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Nonzero-Sum Games of Optimal Stopping for Markov Processes

Natalie Attard[1]

Abstract Two players are observing a right-continuous and quasi-left-continuous strong Markov process X. We study the optimal stopping problem $V_\sigma^1(x) = \sup_\tau \mathsf{M}_x^1(\tau, \sigma)$ for a given stopping time σ (resp. $V_\tau^2(x) = \sup_\sigma \mathsf{M}_x^2(\tau, \sigma)$ for given τ) where $\mathsf{M}_x^1(\tau, \sigma) = \mathsf{E}_x[G_1(X_\tau)I(\tau \leq \sigma) + H_1(X_\sigma)I(\sigma < \tau)]$ with G_1, H_1 being continuous functions satisfying some mild integrability conditions (resp. $\mathsf{M}_x^2(\tau, \sigma) = \mathsf{E}_x[G_2(X_\sigma)I(\sigma < \tau) + H_2(X_\tau)I(\tau \leq \sigma)]$ with G_2, H_2 being continuous functions satisfying some mild integrability conditions). We show that if $\sigma = \sigma_{D_2} = \inf\{t \geq 0 : X_t \in D_2\}$ (resp. $\tau = \tau_{D_1} = \inf\{t \geq 0 : X_t \in D_1\}$) where D_2 (resp. D_1) has a regular boundary, then $V_{\sigma_{D_2}}^1$ (resp. $V_{\tau_{D_1}}^2$) is finely continuous. If D_2 (resp. D_1) is also (finely) closed then $\tau_*^{\sigma_{D_2}} = \inf\{t \geq 0 : X_t \in D_1^{\sigma_{D_2}}\}$ (resp. $\sigma_*^{\tau_{D_1}} = \inf\{t \geq 0 : X_t \in D_2^{\tau_{D_1}}\}$) where $D_1^{\sigma_{D_2}} = \{V_{\sigma_{D_2}}^1 = G_1\}$ (resp. $D_2^{\tau_{D_1}} = \{V_{\tau_{D_1}}^2 = G_2\}$) is optimal for player one (resp. player two). We then derive a partial superharmonic characterisation for $V_{\sigma_{D_2}}^1$ (resp. $V_{\tau_{D_1}}^2$) which can be exploited in examples to construct a pair of first entry times that is a Nash equilibrium.

Keywords Nonzero-sum optimal stopping game · Nash equilibrium · Markov process · Double partial superharmonic characterisation · Principle of double smooth fit · Principle of double continuous fit

Mathematics Subject Classification Primary 91A15 · 60G40; Secondary 60J25 · 60G44

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1 Introduction

Optimal stopping games, often referred to as Dynkin games, are extensions of optimal stopping problems. Since the seminal paper of Dynkin [14], optimal stopping games have been studied extensively. Martingale methods for zero-sum games were studied by Kifer [32], Neveu [44], Stettner [55], Lepeltier and Maingueneau [41] and Ohtsubo [45]. The Markovian framework was initially studied by Frid [22], Gusein-Zade [26], Elbakidze [18] and Bismut [5]. Bensoussan and Friedman [2] and Friedman [20,21] considered zero-sum optimal stopping games for diffusions and developed an analytic approach by relying on variational and quasi-variational inequalities. Ekström and Peskir [16] proved the existence of a value in two-player zero-sum optimal stopping games for right-continuous strong Markov processes and construct a Nash equilibrium point under the additional assumption that the underlying process is quasi-left continuous. Peskir in [51] and [52] extended these results further by deriving a semi-harmonic characterisation of the value of the game without assuming that a Nash equilibrium exists a priori. In particular, a necessary and sufficient condition for the existence of a Nash equilibrium is that the value function coincides with the smallest superharmonic and the largest subharmonic function lying between the gain and the loss function which, in the case of absorbed Brownian motion in [0,1], is equivalent to 'pulling a rope' between 'two obstacles' (that is finding the shortest path between the graphs of two functions). Connections between zero-sum optimal stopping games and singular stochastic control problems were studied in [23,31] and[4]. Cvitanic and Karatzas [11] showed that backward stochastic differential equations are connected with the value function of a zero-sum Dynkin game. Advances in this direction can be found in [27]. Various authors have also studied zero-sum optimal stopping games with randomised strategies. For further details one can refer to [39] and the references therein. Zero-sum optimal stopping games have been used extensively in the pricing of game contingent claims both in complete and incomplete markets (see for example [15,17,24,25,30,33,35,36,38] and [19]).

Literature on nonzero-sum optimal stopping games is mainly concerned with the existence of a Nash equilibrium. Initial studies in discrete time date back to Morimoto [42] wherein a fixed point theorem for monotone mappings is used to derive sufficient conditions for the existence of a Nash equilibrium point. Ohtsubo [46] derived equilibrium values via backward induction and in [47] the same author considers nonzero-sum games in which the lower gain process has a monotone structure, and gives sufficient conditions for a Nash equilibrium point to exist. Shmaya and Solan in [54] proved that every two player nonzero-sum game in discrete time admits an ε-equilibrium in randomised stopping times. In continuous time Bensoussan and Friedman [3] showed that, for diffusions, a Nash equilibrium exists if there exists a solution to a system of quasi-variational inequalities. However, the regularity and uniqueness of the solution remain open problems. Nagai [43] studies a nonzero-sum stopping game of symmetric Markov processes. A system of quasi-variational inequalities is introduced in terms of Dirichlet forms and the existence of extremal solutions of a system of quasi-variational inequalities is discussed. Nash equilibrium points of the stopping game are then obtained from these extremal solutions. Cattiaux and Lepeltier [8] study special right-processes, namely Hunt processes in the Ray topology, and they prove existence

of a quasi-Markov Nash Equlibrium. The authors follow Nagai's idea but use probabilistic tools rather than the theory of Dirichlet forms. Huang and Li in [29] prove the existence of a Nash equilibrium point for a class of nonzero-sum noncyclic stopping games using the martingale approach. Laraki and Solan [40] proved that every two-player nonzero-sum Dynkin game in continuous time admits an $\varepsilon-$equilibrium in randomised stopping times. Hamadène and Zhang in [28] prove existence of a Nash equilibrium using the martingale approach, for processes with positive jumps. One application of nonzero-sum optimal stopping games is seen in the study of game options in incomplete markets, via the consideration of utility-based arguments (see [34]). Nonzero-sum optimal stopping games have also been used to model the interaction between bondholders and shareholders in the study of convertible bonds, when corporate taxes are included and when the company is allowed to claim default (see [9]).

In this work we consider two player nonzero-sum games of optimal stopping for a general strong Markov process. The aim is to use probabilistic tools to study the optimal stopping problem of player one (resp. player two) when the stopping time of player two (resp. player one) is externally given. Although this work does not deal with the question of existence of mutually best responses (that is the existence of a Nash equilibrium) the results obtained can be exploited further in various examples, to show the existence of a pair of first entry times that will be a Nash equilibrium. Indeed, the results derived here will be used in a separate work (see [1]) to construct Nash equilibrium points for one dimensional regular diffusions and for a certain class of payoff functions.

This paper is organised as follows: In Sect. 2 we introduce the underlying setup and formulate the nonzero-sum optimal stopping game. In Sect. 3 we show that if the strategy chosen by player two (resp. player one) is σ_{D_2} (resp. τ_{D_1}), the first entry time into a regular Borel subset D_2 (resp. D_1) of the state space, then the value function of player one associated with σ_{D_2} (resp. the value function of player two associated with τ_{D_1}), which we shall denote by $V^1_{\sigma_{D_2}}$ (resp. $V^2_{\tau_{D_1}}$), is finely continuous. In Section 4 we shall use this regularity property of $V^1_{\sigma_{D_2}}$ (resp. $V^2_{\tau_{D_1}}$) to construct an optimal stopping time for player one (resp. player two). In Sect. 5 we shall use the results obtained in Sects. 3 and 4 to provide a partial superharmonic characterisation for $V^1_{\sigma_{D_2}}$ (resp. $V^2_{\tau_{D_1}}$). More precisely if D_2 (resp. D_1) is also a closed or finely closed subset of the state space then $V^1_{\sigma_{D_2}}$ (resp. $V^2_{\tau_{D_1}}$) can be identified with the smallest finely continuous function that is superharmonic in D_2^c (resp. in D_1^c) and that majorises the lower payoff function. In Section 6 we shall consider stationary one-dimensional Markov processes and we shall assume that there exists a pair of stopping times $(\tau_{A_*}, \sigma_{B_*})$ of the form $\tau_{A_*} = \inf\{t \geq 0 : X_t \leq A_*\}$ and $\sigma_{B_*} = \inf\{t \geq 0 : X_t \geq B_*\}$ where $A_* < B_*$, that is a Nash equilibrium point. We first show that $V^1_{\sigma_{B_*}}$ (resp. $V^1_{\tau_{A_*}}$) is continuous at A_* (resp. at B_*). Then for the special case of one dimensional regular diffusions we shall use the results obtained in Sect. 5 to show that $V^1_{\sigma_{B_*}}$ (resp. $V^2_{\tau_{A_*}}$) is also smooth at A_* (resp. B_*) provided that the payoff functions are smooth. This is in line with the principle of smooth fit observed in standard optimal stopping problems (see for example [49] for further details).

2 Formulation of the Problem

In this section we shall formulate rigorously the nonzero-sum optimal stopping game. For this we shall first set up the underlying framework. This will be similar to the one presented by Ekström and Peskir (cf. [16, p. 3]). On a given filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P}_x\right)$ we define a strong Markov process $X = (X_t)_{t\geq 0}$ with values in a measurable space $(E, \mathcal{B})$, with E being a locally compact Hausdorff space with a countable base (note that since E has a countable base then it is a Polish space) and $\mathcal{B}$ the Borel σ-algebra on E. We shall assume that $\mathsf{P}_x(X_0 = x) = 1$, that the sample paths of X are right-continuous and that X is quasi-left-continuous (that is $X_{\rho_n} \to X_\rho$ P_x-a.s. whenever ρ_n and ρ are stopping times such that $\rho_n \uparrow \rho$ P_x-a.s.). All stopping times mentioned throughout this text are relative to the filtration $(\mathcal{F}_t)_{t\geq 0}$ introduced above, which is also assumed to be right-continuous. This means that entry times in open and closed sets are stopping times. Moreover $\mathcal{F}_0$ is assumed to contain all P_x-null sets from $\mathcal{F}_\infty^X = \sigma(X_t : t \geq 0)$, which further implies that the first entry times to Borel sets are stopping times. We shall also assume that the mapping $x \mapsto \mathsf{P}_x(F)$ is (universally) measurable for each $F \in \mathcal{F}$ so that the mapping $x \mapsto \mathsf{E}_x[Z]$ is (universally) measurable for each (integrable) random variable Z.

Remark 2.1 Note that a subset F of a Polish space E is said to be universally measurable if it is μ-measurable for every finite measure μ on $(E, \mathcal{B})$, where $\mathcal{B}$ is the Borel-σ algebra on E. By μ-measurable we mean that F is measurable with respect to the completion of $\mathcal{B}$ under μ. If $\mathcal{B}^*$ is the collection of universally measurable subsets of E then a function $f : E \to \mathbb{R}$ is said to be universally measurable if $f^{-1}(A) \in \mathcal{B}^*$ for all $A \in \mathcal{B}(\mathbb{R})$, where $\mathcal{B}(\mathbb{R})$ is the Borel-σ algebra on $\mathbb{R}$.

Finally we shall assume that Ω is the canonical space $E^{[0,\infty)}$ with $X_t(\omega) = \omega(t)$ for $\omega \in \Omega$. In this case the shift operator $\theta_t : \Omega \to \Omega$ is well defined by $\theta_t(\omega)(s) = \omega(t+s)$ for $\omega \in \Omega$ and $t, s \geq 0$.

The Markovian version of a nonzero-sum optimal stopping game may now be formally described as follows. Let $G_1, G_2, H_1, H_2 : E \to \mathbb{R}$ be continuous functions satisfying $G_i \leq H_i$ and the following integrability conditions;

$$\mathsf{E}_x[\sup_t |G_i(X_t)|] < \infty, \quad \text{and} \quad \mathsf{E}_x[\sup_t |H_i(X_t)|] < \infty \tag{2.1}$$

for $i = 1, 2$. Suppose that two players are observing X. Player one wants to choose a stopping time τ and player two a stopping time σ in such a way as to maximise their total average gains, which are respectively given by

$$\mathsf{M}_x^1(\tau, \sigma) = \mathsf{E}_x[G_1(X_\tau) I(\tau \leq \sigma) + H_1(X_\sigma) I(\sigma < \tau)] \tag{2.2}$$
$$\mathsf{M}_x^2(\tau, \sigma) = \mathsf{E}_x[G_2(X_\sigma) I(\sigma < \tau) + H_2(X_\tau) I(\tau \leq \sigma)]. \tag{2.3}$$

For a given strategy σ chosen by player two, we let

$$V_\sigma^1(x) = \sup_\tau \mathsf{M}_x^1(\tau, \sigma) \tag{2.4}$$

and for a given strategy τ chosen by player one, we let

$$V_\tau^2(x) = \sup_\sigma \mathsf{M}_x^2(\tau, \sigma). \tag{2.5}$$

We shall refer to V_σ^1 (resp. V_τ^2) as the value function of player one (resp. player two) associated with the given stopping time σ (resp. τ) of player two (resp. player one). We shall assume that the stopping times in (2.4) and (2.5) are finite valued and if the terminal time T is finite we shall further assume that

$$G_i(X_T) = H_i(X_T) \; \mathsf{P}_x\text{-a.s.} \tag{2.6}$$

for $i = 1, 2$. In this case one can think of X as being a two dimensional process $((t, Y_t))_{t \geq 0}$ so that G_i and H_i will be functions on $[0, T] \times E$ (cf. [49, p. 36]). (We note that if the terminal time T is infinite and the stopping times τ and σ are allowed to be infinite our results will still be valid provided that $\limsup_{t \to \infty} G_i(X_t) = \limsup_{t \to \infty} H_i(X_t) \; \mathsf{P}_x$-a.s.)

The game is said to have a solution if there exists a pair of stopping times (τ_*, σ_*) which is a Nash equilibrium point, that is $\mathsf{M}_x^1(\tau, \sigma_*) \leq \mathsf{M}_x^1(\tau_*, \sigma_*)$ and $\mathsf{M}_x^2(\tau_*, \sigma) \leq \mathsf{M}_x^2(\tau_*, \sigma_*)$ for all stopping times τ, σ. This means that none of the players will perform better if they change their strategy independent of each other. In this case $V_{\sigma_*}^1(x) = \mathsf{M}_x^1(\tau_*, \sigma_*)$ is the payoff function of player one and $V_{\tau_*}^2(x) = \mathsf{M}_x^2(\tau_*, \sigma_*)$ the payoff function of player two in this equilibrium. So $V_{\sigma_*}^1$ and $V_{\tau_*}^2$ can be called the value functions of the game (corresponding to (τ_*, σ_*)). In general, as we shall see in Sect. 6, there might be other pairs of stopping times that form a Nash equilibrium point, which can lead to different value functions.

3 Fine continuity property

In this section we show that if the strategy chosen by player two (resp. player one) corresponds to the first entry time into a subset D_2 (resp. D_1) of E, whose boundary ∂D_2 (resp. ∂D_1) is regular, then $V_{\sigma_{D_2}}^1$ (resp. $V_{\tau_{D_1}}^2$) is continuous in the fine topology (i.e. finely continuous). For literature on the fine topology one can refer to [6,10] and [13]. We first define the concept of a finely open set and a regular boundary of a Borel subset of E.

Definition 3.1 An arbitrary set $B \subseteq E$ is said to be finely open if there exists a Borel set $A \subseteq B$ such that $\mathsf{P}_x(\rho_{A^c} > 0) = 1$ for every $x \in A$, where $\rho_{A^c} = \inf\{t > 0 : X_t \in A^c\}$ is the first hitting time in A^c.

Definition 3.2 The boundary ∂D of a Borel set $D \subseteq E$ is said to be regular for D if $\mathsf{P}_x(\rho_D = 0) = 1$ for every point $x \in \partial D$, where $\rho_D = \inf\{t > 0 : X_t \in D\}$.

We now introduce preliminary results which are needed to prove the main theorem of this section.

Lemma 3.3 *For any given stopping time σ (resp. τ), the mapping $x \mapsto V_\sigma^1(x)$, (resp. $x \mapsto V_\tau^2(x)$) is measurable.*

Proof To prove measurability of the mapping $x \mapsto V_\tau^2(x)$ one can follow the proof in [16, p. 5, pt. 3] by replacing G_1 and G_2 with G_2 and H_2 respectively (note that our payoff functions are assumed to be continuous, hence finely-continuous). So we shall only prove the result for V_σ^1. Ekström and Peskir [16, p. 5, pt. 3] proved that for any given stopping time σ, the function $\tilde{V}_\sigma^1$ of the optimal stopping problem $\sup_\tau \tilde{M}_x^1(\tau, \sigma) = \sup_\tau E_x[G_1(X_\tau)I(\tau < \sigma) + H_1(X_\sigma)I(\sigma \leq \tau)]$ is measurable. The same method of proof can be applied in this setting with the following slight modification: Let $G_t^{\sigma,1} = G_1(X_\tau)I(t \leq \sigma) + H_1(X_\sigma)I(\sigma < t)$ and $\tilde{G}_t^{\sigma,1} = G_1(X_t)I(t < \sigma) + H_1(X_\sigma)I(\sigma \leq t)$. Note that the mapping $t \mapsto G_t^{\sigma,1}$ is not right-continuous. Now for any stopping time τ in the optimal stopping problem (2.4) we let $\tau_n = \frac{k}{2^n}$ on $\{\frac{k-1}{2^n} < \tau \leq \frac{k}{2^n}\}$ for each $n \geq 1$. It is well known that τ_n, for each n is a stopping time on the dyadic rationals Q_n of the form $\frac{k}{2^n}$ and that $\tau_n \downarrow \tau$ as $n \to \infty$. Since $G_1(X)$ is right-continuous we have that

$$\lim_{n \to \infty} G_{\tau_n}^{\sigma,1} = \tilde{G}_\tau^{\sigma,1} \quad P_x\text{-a.s.} \tag{3.1}$$

Since $G_1 \leq H_1$ we get, upon using (3.1) and Fatou's lemma (the required integrability condition for using Fatou's lemma can be derived from the integrability assumption (2.1)), that

$$E_x\big[G_\tau^{\sigma,1}\big] \leq E_x\big[\tilde{G}_\tau^{\sigma,1}\big] = E_x\big[\lim_{n \to \infty} G_{\tau_n}^{\sigma,1}\big] \leq \liminf_{n \to \infty} E_x\big[G_{\tau_n}^{\sigma,1}\big]$$
$$\leq \sup_{n \geq 1} \sup_{\tau \in Q_n} E_x\big[G_\tau^{\sigma,1}\big] =: \sup_{n \geq 1} V_n^{\sigma,1}(x). \tag{3.2}$$

Taking the supremum over all τ it follows that $V_\sigma^1(x) \leq \sup_{n \geq 1} V_n^{\sigma,1}(x)$. On the other hand, $V_n^{\sigma,1}(x) \leq V_\sigma^1(x)$ for all $n \geq 1$ so we get that $V_\sigma^1(x) = \sup_{n \geq 1} V_n^{\sigma,1}(x)$ for all $x \in E$. Measurability of $V_\sigma^1(x)$ follows from the measurability property of $\sup_{n \geq 1} V_n^{\sigma,1}(x)$ as in [16]. $\qquad\square$

Lemma 3.4 *Let D be a Borel subset of E and let $x \in \partial D$, where ∂D is a regular boundary for D. Suppose that $(\rho_n)_{n=1}^\infty$ is a sequence of stopping times such that $\rho_n \downarrow 0$ P_x-a.s. as $n \to \infty$. Set $\sigma_{\rho_n} = \inf\{t \geq \rho_n : X_t \in D\}$. Then $\sigma_{\rho_n} \downarrow 0$ P_x-a.s. as $n \to \infty$.*

Proof Let $x \in \partial D$. By regularity of ∂D for any $\varepsilon > 0$ there exists $t \in (0, \varepsilon)$ such that $X_t \in D$ P_x-a.s. Since σ_{ρ_n} is a sequence of decreasing stopping times then $\sigma_{\rho_n} \downarrow \beta$ P_x-a.s. for some stopping time β. So suppose for contradiction that $\beta > 0$. Now $\rho_n \downarrow 0$ P_x-a.s. and for each n we have $\sigma_{\rho_n} \geq \beta$ P_x-a.s. So for any given $\omega \in \Omega \backslash N$ where $P_x(N) = 0$ we have that $X_t(\omega) \notin D$ for all $t \in (0, \beta(\omega))$ and this contradicts the fact that ∂D is regular for D.

The next lemma and theorem, which we shall exploit in this study, provide conditions for fine continuity. The proofs of these results can be found in [13].

Lemma 3.5 *A measurable function $F : E \to \mathbb{R}$ is finely continuous if and only if*

$$\lim_{t \downarrow 0} F(X_t) = F(x) \; \mathsf{P}_x\text{-a.s.} \tag{3.3}$$

for every $x \in E$. This is further equivalent to the fact that the mapping

$$t \mapsto F(X_t(\omega)) \text{ is right-continuous on } \mathbb{R}_+ \tag{3.4}$$

for every $\omega \in \Omega \backslash N$ where $\mathsf{P}_x(N) = 0$ for all $x \in E$.

Theorem 3.6 *Let $F : E \to \mathbb{R}$ be a measurable function and suppose that $K_1 \subseteq K_2 \subseteq K_3 \subseteq \ldots$ is a nested sequence of compact sets in E. Suppose also that $\rho_{K_n} \downarrow 0 \; \mathsf{P}_x$-a.s. as $n \to \infty$, where $\rho_{K_n} = \inf\{t \geq 0 : X_t \in K_n\}$. If $\lim_{n \to \infty} \mathsf{E}_x[F(X_{\rho_{K_n}})] = F(x)$ then F is finely continuous.*

We next state and prove the main result of this section, that is the fine continuity property of $V^1_{\sigma_{D_2}}$ (resp. $V^2_{\tau_{D_1}}$).

Theorem 3.7 *Let $(\rho_n)_{n=1}^{\infty}$ be any sequence of stopping times such that $\rho_n \downarrow 0 \; \mathsf{P}_x$-a.s. as $n \to \infty$. Suppose that D_1, D_2 are Borel subsets of E having regular boundaries ∂D_1 and ∂D_2 respectively. Then*

$$\lim_{n \to \infty} \mathsf{E}_x\left[V^1_{\sigma_{D_2}}(X_{\rho_n})\right] = V^1_{\sigma_{D_2}}(x) \tag{3.5}$$

$$\lim_{n \to \infty} \mathsf{E}_x\left[V^2_{\tau_{D_1}}(X_{\rho_n})\right] = V^2_{\tau_{D_1}}(x) \tag{3.6}$$

where $\sigma_{D_2} = \inf\{t \geq 0 : X_t \in D_2\}$ and $\tau_{D_1} = \inf\{t \geq 0 : X_t \in D_1\}$.

Proof We will only prove (3.5) as (3.6) follows by symmetry.

$1°$ From Lemma 3.3 we know that $V^1_{\sigma_{D_2}}$ is measurable. This implies that

$$V^1_{\sigma_{D_2}}(X_{\rho_n}) = \sup_{\tau} \mathsf{M}^1_{X_{\rho_n}}(\tau, \sigma_{D_2}) \tag{3.7}$$

is a random variable. By the strong Markov property of X we have that

$$\mathsf{M}^1_{X_{\rho_n}}(\tau, \sigma_{D_2}) = \mathsf{E}_x\left[G_1(X_{\tau_{\rho_n}})I(\tau_{\rho_n} \leq \sigma_{\rho_n}) + H_1(X_{\sigma_{\rho_n}})I(\sigma_{\rho_n} < \tau_{\rho_n})|\mathcal{F}_{\rho_n}\right] \tag{3.8}$$

where we set $\tau_{\rho_n} = \rho_n + \tau \circ \theta_{\rho_n}$ and $\sigma_{\rho_n} = \rho_n + \sigma_D \circ \theta_{\rho_n}$. It is well known that τ_{ρ_n} and σ_{ρ_n} are stopping times (see for example [10, Section 1.3, Thoerem 11]). Let us set

$$\mathsf{M}^1_x(\tau, \sigma|\mathcal{F}_\rho) = \mathsf{E}_x\left[G_1(X_\tau)I(\tau \leq \sigma) + H_1(X_\sigma)I(\sigma < \tau)|\mathcal{F}_\rho\right] \tag{3.9}$$

for given stopping times τ, σ and ρ. Then from (3.7) and (3.8) we get that

$$V^1_{\sigma_D}(X_{\rho_n}) = \text{ess sup}_\tau \, \mathsf{M}^1_x(\tau_{\rho_n}, \sigma_{\rho_n}|\mathcal{F}_{\rho_n}) = \text{ess sup}_{\tau \geq \rho_n} \mathsf{M}^1_x(\tau, \sigma_{\rho_n}|\mathcal{F}_{\rho_n}). \tag{3.10}$$

The last equality follows from the fact that for every stopping time $\tau \geq \rho_n$, there exists a function $\tau^{\rho_n} : \Omega \times \Omega \to [0, \infty]$ such that

$$\tau^{\rho_n} \text{ is } \mathcal{F}_{\rho_n} \otimes \mathcal{F}_\infty - \text{ measurable} \tag{3.11}$$

$$\vartheta \mapsto \tau^{\rho_n}(\omega, \vartheta) \text{ is a stopping time} \tag{3.12}$$

$$\tau(\omega) = \rho_n + \tau^{\rho_n}(\omega, \theta_{\rho_n}(\omega)) \tag{3.13}$$

for all $\omega \in \Omega$. Note that the latter assertion can be derived from Galmarino's test. In particular if τ is the first entry time of X into a set, then $\tau = \sigma + \tau \circ \theta_{\rho_n}$ and τ^{ρ_n} can be identified with τ in the sense that $\tau_{\rho_n}(\omega, \vartheta) = \tau(\vartheta)$ for all ω and ϑ. Taking expectations on both sides in (3.10) we get

$$\mathsf{E}_x\big[V^1_{\sigma_D}(X_{\rho_n})\big] = \mathsf{E}_x\big[\operatorname{ess\,sup}_{\tau \geq \rho_n} \mathsf{M}^1_x(\tau, \sigma_{\rho_n}|\mathcal{F}_{\rho_n})\big]. \tag{3.14}$$

$2°$ We next show that the family $\{\mathsf{M}^1_x(\tau, \sigma_{\rho_n}|\mathcal{F}_{\rho_n}) : \tau \geq \rho_n\}$ is upwards directed. For this we show that for any two stopping times $\tau_1, \tau_2 \geq \rho_n$ there exists $\tau_3 \geq \rho_n$ such that $\mathsf{M}^1_x(\tau_3, \sigma_{\rho_n}|\mathcal{F}_{\rho_n}) \geq \mathsf{M}^1_x(\tau_1, \sigma_{\rho_n}|\mathcal{F}_{\rho_n}) \vee \mathsf{M}^1_x(\tau_2, \sigma_{\rho_n}|\mathcal{F}_{\rho_n})$. So let $\tau_1, \tau_2 \geq \rho_n$ be any two stopping times given and fixed and define the set $A = \{\omega : \mathsf{M}^1_x(\tau_1, \sigma_{\rho_n}|\mathcal{F}_{\rho_n})(\omega) \geq \mathsf{M}^1_x(\tau_2, \sigma_{\rho_n}|\mathcal{F}_{\rho_n})(\omega)\}$. Now $A \in \mathcal{F}_{\rho_n}$ because $\mathsf{M}^1_x(\tau_i, \sigma_{\rho_n}|\mathcal{F}_{\rho_n})$ for $i = 1, 2$ are $\mathcal{F}_{\rho_n}$-measurable. Let $\tau_3 = \tau_1 I_A + \tau_2 I_{A^c}$. Since $\tau_1, \tau_2 \geq \rho_n$ it follows that $\tau_3 \geq \rho_n$. Also, $\{\tau_3 \leq t\} = \{\{\tau_1 \leq t\} \cap A\} \cup \{\{\tau_2 \leq t\} \cap A^c\} = \{\{\tau_1 \leq t\} \cap A \cap \{\rho_n \leq t\}\} \cup \{\{\tau_2 \leq t\} \cap A^c \cap \{\rho_n \leq t\}\} \in \mathcal{F}_t$. This follows from the fact that the sets A and A^c belong to $\mathcal{F}_{\rho_n}$ and that $\{\tau_i \leq t\} \subseteq \{\rho_n \leq t\}$ for $i = 1, 2$. So τ_3 is a stopping time and hence,

$$\begin{aligned}
\mathsf{M}^1_x\big(\tau_3, \sigma_{\rho_n}|\mathcal{F}_{\rho_n}\big) &= \mathsf{M}^1_x\big(\tau_1 I_A + \tau_2 I_{A^c}, \sigma_{\rho_n}|\mathcal{F}_{\rho_n}\big) \\
&= \mathsf{M}^1_x\big(\tau_1, \sigma_{\rho_n}|\mathcal{F}_{\rho_n}\big) I_A + \mathsf{M}^1_x\big(\tau_2, \sigma_{\rho_n}|\mathcal{F}_{\rho_n}\big) I_{A^c} \\
&= \mathsf{M}^1_x\big(\tau_1, \sigma_{\rho_n}|\mathcal{F}_{\rho_n}\big) \vee \mathsf{M}^1_x\big(\tau_2, \sigma_{\rho_n}|\mathcal{F}_{\rho_n}\big)
\end{aligned} \tag{3.15}$$

$3°$ We next prove that if ∂D_2 is a regular boundary for D_2 then

$$V^1_{\sigma_{D_2}}(x) \leq \liminf_{n \to \infty} \mathsf{E}_x\big[V^1_{\sigma_{D_2}}(X_{\rho_n})\big]. \tag{3.16}$$

For this we first show that

$$\mathsf{M}^1_x(\tau, \sigma_{\rho_n}) - \mathsf{M}^1_x(\tau \vee \rho_n, \sigma_{\rho_n}) = \mathsf{E}_x[G_1(X_{\tau \wedge \rho_n}) - G_1(X_{\rho_n})] \tag{3.17}$$

Since $\sigma_{\rho_n} \geq \rho_n$ we have that

$$
\begin{aligned}
\mathsf{M}_x^1(\tau, \sigma_{\rho_n}) &= \mathsf{E}_x\big[(G_1(X_\tau)I(\tau \leq \sigma_{\rho_n}) + H_1(X_{\sigma_{\rho_n}})I(\sigma_{\rho_n} < \tau))I(\tau < \rho_n)\big] \\
&\quad + \mathsf{E}_x\big[(G_1(X_{\tau \vee \rho_n})I(\tau \vee \rho_n \leq \sigma_{\rho_n}) \\
&\quad + H_1(X_{\sigma_{\rho_n}})I(\sigma_{\rho_n} < \tau \vee \rho_n))I(\tau \geq \rho_n)\big] \\
&= \mathsf{E}_x\big[G_1(X_\tau)I(\tau < \rho_n)\big] + \mathsf{M}_x^1(\tau \vee \rho_n, \sigma_{\rho_n}) \\
&\quad - \mathsf{E}_x\big[(G_1(X_{\rho_n})I(\rho_n \leq \sigma_{\rho_n}) + H_1(X_{\sigma_{\rho_n}})I(\sigma_{\rho_n} < \rho_n)))I(\tau < \rho_n)\big] \\
&= \mathsf{E}_x\big[(G_1(X_\tau) - G_1(X_{\rho_n}))I(\tau < \rho_n)\big] + \mathsf{M}_x^1(\tau \vee \rho_n, \sigma_{\rho_n}) \\
&= \mathsf{E}_x\big[G_1(X_{\tau \wedge \rho_n}) - G_1(X_{\rho_n})\big] + \mathsf{M}_x^1(\tau \vee \rho_n, \sigma_{\rho_n}) \quad (3.18)
\end{aligned}
$$

from which (3.17) follows. By considering separately the sets $\{\sigma_{D_2} < \rho_n\}, \{\sigma_{D_2} \geq \rho_n\}$ (note that $\sigma_{\rho_n} = \sigma_{D_2}$ on the set $\{\sigma_{D_2} \geq \rho_n\}$), $\{\sigma_{D_2} > 0\}, \{\sigma_{D_2} = 0\}, \{\tau = 0\}$ and $\{\tau > 0\}$, and by using Lemma 3.4 we get

$$
\begin{aligned}
&\mathsf{M}_x^1(\tau, \sigma_{D_2}) - \mathsf{M}_x^1(\tau, \sigma_{\rho_n}) \\
&= \mathsf{E}_x[-G_1(X_\tau)I(\tau \leq \sigma_{\rho_n})I(0 < \rho_n)I(\sigma_{D_2} = 0)I(\tau > 0) \\
&\quad + (H_1(X_0) - H_1(X_{\sigma_{\rho_n}})I(\sigma_{\rho_n} < \tau))I(0 < \rho_n)I(\sigma_{D_2} = 0)I(\tau > 0)] \\
&= \mathsf{E}_x[(H_1(X_0) - H_1(X_{\sigma_{\rho_n}}))I(0 < \rho_n)I(\sigma_{D_2} = 0)I(\tau > 0)] \quad (3.19)
\end{aligned}
$$

for n sufficiently large. The last equality in (3.19) can be seen as follows: If $x \in \mathrm{int}\,D_2$ the interior of D_2 then by the right-continuity property of the sample paths it follows that $\sigma_{\rho_n} = 0$ P_x-a.s. for n sufficiently large. If on the other hand $x \in \partial D_2$ then by Lemma 3.4, we have that $\sigma_{\rho_n} \downarrow 0$ P_x-a.s. Note that in the case $x \notin D_2 \cup \partial D_2$ then $\sigma_{D_2} > 0$ P_x-a.s. and so the terms in the right-hand side of (3.19) vanish. Combining (3.17) and (3.19) we get

$$
\begin{aligned}
&\mathsf{M}_x^1(\tau, \sigma_{D_2}) - \mathsf{M}_x^1(\tau \vee \rho_n, \sigma_{\rho_n}) \\
&= \mathsf{M}_x^1(\tau, \sigma_{D_2}) - \mathsf{M}_x^1(\tau, \sigma_{\rho_n}) + \mathsf{M}_x^1(\tau, \sigma_{\rho_n}) \\
&\quad - \mathsf{M}_x^1(\tau \vee \rho_n, \sigma_{\rho_n}) \leq \mathsf{E}_x[\sup_{t \leq \rho_n} |G_1(X_{t \wedge \rho_n}) - G_1(X_{\rho_n})|] \\
&\quad + \mathsf{E}_x[|H_1(X_0) - H_1(X_{\sigma_{\rho_n}})|I(0 < \rho_n)I(\sigma_{D_2} = 0)I(\tau > 0)]. \quad (3.20)
\end{aligned}
$$

for n sufficiently large. So

$$
\begin{aligned}
V_{\sigma_{D_2}}^1(x) &\\
&\leq \liminf_{n \to \infty} \sup_\tau \mathsf{M}_x^1(\tau \vee \rho_n, \sigma_{\rho_n}) \\
&= \liminf_{n \to \infty} \sup_{\tau \geq \rho_n} \mathsf{M}_x^1(\tau, \sigma_{\rho_n}) = \liminf_{n \to \infty} \mathsf{E}_x[V_{\sigma_{\rho_n}}^1(X_{\rho_n})]. \quad (3.21)
\end{aligned}
$$

The first inequality follows from (3.20). Indeed, since G_1 and H_1 are continuous, the composed processes $G_1(X)$ and $H_1(X)$ are right-continuous and so both terms

on the right hand side of (3.20) tend to zero as $n \to \infty$ (note that if $x \in \partial D_2$ this follows from Lemma 3.4 whereas if $x \in \text{int} D_2$ or $x \notin D \cup \partial D$ the result follows as explained in the text before (3.20)). The last equality in (3.21) follows from the fact that the family $\{M_x(\tau, \sigma_{\rho_n} | \mathcal{F}_{\rho_n}) : \tau \geq \rho_n\}$ is upwards directed (see step 2°), and so we can interchange the expectation and the essential supremum in (3.14).

4° We show that $V^1_{\sigma_{D_2}}(x) \geq \limsup_{n \to \infty} E_x[V^1_{\sigma_{\rho_n}}(X_{\rho_n})]$. From (3.17) and (3.19) we get

$$M^1_x(\tau, \sigma_{D_2}) \geq M^1_x(\tau \vee \rho_n, \sigma_{\rho_n}) - E_x[\sup_{t \leq \rho_n} |G_1(X_t) - G_1(X_{\rho_n})|] \quad (3.22)$$
$$- E_x[|H_1(X_0) - H_1(X_{\sigma_{\rho_n}})|]$$

for any stopping time τ. From this, together with Lebesgue dominated convergence theorem (upon recalling assumption (2.1)) and the continuity property of G_1 and H_1 we conclude that

$$V^1_{\sigma_{D_2}}(x) \geq \limsup_{n \to \infty} \sup_{\tau \geq \rho_n} M^1_x(\tau \vee \rho_n, \sigma_{\rho_n}) = \limsup_{n \to \infty} E_x[V^1_{\sigma_{\rho_n}}(X_{\rho_n})].$$

$\square$

We next present an example to show that if ∂D is not regular for D then $V^1_{\sigma_D}$ may not be finely continuous.

Example 3.8 Let $E = \mathbb{R}$ and $\mathcal{B}$ the Borel σ-algebra on $\mathbb{R}$. Suppose that X is the deterministic motion to the right, that is the process starts at $x \in \mathbb{R}$ and $X_t = x + t$ P_x-a.s. for each $t \geq 0$. In this case the fine topology coincides with the right-topology (cf. [53]), so a function is finely continuous if it is right-continuous. Define the functions $G_1(x) = \frac{e^{x-1}}{2} I(x < 1) + \frac{1}{2x^2} I(x \geq 1)$ and $H_1(x) = 1 I(x < 1) + \frac{1}{x^2} I(x \geq 1)$. Let $D = \{1\}$ and $\sigma_D = \inf\{t \geq 0 : X_t \in D\}$. Then ∂D is not regular for D because if $\rho_D = \inf\{t > 0 : X_t \in D\}$ then $\rho_D = \infty$ P_1-a.s. We show that for any given $\varepsilon > 0$ the stopping time $\tau^\varepsilon = (\tau_{[1,\infty)} + \varepsilon) 1_A + \tau_{[1,\infty)} 1_{\bar{A}}$, where $\tau_{[1,\infty)}$ is the first entry time of X in $[1, \infty)$ and $A = \{\tau_{[1,\infty)} = \sigma_D\}$, is optimal for player one given the strategy σ_D chosen by player two. For each $\varepsilon > 0$ we have that $\tau^\varepsilon = \tau_{[1,\infty)} I(x > 1) + (\tau_{[1,\infty)} + \varepsilon) I(x \leq 1) = 0 I(x > 1) + (1 - x + \varepsilon) I(x \leq 1)$. So $M^1_x(\tau^\varepsilon, \sigma_D) = H_1(1) I(x \leq 1) + G_1(x) I(x > 1)$. On the other hand, for any stopping time τ one can see that

$$M^1_x(\tau, \sigma_D) \leq E_x[G_1(X_\tau) I(\tau < \sigma_D) + H_1(X_{\sigma_D}) I(\sigma_D \leq \tau)]$$
$$< (H_1(1) I(\tau < 1 - x) + H_1(1) I(1 - x \leq \tau)) I(x \leq 1)$$
$$+ G_1(x + \tau) I(x > 1) \leq M^1_x(\tau^\varepsilon, \sigma_D)$$

where the last inequality follows from the fact that G_1 is decreasing in $[1, \infty)$. Taking the supremum over all τ we get that $V^1_{\sigma_D}(x) \leq M^1_x(\tau^\varepsilon, \sigma_D)$ for all x. Since on the other hand $V^1_{\sigma_D}(x) \geq M^1_x(\tau^\varepsilon, \sigma_D)$, it follows that $V^1_{\sigma_D}(x) = M^1_x(\tau^\varepsilon, \sigma_D)$ for all x and so we must have that τ^ε is optimal for player one, provided that player two selects strategy

σ_D. The value function is thus given by $V^1_{\sigma_D}(x) = H_1(x)I(x \leq 1) + G_1(x)I(x > 1)$ which is not right-continuous.

4 Towards a Nash equilibrium

The main result of this section is to show that if σ_{D_2} (resp. τ_{D_1}) is externally given as the first entry time in D_2 (resp. D_1), a set that is either closed or finely closed, and has a regular boundary, then the first entry time $\tau^{\sigma_{D_2}}_* = \inf\{t \geq 0 : X_t \in D_1^{\sigma_{D_2}}\}$ (resp. $\sigma^{\tau_{D_1}}_* = \inf\{t \geq 0 : X_t \in D_1^{\tau_{D_1}}\}$) where $D_1^{\sigma_{D_2}} = \{V^1_{\sigma_{D_2}} = G_1\}$ (resp. $D_2^{\tau_{D_1}} = \{V^2_{\tau_{D_1}} = G_2\}$) solves the optimal stopping problem $V^1_{\sigma_{D_2}}(x) = \sup_\tau \mathsf{M}^1_x(\tau, \sigma_{D_2})$ (resp. $V^2_{\tau_{D_1}}(x) = \sup_\sigma \mathsf{M}^2_x(\tau_{D_1}, \sigma))$. The proof of this result will be divided into several lemmas and propositions.

Proposition 4.1 *Let D_1, D_2 be Borel subsets of E having regularies boundaries ∂D_1 and ∂D_2 respectively. Set $\tau_{D_1} = \inf\{t \geq 0 : X_t \in D_1\}$ and $\sigma_{D_2} = \inf\{t \geq 0 : X_t \in D_2\}$. Then,*

$$V^1_{\sigma_{D_2}}(x) \leq \mathsf{M}^1_x\big(\tau^{\sigma_{D_2}}_\varepsilon, \sigma_{D_2}\big) + \varepsilon \tag{4.1}$$

$$V^2_{\tau_{D_1}}(x) \leq \mathsf{M}^2_x\big(\tau_{D_1}, \sigma^{\tau_{D_1}}_\varepsilon\big) + \varepsilon \tag{4.2}$$

for any $\varepsilon > 0$, where $\tau^{\sigma_{D_2}}_\varepsilon = \inf\{t \geq 0 : X_t \in D_1^{\sigma_{D_2},\varepsilon}\}$ and $\sigma^{\tau_{D_1}}_\varepsilon = \inf\{t \geq 0 : X_t \in D_2^{\tau_{D_1},\varepsilon}\}$ with $D_1^{\sigma_{D_2},\varepsilon} = \{V^1_{\sigma_{D_2}} \leq G_1 + \varepsilon\}$ and $D_2^{\tau_{D_1},\varepsilon} = \{V^2_{\tau_{D_1}} \leq G_2 + \varepsilon\}$.

Proof We shall only prove (4.1) as for (4.2) the result follows by symmetry. The proof will be carried out in several steps.

1° Consider the optimal stopping problem

$$\tilde{V}^1_{\sigma_{D_2}}(x) = \sup_\tau \tilde{\mathsf{M}}^1_x(\tau, \sigma_{D_2}) \tag{4.3}$$

where

$$\tilde{\mathsf{M}}^1_x(\tau, \sigma_{D_2}) = \mathsf{E}_x\big[G_1(X_\tau)I(\tau < \sigma_{D_2}) + H_1(X_{\sigma_{D_2}})I(\sigma_{D_2} \leq \tau)\big]. \tag{4.4}$$

Recall that the mapping $x \mapsto \tilde{V}^1_{\sigma_{D_2}}(x)$ is measurable (cf. [16, p. 5]) and so $\tilde{V}^1_{\sigma_{D_2}}(X_\rho) = \sup_\tau \tilde{\mathsf{M}}^1_{X_\rho}(\tau, \sigma_{D_2})$ is a random variable for any stopping time ρ. By the strong Markov property of X it follows that for any stopping time ρ given and fixed

$$\begin{aligned}
\tilde{\mathsf{M}}^1_{X_\rho}(\tau, \sigma_{D_2}) &= \mathsf{E}_{X_\rho}\big[G_1(X_\tau)I(\tau < \sigma_{D_2}) + H_1(X_{\sigma_{D_2}})I(\sigma_{D_2} \leq \tau)\big] \\
&= \mathsf{E}_x\big[G_1(X_{\rho+\tau\circ\theta_\rho})I(\rho + \tau\circ\theta_\rho < \rho + \sigma_{D_2}\circ\theta_\rho) \\
&\quad + H_1(X_{\rho+\sigma_{D_2}\circ\theta_\rho})I(\rho + \sigma_{D_2}\circ\theta_\rho \leq \rho + \tau\circ\theta_\rho)|\mathcal{F}_\rho\big]
\end{aligned} \tag{4.5}$$

and so we have that

$$\tilde{V}^1_{\sigma_{D_2}}(X_\rho) = \operatorname*{ess\,sup}_{\tau} \mathsf{E}_x\big[G_1(X_{\rho+\tau\circ\theta_\rho})I(\rho+\tau\circ\theta_\rho < \rho+\sigma_{D_2}\circ\theta_\rho)$$
$$+ H_1(X_{\rho+\sigma_{D_2}\circ\theta_\rho})I(\rho+\sigma_{D_2}\circ\theta_\rho \le \rho+\tau\circ\theta_\rho)|\mathcal{F}_\rho\big]$$
$$=: \operatorname*{ess\,sup}_{\tau} \tilde{\mathsf{M}}^1_x(\tau_\rho, \sigma^\rho_{D_2}|\mathcal{F}_\rho) \tag{4.6}$$

where $\sigma^\rho_{D_2} = \inf\{t \ge \rho : X_t \in D_2\}$ and $\tau_\rho = \rho + \tau\circ\theta_\rho$. The gain process $\tilde{G}^{\sigma_{D_2},1}_t = G_1(X_t)I(t < \sigma_{D_2}) + H_1(X_{\sigma_{D_2}})I(\sigma_{D_2} \le t)$ is right-continuous, adapted and satisfies the integrability condition

$$\mathsf{E}_x[\sup_{t\ge0}|\tilde{G}^{\sigma_{D_2},1}_t|] = \mathsf{E}_x[\sup_{t\ge0}|G_1(X_t)I(t < \sigma_{D_2}) + H_1(X_{\sigma_{D_2}})I(\sigma_{D_2} \le t)|]$$
$$\le \mathsf{E}_x[\sup_{t\ge0}|G_1(X_t)I(t < \sigma_{D_2})| + \sup_{t\ge0}|H_1(X_{\sigma_{D_2}})I(\sigma_{D_2} \le t)|] < \infty$$

where the last inequality follows from assumption (2.1). So the martingale approach in the theory of optimal stopping (cf. [49, Theorem 2.2]) can be applied in this setting to deduce that there exists a right-continuous modification of the supermartingale

$$\tilde{S}^{\sigma_{D_2}}_t = \operatorname*{ess\,sup}_{\tau\ge t} \tilde{\mathsf{M}}^1_x(\tau, \sigma_{D_2}|\mathcal{F}_t), \tag{4.7}$$

known as the Snell envelope (for simplicity of exposition we shall still denote the right-continuous modification by $\tilde{S}^{\sigma_{D_2}}_t$), such that the stopping time $\hat{\tau}_t := \hat{\tau}^{\sigma_{D_2}}_t = \inf\{s \ge t : \tilde{S}^{\sigma_{D_2}}_s = \tilde{G}^{\sigma_{D_2},1}_s\}$ is optimal. It is known (cf. [49, Theorem 2 p. 29]) that the stopped process $(\tilde{S}^{\sigma_{D_2}}_{s\wedge\hat{\tau}_t})_{s\ge t}$ is a right-continuous martingale and so

$$\mathsf{E}_x[\tilde{S}^{\sigma_{D_2}}_\rho] = \mathsf{E}_x[\tilde{S}^{\sigma_{D_2}}_{\rho\wedge\hat{\tau}_\varepsilon}] = \mathsf{E}_x[\tilde{S}^{\sigma_{D_2}}_{\rho\wedge\hat{\tau}_\varepsilon\wedge\hat{\tau}_0}] = \tilde{S}^{\sigma_{D_2}}_0 = \tilde{V}^1_{\sigma_{D_2}}(x) \tag{4.8}$$

for every stopping time $\rho \le \hat{\tau}_\varepsilon$ where $\hat{\tau}_\varepsilon := \tau^{\sigma_{D_2}}_\varepsilon = \inf\{t \ge 0 : \tilde{S}^{\sigma_{D_2}}_t \le \tilde{G}^{\sigma_{D_2},1}_t + \varepsilon\}$. Using the fact that $\sigma^\rho_{D_2} = \sigma_{D_2}$ for any stopping time $\rho \le \sigma_{D_2}$, that P_x-a.s., the essential supremum and its right-continuous modification are equivalent at stopping times and that the essential supremum is attained at hitting times (cf. [49]) it follows, from (4.6), that

$$\tilde{V}^1_{\sigma_{D_2}}(X_\rho) = S^{\sigma_{D_2}}_\rho \ \mathsf{P}_x\text{-a.s.} \tag{4.9}$$

for every stopping time $\rho \le \sigma_{D_2}$.

2°. We next show that $V^1_{\sigma_{D_2}}(x) = \tilde{V}^1_{\sigma_{D_2}}(x)$. Since $G_1 \le H_1$ we have that

$$\tilde{V}^1_{\sigma_{D_2}}(x) \ge \tilde{\mathsf{M}}^1_x(\tau, \sigma_{D_2}) \ge \mathsf{M}^1_x(\tau, \sigma_{D_2}) \tag{4.10}$$

for all stopping times τ and for all $x \in E$. Taking the supremum over all τ we get that

$$\tilde{V}^1_{\sigma_{D_2}}(x) \ge V^1_{\sigma_{D_2}}(x). \tag{4.11}$$

To prove the reverse inequality we will show that $\tilde{M}_x^1(\tau, \sigma_{D_2}) \leq V_{\sigma_{D_2}}^1(x)$ for all stopping times τ so that $\sup_\tau \tilde{M}_x^1(\tau, \sigma_{D_2}) = \tilde{V}_{\sigma_{D_2}}^1(x) \leq V_{\sigma_{D_2}}^1(x)$. By definition of $V_{\sigma_{D_2}}^1$ we have that $V_{\sigma_{D_2}}^1(x) \geq M_x^1(\tau, \sigma_{D_2})$ for all stopping times τ. Now take any stopping time τ and set $\tau^\varepsilon = (\tau + \varepsilon)1_A + \tau 1_{A^c}$ where $A = \{\tau = \sigma_{D_2}\}$. (Note that τ^ε is a stopping time since $A \in \mathcal{F}_{\tau \wedge \sigma_{D_2}} \subset \mathcal{F}_\tau$). If the time horizon T is finite, then we shall replace $\tau + \varepsilon$ in the definition of τ^ε with $(\tau + \varepsilon) \wedge T$. Then we have that

$$
\begin{aligned}
& M_x^1(\tau^\varepsilon, \sigma_{D_2}) \\
&= E_x\big[(G_1(X_{\tau^\varepsilon})I(\tau^\varepsilon \leq \sigma_{D_2}) + H_1(X_{\sigma_{D_2}})I(\sigma_{D_2} < \tau^\varepsilon))I(\tau = T)I(\sigma_{D_2} = T) \\
&\quad + (G_1(X_{\tau^\varepsilon})I(\tau^\varepsilon \leq \sigma_{D_2}) + H_1(X_{\sigma_{D_2}})I(\sigma_{D_2} < \tau^\varepsilon))I(\tau < T)I(\sigma_{D_2} = T) \\
&\quad + (G_1(X_{\tau^\varepsilon})I(\tau^\varepsilon \leq \sigma_{D_2}) + H_1(X_{\sigma_{D_2}})I(\sigma_{D_2} < \tau^\varepsilon))I(\tau = T)I(\sigma_{D_2} < T) \\
&\quad + (G_1(X_{\tau^\varepsilon})I(\tau^\varepsilon \leq \sigma_{D_2}) + H_1(X_{\sigma_{D_2}})I(\sigma_{D_2} < \tau^\varepsilon))I(\tau < T)I(\sigma_{D_2} < T)] \\
&= E_x[(G_1(X_\tau)I(\tau < \sigma_{D_2}) + H_1(X_{\sigma_{D_2}})I(\sigma_{D_2} \leq \tau))I(\tau = T)I(\sigma_{D_2} = T) \\
&\quad + (G_1(X_\tau)I(\tau < \sigma_{D_2}) + H_1(X_{\sigma_{D_2}})I(\sigma_{D_2} \leq \tau))I(\tau < T)I(\sigma_{D_2} = T) \\
&\quad + (G_1(X_\tau)I(\tau < \sigma_{D_2}) + H_1(X_{\sigma_{D_2}})I(\sigma_{D_2} \leq \tau))I(\tau = T)I(\sigma_{D_2} < T) \\
&\quad + (G_1(X_\tau)I(\tau < \sigma_{D_2}) + H_1(X_{\sigma_{D_2}})(1_A + I(\sigma_{D_2} < \tau))I(\tau < T)I(\sigma_{D_2} < T)\big] \\
&= \tilde{M}_x^1(\tau, \sigma_{D_2})
\end{aligned}
$$

The first and third expressions in the second equality follow from assumption (2.6). The second expression also follows from assumption (2.6) together with the fact that $I(\tau^\varepsilon < \sigma_{D_2}) = I(\tau < \sigma_{D_2})$ on the set $\{\tau < T\}$. The last expression follows from the fact that $I(\tau^\varepsilon = \sigma_{D_2}) = 0$ and $I(\sigma_{D_2} < \tau^\varepsilon) = 1_A + I(\sigma_{D_2} < \tau)$ on the set $\{\tau < T\} \cap \{\sigma_{D_2} < T\}$. So for any given stopping time τ we have $V_{\sigma_{D_2}}^1(x) \geq M_x^1(\tau^\varepsilon, \sigma_{D_2}) = \tilde{M}_x^1(\tau, \sigma_{D_2})$.

$3°$. We show that $V_{\sigma_{D_2}}^1(x) = E_x[V_{\sigma_{D_2}}^1(X_{\sigma_{D_2} \wedge \tau_\varepsilon})]$ where $\tau_\varepsilon := \tau_\varepsilon^{\sigma_{D_2}} = \inf\{t \geq 0 : X_t \in D_1^{\sigma_{D_2}, \varepsilon}\}$ with $D_1^{\sigma_{D_2}, \varepsilon} = \{V_{\sigma_{D_2}}^1 \leq G_1 + \varepsilon\}$. From step $2°$ it is sufficient to prove that $\tilde{V}_{\sigma_{D_2}}^1(x) = E_x[\tilde{V}_{\sigma_{D_2}}^1(X_{\sigma_{D_2} \wedge \tilde{\tau}_\varepsilon})]$ where $\tilde{\tau}_\varepsilon := \tilde{\tau}_\varepsilon^{\sigma_{D_2}} = \inf\{t \geq 0 : X_t \in \tilde{D}_1^{\sigma_{D_2}, \varepsilon}\}$ with $\tilde{D}_1^{\sigma_{D_2}, \varepsilon} = \{\tilde{V}_{\sigma_{D_2}}^1 \leq G_1 + \varepsilon\}$. By definition of $\tilde{\tau}_\varepsilon$, for any given $t < \sigma_{D_2} \wedge \tilde{\tau}_\varepsilon$, we have

$$
\begin{aligned}
\tilde{V}_{\sigma_{D_2}}^1(X_t) &> G_1(X_t) + \varepsilon \\
&= G_1(X_t)I(t < \sigma_{D_2}) + H_1(X_{\sigma_{D_2}})I(\sigma_{D_2} \leq t) + \varepsilon \\
&= \tilde{G}_t^{\sigma_{D_2}, 1} + \varepsilon
\end{aligned}
\tag{4.12}
$$

where the first equality follows from the fact that $I(\sigma_{D_2} \leq t) = 0$. Since $\sigma_{D_2} \wedge \tilde{\tau}_\varepsilon \leq \sigma_{D_2}$, by (4.9) it follows that

$$
\tilde{V}_{\sigma_{D_2}}^1(X_{\sigma_{D_2} \wedge \tilde{\tau}_\varepsilon}) = \tilde{S}_{\sigma_{D_2} \wedge \tilde{\tau}_\varepsilon}^{\sigma_{D_2}} \quad P_x\text{-a.s.}
\tag{4.13}
$$

Using (4.9) again we get

$$\tilde{V}^1_{\sigma_{D_2}}(X_t) = \tilde{S}^{\sigma_{D_2}}_t \ \mathsf{P}_x\text{-a.s.} \tag{4.14}$$

So by (4.12) we can conclude that

$$\tilde{S}^{\sigma_{D_2}}_t > \tilde{G}^{\sigma_{D_2},1}_t + \varepsilon \ \mathsf{P}_x\text{-a.s.} \tag{4.15}$$

Recall, from step $2°$, that $\hat{\tau}_\varepsilon = \inf\{t \geq 0 : \tilde{S}^{\sigma_{D_2}}_t \leq \tilde{G}^{\sigma_{D_2},1}_t\}$. By using the definition of $\hat{\tau}_\varepsilon$ together with (4.15) one can see that $\sigma_{D_2} \wedge \tilde{\tau}_\varepsilon \leq \hat{\tau}_\varepsilon$. By using (4.8), (4.9) and that $(\tilde{S}^{\sigma_{D_2}}_{s \wedge \hat{\tau}_0})_{s \geq 0}$ is a martingale we further get that

$$\tilde{S}^{\sigma_{D_2}}_0 = \tilde{V}^1_{\sigma_{D_2}}(x) = \mathsf{E}_x\big[\tilde{S}^{\sigma_{D_2}}_{\sigma_{D_2} \wedge \tilde{\tau}_\varepsilon \wedge \hat{\tau}_\varepsilon}\big] = \mathsf{E}_x\big[\tilde{S}^{\sigma_{D_2}}_{\sigma_{D_2} \wedge \tilde{\tau}_\varepsilon}\big] = \mathsf{E}_x\big[\tilde{V}^1_{\sigma_{D_2}}(X_{\sigma_{D_2} \wedge \tilde{\tau}_\varepsilon})\big]. \tag{4.16}$$

From (4.16) together with the fine continuity property of $V^1_{\sigma_{D_2}}$ (upon using Lemma 3.5), the right-continuity property of the composite process $\tilde{G}(X)$ and the fact that $\tilde{V}^1_{\sigma_{D_2}} = V^1_{\sigma_{D_2}}$, we have

$$\begin{aligned}
V^1_{\sigma_{D_2}}(x) &= \mathsf{E}_x\big[\tilde{S}^{\sigma_{D_2}}_{\sigma_{D_2} \wedge \tau_\varepsilon}\big] = \mathsf{E}_x\big[\tilde{S}^{\sigma_{D_2}}_{\tau_\varepsilon} I(\tau_\varepsilon \leq \sigma_{D_2}) + \tilde{S}^{\sigma_{D_2}}_{\sigma_{D_2}} I(\sigma_{D_2} < \tau_\varepsilon)\big] \\
&= \mathsf{E}_x\big[\tilde{V}^1_{\sigma_{D_2}}(X_{\tau_\varepsilon}) I(\tau_\varepsilon \leq \sigma_{D_2}) + H_1(X_{\sigma_{D_2}}) I(\sigma_{D_2} < \tau_\varepsilon)\big] \\
&\leq \mathsf{E}_x\big[(G_1(X_{\tau_\varepsilon}) + \varepsilon) I(\tau_\varepsilon \leq \sigma_{D_2}) + H_1(X_{\sigma_{D_2}}) I(\sigma_{D_2} < \tau_\varepsilon)\big] \\
&\leq \mathsf{M}^1_x(\tau_\varepsilon, \sigma_{D_2}) + \varepsilon \tag{4.17}
\end{aligned}$$

for any $\varepsilon > 0$, where the third equality follows from the fact that

$$\begin{aligned}
\tilde{S}^{\sigma_{D_2}}_{\sigma_{D_2}} &= \operatorname*{ess\,sup}_{\tau \geq \sigma_{D_2}} \mathsf{E}_x[G_1(X_\tau) I(\tau < \sigma_{D_2}) + H_1(X_{\sigma_{D_2}}) I(\sigma_{D_2} \leq \tau) | \mathcal{F}_{\sigma_{D_2}}] \\
&= \operatorname*{ess\,sup}_{\tau \geq \sigma_{D_2}} \mathsf{E}_x[H_1(X_{\sigma_{D_2}}) | \mathcal{F}_{\sigma_{D_2}}] = H_1(X_{\sigma_{D_2}}) \tag{4.18}
\end{aligned}$$

$\square$

Lemma 4.2 *Let* $\{\rho_n\}_{n=1}^\infty$ *be a sequence of stopping times such that* $\rho_n \uparrow \rho \ \mathsf{P}_x$-*a.s. For a given Borel set* $D \subseteq E$ *define the entry times* $\sigma_{\rho_n} = \inf\{t \geq \rho_n : X_t \in D\}$ *and* $\sigma_\rho = \inf\{t \geq \rho : X_t \in D\}$. *If either*

$$D \text{ is closed, or} \tag{4.19}$$

$$D \text{ is finely closed with regular boundary } \partial D, \tag{4.20}$$

then $\sigma_{\rho_n} \uparrow \sigma_\rho \ \mathsf{P}_x$-*a.s.*

Proof Since ρ_n is an increasing sequence of stopping times, σ_{ρ_n} is increasing and thus $\sigma_{\rho_n} \uparrow \beta \ \mathsf{P}_x$-a.s. for some stopping time β. We need to prove that $\beta = \sigma_\rho \ \mathsf{P}_x - a.s.$

1° We first show that $\beta \geq \rho$ P_x-a.s. Suppose, for contradiction, that the set $\hat{\Omega} := \{\omega \in \Omega : \beta(\omega) < \rho(\omega)\}$ is of positive measure. Since $\rho_n \uparrow \rho$ P_x-a.s. then for every $\omega \in \hat{\Omega} \backslash N_1$, where $P_x(N_1) = 0$-a.s., there exists $n_0(\omega) \in \mathbb{N}$ such that

$$\rho_n(\omega) > \beta(\omega) \tag{4.21}$$

for all $n \geq n_0(\omega)$. But for each $n \in \mathbb{N}$ we have that $\sigma_{\rho_n} \leq \beta$ P_x-a.s., so for every $\omega \in \hat{\Omega} \backslash N_2$ where $P_x(N_2) = 0$ there exists $n_1(\omega) \in \mathbb{N}$ such that

$$\sigma_{\rho_n}(\omega) < \rho_n(\omega) \tag{4.22}$$

for all $n \geq n_1(\omega)$. Combining (4.21) and (4.22) it follows that for every $\omega \in \hat{\Omega} \backslash (N_1 \cup N_2)$ there exists $\hat{n}(\omega) \in \mathbb{N}$ such that $\sigma_{\rho_n}(\omega) < \rho_n(\omega)$ for all $n \geq \hat{n}(\omega)$. But this contradicts the fact that $\sigma_{\rho_n} \geq \rho_n$ P_x-a.s. So we must have that $\beta \geq \rho$ P_x-a.s.

2° Let $\Omega_1 = \{\omega \in \Omega : \beta(\omega) > \rho(\omega)\}$ and $\Omega_2 = \{\omega \in \Omega : \beta(\omega) = \rho(\omega)\}$. We prove that there exists a set N with $P_x(N) = 0$ such that $\beta(\omega) = \sigma_\rho(\omega)$ for every $\omega \in (\Omega_1 \cup \Omega_2) \backslash N$.

(i.) Suppose first that $P_x(\Omega_1) > 0$. Since $\sigma_{\rho_n} \uparrow \beta$ P_x-a.s. then for every $\omega \in \Omega_1 \backslash N_3$, where $P_x(N_3) = 0$, there exists $n_2(\omega) \in \mathbb{N}$ such that $\sigma_{\rho_n}(\omega) > \rho(\omega)$ for all $n \geq n_2(\omega)$. Moreover, since $\sigma_\rho(\omega) \geq \rho(\omega)$ for every $\omega \in \Omega_1 \backslash N_4$ where $P_x(N_4) = 0$ it follows that for every $\omega \in \Omega_1 \backslash N$, where $N = N_3 \cup N_4$, there exists $n_3(\omega) \in \mathbb{N}$ such that $\sigma_{\rho_n}(\omega) = \sigma_\rho(\omega)$ for all $n \geq n_3(\omega)$. From this it follows that $\beta = \sigma_\rho$ P_x-a.s. on Ω_1.

(ii.) Now suppose that $P_x(\Omega_2) > 0$. Let us consider first the case when (4.19) holds. So we have that $\sigma_{\rho_n}(\omega) \uparrow \rho(\omega)$ for every $\omega \in \Omega_2 \backslash N_5$ where $P_x(N_5) = 0$. The fact that D is closed implies that $X_{\sigma_{\rho_n}} \in D$ for all $n \in \mathbb{N}$. Moreover, by the quasi-left-continuity property of X it follows that $X_{\sigma_{\rho_n}(\omega)}(\omega) \to X_{\rho(\omega)}(\omega)$ for every $\omega \in \Omega_2 \backslash N_6$ where $P_x(N_6) = 0$. Again using the fact that D is closed we have that $X_\rho(\omega) \in D$ for every $\omega \in \Omega_2 \backslash N_7$ where $P_x(N_7) = 0$. From this it follows that $\sigma_\rho(\omega) = \rho(\omega)$ for every $\omega \in \Omega_2 \backslash N$ with $N = N_6 \cup N_7$. By definition of Ω_2, this implies that $\sigma_\rho = \beta$ P_x-a.s. on Ω_2. Now let us consider the case when (4.20) holds. Again by quasi-left-continuous of X it follows that $X_{\sigma_{\rho_n}(\omega)}(\omega) \to X_{\rho(\omega)}(\omega)$ for each $\omega \in \Omega_2 \backslash N_8$ with $P_x(N_8) = 0$. Since D is not necessarily closed we have that $X_{\rho(\omega)}(\omega) \in D \cup \partial D$. Suppose first that $X_{\rho(\omega)}(\omega) \in D$. This means that $\sigma_{\rho(\omega)}(\omega) = \rho(\omega)$ and so we have that $\sigma_{\rho_n(\omega)}(\omega) \uparrow \sigma_{\rho(\omega)}(\omega)$. From this we can conclude that $\beta(\omega) = \sigma_{\rho(\omega)}(\omega)$. To prove that $\sigma_\rho = \rho$ P_x-a.s. on the set $\Omega' := \{\omega \in \Omega : X_\rho(\omega) \in \partial D\}$ it is sufficient to show that $P_x(\{\sigma_\rho > \rho\} \cap \Omega')$. Let $\eta_D = \inf\{t > 0 : X_t \in D\}$. By the strong Markov property of X we have that

$$E_{X_\rho}[I(\eta_D > 0)] = E_x[I(\rho + \eta_D \circ \theta_\rho > \rho)|\mathcal{F}_\rho] \tag{4.23}$$

Multiplying both sides in (4.23) by $I_{\Omega'}$ and taking E_x expectation on both sides (note that X_ρ is $\mathcal{F}_\rho$ measurable) we get that $P_x(\{\sigma_\rho > \rho\} \cap \Omega') = E_x[I_{\Omega'} E_{X_\rho}[I(\eta_D > 0)]] = 0$ by the regularity property of X. □

Proposition 4.3 *Let D_1, D_2 be either closed or finely closed subsets of E. Suppose also that their respective boundaries ∂D_1 and ∂D_2 are regular. Let τ_{D_1} and σ_{D_2} be the first entry times into D_1 and D_2 respectively. Set $\tau_\varepsilon^{\sigma_{D_2}} = \inf\{t \geq 0 : X_t \in D_1^{\sigma_{D_2},\varepsilon}\}$ and $\sigma_\varepsilon^{\tau_{D_1}} = \inf\{t \geq 0 : X_t \in D_2^{\tau_{D_1},\varepsilon}\}$ where $D_1^{\sigma_{D_2},\varepsilon} = \{V_{\sigma_{D_2}}^1 \leq G_1 + \varepsilon\}$ and $D_2^{\tau_{D_1},\varepsilon} = \{V_{\tau_{D_1}}^2 \leq G_2 + \varepsilon\}$. Then $\tau_\varepsilon^{\sigma_{D_2}} \uparrow \tau_*^{\sigma_{D_2}}$ P_x-a.s. and $\sigma_\varepsilon^{\tau_{D_1}} \uparrow \sigma_*^{\tau_{D_1}}$ P_x-a.s., where $\tau_*^{\sigma_{D_2}} = \inf\{t \geq 0 : X_t \in D_1^{\sigma_{D_2}}\}$ with $D_1^{\sigma_{D_2}} = \{V_{\sigma_{D_2}}^1 = G_1\}$ and $\sigma_*^{\tau_{D_1}} = \inf\{t \geq 0 : X_t \in D_2^{\tau_{D_1}}\}$ with $D_2^{\tau_{D_1}} = \{V_{\tau_{D_1}}^2 = G_2\}$.*

Proof We shall only prove that $\tau_\varepsilon^{\sigma_{D_2}} \uparrow \tau_*^{\sigma_{D_2}}$ P_x-a.s. as the other assertion follows by symmetry. Recall the definition of $\tilde{V}_{\sigma_{D_2}}^1$ from (4.3)–(4.4). From step 2° in the proof of Proposition 4.1 it is sufficient to prove that $\tilde{\tau}_\varepsilon \uparrow \tilde{\tau}_*$, where we recall that $\tilde{\tau}_\varepsilon = \inf\{t \geq 0 : X_t \in \tilde{D}_1^{\sigma_{D_2},\varepsilon}\}$ with $\tilde{D}_1^{\sigma_{D_2},\varepsilon} = \{\tilde{V}_{\sigma_{D_2}}^1 \leq G_1 + \varepsilon\}$ and $\tilde{\tau}_*^{\sigma_{D_2}} = \inf\{t \geq 0 : X_t \in \tilde{D}_1^{\sigma_{D_2}}\}$ with $\tilde{D}_1^{\sigma_{D_2}} = \{\tilde{V}_{\sigma_{D_2}}^1 = G_1\}$. For each $\varepsilon > 0$ we have that $\tilde{\tau}_\varepsilon \leq \tilde{\tau}_*^{\sigma_{D_2}}$. Since $\tilde{\tau}_\varepsilon$ increases as ε decreases, $\tilde{\tau}_\varepsilon \uparrow \beta$ as $\varepsilon \downarrow 0$ where $\beta \leq \tilde{\tau}_*^{\sigma_{D_2}}$ P_x-a.s. To prove that $\beta = \tilde{\tau}_*^{\sigma_{D_2}}$ we first show that

$$\mathsf{E}_x\left[\tilde{V}_{\sigma_{D_2}}^1(X_\beta)\right] \leq \liminf_{\varepsilon \downarrow 0} \mathsf{E}_x\left[\tilde{V}_{\sigma_{D_2}}^1(X_{\tilde{\tau}_\varepsilon})\right] \tag{4.24}$$

For stopping times $\sigma_\beta = \inf\{t \geq \beta : X_t \in D_2\}$ and $\sigma_{\tilde{\tau}_\varepsilon} = \inf\{t \geq \tilde{\tau}_\varepsilon : X_t \in D_2\}$ we have

$$
\begin{aligned}
&\tilde{\mathsf{M}}_x^1(\tau, \sigma_\beta) - \tilde{\mathsf{M}}_x^1(\tau, \sigma_{\tilde{\tau}_\varepsilon}) \\
&= \mathsf{E}_x\big[G_1(X_\tau)I(\tau < \sigma_\beta) + H_1(X_{\sigma_\beta})I(\sigma_\beta < \tau) \\
&\quad + H_1(X_{\sigma_\beta})I(\sigma_\beta = \tau, \sigma_\beta \neq \sigma_{\tilde{\tau}_\varepsilon}) - G_1(X_\tau)I(\tau < \sigma_{\tilde{\tau}_\varepsilon}) \\
&\quad - H_1(X_{\sigma_{\tilde{\tau}_\varepsilon}})I(\sigma_{\tilde{\tau}_\varepsilon} < \tau) - H_1(X_{\sigma_{\tilde{\tau}_\varepsilon}})I(\sigma_{\tilde{\tau}_\varepsilon} = \tau, \sigma_{\tilde{\tau}_\varepsilon} \neq \sigma_\beta) \\
&\quad + H_1(X_{\sigma_\beta})I(\sigma_{\tilde{\tau}_\varepsilon} < \tau) - H_1(X_{\sigma_\beta})I(\sigma_{\tilde{\tau}_\varepsilon} < \tau)\big] \\
&\leq \mathsf{E}_x\big[G_1(X_\tau)(I(\tau < \sigma_\beta) - I(\tau < \sigma_{\tilde{\tau}_\varepsilon}) - I(\sigma_{\tilde{\tau}_\varepsilon} = \tau, \sigma_{\tilde{\tau}_\varepsilon} \neq \sigma_\beta))\big] \\
&\quad + \mathsf{E}_x\big[H_1(X_{\sigma_\beta})(I(\sigma_\beta < \tau) - I(\sigma_{\tilde{\tau}_\varepsilon} < \tau) + I(\sigma_\beta = \tau, \sigma_\beta \neq \sigma_{\tilde{\tau}_\varepsilon}))\big] \\
&\quad + \mathsf{E}_x\big[(H_1(X_{\sigma_\beta}) - H_1(X_{\sigma_{\tilde{\tau}_\varepsilon}}))I(\sigma_{\tilde{\tau}_\varepsilon} < \tau)\big] \\
&= \mathsf{E}_x\big[(G_1(X_\tau) - H_1(X_{\sigma_\beta}))I(\sigma_{\tilde{\tau}_\varepsilon} < \tau < \sigma_\beta)\big] \\
&\quad + \mathsf{E}_x\big[(H_1(X_{\sigma_\beta}) - H_1(X_{\sigma_{\tilde{\tau}_\varepsilon}}))I(\sigma_{\tilde{\tau}_\varepsilon} < \tau)\big] \\
&\leq \mathsf{E}_x\big[(\sup_t |G_1(X_t)| + \sup_t |H_1(X_t)|)I(\sigma_{\tilde{\tau}_\varepsilon} < \tau < \sigma_\beta)\big] \\
&\quad + \mathsf{E}_x\big[|H_1(X_{\sigma_\beta}) - H_1(X_{\sigma_{\tilde{\tau}_\varepsilon}})|\big] \tag{4.25}
\end{aligned}
$$

where the first inequality follows from the fact that $G_1 \leq H_1$. By Lemma 4.2 we have that $\sigma_{\tilde{\tau}_\varepsilon} \uparrow \sigma_\beta$ as $\varepsilon \downarrow 0$ and so the first term on the right hand side of the above expression tends to zero uniformly over all τ. Since $H_1(X)$ is quasi-left-continuous, the second expression also tends to zero. By the strong Markov property of X (recall

that the expectation and the essential supremum in (3.14) can be interchanged) it follows that $\mathsf{E}_x[\tilde{V}^1_{\sigma_{D_2}}(X_\beta)] = \sup_{\tau \geq \beta} \tilde{\mathsf{M}}^1_x(\tau, \sigma_\beta)$. By (4.25) we have

$$
\begin{aligned}
\sup_{\tau \geq \beta} \tilde{\mathsf{M}}^1_x(\tau, \sigma_\beta) &\leq \liminf_{\varepsilon \downarrow 0} \sup_{\tau \geq \beta} \tilde{\mathsf{M}}^1_x(\tau, \sigma_{\tilde{\tau}_\varepsilon}) \leq \liminf_{\varepsilon \downarrow 0} \sup_{\tau \geq \tilde{\tau}_\varepsilon} \tilde{\mathsf{M}}^1_x(\tau, \sigma_{\tilde{\tau}_\varepsilon}) \\
&= \liminf_{\varepsilon \downarrow 0} \mathsf{E}_x[\tilde{V}^1_{\sigma_{D_2}}(X_{\tilde{\tau}_\varepsilon})]
\end{aligned}
\tag{4.26}
$$

Using again the fact that $\tilde{V}^1_{\sigma_{D_2}} = V^1_{\sigma_{D_2}}$ so that $\tilde{V}^1_{\sigma_{D_2}}$ is finely continuous, together with the fact that $G_1(X)$ is left-continuous over stopping times, we get, from Lemma 3.5, that $\tilde{V}^1_{\sigma_{D_2}}(X_{\tilde{\tau}_\varepsilon}) \leq G_1(X_{\tilde{\tau}_\varepsilon}) + \varepsilon$ P_x-a.s. Hence it follows that

$$
\begin{aligned}
\mathsf{E}_x[\tilde{V}^1_{\sigma_{D_2}}(X_\beta)] &\leq \liminf_{\varepsilon \downarrow 0} \mathsf{E}_x[\tilde{V}^1_{\sigma_{D_2}}(X_{\tilde{\tau}^{\sigma_{D_2}}_\varepsilon})] \\
&\leq \liminf_{\varepsilon \downarrow 0} \mathsf{E}_x[G_1(X_{\tilde{\tau}^{\sigma_{D_2}}_\varepsilon}) + \varepsilon] = \mathsf{E}_x[G_1(X_\beta)].
\end{aligned}
\tag{4.27}
$$

Combining (4.27) with the fact that $\tilde{V}^1_{\sigma_{D_2}}(X_\beta) \geq G_1(X_\beta)$ P_x-a.s. we conclude that

$$
\tilde{V}^1_{\sigma_{D_2}}(X_\beta) = G_1(X_\beta) \ \mathsf{P}_x\text{-a.s.}
\tag{4.28}
$$

But $\tilde{\tau}^{\sigma_{D_2}}_* = \inf\{t \geq 0 : X_t \in \tilde{D}^{\sigma_{D_2}}_1\}$ and so we must have that $\beta \geq \tilde{\tau}^{\sigma_{D_2}}_0$. This fact together with $\beta \leq \tilde{\tau}^{\sigma_{D_2}}_*$ proves the required result. $\square$

We now state and prove the main result of this section.

Theorem 4.4 *Given the setting in Proposition 4.3, we have*

$$
V^1_{\sigma_{D_2}}(x) = \mathsf{M}^1_x(\tau^{\sigma_{D_2}}_*, \sigma_{D_2}),
\tag{4.29}
$$

$$
V^2_{\tau_{D_1}}(x) = \mathsf{M}^2_x(\tau_{D_1}, \sigma^{\tau_{D_1}}_*).
\tag{4.30}
$$

Proof We shall only prove (4.29) as the proof of (4.30) follows by symmetry. Recall, from Proposition 4.1, that $V^1_{\sigma_{D_2}}(x) \leq \mathsf{M}_x(\tau^{\sigma_{D_2}}_\varepsilon, \sigma_{D_2}) + \varepsilon$. We show that $\limsup_{\varepsilon \downarrow 0} \mathsf{M}^1_x(\tau^{\sigma_{D_2}}_\varepsilon, \sigma_{D_2}) \leq \mathsf{M}^1_x(\tau^{\sigma_{D_2}}_*, \sigma_{D_2})$ so that $V^1_{\sigma_{D_2}}(x) \leq \limsup_{\varepsilon \downarrow 0}(\mathsf{M}^1_x(\tau^{\sigma_{D_2}}_\varepsilon, \sigma_{D_2}) + \varepsilon) \leq \mathsf{M}^1_x(\tau^{\sigma_{D_2}}_*, \sigma_{D_2})$. For simplicity of exposition let us set $\tau_\varepsilon := \tau^{\sigma_{D_2}}_\varepsilon$ and

$\tau_* := \tau_*^{\sigma_{D_2}}$. Now

$$
\begin{aligned}
&\mathsf{M}_x^1(\tau_*, \sigma_{D_2}) - \mathsf{M}_x^1(\tau_\varepsilon, \sigma_{D_2}) \\
&= \mathsf{E}_x\big[G_1(X_{\tau_*})I(\tau_* < \sigma_{D_2}) + H_1(X_{\sigma_{D_2}})I(\sigma_{D_2} < \tau_*) \\
&\quad + G_1(X_{\sigma_{D_2}})I(\sigma_{D_2} = \tau_*, \tau_* \neq \tau_\varepsilon) \\
&\quad - G_1(X_{\tau_\varepsilon})I(\tau_\varepsilon < \sigma_{D_2}) - H_1(X_{\sigma_{D_2}})I(\sigma_{D_2} < \tau_\varepsilon) \\
&\quad - G_1(X_{\sigma_{D_2}})I(\sigma_{D_2} = \tau_\varepsilon, \tau_\varepsilon \neq \tau_*) + G_1(X_{\tau_\varepsilon})I(\tau_* < \sigma_{D_2}) \\
&\quad - G_1(X_{\tau_\varepsilon})I(\tau_* < \sigma_{D_2}) + G_1(X_{\tau_\varepsilon})I(\tau_* = \sigma_{D_2}, \tau_* \neq \tau_\varepsilon) \\
&\quad - G_1(X_{\tau_\varepsilon})I(\tau_* = \sigma_{D_2}, \tau_* \neq \tau_\varepsilon)\big] \geq \mathsf{E}_x\big[(G_1(X_{\tau_*}) \\
&\quad - G_1(X_{\tau_\varepsilon}))I(\tau_* < \sigma_{D_2})\big] + G_1(X_{\tau_\varepsilon})(I(\tau_* < \sigma_{D_2}) \\
&\quad - I(\tau_\varepsilon < \sigma_{D_2})I(\tau_* = \sigma_{D_2}, \tau_* \neq \tau_\varepsilon)) \\
&\quad + G_1(X_{\tau_*})I(\tau_* = \sigma_{D_2}, \tau_* \neq \tau_\varepsilon) \\
&\quad - G_1(X_{\tau_\varepsilon})I(\tau_* = \sigma_{D_2}, \tau_* \neq \tau_\varepsilon) + H_1(X_{\sigma_{D_2}})(I(\sigma_{D_2} < \tau_*) \\
&\quad - I(\sigma_{D_2} < \tau_\varepsilon) - I(\tau_\varepsilon = \sigma_{D_2}, \tau_\varepsilon \neq \tau_*))\big] \\
&= \mathsf{E}_x\big[(G_1(X_{\tau_*}) - G_1(X_{\tau_\varepsilon}))I(\tau_* < \sigma_{D_2}) + (H_1(X_{\sigma_{D_2}}) \\
&\quad - G_1(X_{\tau_\varepsilon}))I(\tau_\varepsilon < \sigma_{D_2} < \tau_*) + G_1(X_{\tau_*})I(\tau_* = \sigma_{D_2}, \tau_* \neq \tau_\varepsilon) \\
&\quad - G_1(X_{\tau_\varepsilon})I(\tau_* = \sigma_{D_2}, \tau_* \neq \tau_\varepsilon)\big] \geq -2\mathsf{E}_x\big[|G_1(X_{\tau_*}) \\
&\quad - G_1(X_{\tau_\varepsilon})|\big] + \mathsf{E}_x\big[(H_1(X_{\sigma_{D_2}}) - G_1(X_{\tau_\varepsilon}))I(\tau_\varepsilon < \sigma_{D_2} < \tau_*)\big] \\
&\geq -2\mathsf{E}_x\big[|G_1(X_{\tau_*}) - G_1(X_{\tau_\varepsilon})|\big] - \mathsf{E}_x\big[(\sup_t |H_1(X_t)| \\
&\quad + \sup_t |G_1(X_t)|)I(\tau_\varepsilon < \sigma_{D_2} < \tau_*)\big]
\end{aligned}
\tag{4.31}
$$

The first inequality follows from the assumption $-G_1 \geq -H_1$ whereas the second equality follows from the fact that

$$
\begin{aligned}
I(\sigma_{D_2} < \tau_*) - I(\sigma_{D_2} < \tau_\varepsilon) - I(\tau_\varepsilon = \sigma_{D_2}, \tau_\varepsilon \neq \tau_*) &= -(I(\tau_* < \sigma_{D_2}) \\
- I(\tau_\varepsilon < \sigma_{D_2}) + I(\tau_* = \sigma_{D_2}, \tau_* \neq \tau_\varepsilon)) &= I(\tau_\varepsilon < \sigma_{D_2} < \tau_*)
\end{aligned}
\tag{4.32}
$$

The penultimate inequality follows from the fact that

$$
\begin{aligned}
(G_1(X_{\tau_*}) - G_1(X_{\tau_\varepsilon}))I(\tau_* < \sigma_{D_2}) &+ (G_1(X_{\tau_*}) - G_1(X_{\tau_\varepsilon}))I(\tau_* = \sigma_{D_2}, \tau_* \neq \tau_\varepsilon) \\
&\geq -2|G_1(X_{\tau_*}) - G_1(X_{\tau_\varepsilon})|
\end{aligned}
\tag{4.33}
$$

whereas the last inequality follows from the fact that

$$
\begin{aligned}
(H_1(X_{\sigma_{D_2}}) - G_1(X_{\tau_\varepsilon}))I(\tau_\varepsilon < \sigma_{D_2} < \tau_*) &\geq -((\sup_t |H_1(X_t)| \\
&+ \sup_t |G_1(X_t)|)I(\tau_\varepsilon < \sigma_{D_2} < \tau_*))
\end{aligned}
\tag{4.34}
$$

Letting $\varepsilon \downarrow 0$ in (4.31) we get, from Proposition 4.3, that $I(\tau_\varepsilon < \sigma_{D_2} < \tau_*)$ converges to zero uniformly over σ_{D_2}. Moreover, by using the quasi-left-continuity property of

$G_1(X)$ we conclude that $\mathsf{M}_x^1(\tau_*, \sigma_{D_2}) \geq \limsup_{\varepsilon \downarrow 0} \mathsf{M}_x^1(\tau_\varepsilon, \sigma_{D_2})$ and this completes the proof. $\square$

5 Partial superharmonic characterisation

The purpose of the current section is to utilise the results derived in Sects. 3 and 4 to provide a partial superharmonic characterisation of $V_{\sigma_{D_2}}^1$ (resp. $V_{\tau_{D_1}}^2$) when the stopping time σ_{D_2} (resp. τ_{D_1}) of player two (resp. player one) is externally given. This characterisation attempts to extend the semiharmonic characterisation of the value function in zero-sum games (see [51] and [52]) and can informally be described as follows: Suppose that $G_2 \equiv -\infty$ in (2.3). Then the second player has no incentive of stopping the process and so (2.4) reduces to the optimal stopping problem $V_\infty^1(x) = \sup_\tau \mathsf{E}_x[G_1(X_\tau)]$. By results in optimal stopping theory V_∞^1 admits a superharmonic characterisation. More precisely V_∞^1 can be identified with the smallest superharmonic function that dominates G_1 (see [49, p. 37 Theorem 2.4]). However, if G_2 is finite valued then there might be an incentive for the second player to stop the process. This raises two questions: (i) is the superharmonic characterisation of V_σ^1 still valid before the second player stops the process, (ii) does V_σ^1 coincide with H_1 at the time the second player stops the process? If the second player selects the stopping time $\sigma := \sigma_{D_2} = \inf\{t \geq 0 : X_t \in D_2\}$ where D_2 is a closed or finely closed subset of the state space E having a regular boundary ∂D_2 then the above questions can be answered affirmatively and we will say that the value function of player one associated with the stopping time σ_{D_2} admits a partial superharmonic characterisation. To be more precise let us consider the set

$$\mathbf{Sup}_{D_2}^1(G_1, K_1) = \{F : E \to [G_1, K_1] : F \text{ is finely continuous, } F = H_1 \text{ in } D_2,$$
$$F \text{ is superharmonic in } D_2^c\} \tag{5.1}$$

where K_1 is the smallest superharmonic function that dominates H_1 and $[G_1, K_1]$ means that $G_1(x) \leq F(x) \leq K_1(x)$ for all $x \in E$. Then the value function of player one can be identified with the smallest finely continuous function from $\mathbf{Sup}_{D_2}^1(G_1, K_1)$.

Likewise, suppose that player one selects the stopping time $\tau_{D_1} = \inf\{t \geq 0 : X_t \in D_1\}$ where D_1 is a closed or finely closed set having a regulary boundary ∂D_1, and consider the set

$$\mathbf{Sup}_{D_1}^2(G_2, K_2) = \{F : E \to [G_2, K_2] : F \text{ is finely continuous, } F = H_2 \text{ in } D_1,$$
$$F \text{ is superharmonic in } D_1^c\} \tag{5.2}$$

where K_2 is the smallest superharmonic function that dominates H_2 and $[G_2, K_2]$ means that $G_2(x) \leq F(x) \leq K_2(x)$ for all $x \in E$. Then the value function of player two associated to τ_{D_1} can be identified with the smallest finely continuous function from $\mathbf{Sup}_{D_1}^2(G_2, K_2)$.
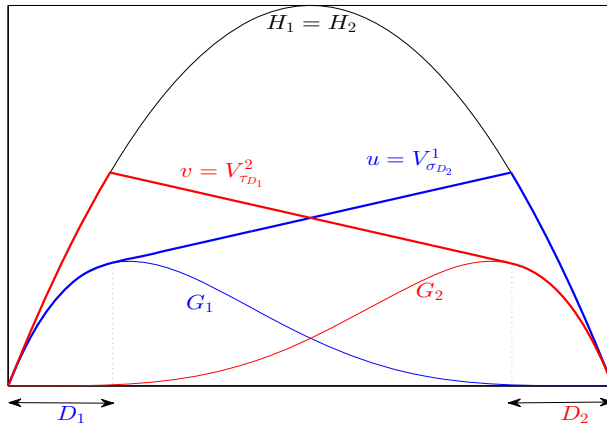
Fig. 1 The double partial superharmonic characterisation of the value functions in a nonzero-sum optimal stopping game for absorbed Brownian motion

The above characterisation of $V^1_{\sigma_{D_2}}$ and $V^2_{\tau_{D_1}}$ can be used to study the existence of a Nash equilibrium. Indeed suppose that one can show the existence of finely continuous functions u and v such that:

(i.) u lies between G_1 and K_1, u is identified with H_1 in the region $D_2 = \{v = G_2\}$ and u is the smallest superharmonic function that dominates G_1 in the region $\{v > G_2\}$

(ii.) v lies between G_2 and K_2, v is identified with H_2 in the region $D_1 = \{u = G_1\}$ and v is the smallest superharmonic function that dominates G_2 in the region $\{u > G_1\}$.

Then under the assumption that D_1 and D_2 have regular boundaries and are either closed or finely closed, u and v coincide with $V^1_{\sigma_{D_2}}$ and $V^2_{\tau_{D_1}}$ respectively. In this case we shall say that together, $V^1_{\sigma_{D_2}}$ and $V^2_{\tau_{D_1}}$ admit a double partial superharmonic characterisation (see Fig. 1) and can be called the value functions of the game (2.4)–(2.5). Moreover, the pair $(\tau_{D_1}, \sigma_{D_2})$ will form a Nash equilibrium point.

To prove the partial superharmonic characterisation of $V^1_{\sigma_{D_2}}$ (resp. $V^2_{\tau_{D_1}}$) we first show that for any stopping time σ (resp. τ), V^1_σ (resp. V^2_τ) is bounded above by K_1 (resp. K_2). For this we define the concept of superharmonic functions.

Definition 5.1 Let C be a measurable subset of E and $D = E \backslash C$. A measurable function $F : E \to \mathbb{R}$ is said to be superharmonic in C if $\mathsf{E}_x[F(X_{\rho \wedge \sigma_D})] \leq F(x)$ for every stopping time ρ and for all $x \in E$, where $\sigma_D = \inf\{t \geq 0 : X_t \in D\}$. F is said to be superharmonic if $\mathsf{E}_x[F(X_\rho)] \leq F(x)$ for every stopping time ρ and for all $x \in E$.

Lemma 5.2 *Let* $\boldsymbol{Sup}(H_1) = \{F : E \to \mathbb{R} : F \geq H_1, F$ *is superharmonic*$\}$ *be the collection of superharmonic functions that majorise* H_1. *Then for any given stopping time* σ *we have*

$$V_\sigma^1 \leq \inf_{F \in \mathbf{Sup}(H_1)} F. \tag{5.3}$$

Similarly let $\mathbf{Sup}(H_2) = \{F : E \to \mathbb{R} : F \geq H_2, F \text{ is superharmonic}\}$. *Then for any given stopping time* τ *we have*

$$V_\tau^2 \leq \inf_{F \in \mathbf{Sup}(H_2)} F. \tag{5.4}$$

Proof We shall only prove (5.3) as (5.4) can be proved in exactly the same way. Take any stopping time σ and any $F \in \mathbf{Sup}(H_1)$. Then

$$\mathsf{M}_x^1(\tau, \sigma) \leq \mathsf{E}_x[F(X_\tau)I(\tau \leq \sigma) + F(X_\sigma)I(\sigma < \tau)] = \mathsf{E}_x[F(X_{\tau \wedge \sigma})] \tag{5.5}$$

for all stopping times τ. The first two inequalities follow from the fact that $G_1 \leq H_1 \leq F$. F is superharmonic, so $\mathsf{E}_x[F(X_\rho)] \leq F(x)$ for any stopping time ρ and for all $x \in E$. In particular $\mathsf{E}_x[F(X_{\tau \wedge \sigma})] \leq F(x)$ for every stopping time τ. Thus

$$\mathsf{M}_x^1(\tau, \sigma) \leq F(x) \tag{5.6}$$

for all stopping times τ and for all $x \in E$. Taking the infimum over all F in $\mathbf{Sup}(H_1)$ on the right hand side and the supremum over all τ on the left hand side of (5.6) we get the required result. □

Theorem 5.3 *i. Suppose that* K_1 *is the smallest superharmonic function that dominates* H_1. *Let* $v \geq G_2$ *be a finely continuous function, with* $D_2 = \{v = G_2\}$, *such that* $u := \inf_{F \in \mathbf{Sup}_{D_2}^1(G_1, K_1)} F$ *where* $\mathbf{Sup}_{D_2}^1(G_1, K_1)$ *is the collection of functions given in* (5.1), *exists. If the boundary* ∂D_2 *of* D_2 *is regular for* D_2, *then*

$$u(x) = V_{\sigma_{D_2}}^1(x) \tag{5.7}$$

for all $x \in E$ *where* $\sigma_{D_2} = \{t \geq 0 : X_t \in D_2\}$.

ii. Similarly suppose that K_2 *be the smallest superharmonic function that dominates* H_2. *Let* $u \geq G_1$ *be a finely continuous function, with* $D_1 = \{u = G_1\}$, *such that* $v := \inf_{F \in \mathbf{Sup}_{D_1}^2(G_2, K_2)} F$ *where* $\mathbf{Sup}_{D_1}^2(G_2, K_2)$ *is the collection of functions defined in* (5.2), *exists. If the boundary* ∂D_1 *of* D_1 *is regular for* D_1, *then*

$$v(x) = V_{\tau_{D_1}}^2(x) \tag{5.8}$$

for all $x \in E$ *where* $\tau_{D_1} = \{t \geq 0 : X_t \in D_1\}$.

Proof We shall only prove (i.) as (ii.) follows by symmetry. We first show that $u \geq V_{\sigma_{D_2}}^1$. Take any $F \in \mathbf{Sup}_{D_2}^1(G_1, K_1)$. We know that F is superharmonic in D_2^c so

$$F(x) \geq \mathsf{E}_x\left[F\left(X_{\tau \wedge \sigma_{D_2}}\right)\right] = \mathsf{E}_x[F(X_\tau)I(\tau \leq \sigma_{D_2}) + F(X_{\sigma_{D_2}})I(\sigma_{D_2} < \tau)]$$

$$\geq \mathsf{E}_x\left[G_1(X_\tau)I(\tau \leq \sigma_{D_2}) + F\left(X_{\sigma_{D_2}}\right)I(\sigma_{D_2} < \tau)\right] \tag{5.9}$$

for every stopping time τ and for all $x \in E$, where the last inequality follows from the fact that $F \geq G_1$. Since v is finely continuous and G_2 is continuous (hence finely continuous), D_2 is finely closed and thus by the definition of $\mathbf{Sup}^1_{D_2}(G_1, K_1)$, upon using (3.4) (since F is finely continuous and H_1 is continuous hence finely continuous) it can be seen that $F(X_{\sigma_{D_2}}) = H_1(X_{\sigma_{D_2}})$. Thus for any $F \in \mathbf{Sup}^1_{D_2}(G_1, K_1)$ we have

$$F(x) \geq \mathsf{E}_x \left[G_1(X_\tau) I(\tau \leq \sigma_{D_2}) + H_1\left(X_{\sigma_{D_2}}\right) I(\sigma_{D_2} < \tau) \right] = \mathsf{M}^1_x\left(\tau, \sigma_{D_2}\right) \tag{5.10}$$

for every stopping time τ and for all $x \in E$. Taking the infimum over all F and the supremum over all τ we get that $u(x) \geq V^1_{\sigma_{D_2}}(x)$ for all $x \in E$. We next show that $u \leq V^1_{\sigma_{D_2}}$. For this it is sufficient to prove that $V^1_{\sigma_{D_2}} \in \mathbf{Sup}^1_{D_2}(G_1, K_1)$ as the result will follow by definition of u. Recall, from Theorem 3.7, that $V^1_{\sigma_{D_2}}$ is finely continuous since the boundary ∂D_2 is assumed to be regular for D_2. The fact that $V^1_{\sigma_{D_2}} \leq K_1$ follows from Lemma 5.2. To show that $V^1_{\sigma_{D_2}}$ is bounded below by G_1 we note that $V^1_{\sigma_{D_2}} \geq \mathsf{M}^1_x(\tau, \sigma_{D_2})$ for any τ in particular for $\tau = 0$. Since $\mathsf{M}^1_x(0, \sigma_{D_2}) = G_1(x)$ the result follows. To prove that $V^1_{\sigma_{D_2}} = H_1$ in D_2 we take any $x \in D_2$ so that $\sigma_{D_2} = 0$. Then by selecting any stopping time $\tau > 0$ we get that $V^1_0(x) \geq \mathsf{M}^1_x(\tau, 0) = H_1(x)$. On the other hand

$$V^1_0(x) \leq \sup_\tau (H_1(x) \mathsf{P}_x(\tau = 0) + H_1(x) \mathsf{P}_x(\tau > 0)) = H_1(x). \tag{5.11}$$

From this we conclude that $V^1_0(x) = H_1(x)$. It remains to prove that $V^1_{\sigma_{D_2}}$ is superharmonic in D_2^c By the strong Markov property of X we have

$$\begin{aligned}
&\mathsf{E}_x\left[V^1_{\sigma_{D_2}}(X_{\rho \wedge \sigma_{D_2}})\right] \\
&= \mathsf{E}_x\left[\sup_\tau \mathsf{M}^1_{X_{\rho \wedge \sigma_{D_2}}}(\tau, \sigma_{D_2})\right] \\
&= \mathsf{E}_x\left[\operatorname*{ess\,sup}_\tau \mathsf{M}^1_x(\rho \wedge \sigma_{D_2} + \tau \circ \theta_{\rho \wedge \sigma_{D_2}}, \rho \wedge \sigma_{D_2} + \sigma_{D_2} \circ \theta_{\rho \wedge \sigma_{D_2}} | \mathcal{F}_{\rho \wedge \sigma_{D_2}})\right] \\
&= \mathsf{E}_x\left[\operatorname*{ess\,sup}_{\tau \geq \rho \wedge \sigma_{D_2}} \mathsf{M}^1_x(\tau, \sigma_{D_2} | \mathcal{F}_{\rho \wedge \sigma_{D_2}})\right] \\
&= \sup_{\tau \geq \rho \wedge \sigma_{D_2}} \mathsf{M}^1_x(\tau, \sigma_{D_2}) \\
&\leq \sup_\tau \mathsf{M}^1_x(\tau, \sigma_{D_2}) = V^1_{\sigma_{D_2}}(x) \tag{5.12}
\end{aligned}$$

for any stopping time ρ where we recall that $\mathsf{M}^1_x(\tau, \sigma | \mathcal{F}_\rho) = \mathsf{E}_x[G_1(X_\tau) I(\tau \leq \sigma) + H_1(X_\sigma) I(\sigma < \tau) | \mathcal{F}_\rho]$ for stopping times τ, σ and ρ. $\qquad\square$

6 The case of stationary one-dimensional Markov processes

In this section we shall assume that the Markov process X takes values in $\mathbb{R}$ and is such that $\mathrm{Law}(X | \mathsf{P}_x) = \mathrm{Law}(X^x | \mathsf{P})$. We shall also assume that there exist points A_*

and B_* satisfying $-\infty < A_* < B_* < \infty$ such that (i.) for given D_2 of the form $[B_*, \infty)$, the first entry time $\tau_{A_*} = \inf\{t \geq 0 : X_t \leq A_*\}$ (as obtained from Theorem 4.4) is optimal for player one and (ii.) for given D_1 of the form $(-\infty, A_*]$, the first entry time $\sigma_{B_*} = \inf\{t \geq 0 : X_t \geq B_*\}$ (as obtained from Theorem 4.4) is optimal for player two.

So in this section we will assume the existence of a pair $(\tau_{A_*}, \sigma_{B_*})$ that is a Nash equilibrium.

6.1 The principle of double continuous fit

We prove that $V^1_{\sigma_{B_*}}$ (resp. $V^2_{\tau_{A_*}}$) is continuous at A_* (resp. B_*). We shall refer to this result as the principle of double continuous fit. For this we shall further assume that the following time-space conditions hold:

$$X^{A_*+\varepsilon}_{t+h} \to X^{A_*}_t \quad \mathsf{P} - \text{a.s.} \tag{6.1}$$

$$X^{B_*-\varepsilon}_{t+h} \to X^{B_*}_t \quad \mathsf{P} - \text{a.s.} \tag{6.2}$$

as $\varepsilon, h \downarrow 0$ and

$$X^{A_*+\varepsilon}_{\rho_n} \to X^{A_*}_\rho \quad \mathsf{P} - \text{a.s.} \tag{6.3}$$

$$X^{B_*-\varepsilon}_{\rho_n} \to X^{B_*}_\rho \quad \mathsf{P} - \text{a.s.} \tag{6.4}$$

whenever ρ_n is a sequence of stopping times such that $\rho_n \uparrow \rho$. Conditions (6.1)-(6.4) imply that the mapping $x \mapsto X^x$ (stochastic flow) is continuous at A_* and B_*. Stochastic differential equations driven by Lévy processes, for example, satisfy this property under regularity assumptions on the drift and diffusion coefficients (see for example ([37, p. 340])).

We shall first state and prove the following lemma.

Lemma 6.1 Let $\sigma^{A_*}_{B_*} = \inf\{t \geq 0 : X_t \geq B_*\}$ be the optimal stopping time for player two under P_{A_*} and let $\sigma^{A_*+\varepsilon}_{B_*} = \inf\{t \geq 0 : X_t \geq B_*\}$ be the optimal stopping time of player two under $\mathsf{P}_{A_*+\varepsilon}$, for given $\varepsilon > 0$. Then, if condition (6.3) is satisfied we have that $\sigma^{A_*+\varepsilon}_{B_*} \uparrow \sigma^{A_*}_{B_*}$ as $\varepsilon \downarrow 0$. Similarly, if $\tau^{B_*}_{A_*} = \inf\{t \geq 0 : X_t \leq A_*\}$ is the optimal stopping time for player two under P_{B_*} and $\tau^{B_*-\varepsilon}_{A_*} = \inf\{t \geq 0 : X_t \leq A_*\}$ is the optimal stopping time of player one under $\mathsf{P}_{B_*-\varepsilon}$, for given $\varepsilon > 0$, then if condition (6.4) is satisfied we have that $\tau^{B_*-\varepsilon}_{A_*} \uparrow \tau^{B_*}_{A_*}$ as $\varepsilon \downarrow 0$.

Proof We shall only prove that $\sigma^{A_*+\varepsilon}_{B_*} \uparrow \sigma^{A_*}_{D_2}$ as $\varepsilon \downarrow 0$. The fact that $\tau^{B_*-\varepsilon}_{A_*} \uparrow \tau^{B_*}_{A_*}$ as $\varepsilon \downarrow 0$ can be proved in the same way. Since $\text{Law}(X|\mathsf{P}_x) = \text{Law}(X^x|\mathsf{P})$ we have that $\sigma^{A_*+\varepsilon}_{B_*}$ is equally distributed as $\hat{\sigma}^{A_*+\varepsilon}_{B_*} := \inf\{t \geq 0 : X^{A_*+\varepsilon}_t \geq B_*\}$ under P whereas $\sigma^{A_*}_{B_*}$ is equally distributed as $\hat{\sigma}^{A_*}_{B_*} := \inf\{t \geq 0 : X^{A_*}_t \geq B_*\}$ under P. Now $\hat{\sigma}^{A_*+\varepsilon}_{B_*} \uparrow \gamma$ as $\varepsilon \downarrow 0$ for some stopping time $\gamma \leq \hat{\sigma}^{A_*}_{B_*}$. So to prove the result it remains to show that $\gamma \geq \hat{\sigma}^{A_*}_{B_*}$. By the time-space condition (6.3) we have that $X^{A_*+\varepsilon}_{\hat{\sigma}^{A_*+\varepsilon}_{B_*}} \to X^{A_*}_\gamma$

P-a.s. as $\varepsilon \downarrow 0$. Now since $X^{A_*+\varepsilon}_{\hat{\sigma}^{A_*+\varepsilon}_{B_*}} \geq B_*$ for each $\varepsilon > 0$ it follows that $X^{A_*}_\gamma \geq B_*$.
But this implies that $\gamma \geq \hat{\sigma}^{A_*}_{B_*}$ and this proves the required result.   □

Proposition 6.2 *Suppose that the payoff functions* G_i, H_i *for* $i = 1, 2$ *are also assumed to be bounded. Then the value functions* $V^1_{\sigma_{B_*}}$ *and* $V^2_{\tau_{A_*}}$ *are continuous at* A_* *and* B_* *respectively.*

Proof We shall only prove the result for $V^1_{\sigma_{B_*}}$ as for $V^2_{\tau_{A_*}}$ the result will follow by symmetry. To prove this it is sufficient to show that

$$\lim_{\varepsilon \downarrow 0} \left(V^1_{\sigma_{B_*}}(A_* + \varepsilon) - V^1_{\sigma_{B_*}}(A_*) \right) = 0 \tag{6.5}$$

because $\lim_{\varepsilon \downarrow 0}(V^1_{\sigma_{B_*}}(A_* - \varepsilon) - V^1_{\sigma_{B_*}}(A_*)) = \lim_{\varepsilon \downarrow 0}(G_1(A_* - \varepsilon) - G_1(A_*)) = 0$
by continuity of G_1. Since $V^1_{\sigma_{B_*}}(x) \geq G_1(x)$ for all $x \in \mathbb{R}$ and $V^1_{\sigma_{B_*}}(A_*) = G_1(A_*)$
we get that $V^1_{\sigma_{B_*}}(A_* + \varepsilon) - V^1_{\sigma_{B_*}}(A_*) \geq G_1(A_* + \varepsilon) - G_1(A_*)$ for every $\varepsilon > 0$.
So by continuity of G_1 we have that $\liminf_{\varepsilon \downarrow 0}(V^1_{\sigma_{B_*}}(A_* + \varepsilon) - V^1_{\sigma_{B_*}}(A_*)) \geq 0$. We
next show that $\limsup_{\varepsilon \downarrow 0}(V^1_{\sigma_{B_*}}(A_* + \varepsilon) - V^1_{\sigma_{B_*}}(A_*)) \leq 0$ so that we get the required
result. Given $\varepsilon > 0$, let $\tau^{A_*+\varepsilon}_{A_*} = \inf\{t \geq 0 : X_t \leq A_*\}$ be the optimal stopping
time for player one under $\mathsf{P}_{A_*+\varepsilon}$. Then we have that $\tau^{A_*+\varepsilon}_{A_*}$ is equally distributed as
$\hat{\tau}^{A_*+\varepsilon}_{A_*} = \inf\{t \geq 0 : X^{A_*+\varepsilon}_t \leq A_*\}$ under P. Now from the optimality of $\tau^{A_*+\varepsilon}_{A_*}$ under
$\mathsf{P}_{A_*+\varepsilon}$ we have that

$$
\begin{aligned}
&V^1_{\sigma_{B_*}}(A_* + \varepsilon) - V^1_{\sigma_{B_*}}(A_*) \\
&\leq \mathsf{M}^1_{A_*+\varepsilon}(\tau^{A_*+\varepsilon}_{A_*}, \sigma^{A_*+\varepsilon}_{B_*}) - \mathsf{M}^1_{A_*}(\tau^{A_*+\varepsilon}_{A_*}, \sigma^{A_*}_{B_*}) \\
&= \mathsf{E}_{A_*+\varepsilon}\left[G_1(X_{\tau^{A_*+\varepsilon}_{A_*}})I\left(\tau^{A_*+\varepsilon}_{A_*} \leq \sigma^{A_*+\varepsilon}_{B_*}\right)\right] - \mathsf{E}_{A_*}\left[G_1(X_{\tau^{A_*+\varepsilon}_{A_*}})I\left(\tau^{A_*+\varepsilon}_{A_*} \leq \sigma^{A_*}_{B_*}\right)\right] \\
&\quad + \mathsf{E}_{A_*+\varepsilon}\left[H_1(X_{\sigma^{A_*+\varepsilon}_{B_*}})I\left(\sigma^{A_*+\varepsilon}_{B_*} < \tau^{A_*+\varepsilon}_{A_*}\right)\right] - \mathsf{E}_{A_*}\left[H_1(X_{\sigma^{A_*}_{B_*}})I\left(\sigma^{A_*}_{B_*} < \tau^{A_*+\varepsilon}_{A_*}\right)\right] \\
&= \mathsf{E}\left[G_1(X^{A_*+\varepsilon}_{\hat{\tau}^{A_*+\varepsilon}_{A_*}})I\left(\hat{\tau}^{A_*+\varepsilon}_{A_*} \leq \hat{\sigma}^{A_*+\varepsilon}_{B_*}\right) - G_1(X^{A_*}_{\hat{\tau}^{A_*+\varepsilon}_{A_*}})I\left(\hat{\tau}^{A_*+\varepsilon}_{A_*} \leq \hat{\sigma}^{A_*}_{B_*}\right)\right] \\
&\quad + \mathsf{E}\left[H_1(X^{A_*+\varepsilon}_{\hat{\sigma}^{A_*+\varepsilon}_{B_*}})I\left(\hat{\sigma}^{A_*+\varepsilon}_{B_*} < \hat{\tau}^{A_*+\varepsilon}_{A_*}\right) - H_1(X^{A_*}_{\hat{\sigma}^{A_*}_{B_*}})I\left(\hat{\sigma}^{A_*}_{B_*} < \hat{\tau}^{A_*+\varepsilon}_{A_*}\right)\right] \tag{6.6}
\end{aligned}
$$

The first expectation in the last expression on the right hand side of (6.6) can be written as

$$
\begin{aligned}
&\mathsf{E}\big[(G_1(X^{A_*+\varepsilon}_{\hat{\tau}^{A_*+\varepsilon}_{A_*}})I(\hat{\tau}^{A_*+\varepsilon}_{A_*} \leq \hat{\sigma}^{A_*+\varepsilon}_{B_*}) - G_1(X^{A_*}_{\hat{\tau}^{A_*+\varepsilon}_{A_*}})I(\hat{\tau}^{A_*+\varepsilon}_{A_*} \leq \hat{\sigma}^{A_*}_{B_*}))I(\hat{\tau}^{A_*+\varepsilon}_{A_*} < \hat{\sigma}^{A_*}_{B_*}) \\
&\quad + (G_1(X^{A_*+\varepsilon}_{\hat{\tau}^{A_*+\varepsilon}_{A_*}})I(\hat{\tau}^{A_*+\varepsilon}_{A_*} \leq \hat{\sigma}^{A_*+\varepsilon}_{B_*}) - G_1(X^{A_*}_{\hat{\tau}^{A_*+\varepsilon}_{A_*}})I(\hat{\tau}^{A_*+\varepsilon}_{A_*} \leq \hat{\sigma}^{A_*}_{B_*}))I(\hat{\tau}^{A_*+\varepsilon}_{A_*} = \hat{\sigma}^{A_*}_{B_*}) \\
&\quad + (G_1(X^{A_*+\varepsilon}_{\hat{\tau}^{A_*+\varepsilon}_{A_*}})I(\hat{\tau}^{A_*+\varepsilon}_{A_*} \leq \hat{\sigma}^{A_*+\varepsilon}_{B_*}) - G_1(X^{A_*}_{\hat{\tau}^{A_*+\varepsilon}_{A_*}})I(\hat{\tau}^{A_*+\varepsilon}_{A_*} \leq \hat{\sigma}^{A_*}_{B_*}))I(\hat{\tau}^{A_*+\varepsilon}_{A_*} > \hat{\sigma}^{A_*}_{B_*})\big] \\
&= \mathsf{E}\big[(G_1(X^{A_*+\varepsilon}_{\hat{\tau}^{A_*+\varepsilon}_{A_*}}) - G_1(X^{A_*}_{\hat{\tau}^{A_*+\varepsilon}_{A_*}}))I(\hat{\tau}^{A_*+\varepsilon}_{A_*} < \hat{\sigma}^{A_*}_{B_*})\big] \tag{6.7}
\end{aligned}
$$

The last expression in (6.7) follows from the fact that $\mathsf{P}(\hat{\tau}_{A_*}^{A_*+\varepsilon} = \hat{\sigma}_{B_*}^{A_*}) = 0$ for all $\varepsilon > 0$ sufficiently small (upon assuming that the hitting times considered are finite). and from the fact that $\sigma_{B_*}^{\hat{A}_*+\varepsilon} \uparrow \sigma_{B_*}^{\hat{A}_*}$ as $\varepsilon \downarrow 0$ (see Lemma 6.1).

In a similar way one can show that the second expectation in the last expression on the right hand side of (6.6) can be written as

$$\mathsf{E}\big[\big(H_1\big(X_{\hat{\sigma}_{B_*}^{A_*+\varepsilon}}^{A_*+\varepsilon}\big) - H_1\big(X_{\hat{\sigma}_{B_*}^{A_*}}^{A_*}\big)\big)I\big(\hat{\sigma}_{B_*}^{A_*} < \hat{\tau}_{A_*}^{A_*+\varepsilon}\big)\big] \qquad (6.8)$$

Since G_1 and H_1 are continuous and bounded then by the time space conditions (6.1) and (6.3) together with Lemma 6.1 and Fatou's lemma we get the required result. □

Remark 6.3 The assumption of boundedness on G_1 and H_1 in Proposition 6.2 can be relaxed. For example, the result will also hold provided that $G_1(X_{\hat{\tau}_{A_*}^{A_*+\varepsilon}}^{A_*+\varepsilon}) - G_1(X_{\hat{\tau}_{A_*}^{A_*+\varepsilon}}^{A_*})$ and $H_1(X_{\hat{\sigma}_{B_*}^{A_*+\varepsilon}}^{A_*+\varepsilon}) - H_1(X_{\hat{\sigma}_{B_*}^{A_*}}^{A_*})$ are bounded above by some integrable random variables $\tilde{Z}_1$ and $\tilde{Z}_2$ respectively. Similarly the boundedness assumption on G_2 and H_2 can be relaxed.

6.2 The principle of double smooth fit

In this section we will consider the special case when X is a one dimensional regular diffusion process and we shall assume that $V_{\sigma_{B_*}}^1$ and $V_{\tau_{A_*}}^2$ are obtained from the double partial superharmonic characterisation as explained in Sect. 5. More precisely we shall assume that the functions u, v introduced in Theorem 5.3 (i.) coincide with those from Theorem 5.3 (ii.) so that a mutual response is assumed to exist. The aim is to use this characterisation to derive the so-called principle of double smooth fit. This principle is an extension of the principle of smooth fit observed in standard optimal stopping problems (see [49]). We note that in the case of more general strong Markov processes in $\mathbb{R}$ this principle may break down. As observed in standard optimal stopping problems this may happen for example when the scale function of X is not differentiable (see [50]) or in the case of Poisson process (see [48]). Carr et. al in [7], for example, also showed that this principle breaks down in a CGMY model.

Remark 6.4 Examples of nonzero-sum optimal stopping games for one dimensional regular diffusion processes, for which the optimal stopping regions are of the threshold type are given in [1] and [12][1]. In particular the authors therein provide sufficient conditions for existence and uniqueness of Nash equilibria.

So suppose that X is a regular diffusion process with values in $\mathbb{R}$. We shall also assume that the fine topology coincides with Euclidean topology so that fine continuity is equivalent to continuity in the classical sense. In this context we can define the scale

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[1] The second manuscript was available to the author after the first draft of the paper was published on The University of Manchester Website http://www.maths.manchester.ac.uk/our-research/research-groups/statistics-and-its-applications/research-reports/.

function S of the process X, that is the mapping $S : \mathbb{R} \to \mathbb{R}$ which is a strictly increasing continuous function satisfying

$$\mathsf{P}_x(\tau_c < \tau_d) = \frac{S(d) - S(x)}{S(d) - S(c)} \quad \text{and} \quad \mathsf{P}_x(\tau_d < \tau_c) = \frac{S(x) - S(c)}{S(d) - S(c)} \qquad (6.9)$$

for any $c < x < d$ where $\tau_y = \inf\{t \geq 0 : X_t = y\}$ for $y \in \mathbb{R}$. Since we are assuming that $D_2 = [B_*, \infty)$ then for any given $a, b \in (-\infty, B_*)$ such that $a < b$ we have

$$u(x) \geq \mathsf{E}_x[u(X_{\tau_{a,b} \wedge \sigma_{B_*}})] = \mathsf{E}_x[u(X_{\tau_{a,b}})] = u(a)\frac{S(b) - S(x)}{S(b) - S(a)} + u(b)\frac{S(x) - S(a)}{S(b) - S(a)} \tag{6.10}$$

for all $a < x < b$, where $\tau_{a,b} = \inf\{t \geq 0 : X_t \notin (a, b)\}$. The first inequality follows from the fact that u is superharmonic in D_2^c (recall Definition 5.1). This means that u is S-concave in every interval in $(-\infty, B_*)$ and as for concave functions this implies that the mapping

$$y \mapsto \frac{u(y) - u(x)}{S(y) - S(x)} \tag{6.11}$$

is decreasing provided that $y \neq x$. By symmetry we have that v is S-concave in every interval in (A_*, ∞) and that the mapping $y \mapsto \frac{v(y) - v(x)}{S(y) - S(x)}$ is decreasing provided $y \neq x$.

From the results for the Dirichlet problem (see for example [49, (7.1.2)-(7.1.3)]) one can show that

$$\mathbb{L}_X u = 0 \text{ and } \mathbb{L}_X v = 0 \text{ in } C_1 \cap C_2 u \big|_{\partial C_1} = G_1 \text{ and } u \big|_{\partial C_2} = H_1$$
$$v \big|_{\partial C_1} = H_2 \text{ and } v \big|_{\partial C_2} = G_2$$

where $C_1 = D_1^c$ and $C_2 = D_2^c$. The aim is to show that $u' \big|_{A_*} = G_1' \big|_{A_*}$ and $v' \big|_{B_*} = G_2' \big|_{B_*}$ These two conditions will be referred to as the principle of double smooth fit. Informally this principle states that the optimal stopping boundary points A_* and B_* must be selected in such a way that u and v are respectively smooth at these points. The proof of this result follows in a similar way as the proof of Theorem 2.3 in [50]. We shall first state the following lemma, the proof of which can be found in [50].

Lemma 6.5 *Suppose that $f, g : \mathbb{R}_+ \to \mathbb{R}$ are two continuous functions such that $f(0) = g(0) = 0$, $f(\varepsilon) > 0$ whenever $\varepsilon > 0$, and $g(\delta) > 0$ whenever $\delta > 0$. Then for every $\varepsilon_n \downarrow 0$ as $n \to \infty$, there exists $\varepsilon_{n_k} \downarrow 0$ and $\delta_k \downarrow 0$ as $k \to \infty$ such that $\lim_{k \to \infty} \frac{f(\varepsilon_{n_k})}{g(\delta_k)} = 1$.*

Proposition 6.6 *Suppose that D_1 is of the form $(-\infty, A_*]$ and D_2 of the form $[B_*, \infty)$ for some points A_*, B_* such that $A_* < B_*$. Suppose that G_1 is differentiable at A_* and G_2 is differentiable at B_*. If the scale function S of X is differentiable at A_* and B_*, then $u'(A_*) = G_1'(A_*)$ and $v'(B_*) = G_2'(B_*)$.*

Proof We shall first consider the case when $S'(A_*) \neq 0$. Since u is superharmonic in $(-\infty, B_*)$ we have that $\mathsf{E}_x[u(X_{\rho \wedge \sigma_{B_*}})] \leq u(x)$ for all stopping times ρ and for all $x \in \mathbb{R}$. Define the exit time $\tau_\varepsilon = \inf \{t \geq 0 : X_t \notin (A_* - \varepsilon, A_* + \varepsilon)\}$ for $\varepsilon > 0$ such that $A_* + \varepsilon < B_*$. Then

$$\mathsf{E}_{A_*}[u(X_{\tau_\varepsilon \wedge \sigma_{D_2}})] = \mathsf{E}_{A_*}[u(X_{\tau_\varepsilon})]$$
$$\leq u(A_*) = u(A_*)\mathsf{P}_{A_*}(X_{\tau_\varepsilon} = A_* + \varepsilon) + G_1(A_*)\mathsf{P}_{A_*}(X_{\tau_\varepsilon} = A_* - \varepsilon) \quad (6.12)$$

where the last equality follows from the fact that $u(A_*) = G_1(A_*)$. On the other hand we have

$$\mathsf{E}_{A_*}[u(X_{\tau_\varepsilon})] = u(A_* + \varepsilon)\mathsf{P}_{A_*}(X_{\tau_\varepsilon} = A_* + \varepsilon) + G_1(A_* - \varepsilon)\mathsf{P}_{A_*}(X_{\tau_\varepsilon} = A_* - \varepsilon).$$
$$(6.13)$$

By combining (6.12) and (6.13) it follows that

$$(u(A_* + \varepsilon) - u(A_*))\mathsf{P}_{A_*}\left(X_{\tau_\varepsilon} = A_* + \varepsilon\right)$$
$$\leq (G_1(A_*) - G_1(A_* - \varepsilon))\mathsf{P}_{A_*}\left(X_{\tau_\varepsilon} = A_* - \varepsilon\right). \quad (6.14)$$

Since we are assuming that S and G_1 are differentiable at A_* and that $S'(A_*) \neq 0$ we get, upon using the facts $\mathsf{P}_{A_*}(X_{\tau_\varepsilon} = A_* - \varepsilon) = \frac{S(A_* + \varepsilon) - S(A_*)}{S(A_* + \varepsilon) - S(A_* - \varepsilon)}$ and $\mathsf{P}_{A_*}(X_{\tau_\varepsilon} = A_* + \varepsilon) = \frac{S(A_*) - S(A_* - \varepsilon)}{S(A_* + \varepsilon) - S(A_* - \varepsilon)}$, that

$$\frac{(u(A_* + \varepsilon) - u(A_*))(S(A_*) - S(A_* - \varepsilon))}{S(A_* + \varepsilon) - S(A_* - \varepsilon)}$$
$$\leq \frac{(G_1(A_*) - G_1(A_* - \varepsilon))(S(A_* + \varepsilon) - S(A_*))}{S(A_* + \varepsilon) - S(A_* - \varepsilon)}$$

which is equivalent to (since the scale function is increasing)

$$\frac{u(A_* + \varepsilon) - u(A_*)}{\varepsilon} \leq \frac{(G_1(A_*) - G_1(A_* - \varepsilon))}{\varepsilon} \frac{\frac{(S(A_* + \varepsilon) - S(A_*))}{\varepsilon}}{\frac{(S(A_*) - S(A_* - \varepsilon))}{\varepsilon}}. \quad (6.15)$$

Taking the limit as $\varepsilon \downarrow 0$ on both sides of (6.15) and using the fact that G_1 and S are differentiable at A_* and that $S'(A_*) \neq 0$ we get that $u'(A_*+) = \lim_{\varepsilon \downarrow 0} \frac{u(A_* + \varepsilon) - u(A_*)}{\varepsilon} \leq G_1'(A_*)$. On the other hand we have

$$u(A_* + \varepsilon) - u(A_*) \geq \frac{G_1(A_* + \varepsilon) - G_1(A_*)}{\varepsilon}. \quad (6.16)$$

Taking limits on both sides of (6.16) as $\varepsilon \downarrow 0$ we get that $u'(A_*+) \geq G_1'(A_*)$. So we can conclude that $u'(A_*+) = G_1'(A_*)$. On the other hand, since $u(A_* - \varepsilon) = G_1(A_* - \varepsilon)$ for all $\varepsilon > 0$ sufficiently small we have that $u'(A_*-) = \lim_{\varepsilon \downarrow 0} \frac{u(A_* - \varepsilon) - u(A_*)}{\varepsilon} = G_1'(A_*)$. So the result is proved in the case $S'(A_*) \neq 0$. Now suppose that $S'(A_*) = 0$.

Since $G_1 \leq u$, we have, for any $\varepsilon, \delta > 0$ sufficiently small, that

$$\frac{G_1(A_* + \varepsilon) - G_1(A_*)}{S(A_* + \varepsilon) - S(A_*)} \leq \frac{u(A_* + \varepsilon) - u(A_*)}{S(A_* + \varepsilon) - S(A_*)} \leq \frac{u(A_* - \delta) - u(A_*)}{S(A_* - \delta) - S(A_*)}$$
$$= \frac{G_1(A_* - \delta) - G_1(A_*)}{S(A_* - \delta) - S(A_*)} \tag{6.17}$$

where the second inequality follows from the fact that the mapping $y \mapsto \frac{u(y)-u(x)}{S(y)-S(x)}$ is decreasing. Multiplying both sides of (6.17) by $\frac{S(A_*+\varepsilon)-S(A_*)}{\varepsilon}$ and using the fact that the scale function is increasing we get

$$\frac{G_1(A_* + \varepsilon) - G_1(A_*)}{\varepsilon} \leq \frac{u(A_* + \varepsilon) - u(A_*)}{\varepsilon}$$
$$\leq \frac{\frac{G_1(A_*-\delta)-G_1(A_*)}{-\delta}}{\frac{S(A_*-\delta)-S(A_*)}{-\delta}} \frac{S(A_* + \varepsilon) - S(A_*)}{\varepsilon} \tag{6.18}$$

Setting $f(\varepsilon) := \frac{S(A_*+\varepsilon)-S(A_*)}{\varepsilon}$ and $g(\delta) := \frac{S(A_*-\delta)-S(A_*)}{-\delta}$ in Lemma 6.5 we get that for any $\varepsilon_n \downarrow 0$, there exists $\varepsilon_{n_k} \downarrow 0$ and $\delta_k \downarrow 0$ as $k \to \infty$ such that

$$G_1'(A_*) \leq \lim_{k \to \infty} \frac{u(A_* + \varepsilon_{n_k}) - u(A_*)}{\varepsilon_{n_k}} \leq \lim_{k \to \infty} \frac{f(\varepsilon_{n_k})}{g(\delta_k)} \frac{G_1(A_* - \delta_k) - G_1(A_*)}{-\delta_k}$$
$$= G_1'(A_*). \tag{6.19}$$

From this it follows that that $u'(A_*+) = G_1'(A_*)$. On the other hand if we multiply (6.17) by $\frac{S(A_*-\delta)-S(A_*)}{-\delta}$ we get

$$\frac{\frac{G_1(A_*+\varepsilon)-G_1(A_*)}{\varepsilon}}{\frac{S(A_*+\varepsilon)-S(A_*)}{\varepsilon}} \frac{S(A_* - \delta) - S(A_*)}{-\delta} \leq \frac{u(A_* - \delta) - u(A_*)}{-\delta} = \frac{G_1(A_* - \delta) - G_1(A_*)}{-\delta}$$

Interchanging ε and δ in (6.20) and using Lemma 6.5 again with $f(\varepsilon) := \frac{S(A_*)-S(A_*-\varepsilon)}{\varepsilon}$ and $g(\delta) := \frac{S(A_*+\delta)-S(A_*)}{\delta}$ we get that $\frac{d^- u(A_*)}{dx} = G_1'(A_*)$. This assertion together with $\frac{d^+ u(A_*)}{dx} = G_1'(A_*)$ proves the required result. By symmetry one can show that $v'(B_*) = G_1'(B_*)$.   □

Figure 2a, b show that there can exist more than one Nash equilibrium point. In this example, if both players cooperate and decide to stop the process in the regions D_1 and D_2 given in Fig. 2a, then their expected gains are higher than those earned if they stop the process in the regions given in Fig. 2b. However this is not the case in the example presented in Fig. 2c, d, where it is evident that nothing is gained if the players cooperate. For a more detailed study on the existence and uniqueness of a Nash equilibrium in the case of absorbed Brownian motion in [0,1] and when the class of payoff functions G_i are of a similar form to the one presented in Fig. 2, the reader is referred to [1].
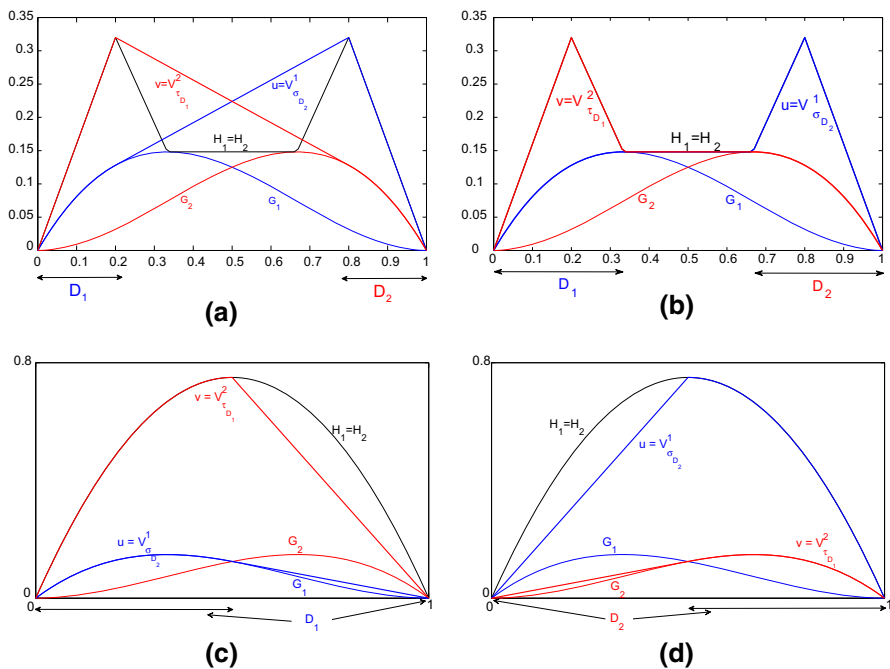
Fig. 2 Examples for absorbed Brownian motion which show that uniqueness of Nash equilibrium fails in general

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