Letter

What Could "Effective" Mean for Proofs?

MICHAEL NEUBRAND

The Mathematical Intelligencer encourages comments about the material in this issue. Letters to the editor should be sent to the editor-in-chief.

In the preceding letter, Artin Sharifi claimed to have detected "effective tools that might lead us to ineffective proofs" in a proof I presented in this journal (*The Mathematical Intelligencer* 38(1), 2016; published online 25 January 2016). Of course, he is right in that a proof of the Law of Cosines actually needs only two lines, after noting the relation $a = b \cos C + c \cos B$ at any of the three sides of the triangle: Just multiply the three resp. equations by $(-a), b, c$ and add, then the formula stands. Yes, that is really effective, and of course I know that proof, and have used it also in my lectures on Elementary Geometry. Sometimes I commented that this proof is a quite formal exercise; it is nevertheless valuable and one way to get the result.

However, can "effective" in connection with proofs mean something other than just being short? "A good proof is one that makes us wiser," Yuri Manin once said (this journal, 2(1), 1979, 17–18). In mathematics, becoming wiser means becoming aware of some inner coherence. The short proof sketched by Sharifi promotes the basic idea of the trigonometric functions: They allow us to describe various relations in the triangle (and in other geometric figures) in a very compact form, the Law of Cosines included.

But in this "short proof," the assertion of the theorem just comes out, and is not embedded into some geometric intuition and coherence. Only further observation of the formula shows that we are dealing with a generalization of Pythagoras's Theorem. Inserting the relations $b = a \cos C$ and $c = a \cos B$ into the formula above, the "short proof" eventually can emerge from the wish to make things symmetric (in a formal way) on all three sides. Coherence then could appear on the level of mathematical behavior, that is, while noting that symmetrization is one of the big principles of doing mathematics.

In my article I also mentioned other kinds of coherence: appropriateness of the tools; employing analogies; thinking in structural relation to the situation. Those thoughts also are reactions to the question of motivation that Sharifi posed: If the height is a good tool for proving Pythagoras, why not take a generalization, for example the nonperpendiculars, for proving a more general situation?

Thus, Sharifi is right: Always think from the end, and consider what the end can mean. But the end is not necessarily the theorem as such, that is, the formula to be produced. It also makes sense to say the end is embedding, creating coherence, searching for outlooks, reflecting mathematical behavior, consciously generalizing a mathematical situation, and much more beyond. But then we must consider going different ways, taking ways that are not just short but meet interesting points, surprising connections, and interesting views. The problem then is not just to prove, but to reflect on the (various issues of the) quality of the proofs.

Institute of Mathematics
Carl-von-Ossietzky-University
26111 Oldenburg
Germany
e-mail: michael.neubrand@uni-oldenburg.de

DOI 10.1007/s00283-016-9638-1