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Erratum to "When is the Albanese morphism an algebraic fiber space in positive characteristic?"

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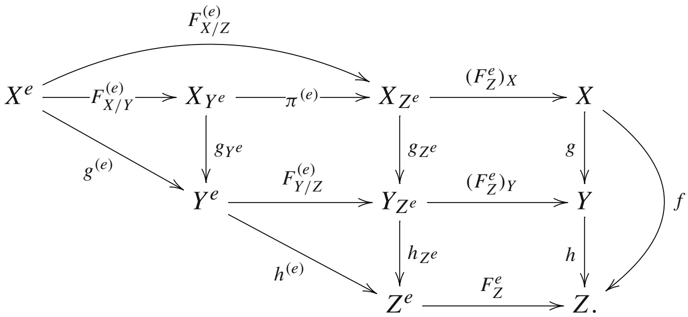
An error in Proposition 5.11 was pointed out by professor Adrian Langer. In the statement of the proposition, the variety Z must be projective. Therefore, Proposition 5.11 should have been stated as follows:

Proposition 5.11. *Let X, Δ, Z and f be as in Definition 5.1, and Y be a normal variety. Assume that $f : X \to Z$ can be factored into projective morphisms $g : X \to Y$ with $g_*\mathcal{O}_X \cong \mathcal{O}_Y$ and $h : Y \to Z$. Suppose that Z is projective.*

(1) *If (f, Δ) is F-split, then so is h.*
(2) *Assume that Y is smooth. If (g, Δ) and h are F-split, then so is (f, Δ).*
(3) *The converse of* (2) *holds if $K_Y \sim_{\mathbb{Z}_{(p)}} h^*K_Z$.*

Furthermore, the proof of statements (2) and (3) of the proposition includes an unclear part and an unsuitable part, so the proof should be modified as follows:

Proof of Proposition 5.11. Let $e > 0$ be an integer. Now we have the following commutative diagram:



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Here, $\pi^{(e)} := (F_{Y/Z}^{(e)})_X$. We first show (1). The above diagram induces the commutative diagram of $\mathcal{O}_{Y_{Z^e}}$-modules

$$\begin{array}{ccc} \mathcal{O}_{Y_{Z^e}} & \longrightarrow & F_{Y/Z*}^{(e)}\mathcal{O}_{Y^e} \\ \cong\downarrow & & \downarrow\cong \\ g_{Z^e*}\mathcal{O}_{X_{Z^e}} & \longrightarrow & g_{Z^e*}F_{X/Z*}^{(e)}\mathcal{O}_{X^e}, \end{array}$$

where the left vertical morphism is an isomorphism because of the flatness of $(F_Z^e)_Y$. Since the lower horizontal morphism splits, so does the upper one.

Next, we show (2) and (3). As explained in Observation 5.4, if (g, Δ) is F-split, then there exists an effective $\mathbb{Z}_{(p)}$-Weil divisor $\Delta' \geq \Delta$ on X such that $K_{X/Y}+\Delta'$ is $\mathbb{Z}_{(p)}$-linearly trivial and that (g, Δ') is also F-split. Therefore, when we prove (2), we may assume that Δ is a $\mathbb{Z}_{(p)}$-Weil divisor and that $(p^e-1)(K_{X/Y}+\Delta) \sim 0$ for every $e > 0$ divisible enough. In this case, $(p^e-1)(K_{X/Z}+\Delta) \sim (p^e-1)g^*K_{Y/Z}$, so $\mathcal{L}_{(X/Z,\Delta)}^{(e)} \cong g^{(e)*}\mathcal{N}_1^{(e)}$ for a line bundle $\mathcal{N}_1^{(e)}$ on Y^e, and $\mathcal{L}_{(X/Y,\Delta)}^{(e)} \cong \mathcal{O}_{X^e} \cong g^{(e)*}\mathcal{O}_{Y^e}$. By an argument similar to the above, when we show (3), we may suppose that for every $e > 0$ divisible enough, $\mathcal{L}_{(X/Y,\Delta)}^{(e)} \cong g^{(e)*}\mathcal{M}_1^{(e)}$ for a line bundle $\mathcal{M}_1^{(e)}$ on Y^e, and $\mathcal{L}_{(X/Z,\Delta)}^{(e)} \cong \mathcal{O}_{X^e} \cong g^{(e)*}\mathcal{O}_{Y^e}$. In summary, since we now prove (2) and (3), we may assume that for every $e > 0$ divisible enough, $\mathcal{L}_{(X/Y,\Delta)}^{(e)} \cong g^{(e)*}\mathcal{M}^{(e)}$ and $\mathcal{L}_{(X/Z,\Delta)}^{(e)} \cong g^{(e)*}\mathcal{N}^{(e)}$ for line bundles $\mathcal{M}^{(e)}$ and $\mathcal{N}^{(e)}$ on Y^e.

Let $V \subseteq Y$ be an open subset such that g is flat at every point in $X_V := g^{-1}(V)$ and $\mathrm{codim}(Y \setminus V) \geq 2$. Let $u : U \to X_V$ be the open immersion of the regular locus of X_V. Set $g' := g \circ u : U \to Y$. We then have $g'_*\mathcal{O}_U \cong \mathcal{O}_Y$ because of the assumptions. Therefore, for every line bundle $\mathcal{N}$ on Y, we see that

$$H^0\left(U, (g^*\mathcal{N})|_U\right) = H^0(U, g'^*\mathcal{N}) \cong H^0(Y, g'_*g'^*\mathcal{N}) \cong H^0(Y, \mathcal{N}) \cong H^0(X, g^*\mathcal{N})$$

by the projection formula. In addition, by the flatness of F_Z^e, we get $g'_{Z^e*}\mathcal{O}_{U_{Z^e}} \cong \mathcal{O}_{Y_{Z^e}}$, and so $H^0\left(U_{Z^e}, \mathcal{O}_{U_{Z^e}}\right) \cong H^0\left(X_{Z^e}, \mathcal{O}_{X_{Z^e}}\right)$ by an argument similar to the above. Hence, we have the following commutative diagram:

$$\begin{array}{ccc} H^0\left(X^e, \mathcal{L}_{(X/Z,\Delta)}^{(e)}\right) & \xrightarrow{H^0\left(X_{Z^e}, \phi_{(X/Z,\Delta)}^{(e)}\right)} & H^0(X_{Z^e}, \mathcal{O}_{X_{Z^e}}) \\ \cong\downarrow & & \downarrow\cong \\ H^0\left(U^e, \mathcal{L}_{(X/Z,\Delta)}^{(e)}|_{U^e}\right) & \xrightarrow{H^0\left(U_{Z^e}, \phi_{(U/Z,\Delta|_U)}^{(e)}\right)} & H^0(U_{Z^e}, \mathcal{O}_{U_{Z^e}}). \end{array}$$

Note that we are assuming that $\mathcal{L}_{(X/Z,\Delta)}^{(e)} \cong g^{(e)*}\mathcal{N}$ for a line bundle $\mathcal{N}$ on Y^e. Since the splitting of $\phi_{(X/Z,\Delta)}^{(e)}$ is clearly equivalent to the surjectivity of $H^0\left(X_{Z^e}, \phi_{(X/Z,\Delta)}^{(e)}\right)$, we see that the F-splitting of (f, Δ) and that of $(f|_U : U \to Z, \Delta|_U)$ are equivalent. By an argument similar to the above, we find that the F-splitting of (g, Δ) and that of $(g|_U, \Delta|_U)$ are also equivalent.

Assume that we can choose $V = Y$ and $U = X$, i.e. X and Y are regular and g is flat. Let $e > 0$ be an integer. By the flatness of g, we have the following commutative diagram:

$$\begin{array}{ccc} {g_{Z^e}}^*\mathcal{O}_{Y_{Z^e}} & \xrightarrow{{g_{Z^e}}^*\left({F_{Y/Z}^{(e)}}^\sharp\right)} & {g_{Z^e}}^*{F_{Y/Z}^{(e)}}_*\mathcal{O}_{Y^e} \\ \cong\downarrow & & \downarrow\cong \\ \mathcal{O}_{X_{Z^e}} & \xrightarrow{\pi^{(e)\sharp}} & {\pi^{(e)}}_*\mathcal{O}_{X_{Y^e}}. \end{array}$$

This implies that

$$\mathcal{H}om\left(\pi^{(e)\sharp}, \mathcal{O}_{X_{Z^e}}\right) \cong {g_{Z^e}}^*\mathcal{H}om\left({F_{Y/Z}^{(e)}}^\sharp, \mathcal{O}_{V_{Z^e}}\right) = {g_{Z^e}}^*\phi_{Y/Z}^{(e)}.$$

Applying the functor $\mathcal{H}om(\_, \mathcal{O}_{X_{Z^e}})$ and the Grothendieck duality to the natural morphism

$$\mathcal{O}_{X_{Z^e}} \xrightarrow{\pi^{(e)\sharp}} {\pi^{(e)}}_*\mathcal{O}_{X_{Y^e}} \to {F_{X/Z}^{(e)}}_*\mathcal{O}_{X^e}(\lceil(p^e - 1)\Delta\rceil),$$

we obtain the morphism

$$\phi_{(X/Z,\Delta)}^{(e)} : {F_{X/Z}^{(e)}}_*\mathcal{L}_{(X/Z,\Delta)}^{(e)} \xrightarrow{{\pi^{(e)}}_*\phi_{(X/Y,\Delta)}^{(e)}\otimes\omega_{\pi^{(e)}}} {g_{Z^e}}^*{F_{Y/Z}^{(e)}}_*\mathcal{L}_{Y/Z}^{(e)} \xrightarrow{{g_{Z^e}}^*\phi_{Y/Z}^{(e)}} \mathcal{O}_{X_{Z^e}}.$$

Note that

$$\omega_{\pi^{(e)}} \cong \omega_{X_{Y^e}} \otimes {\pi^{(e)}}^*\omega_{X_{Z^e}} \cong {g_{Z^e}}^*\omega_{Y^e/Z^e}^{1-p^e} \quad \text{and} \quad {g_{Z^e}}^*{F_{Y/Z}^{(e)}}_*\mathcal{L}_{Y/Z}^{(e)} \cong {\pi^{(e)}}_*{g_{Y^e}}^*\mathcal{L}_{Y/Z}^{(e)}.$$

We now prove the assertion. If (g, Δ) is F-split and h is F-split, then both of $\phi_{(X/Y,\Delta)}^{(e)}$ and $\phi_{Y/Z}^{(e)}$ split for every $e > 0$ divisible enough. Therefore, $\phi_{(X/Z,\Delta)}^{(e)}$ also splits, i.e. (f, Δ) is F-split. Conversely, suppose that (f, Δ) is F-split and that $(p^e - 1)K_{Y/Z} \sim 0$ for an $e > 0$. Then, $\mathcal{L}_{Y/Z}^{(e)} \cong \mathcal{O}_{Y_{Z^e}}$ and $\omega_{\pi^{(e)}} \cong \mathcal{O}_{X_{Y^e}}$. Fix an $e > 0$ divisible enough. Since $H^0\left(X_{Z^e}, \phi_{(X/Z,\Delta)}^{(e)}\right)$ is surjective, $H^0\left(X_{Z^e}, {\pi^{(e)}}_*\phi_{(X/Y,\Delta)}^{(e)}\right)$ is a nonzero morphism, and hence so is $H^0\left(X_{Y^e}, \phi_{(X/Y,\Delta)}^{(e)}\right)$. This morphism is surjective because of $H^0(X_{Y^e}, \mathcal{O}_{X_{Y^e}}) \cong H^0(Y^e, \mathcal{O}_{Y^e}) \cong k$. Thus, $\phi_{(X/Y,\Delta)}^{(e)}$ splits, and so (g, Δ) is F-split. Note that the F-splitting of h follows directly from (1). □

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