thus  $q + 1$  isotropic 1-subspaces of  $C_V(t)$ .

Suppose now  $v$  is  $C_G(t)$ -conjugate to  $v_3 + v_5$ , which is non-isotropic. Note that  $\langle v_3 + v_5 \rangle^\perp \cap C_V(t)$  is a 2-subspace of  $V$  of  $+$ -type. If  $\delta = 1$ , then by Lemma 4.1(ii),  $\langle v_3 + v_5 \rangle^\perp$  is a 4-subspace of  $V$  of  $+$ -type and so  $C_G(\langle v_3 + v_5 \rangle) \sim SL_2(q) \circ SL_2(q)$ . While  $\delta = -1$  gives that  $\langle v_3 + v_5 \rangle^\perp$  is a 4-subspace of  $V$  of  $-$ -type and so  $C_G(\langle v_3 + v_5 \rangle) \sim L_2(q^2)$ . A quick check shows that  $(v_3 + v_5, v_3 + v_5) = 2$  and so  $(v, v) = 2\alpha\beta + 2\gamma^2 = 2\lambda^2$  for some  $\lambda \in GF(q)^*$ . Thus,  $\alpha\beta + \gamma^2 = \lambda^2$  for some  $\lambda \in GF(q)^*$ . If  $\gamma = 0$ , then  $\alpha = \beta^{-1}\lambda^2$  and so there are  $q - 1$  such vectors that satisfy this. If  $\gamma = \pm\lambda$ , then  $\alpha\beta = 0$  and so for both values of  $\gamma$ , there are  $2(q - 1) + 1$  vectors that satisfy this. Finally, if  $\gamma \in GF(q) \setminus \{0, \lambda, -\lambda\}$ , then  $\alpha\beta = 1 - \gamma^2 \neq 0$  and so  $\alpha = \beta^{-1}(1 - \gamma^2)$ . There are  $(q - 1)(q - 3)$  such vectors that satisfy this. Hence for any given  $\lambda$ , there exist  $(q - 1) + 4(q - 1) + 2 + (q - 1)(q - 3) = q(q + 1)$  vectors that satisfy  $\alpha\beta + \gamma^2 = \lambda^2$ . Since there are  $\frac{1}{2}(q - 1)$  squares in  $GF(q)$ , there are  $q(q + 1)(q - 1)$  vectors that are  $C_G(t)$ -conjugate to  $v_3 + v_5$  and hence  $\frac{1}{2}q(q + 1)$  1-subspaces of  $C_V(t)$  that are  $C_G(t)$ -conjugate to  $\langle v_3 + v_5 \rangle$ .

This leaves the remaining orbit  $\mathcal{U}_1^\delta$ . Recall there are  $q^2 + q + 1$  subspaces of  $C_V(t_1)$  of dimension 1, and hence the size of the remaining orbit is  $q^2 + q + 1 - (q + 1) - \frac{1}{2}q(q + 1) = \frac{1}{2}q(q - 1)$ , so proving the lemma.  $\square$

**Corollary 4.17** *The sets  $\mathcal{U}_2^+$ ,  $\mathcal{U}_2^-$  and  $\mathcal{U}_2^0$  are single  $C_G(t)$ -orbits. Moreover,*

$$\begin{aligned} |\mathcal{U}_2^0| &= q + 1; \\ |\mathcal{U}_2^+| &= \frac{1}{2}q(q + 1); \text{ and} \\ |\mathcal{U}_2^-| &= \frac{1}{2}q(q - 1). \end{aligned}$$

**Proof** Since  $C_V(t)$  is 3-dimensional,  $U^\perp \cap C_V(t) \in \mathcal{U}_1$  for any  $U \in \mathcal{U}_2$ , and so the result is immediate by Lemma 4.16.  $\square$

**Lemma 4.18** *Let  $U, U' \in \mathcal{U}_2$  be such that  $U \neq U'$ . Then  $C_G(U) \cap C_G(U') \cap Y_2 = \{t\}$ .*

**Proof** Suppose  $x \in C_G(U) \cap C_G(U') \cap Y_2$ . Since  $U \neq U'$  and  $x$  fixes each 2-subspace pointwise,  $U + U' = C_V(t)$  and so  $x$  fixes  $C_V(t)$  pointwise. That is to say,  $C_V(x) = C_V(t)$  and so  $t = x$  by Lemma 4.2(i).  $\square$

**Lemma 4.19** *Let  $U_0 \in \mathcal{U}_2^0$ , and  $G^0 = QL$ ,  $Y^0$  be as defined in the discussion prior to Lemma 4.9. Let  $\rho : C_G(U_0^\perp \cap C_V(t)) \rightarrow G^0$  be an isomorphism such that*

$$t^\rho = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then  $C_G(U_0)$  is totally disconnected and  $(C_G(U_0) \cap Y_2)^\rho = Qt \cap Y^0$ .

**Proof** Since  $U_0^\perp \cap C_V(t)$  is isotropic, it must lie inside of  $U_0$  and so  $C_G(U_0) \leq C_G(U_0^\perp \cap C_V(t))$ . As  $t$  fixes  $U_0$  pointwise,  $t^\rho \in (C_G(U_0))^\rho \sim q^2 : \frac{q-\delta}{2}$  by Lemma 4.5(i). The subgroup of  $L$  with shape  $\frac{q-\delta}{2}$  contains one single involution which must necessarily be  $t^\rho$ . For all  $x \in Y^0$ , we have  $x_L^2 = 1$  and  $x_L$  inverts  $x_Q$ , so  $(C_G(U_0) \cap Y_2)^\rho \subseteq Qt \cap Y^0$ . By comparing the orders of both sides, we get equality. By Lemma 4.9(ii)  $C_G(U_0) \cap C_G(t) \cap Y_2 = \{t\}$ , hence  $C_G(U_0)$  is totally disconnected.  $\square$

**Lemma 4.20**

$$|\Delta_1(t)| = \frac{1}{2}q(q^2 + (1 - \delta)q + \delta).$$

**Proof** Clearly,  $x \in \Delta_1(t)$  if and only if  $x \in \Delta_1(t) \cap C_G(U)$  for  $U = C_V(t) \cap C_V(x)$ , so

$$\Delta_1(t) = \bigcup_{U \in \mathcal{U}_1 \cup \mathcal{U}_2} (\Delta_1(t) \cap C_G(U)).$$

If  $W, W' \in \mathcal{U}_1$  with  $W \neq W'$ , then  $W \oplus W' \in \mathcal{U}_2$  and if  $y \in C_G(W) \cap C_G(W')$ , then  $y \in C_G(W \oplus W')$  and hence  $y \in C_G(W'')$  for any 1-subspace  $W''$  of  $W \oplus W'$ . Since there are  $q + 1$  subspaces of  $W''$  of dimension 1, any such  $y$  will lie in exactly  $q + 1$  such  $C_G(U)$  for  $U \in \mathcal{U}_1$ . Together with  $C_G(W'')$  and Lemma 4.18,

$$|\Delta_1(t)| = \sum_{U \in \mathcal{U}_1} |\Delta_1(t) \cap C_G(U)| - q \sum_{U \in \mathcal{U}_2} |\Delta_1(t) \cap C_G(U)|.$$

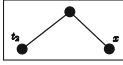
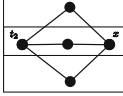
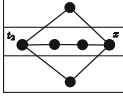
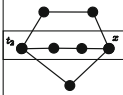
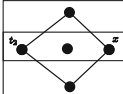
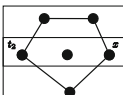
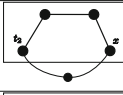
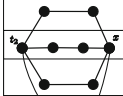
Combining Lemmas 4.16, 4.19 and Corollary 4.17 with Theorems 4.6, 4.8, 4.14 and [11], we have

$$\begin{aligned} |\Delta_1(t)| &= \frac{1}{2}q(q+1)(q-\delta) + \frac{1}{2}q(q+\delta) \left[ \frac{1}{2}(q-\delta)^2 + 1 \right] + \frac{1}{4}q(q-\delta)(q^2-1) \\ &\quad - \frac{1}{2}q(q-\delta) \left[ \frac{1}{2}q(q+1) + \frac{1}{2}q(q-1) \right] \\ &= \frac{1}{2}q(q^2 + (1 - \delta)q + \delta) \end{aligned}$$

as required.  $\square$

We now consider the second disc  $\Delta_2(t)$ . Here, we must be careful as elements that are distance 2 from  $t$  in some subgroup  $C_G(U)$  may not be distance 2 from  $t$  in

**Table 1** List of cases in  $\Delta_2(t)$

Case	Configuration	Properties	Description as Set
1	 $C_G(U_1)$	$x \in \Delta_2^{C_G(U_1)}(t)$ $U_1 = C_V(\langle t, x \rangle) \in \mathcal{U}_1$	$\bigcup_{U \in \mathcal{U}_1} \Gamma_1(C_G(U))$
2	 $C_G(U_1)$ $C_G(U_1 \oplus U_2)$ $C_G(U_2)$	$x \in \Delta_2^{C_G(U_1 \oplus U_2)}(t)$ $U_1 \oplus U_2 = C_V(\langle t, x \rangle)$ $U_i \in \mathcal{U}_1$	$\bigcup_{W \in \mathcal{U}_2} \Gamma_2(C_G(W))$
3	 $C_G(U_1)$ $C_G(U_1 \oplus U_2)$ $C_G(U_2)$  $C_G(U_1)$ $C_G(U_1 \oplus U_2)$ $C_G(U_2)$  $C_G(U_1)$ $C_G(U_1 \oplus U_2)$ $C_G(U_2)$  $C_G(U_1)$ $C_G(U_1 \oplus U_2)$ $C_G(U_2)$	$x \in \Delta_2^{C_G(U_2)}(t)$ for some $U_2 \leq C_V(\langle t, x \rangle)$ $x \notin \Delta_2^{C_G(U_1 \oplus U_2)}(t)$ $U_1 \oplus U_2 = C_V(\langle t, x \rangle)$ $U_i \in \mathcal{U}_1$	$\bigcup_{U \in \mathcal{U}_1} \Gamma_2(C_G(U)) \setminus \bigcup_{W \in \mathcal{U}_2} \Gamma_2(C_G(W))$
4	 $C_G(U_1)$	$x \in \Delta_2^G(t)$ $x \notin \Delta_2^{C_G(U_1)}(t)$ $U_1 = C_V(\langle t, x \rangle) \in \mathcal{U}_1$	$\Gamma_1(G) \setminus \bigcup_{U \in \mathcal{U}_1} \Gamma_1(C_G(U))$
5	 $C_G(U_1)$ $C_G(U_1 \oplus U_2)$ $C_G(U_2)$	$x \in \Delta_2^G(t)$ $x \notin \Delta_2^{C_G(U_i)}(t)$ for any $U_i \leq C_V(\langle t, x \rangle)$ $U_1 \oplus U_2 = C_V(\langle t, x \rangle)$ $U_i \in \mathcal{U}_1$	$\Gamma_2(G) \setminus \bigcup_{U \in \mathcal{U}_1} \Gamma_2(C_G(U))$

another subgroup  $C_G(U')$ . Moreover, there may be elements that are distance 3 from  $t$  in every such subgroup centralizing an element of  $\mathcal{U}_1$ , but actually are distance 2 from  $t$  in  $G$ . We introduce the following notation. Let  $\Delta_2^K(t)$  be the second disc in the commuting involution graph  $\mathcal{C}(K, K \cap Y_2)$  and

$$\Gamma_i(K) = \{x \in \Delta_2^K(t) \mid \dim C_V(\langle t, x \rangle) = i\}$$

for  $K = C_G(U)$ ,  $U \in \mathcal{U}_1 \cup \mathcal{U}_2$ . Clearly,  $\Delta_2(t) = \Gamma_1(G) \cup \Gamma_2(G)$ . A full list of cases with corresponding notation is found in Table 1. Also we use the following notation: for any  $U \leq C_V(t)$ , define  $\mathcal{U}_i(U)$  to be the totality of  $i$ -dimensional subspaces of  $U$  and  $\mathcal{W}_i(U)$  to be the totality of  $i$ -dimensional subspaces of  $C_V(t)$  containing  $U$ . Note that  $\mathcal{U}_i = \mathcal{U}_i(C_V(t))$ .

**Lemma 4.21**

- (i) If  $W \in \mathcal{U}_2^0$ , then  $|\mathcal{U}_1^0 \cap \mathcal{U}_1(W)| = 1$  and  $|\mathcal{U}_1^+ \cap \mathcal{U}_1(W)| = q$ .
- (ii) If  $W \in \mathcal{U}_2^+$ , then  $|\mathcal{U}_1^0 \cap \mathcal{U}_1(W)| = 2$  and  $|\mathcal{U}_1^+ \cap \mathcal{U}_1(W)| = |\mathcal{U}_1^- \cap \mathcal{U}_1(W)| = \frac{q-1}{2}$ .
- (iii) If  $W \in \mathcal{U}_2^-$ , then  $|\mathcal{U}_1^+ \cap \mathcal{U}_1(W)| = |\mathcal{U}_1^- \cap \mathcal{U}_1(W)| = \frac{q+1}{2}$ .

**Proof** Recall the Gram matrix  $J$ , with respect to the ordered basis  $\{v_i\}, i = 1, \dots, 5$ . Suppose  $W^\perp \cap C_V(t) = U_0 \in \mathcal{U}_1^0$ . Without loss of generality, choose  $W = \langle v_1, v_3 + v_5 \rangle$ . Clearly  $\langle v_1 \rangle \in \mathcal{U}_1^0$ , and  $\langle v_3 + v_5 \rangle^\perp \cap C_V(t) \in \mathcal{U}_2^+$ . Since

$$(v_1 + \lambda(v_3 + v_5), v_1 + \lambda(v_3 + v_5)) = \lambda^2(v_3 + v_5, v_3 + v_5),$$

$v_1 + \lambda(v_3 + v_5)$  lies in the same  $C_G(t)$ -orbit as  $v_3 + v_5$  and so  $\langle v_1 + \lambda(v_3 + v_5) \rangle^\perp \cap C_V(t) \in \mathcal{U}_2^+$ , proving (i).

Suppose now  $W \in \mathcal{U}_2^+$ . Without loss of generality, choose  $W = \langle v_1, v_2 \rangle$ . Clearly  $\langle v_1 \rangle, \langle v_2 \rangle \in \mathcal{U}_1^0$ . Let  $U_\lambda = \langle v_1 + \lambda v_2 \rangle$  for  $\lambda \neq 0$  and note that  $(v_1 + \lambda v_2, v_1 + \lambda v_2) = 2\lambda = \mu \neq 0$ . Since the type of  $U_\lambda^\perp$  is determined by whether  $\mu$  is a square or a non-square in  $GF(q)$ , and there are  $\frac{q-1}{2}$  of each, it is clear that there exist  $\frac{q-1}{2}$  such  $U_\lambda$  for which  $U_\lambda^\perp$  is of  $+$ -type, and similarly for  $-$ -type, proving (ii).

Finally suppose  $W \in \mathcal{U}_2^-$ , so for all  $v \in W, (v, v) \neq 0$ . The simple orthogonal group on  $W$  is cyclic of order  $\frac{q+1}{2}$  and acts on the 1-subspaces of  $W$  in exactly two orbits with representatives  $\langle u_1 \rangle$  and  $\langle u_2 \rangle$  where  $(u_1, u_1)$  is a square and  $(u_2, u_2)$  is a non-square in  $GF(q)$ . Since  $|\mathcal{U}_1(W)| = q + 1$ , both orbits must be of size  $\frac{q+1}{2}$ . This proves (iii) and hence the lemma follows. □

**Corollary 4.22** *Let  $U \in \mathcal{U}_1$ . Then the following:*

- (i)  $|\mathcal{W}_2(U)| = q + 1$ .
- (ii) If  $U \in \mathcal{U}_1^0$ , then  $|\mathcal{U}_2^0 \cap \mathcal{W}_2(U)| = 1$  and  $|\mathcal{U}_2^+ \cap \mathcal{W}_2(U)| = q$ .
- (iii) If  $U \in \mathcal{U}_1^+$ , then  $|\mathcal{U}_2^0 \cap \mathcal{W}_2(U)| = 2$  and  $|\mathcal{U}_2^+ \cap \mathcal{W}_2(U)| = |\mathcal{U}_2^- \cap \mathcal{W}_2(U)| = \frac{q-1}{2}$ .
- (iv) If  $U \in \mathcal{U}_2^\delta$ , then  $|\mathcal{U}_2^+ \cap \mathcal{W}_2(U)| = |\mathcal{U}_2^- \cap \mathcal{W}_2(U)| = \frac{q+1}{2}$ .

**Proof** Let  $U \leq W \leq C_V(t)$ . Then  $W^\perp \cap C_V(t) \leq U^\perp \cap C_V(t) \leq C_V(t)$ . The result follows from Lemma 4.21. □

**Lemma 4.23** *Let  $U \in \mathcal{U}_1^0$  and  $W \in \mathcal{U}_2^+ \cap \mathcal{W}_2(U)$ . If  $x \in Y_2 \cap C_G(W)$  is such that  $d^{C_G(W)}(t, x) = 3$ , then  $d^{C_G(U)}(t, x) = 3$ . Moreover,*

$$|\Gamma_1(C_G(U))| = \begin{cases} \frac{1}{4}q(q-3)(q-1)^2 & q \equiv 1 \pmod{4} \\ \frac{1}{4}q(q-1)^2(q+1) & q \equiv -1 \pmod{4}. \end{cases}$$

**Proof** Recall that  $C_G(U) = QL \sim G^0$  where  $G^0$  is defined as in the discussion prior to Lemma 4.9. By conjugacy, we may assume  $L = C_G(W)$ . Now  $C_G(U) \cap C_G(t) = Q_0C_L(t) \sim q : Dih(q - \delta)$  where  $Q_0 \leq Q$  is elementary abelian of order  $q$ . Let  $x \in Q_0C_L(t) \cap Y_2$ , so  $x_L^2 = 1$  and  $x_L$  inverts  $x_Q$ . Clearly,  $x_L^{x_Q} = x_Lx_Q^2 \notin L$  since  $Q_0$  is of odd order. Hence,  $C_L(t)$  is self-normalizing in  $Q_0C_L(t)$  and thus there are  $q$  distinct conjugates of  $C_L(t)$  in  $Q_0C_L(t)$ . Let  $g \in Q_0C_L(t) \setminus C_L(t)$ , so  $C_L(t)^g \neq C_L(t)$ . Now  $[C_L(t), t] = [C_L(t)^g, t] = 1$  and so  $\langle C_L(t), C_L(t)^g \rangle$  centralizes  $t$ . If  $C_L(t), C_L(t)^g \leq L^h$  for some  $h \in QL$ , then  $\langle C_L(t), C_L(t)^g \rangle \leq L^h$ . However,  $C_L(t) \not\leq \langle C_L(t), C_L(t)^g \rangle \leq C_L(t)$ , a contradiction. Hence every conjugate of  $C_L(t)$  lies in a different conjugate of  $L$  and so there are  $q$  distinct  $Q_0C_L(t)$ -conjugates of  $L$ . Therefore,  $U_2^+ \cap W_2(U)$  is contained in the same  $C_G(U) \cap C_G(t)$ -orbit, and  $|U_2^+ \cap W_2(U)| = q$  by Corollary 4.22. There are exactly  $q$   $+$ -type 2-subspaces of  $C_V(t)$  containing  $U$ , all of which lie in the same  $C_G(U) \cap C_G(t)$  orbit.

Let  $x \in C_G(W) \cap Y_2$  be such that  $d^{C_G(W)}(t, x) = 3$ . Suppose  $W^g \in U_2^+ \cap W_2(U)$  for some  $g \in C_G(U) \cap C_G(t)$ ,  $W \neq W^g$ . If  $d^{C_G(U)}(t, x) = 2$  then  $d^{C_G(U)}(t^g, x^g) = d^{C_G(U)}(t, x^g) = 2$ , and  $d^{C_G(W)}(t, x) = d^{C_G(W^g)}(t, x^g) = 3$ . Hence it suffices to prove the lemma for  $C_G(W)$ . By Theorem 4.14, any involution distance 3 away from  $t$  in  $L$  is necessarily distance 3 away from  $t$  in  $C_G(U)$ , proving the first statement.

Let  $W_0 \in U_2^0 \cap W_2(U)$ , so  $C_G(W_0) \sim q^2 : \frac{q-\delta}{2}$ . By Lemma 4.19,  $\Delta_2^{C_G(U)}(t) \cap C_G(W_0) = Qt \cap \Delta_2^{C_G(U)}(t)$ . Let  $W_i, i = 1, \dots, q$  be the subspaces in  $U_2^+ \cap W_2(U)$ . From Lemma 4.18,  $C_G(W_i) \cap C_G(W_j) \cap Y_2 = \{t\}$  if and only if  $i = j$ . Using Corollary 4.22(i) with [11], we have

$$\left| \bigcup_{i=1}^q \Delta_2^{C_G(W_i)}(t) \right| = \frac{1}{4}q(q - \delta)(q - 4 - \delta). \tag{4.23.1}$$

Combining Lemma 4.12 with (4.23.1),

$$\begin{aligned} |\Gamma_2(C_G(U))| &= \left| \bigcup_{i=1}^q \Delta_2^{C_G(W_i)}(t) \right| + \left| \Delta_2^{C_G(U)}(t) \cap C_G(W_0) \right| \\ &= \frac{1}{4}(q^3 - 2(1 + \delta)q^2 + (2\delta - 1)q + 2\delta). \end{aligned} \tag{4.23.2}$$

Together, (4.23.2) and Theorem 4.14 give

$$\begin{aligned} |\Gamma_1(C_G(U))| &= \left| \Delta_2^{C_G(U)}(t) \right| - |\Gamma_2(C_G(U))| \\ &= \frac{1}{4}q(q^3 - (2\delta + 3)q^2 + (4\delta + 3)q - 2\delta - 1) \end{aligned}$$

as required. □

**Lemma 4.24** *Let  $t, x \in L_2(q)$ . Then  $d^{L_2(q)}(t, x) \leq 2$  if and only if the order of  $tx$  divides  $\frac{1}{2}(q - \delta)$ .*

**Proof** See Lemma 2.11 of [11]. □

**Lemma 4.25** *Let  $U \in \mathcal{U}_1^+$ , and  $W \in (\mathcal{U}_2^+ \cup \mathcal{U}_2^-) \cap \mathcal{W}_2(U)$ .*

- (i) If  $\delta = 1$  and  $W_0 \in \mathcal{U}_2^0 \cap \mathcal{W}_2(U)$ , then  $Y_2 \cap C_G(W_0) \setminus \{t\} \subseteq \Delta_3^{C_G(U)}(t)$ .
- (ii) If  $x \in Y_2 \cap C_G(W)$  is such that  $d^{C_G(W)}(t, x) = 3$ , then  $d^{C_G(U)}(t, x) = 3$  and

$$|\Gamma_1(C_G(U))| = \begin{cases} \frac{1}{8}(q-1)(q-3)(q^2-6q+13) & q \equiv 1 \pmod{4} \\ \frac{1}{8}(q^2-1)(q^2-2q+5) & q \equiv -1 \pmod{4}. \end{cases}$$

**Proof** Recall that  $C_G(U) \sim G^+ \sim L_1 \circ L_2$  for  $L_1 \sim SL_2(q) \sim L_2$ . Suppose  $y \in C_G(W)$  is such that  $d^{C_G(W)}(t, y) = 3$ . Since  $C_G(W) \sim L_2(q)$  is simple, then  $y = gg^\varphi$  for some  $g \in L_1$  and  $\varphi : L_1 \rightarrow L_2$ . Since  $t \in C_G(W)$ , write  $t = ss^\varphi$  for some  $s \in L_1$ . Then  $d^{L_1}(s, g) = 3$ , so  $d^{C_G(U)}(t, y) = 3$  by Theorem 4.8, and thus

$$\Delta_3^{C_G(W)}(t) \subseteq \Delta_3^{C_G(U)}(t) \quad \text{for all } W \in (\mathcal{U}_2^+ \cup \mathcal{U}_2^-) \cap \mathcal{W}_2(U). \tag{4.25.1}$$

If  $\delta = -1$ , then  $\mathcal{U}_2^0 \cap \mathcal{W}_2(U) = \emptyset$  by Corollary 4.22. If  $\delta = 1$ , there exists  $W_0 \in \mathcal{U}_2^0 \cap \mathcal{W}_2(U)$ . Recall that  $W_0^\perp \cap C_V(t) \in \mathcal{U}_1^0$  so  $C_G(W_0) \leq C_G(W_0^\perp \cap C_V(t)) \sim G^0 = QL$ . By Lemma 4.19, if  $x \in C_G(W_0) \cap Y_2$  then  $x = x_Q t$  and  $x_Q$  is inverted by  $t$  and has order  $p$ . Since  $x_Q$  also lies in  $C_G(U)$ , we can write  $x_Q = hh^\varphi$  for some  $h \in L_1$ . Now  $x_Q^{-1} = h^{-1}h^{-1\varphi}$  and so  $x_Q^t = x_Q^{ss^\varphi} = h^s(h^\varphi)^{s^\varphi} = h^{-1}h^{-1\varphi}$ . Therefore,  $h^s = h^{-1}$  and  $h^{\varphi s^\varphi} = h^{-1\varphi}$ . Moreover,  $x = x_Q t = (hs)(hs)^\varphi$  where  $hs \in L_1$  is an element of order 4 squaring to the non-trivial element of  $Z(L_1)$ , and  $h = (hs)s$  has order  $p$ . By Lemma 4.24 and [11],  $d^{L_1}(hs, s) = 3$  and so  $d^{C_G(U)}(t, x_Q t) = 3$  by Theorem 4.8. Therefore,

$$C_G(W_0) \cap \Delta_2^{C_G(U)}(t) = \emptyset \quad \text{for all } W_0 \in \mathcal{U}_2^0 \cap \mathcal{W}_2(U). \tag{4.25.2}$$

Hence combining (4.25.1) with Lemma 4.21, [11] and, if  $\delta = 1$ , (4.25.2) we get

$$\left| \bigcup_{U \leq W} \Delta_2^{C_G(W)}(t) \right| = |\Gamma_2(C_G(U))| = \frac{1}{4}(q-\delta)^2(q-4-\delta).$$

This, together with Theorem 4.8 yields

$$\begin{aligned} |\Gamma_1(C_G(U))| &= \left| \Delta_2^{C_G(U)}(t) \right| - |\Gamma_2(C_G(U))| \\ &= \frac{1}{8}(q-1)(q-1-2\delta)(q^2-(4+2\delta)q+9+4\delta), \end{aligned}$$

which proves the lemma. □

**Lemma 4.26** *Let  $U \in \mathcal{U}_1^-$ , and  $W \in (\mathcal{U}_2^+ \cup \mathcal{U}_2^-) \cap \mathcal{W}_2(U)$ .*

- (i) If  $\delta = -1$  and  $W_0 \in \mathcal{U}_2^0 \cap \mathcal{W}_2(U)$ , then  $Y_2 \cap C_G(W_0) \setminus \{t\} \subseteq \Delta_3^{C_G(U)}(t)$ .
- (ii) We have

$$\left| \Gamma_2(C_G(U)) \setminus \bigcup_{W \in \mathcal{W}_2(U)} \Gamma_2(C_G(W)) \right| = \frac{1}{4}(q - 2 + \delta)(q^2 - 1)$$

and  $|\Gamma_1(C_G(U))| = \frac{1}{4}(q - 1)^3(q + 1)$ .

**Proof** First assume  $\delta = -1$ , and consider  $C_G(W_0)$ . By Lemma 4.19, every involution in  $C_G(W_0)$  can be written as  $xt$  where  $x$  has order  $p$ . But  $(xt)t = x$  has order  $p$ , which does not divide  $\frac{1}{2}(q^2 - 1)$ , and hence  $d^{C_G(U)}(xt, t) = 3$ . In other words,  $Y_2 \cap C_G(W_0) \setminus \{t\} \subseteq \Delta_3^{C_G(U)}(t)$ , so proving (i).

Consider then  $C_G(W) \sim L_2(q)$ . We utilize the character table of  $L_2(q)$  from Chapter 38 of [22] (see also Schur [42]). Recall that  $L_2(q)$  contains one conjugacy class of involutions, and two conjugacy classes of elements of order  $p$ . The remaining conjugacy classes partition into two cases: those whose order divides  $\frac{1}{2}(q - 1)$  and those whose order divides  $\frac{1}{2}(q + 1)$ . Let  $C$  be a conjugacy class of elements in  $C_G(W)$  and define  $X_C = \{x \in Y_2 \cap C_G(W) | tx \in C\}$ . It is a well-known character theoretic result (see, for example, Theorem 4.2.12 of [28]) that

$$|X_C| = \frac{|C|}{|C_{C_G(W)}(t)|} \sum_{\chi} \frac{\chi(tx)|\chi(t)|^2}{\chi(1)} \tag{4.26.1}$$

Irreducible

and all  $X_C$  are pairwise disjoint. Let  $x \in Y_2 \cap C_G(W)$ . If the order of  $tx$  divides  $\frac{1}{2}(q^2 - 1)$  but not  $\frac{1}{2}(q - \delta)$  then it must necessarily divide  $\frac{1}{2}(q + \delta)$ . Hence, if  $C$  is a conjugacy class of elements of order dividing  $\frac{q+\delta}{2}$ , then any  $y \in X_C$  has the property that  $d^{C_G(W)}(t, y) = 3$  but  $d^{C_G(U)}(t, y) = 2$ , by Lemma 4.24. Recall that  $\Gamma_2(C_G(U)) \setminus \bigcup_{W \in \mathcal{W}_2(U)} \Gamma_2(C_G(W))$  is the set consisting of all such involutions. Therefore, it suffices to calculate the sizes of all such relevant  $X_C$ . We use  $\mathcal{F}$  to denote to be the set of all conjugacy classes of elements with order dividing  $\frac{q+\delta}{2}$ .

By [22], we see that for any  $C \in \mathcal{F}$ ,  $|C| = q(q - \delta)$  and so for any  $x \in C$   $|C_{C_G(W)}(x)| = (q - \delta)$ . Hence (4.26.1) and [22] gives  $|X_C| = q - \delta$ . Now if  $\delta = 1$ , then  $|\mathcal{F}| = \frac{q-1}{4}$  by [22]. If  $\delta = -1$ , then  $|\mathcal{F}| = \frac{q-3}{4}$ . Since  $|\Delta_3^{C_G(W)}(t) \cap \Delta_2^{C_G(U)}| = |X_C||\mathcal{F}|$ , and by Corollary 4.22,  $|\mathcal{W}_2(U) \cap (\mathcal{U}_2^+ \cup \mathcal{U}_2^-)| = q + \delta$ , we obtain



$$\begin{aligned} \left| \Gamma_2(C_G(U)) \setminus \bigcup_{W \in \mathcal{W}_2(U)} \Gamma_2(C_G(W)) \right| &= |\mathcal{W}_2(U) \cap (\mathcal{U}_2^+ \cup \mathcal{U}_2^-)| |X_C| |\mathcal{F}| \\ &= \begin{cases} \frac{1}{4}(q-1)(q^2-1) & q \equiv 1 \pmod{4} \\ \frac{1}{4}(q-3)(q^2-1) & q \equiv -1 \pmod{4} \end{cases} \end{aligned}$$

which proves the first part of (ii). We now prove the last part of (ii). Recall that

$$\left| \bigcup_{W \in \mathcal{W}_2(U)} \Gamma_2(C_G(W)) \right| = (q + \delta) \left| \Delta_2^{C_G(W)}(t) \right| = \frac{1}{4}(q^2 - 1)(q - 4 - \delta)$$

by [11] and Corollary 4.22. Together with the above statement, we have

$$\begin{aligned} |\Gamma_2(C_G(U))| &= \left| \bigcup_{W \in \mathcal{W}_2(U)} \Gamma_2(C_G(W)) \right| + \left| \Gamma_2(C_G(U)) \setminus \bigcup_{W \in \mathcal{W}_2(U)} \Gamma_2(C_G(W)) \right| \\ &= \frac{1}{4}(q^2 - 1)(q - 4 - \delta) + \frac{1}{4}(q^2 - 1)(q - 2 + \delta) \\ &= \frac{1}{2}(q^2 - 1)(q - 3). \end{aligned}$$

Hence

$$\begin{aligned} |\Gamma_1(C_G(U))| &= \left| \Delta_2^{C_G(U)}(t) \right| - |\Gamma_2(C_G(U))| \\ &= \frac{1}{4}(q - 1)^3(q + 1), \end{aligned}$$

and Lemma 4.26 holds. □

**Lemma 4.27**

$$\left| \bigcup_{U \in \mathcal{U}_1} \Gamma_1(C_G(U)) \right| = \begin{cases} \frac{1}{16}q(q^2 - 1)(3q^3 - 11q^2 + 21q - 29) & q \equiv 1 \pmod{4} \\ \frac{1}{16}q(q^2 - 1)(q - 1)(3q^2 + 2q + 7) & q \equiv -1 \pmod{4}. \end{cases}$$

**Proof** Since  $\mathcal{U}_1 = \mathcal{U}_1^0 \cup \mathcal{U}_1^+ \cup \mathcal{U}_1^-$ , with each orbit size given in Lemma 4.16, the result follows immediately from Lemmas 4.23, 4.25 and 4.26. □

Recall the list of cases in Table 1. The next lemma concerns Cases 2 and 3, in other words,  $\bigcup_{U \in \mathcal{U}_1} \Gamma_2(C_G(U))$ .

**Lemma 4.28**

$$\left| \bigcup_{U \in \mathcal{U}_1} \Gamma_2(C_G(U)) \right| = \frac{1}{2}(q - \delta)(q^3 - 2q^2 - 1).$$

**Proof** By Lemmas 4.12 and 4.19, for any  $W_0 \in \mathcal{U}_2^0$  we have  $|\Delta_2^{C_G(U)}(t) \cap C_G(W_0)| = \frac{1}{2}(q - 1)(q - \delta)$  for some  $U \in \mathcal{U}_1(W_0)$ . Additionally, for any  $W \in (\mathcal{U}_2^+ \cup \mathcal{U}_2^-)$  we have

$$\begin{aligned} \left| \Delta_2^{C_G(U)}(t) \cap C_G(W) \right| &= \left| \Delta_2^{C_G(W)}(t) \right| + \left| \Delta_3^{C_G(W)}(t) \cap \Delta_2^{C_G(U)}(t) \right| \\ &= \frac{1}{2}(q - \delta)(q - 3), \end{aligned}$$

for some  $U \in \mathcal{U}_1(W)$ , by [11] and Lemma 4.26. Since  $\mathcal{U}_2 = \mathcal{U}_2^0 \cup \mathcal{U}_2^+ \cup \mathcal{U}_2^-$ , with the orbit sizes given in Corollary 4.17, this covers every involution in  $\bigcup_{U \in \mathcal{U}_1} \Gamma_2(C_G(U))$ , and the lemma follows.  $\square$

We now concern ourselves with the final two cases. These concern involutions that are distance 3 from  $t$  in every  $C_G(U)$  that they appear in, but actually are distance 2 from  $t$  in  $G$ . Recall that for any involution  $y \in Y_2$ ,  $C_G(y) = \text{Stab}_G C_V(y) = L_y K_y$  where  $L_y = C_G(y) \cap C_G([V, y]) \sim L_2(q)$  and  $|K_y| = 2(q - \delta)$ . Also note that  $L_y \trianglelefteq C_G(y)$  acts faithfully on  $C_V(y)$ , and  $\text{Syl}_p C_G(y) = \text{Syl}_p L_y$ . The following three lemmas concern Case 5.

**Lemma 4.29** *Let  $W \in \mathcal{U}_2^0 \cup \mathcal{U}_2^{-\delta}$  and  $x \in C_G(W)$  be such that  $d^{C_G(U)}(t, x) = 3$  for all  $U \in \mathcal{U}_1(W)$ . Then  $d(t, x) = 3$ .*

**Proof** If  $W \in \mathcal{U}_2^0$ , then any involution in  $C_G(W)$  can be written as  $x = x_Q t$  where  $x_Q = xt$  has order  $p$ . If  $W \in \mathcal{U}_2^{-\delta}$ , then, from Lemma 4.26, any involution  $x \in C_G(W)$  such that  $tx$  has order dividing  $\frac{1}{2}(q^2 - 1)$  must be distance 2 from  $t$  in  $C_G(U)$  for some  $U \in \mathcal{U}_1(W)$ . Hence, any  $x$  satisfying the hypothesis must have the property that the order of  $tx$  is  $p$ .

Let  $W \in \mathcal{U}_2^0 \cup \mathcal{U}_2^{-\delta}$  and suppose  $d(t, x) = 2$ , then there exists  $y \in Y_2$  such that  $t, x \in C_G(y) = L_y K_y$ . Since  $tx$  has order  $p$ ,  $tx \in L_y$  and so  $tx \in C_G([V, y])$ . As  $L_y$  acts faithfully on  $C_V(y)$ , any element of order  $p$  must fix a 1-subspace of  $C_V(y)$ , say  $U_y$ . Therefore,  $tx \in C_G(U_y \oplus [V, y])$ . But  $tx \in C_G(W + [V, y])$  and since  $[V, y] \in \mathcal{U}_2^\delta$ , we have  $W \neq [V, y]$ . Set  $W + [V, y] = U_y \oplus [V, y]$ .

Suppose  $U_y \leq W$ . Then  $t, x, y \in C_G(U_y)$  and so  $d^{C_G(U_y)}(t, x) = 2$ , contradicting our assumption. Hence  $U_y \not\leq W$  and so  $U_y = \langle u_1 + u_2 \rangle$  for  $u_1 \in W \setminus [V, y]$  and  $u_2 \in [V, y]$ . Since  $y \in C_G(y)$ ,  $(u_1 + u_2)^y = u_1 + u_2$ . However,  $(u_1 + u_2)^y = u_1^y + u_2^y = u_1^y - u_2$  and so  $u_2 = -2^{-1}u_1 + 2^{-1}u_1^y$ . Thus  $u_1 + u_2 = 2^{-1}(u_1 + u_1^y)$  and so  $U_y = \langle u_1 + u_1^y \rangle$ . Recall that  $t, x \in C_G(y)$  and  $u_1 \in W \setminus [V, y]$ , so  $u_1^t = u_1^x = u_1$ . Hence  $u_1 + u_1^y$  is centralized by both  $t$  and  $x$  and so  $U_y \leq W = C_V(\langle t, x \rangle)$ , a contradiction. Therefore,  $d(t, x) \neq 2$  and the lemma holds.  $\square$

**Lemma 4.30** *Let  $W \in \mathcal{U}_2^\delta$ . Then  $\Delta_3^{C_G(W)}(t) \subseteq \Delta_2(t)$ . In particular,*

$$\left| \Gamma_2(G) \setminus \bigcup_{U \in \mathcal{U}_1} \Gamma_2(C_G(U)) \right| = \begin{cases} q(q^2 - 1) & q \equiv 1 \pmod{4} \\ 0 & q \equiv -1 \pmod{4}. \end{cases}$$

**Proof** We deal first with the case when  $\delta = -1$ . From Lemma 4.26, the number of involutions distance 3 from  $t$  in  $C_G(W)$  that are actually distance 2 from  $t$  in some  $U \in \mathcal{U}_1(W)$  is  $\frac{1}{4}(q + 1)(q - 3) = \left| \Delta_3^{C_G(W)}(t) \right|$ . That is to say all elements in  $\Delta_3^{C_G(W)}(t)$  are distance 2 from  $t$  in  $C_G(U)$  for some  $U \in \mathcal{U}_2(W)$ . This occurs for every such  $W \in \mathcal{U}_2^\delta$  and so  $\Gamma_2(G) = \bigcup_{U \in \mathcal{U}_1} \Gamma_2(C_G(U))$ .

Assume now that  $\delta = 1$ . As before, any element  $x$  in  $\Gamma_2(G) \setminus \bigcup_{U \in \mathcal{U}_1} \Gamma_2(C_G(U))$  must have the property that the order of  $tx$  is  $p$ . Suppose  $d(t, x) = 2$ , and so there exists  $y \in Y_2$  such that  $t, x \in C_G(y)$ . If  $W \neq [V, y]$  then the argument from Lemma 4.29 holds and results in a contradiction. So we must have  $W = [V, y]$ . Since  $\text{Stab}_G C_V(y) = \text{Stab}_G [V, y] = C_G(y)$ ,  $C_G([V, y]) \leq C_G(y)$  and so any element in  $C_G([V, y]) = C_G(W)$  centralizes  $y$ . In particular,  $\Delta_3^{C_G(W)}(t) \subseteq \Delta_2(t)$ , establishing the first statement. By Lemma 4.26, the number of involutions distance 3 from  $t$  in  $C_G(W)$  that are actually distance 2 from  $t$  in some  $U \in \mathcal{U}_1(W)$  is  $\frac{1}{4}(q - 1)^2$ . By [11],  $\left| \Delta_3^{C_G(W)}(t) \right| = \frac{1}{4}(q - 1)(q + 7)$  and so by subtracting the two, there are  $2(q - 1)$  involutions in  $\Delta_3^{C_G(W)}(t)$  that are distance 3 from  $t$  in  $C_G(U)$  for all  $U \in \mathcal{U}_1(W)$ , but are actually distance 2 from  $t$  in  $\mathcal{C}(G, Y_2)$ . Since  $|\mathcal{U}_2^\delta| = \frac{1}{2}q(q + \delta)$  by Corollary 4.17, the lemma follows.  $\square$

Finally we turn to Case 4,  $\Gamma_1(G) \setminus \bigcup_{U \in \mathcal{U}_1} \Gamma_1(C_G(U))$ .

**Lemma 4.31** *Let  $U \in \mathcal{U}_1^- \cup \mathcal{U}_1^0$  and  $x \in C_G(U)$  be such that  $C_V(\langle t, x \rangle) = U$  and  $d^{C_G(U)}(t, x) = 3$ . Then  $d(t, x) = 3$ .*

**Proof** Assume first that  $U \in \mathcal{U}_1^-$ . By Lemma 4.24,  $tx$  has order  $p$  or divides  $\frac{1}{2}(q^2 + 1)$ . Suppose  $d(t, x) = 2$ , then there exists  $y \in Y_2$  such that  $t, x \in C_G(y)$ . Since  $\frac{1}{2}(q^2 + 1)$  is coprime to  $|C_G(y)| = q(q^2 - 1)(q - \delta)$ ,  $tx$  must have order  $p$ . Indeed, clearly  $\frac{1}{2}(q^2 + 1)$  is coprime to both  $q$  and  $q^2 - 1$ , and any factor dividing  $q - \delta$  must divide  $q^2 - 1$  and so  $\frac{1}{2}(q^2 + 1)$  is coprime to  $q - \delta$ . Since  $tx$  has order  $p$ , then  $tx \in L_y$ .

Assume now that  $U \in \mathcal{U}_1^0$ . Let  $x$  be an involution in  $C_G(U) = QL \sim G^0$  as defined in the discussion prior to Lemma 4.9. Then  $tx \in Qtx_L$  which has order  $n$  dividing  $\frac{1}{2}(q + \delta)$  in  $QLL$ . Therefore,  $(Qtx_L)^n \in Q$  and so  $(tx)^n$  has order  $p$ . Therefore,  $tx$  has order dividing  $\frac{1}{2}q(q + \delta)$ . Suppose  $d(t, x) = 2$ . Then there exists  $y \in Y_2$  such that  $t, x \in C_G(y)$ . By the structure of  $C_G(y) \sim (L_2(q) \times \frac{q-\delta}{2}) : 2^2$ , the order of  $tx$  forces  $tx \in L_y$ .

We may now assume  $U \in \mathcal{U}_1^- \cup \mathcal{U}_1^0$ , so  $tx \in L_y = C_G([V, y])$  and hence  $tx \in C_G(U + [V, y])$ . Suppose  $U \not\leq [V, y]$ , then  $tx \in C_G(U \oplus [V, y])$ . Also,  $tx \in C_G(U_y \oplus [V, y])$  for some  $U_y \leq C_V(y)$ . However, if  $U = U_y$  then  $t, x, y \in C_G(U)$  and  $d^{C_G(U)}(t, x) = 2$ . While  $U_y \neq U$  results in a contradiction using an analogous argument from Lemma 4.29. Hence  $U \leq [V, y]$ .

As  $t, x \in C_G(y) = \text{Stab}_G([V, y])$ ,  $tx \in L_y = C_G([V, y])$  and  $[V, y] = U \perp U'$  where  $U' = U^\perp \cap [V, y]$ . Then for  $u \in [V, y]$  we have  $u^{tx} = u$  and so  $u^t = u^x$ . In particular, if  $u \in U'$  then  $u^t = u^x = -u$ . Hence  $[V, y] = U \perp ([V, t] \cap [V, x])$ . If  $C_V(\langle t, y \rangle)$  is 1-dimensional, then  $C_V(y) = C_V(\langle t, y \rangle) \perp [V, t]$  since  $t$  stabilizes  $C_V(y)$ . However, then  $[V, t] \oplus ([V, t] \cap [V, x])$  is 3-dimensional, a contradiction. A similar argument holds for  $C_V(\langle x, y \rangle)$ . Therefore both  $C_V(\langle t, y \rangle)$  and  $C_V(\langle x, y \rangle)$  are 2-dimensional. But since  $\dim C_V(y) = 3$ , this means  $C_V(\langle t, y \rangle)$  and  $C_V(\langle x, y \rangle)$  intersect non-trivially, that is  $C_V(\langle t, x, y \rangle) \neq 0$ , contradicting our assumption. Therefore,  $d(t, x) \neq 2$ , and consequently  $d(t, x) = 3$ .  $\square$

The final case when  $U \in \mathcal{U}_1^+$  is slightly trickier. Recall the definition of  $Y_1$ . For any  $z \in Y_1$ , we have  $C_G(z) \sim SL_2(q) \circ SL_2(q) : 2$  and  $C_V(z)$  is 1-dimensional. We choose  $z$  such that  $t \in C_G(z)$  and  $C_V(z) = U$ , and return to work in the setting of  $Sp(4, q) / \langle -I_4 \rangle = G^\tau \sim G$ . We denote the image of any subgroup  $K \leq G$  by  $K^\tau$ . Choose

$$z = \left( \begin{array}{c|c} -I_2 & \\ \hline & I_2 \end{array} \right) \in G^\tau$$

and note that  $C_{G^\tau}(z) \sim C_G(U) : 2$ . Hence,

$$C_{G^\tau}(U) \sim \left\{ \left( \begin{array}{c|c} A & \\ \hline & B \end{array} \right) \mid A, B \in SL_2(q) \right\} / \langle -I_4 \rangle = C_G(U)^\tau.$$

Let  $t^\tau$  be the image of  $t$  in  $G^\tau$ . We start with a preliminary lemma concerning the commuting involution graph  $\mathcal{C}(L_2(q), X)$  where  $X$  is the sole conjugacy class of involutions. Denote by  $L \sim L_2(q)$  and  $\widehat{L} \sim PGL_2(q)$ .

**Lemma 4.32** *Let  $x$  be an involution in  $L$ . Then  $\Delta_3^L(x)$  splits into  $\frac{1}{4}(q + 2 + 5\delta)$   $C_L(x)$ -orbits of length  $q - \delta$ . Moreover, every  $C_L(x)$ -orbit in  $\Delta_3^L(x)$  is  $C_{\widehat{L}}(x)$ -invariant.*

**Proof** Assume first that  $\delta = -1$ . Choose  $x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and let  $x_\lambda = \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix}$  for some  $\lambda \in GF(q) \setminus \{\pm 1\}$ . There are two possibilities for an element of  $C_L(x)$ :

$$g_1 = \begin{pmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} b_1 & b_2 \\ b_2 & -b_1 \end{pmatrix}.$$

By direct calculation, if  $g_1^{-1}x_\lambda g_1 = x_\mu$  for some  $\lambda, \mu \in GF(q) \setminus \{\pm 1\}$  then  $(-\lambda^{-1} + \lambda)a_1 a_2 = 0$ . Note that since  $\lambda \neq \pm 1$ , then  $\lambda \neq \lambda^{-1}$ . If  $a_1 = 0$ , then  $a_2^2 = 1$ , and so  $\mu = \lambda^{-1}$ . On the other hand, if  $a_2 = 0$  then  $a_1^2 = 1$  and so  $\mu = \lambda$ . Note that in the case of  $g_2$ , neither  $b_1$  or  $b_2$  can be 0 and so  $g_2^{-1}x_\lambda g_2 = x_\mu$  requires  $xy(\lambda - \lambda^{-1}) = 0$ , a contradiction. Hence for  $\lambda, \mu \in GF(q) \setminus \{\pm 1\}$ ,  $x_\lambda$  and  $x_\mu$  lie in different  $C_L(x)$  orbits if and only if  $\mu \notin \{\lambda, \lambda^{-1}\}$ . As we work modulo  $\langle -I_4 \rangle$ , there are at least  $\frac{1}{4}(q - 3)$   $C_L(x)$ -orbits in  $\Delta_3^L(x)$ . However for any  $\lambda \neq \pm 1$ ,  $C_L(x, x_\lambda) = 1$  and so, each  $C_L(x)$ -orbit containing an  $x_\lambda$  is of length  $q + 1$ . But  $|\Delta_3^L(x)| = \frac{1}{4}(q - 3)(q + 1)$  and so all involutions in  $\Delta_3^L(x)$  are accounted for. Hence the first statement holds for  $\delta = -1$ , and each  $C_L(x)$ -orbit has representative  $x_\lambda$  for some  $\lambda \neq \pm 1$ . Let

$$e = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \widehat{L} \setminus L$$

and note that  $C_{\widehat{L}}(x) = \langle e \rangle C_L(x)$ , and an easy check shows  $[e, x_\lambda] = 1$  for all  $\lambda \neq \pm 1$ . Let  $y \in \Delta_3^L(x)$ , then  $y = x_\lambda^s$  for some  $s \in C_L(x)$ . Let  $g = er \in C_{\widehat{L}}(x)$  for some  $r \in C_L(x)$ . Then  $y^g = x_\lambda^{s^e r}$  and since  $C_L(x) \trianglelefteq C_{\widehat{L}}(x)$ ,  $s^e r \in C_L(x)$ . That is, every  $C_L(x)$ -orbit in  $\Delta_3^L(x)$  is  $C_{\widehat{L}}(x)$ -invariant.

Assume now that  $\delta = 1$ . Choose  $x = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$  where  $i^2 = -1$  and let  $y = \begin{pmatrix} \sigma & \mu\tau \\ \tau & \sigma \end{pmatrix}$  for some  $\sigma, \mu, \tau \in GF(q)$ ,  $\sigma \neq 0$  and  $\mu$  a non-square in  $GF(q)$ . By [11],  $y \in \Delta_3^L(x)$ . There are two possibilities for an element of  $C_L(x)$ :

$$g_1 = \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix}.$$

By direct calculation, if  $g_1^{-1}y g_1 = y$  then  $a = 1$ . Note that  $g_2^{-1}y g_2 \neq y$  as  $\pm b^2 \neq \mu$  for any non-square  $\mu$ . Hence  $C_L(\langle x, y \rangle) = 1$ . Since  $y$  was arbitrary, each  $C_L(x)$ -orbit has length  $q - 1$ . Now  $|\Delta_3^L(x)| = \frac{1}{4}(q + 7)(q - 1)$  and so the first statement holds for  $\delta = 1$ . Let

$$e_v = \begin{pmatrix} 0 & v \\ 1 & 0 \end{pmatrix} \in \widehat{L} \setminus L$$

and note that  $C_{\widehat{L}}(x) = \langle e_v \rangle C_L(x)$  for any non-square  $v$ . It is easy to check that  $y^{e_v} = y$ . Let  $g = e_\mu r \in C_{\widehat{L}}(x)$  for some  $r \in C_L(x)$ . Then  $y^g = y^{e_\mu r} = y^r$  and since  $y$  was arbitrary and  $r \in C_L(x)$ , every  $C_L(x)$ -orbit in  $\Delta_3^L(x)$  is  $C_{\widehat{L}}(x)$ -invariant.  $\square$

**Lemma 4.33**

$$|\Delta_3^{C_G(U)}(t) \cap \Gamma_1(G)| = \frac{1}{4}(q - \delta)^2(q + 2 + 5\delta).$$

**Proof** We first work in the setting of  $G^\tau$ . Choose

$$t^\tau = \left( \begin{array}{cc|cc} 0 & -1 & & \\ 1 & 0 & & \\ \hline & & 0 & -1 \\ & & 1 & 0 \end{array} \right) = \left( \begin{array}{c|c} J_0 & \\ \hline & J_0 \end{array} \right).$$

By direct calculation, it is easily seen that

$$C_{G^\tau}(t^\tau) \subseteq \left\{ \left( \begin{array}{cc|cc} A_1 & A_2 & & \\ A_3 & A_4 & & \\ \hline & & & \end{array} \right) \mid A_i^{-1} J_0 A_i = J_0 \pmod{\langle -I_4 \rangle} \right\}$$

and any involution  $y \in C_{G^\tau}(t^\tau)$  has the additional properties that

$$\begin{aligned} \det A_1 + \det A_3 &= \det A_2 + \det A_4 = 1 \quad \text{and} \\ A_1^2 + A_2 A_3 &= A_3 A_2 + A_4^2 = -I_2. \end{aligned} \tag{4.33.1}$$

Recall that if  $x \in C_G(U)^\tau$  then  $x = \left( \begin{array}{c|c} A & \\ \hline & B \end{array} \right)$  for some  $A, B \in SL_2(q)$  and by

Theorem 4.8,  $x \in \Delta_3^{C_G(U)^\tau}(t^\tau)$  if and only if  $A, B$  are involutions in  $L$  and either  $d^L(A, J_0) = 3$  or  $d^L(B, J_0) = 3$ . So without loss of generality, set  $A = B_i$  where  $d^L(B_i, J_0) = i$  and choose  $B \in \Delta_3^L(J_0)$ .

If  $x \in \Delta_2^{G^\tau}(t^\tau)$  then there exists  $y = \left( \begin{array}{cc|cc} A_1 & A_2 & & \\ A_3 & A_4 & & \\ \hline & & & \end{array} \right) \in C_{G^\tau}(t^\tau)$  such that  $y^2 = 1$

and  $[x, y] = 1$ . Suppose  $\det A_2 = 0$ . Then  $\det A_4 = 1$  by (4.33.1), and so  $A_4 \in C_L(J_0)$ . As  $[x, y] = 1, [A_4, B] = 1$ . However  $C_L(\langle J_0, B \rangle) = 1$ , by Lemma 4.32 and so  $A_4 = \pm I_2$ . But then  $A_3 A_2 = -2I_2$  by (4.33.1), which is impossible as  $\det A_2 = 0$ . An analogous argument holds for  $\det A_3$ . Hence  $\det A_2, \det A_3 \neq 0$ . Since  $[x, y] = 1, B_i A_2 B = \pm A_2$  and so  $B_i$  and  $B$  must be  $C_{\widehat{L}}(J_0)$ -conjugate. In other words, if  $B_i$  and  $B$  are not  $C_{\widehat{L}}(J_0)$ -conjugate, then  $[x, y] \neq 1$ . By Lemma 4.32, every  $C_L(J_0)$  orbit is an  $C_{\widehat{L}}(J_0)$ -orbit and so if  $[x, y] = 1$  then  $B_i$  and  $B$  must be  $C_L(J_0)$ -conjugate. Assume then  $B_i$  and  $B$  are  $C_L(J_0)$ -conjugate and let  $A \in C_L(J_0)$  be such

that  $B_i^A = B$ . Hence if  $y_A = \left( \begin{array}{c|c} & A \\ \hline -A^{-1} & \end{array} \right) \in C_{G^\tau}(t^\tau)$ , then  $[y_A, x] = 1$  and so

$d^{G^\tau}(t^\tau, x) = 2$ . By Lemma 4.32, each  $C_L(J_0)$ -orbit of  $\Delta_3^L(J_0)$  is of length  $q - \delta$ , and there are  $\frac{1}{4}(q + 2 + 5\delta)$  such orbits. Moreover, for any involution  $x_0 \in C_G(U)^\tau$  conjugate to  $t^\tau, z x_0$  is also an involution in  $C_G(U)^\tau$  conjugate to  $t^\tau$  which has not

been accounted for. Therefore, the number of involutions in  $\Delta_3^{C_G(U)^\tau}(t^\tau)$  that are actually distance 2 from  $t^\tau$  in  $G^\tau$  is  $\frac{1}{2}(q - \delta)^2(q + 2 + 5\delta)$ .

We now return to the setting of  $G$ , and first assume that  $\delta = -1$  and so by Corollary 4.22(i),  $|\mathcal{W}_2(U)| = q + 1$ , and for every  $W \in \mathcal{W}_2(U)$ ,  $C_G(W) \sim L_2(q)$ . For each  $W$ , there exists  $U_W \in \mathcal{U}_1^+$  such that  $C_G(W) \leq C_G(U_W) \sim L_2(q^2)$  by Lemma 4.21, and  $\Delta_3^{C_G(W)}(t) \subseteq \Delta_2^{C_G(U_W)}(t)$  by Lemma 4.30. Hence, there are  $\frac{1}{4}(q + 1)^2(q - 3)$  involutions already counted (from Case 3) and the remaining involutions do not fix a 2-subspace of  $C_V(t)$ . Therefore

$$\begin{aligned} \left| \Delta_3^{C_G(U)}(t) \cap \Gamma_1(G) \right| &= \frac{1}{2}(q + 1)^2(q - 3) - \frac{1}{2}(q + 1)^2(q - 3) \\ &= \frac{1}{4}(q + 1)^2(q - 3), \end{aligned}$$

as required. Now assume that  $\delta = 1$  and so by Corollary 4.22. For each  $W$ , there exists  $U_W \in \mathcal{U}_1^-$  such that  $C_G(W) \leq C_G(U_W) \sim L_2(q^2)$  by Lemma 4.21 and  $\left| \Delta_3^{C_G(W)}(t) \cap \Delta_2^{C_G(U_W)}(t) \right| = \frac{1}{4}(q - 1)^2$  by Lemma 4.26. Since  $|\mathcal{W}_2(U) \cap (\mathcal{U}_1^+ \cup \mathcal{U}_1^-)| = q - 1$  by Corollary 4.22(iii), this accounts for  $\frac{1}{4}(q - 1)^3$  involutions. Suppose now  $W_0 \in \mathcal{W}_2(U) \cap \mathcal{U}_2^0$ . By Lemma 4.21, there exists  $U_0 \in \mathcal{U}_1^0$  such that  $C_G(W_0) \leq C_G(U_0)$ . From Lemmas 4.12 and 4.19,  $\left| C_G(W) \cap \Delta_2^{C_G(U_0)}(t) \right| = \frac{1}{2}(q - 1)^2$ . Since  $|\mathcal{W}_2(U) \cap \mathcal{U}_2^0| = 2$  by Corollary 4.22(iii), this yields a further  $(q - 1)^2$  involutions. Finally, if  $W \in \mathcal{U}_2^+$ , then by Lemma 4.30,  $\Delta_3^{C_G(W)}(t) \subseteq \Delta_2(t)$  and there are  $2(q - 1)$  involutions in  $\Delta_3^{C_G(W)}(t)$  not already enumerated. Now  $|\mathcal{U}_2^+ \cap \mathcal{W}_2(U)| = \frac{1}{2}(q - 1)$  by Corollary 4.22(iii), and this yields another  $(q - 1)^2$  involutions. Hence, there are  $\frac{1}{4}(q - 3)^2 + 2(q - 1)^2 = \frac{1}{4}(q - 1)^2(q + 7)$  involutions already counted (from Cases 3 and 5) and the remaining involutions do not fix a 2-subspace of  $C_V(t)$ . Consequently

$$\begin{aligned} \left| \Delta_3^{C_G(U)}(t) \cap \Gamma_1(G) \right| &= \frac{1}{2}(q - 1)^2(q + 7) - \frac{1}{2}(q - 1)^2(q + 7) \\ &= \frac{1}{4}(q - 1)^2(q + 7), \end{aligned}$$

as required. □

**Corollary 4.34**

$$\left| \Gamma_1(G) \setminus \bigcup_{U \in \mathcal{U}_1} \Gamma_1(C_G(U)) \right| = \frac{1}{8}q(q - \delta)(q^2 - 1)(q + 2 + 5\delta).$$

**Proof** Since  $|\mathcal{U}_1^+| = \frac{1}{2}q(q + \delta)$ , the result holds by Lemmas 4.32 and 4.33. □

**Lemma 4.35** *If  $q \equiv -1 \pmod{4}$ , then*

- (i)  $|\Delta_2(t)| = \frac{1}{16}(q+1)(3q^5 - 2q^4 + 8q^3 - 30q^2 + 13q - 8)$ ; and  
 (ii)  $|\Delta_3(t)| = \frac{1}{16}(q^2 - 1)(5q^4 - 9q^3 + 7q^2 - 3q + 8)$ .

If  $q \equiv 1 \pmod{4}$ , then

- (iii)  $|\Delta_2(t)| = \frac{1}{16}(q-1)(3q^5 - 6q^4 + 32q^3 - 10q^2 - 27q - 8)$ ; and  
 (iv)  $|\Delta_3(t)| = \frac{1}{16}(q-1)(5q^5 + 22q^4 - 8q^3 + 34q^2 + 51q + 24)$ .

**Proof** The cases listed in Table 1 are disjoint. Hence  $|\Delta_2(t)|$  is determined by summing the values calculated in Lemmas 4.27, 4.28, 4.30 and 4.34. By Theorem 4.15,  $\mathcal{C}(G, Y_2)$  has diameter 3 and so  $|\Delta_3(t)| = |Y_2| - |\Delta_1(t)| - |\Delta_2(t)| - 1$ . Since  $|G| = \frac{1}{2}q^4(q^2 - 1)(q^4 - 1)$  and  $|C_G(t)| = q(q^2 - 1)(q - \delta)$ ,  $|Y_2| = \frac{1}{2}q^3(q + \delta)(q^2 + 1)$ . Together with Lemma 4.20, this proves the lemma.  $\square$

Together, Theorem 4.15 and Lemmas 4.20 and 4.35 complete the proof of Theorem 1.4.

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