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K. Leschke · K. Moriya

The μ-Darboux transformation of minimal surfaces

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Abstract. The classical notion of the Darboux transformation of isothermic surfaces can be generalised to a transformation for conformal immersions. Since a minimal surface is Willmore, we can use the associated $\mathbb{C}_*$-family of flat connections of the harmonic conformal Gauss map to construct such transforms, the so-called μ-Darboux transforms. We show that a μ-Darboux transform of a minimal surface is not minimal but a Willmore surface in 4-space. More precisely, we show that a μ-Darboux transform of a minimal surface f is a twistor projection of a holomorphic curve in $\mathbb{CP}^3$ which is canonically associated to a minimal surface $f_{p,q}$ in the right-associated family of f. Here we use an extension of the notion of the associated family $f_{p,q}$ of a minimal surface to allow quaternionic parameters. We prove that the pointwise limit of Darboux transforms of f is the associated Willmore surface of f at $\mu = 1$. Moreover, the family of Willmore surfaces μ-Darboux transforms, $\mu \in \mathbb{C}_*$, extends to a $\mathbb{CP}^1$ family of Willmore surfaces $f^\mu : M \to S^4$ where $\mu \in \mathbb{CP}^1$.

1. Introduction

A classical Darboux pair is given geometrically by a pair of conformal immersions $(f, f^\sharp)$ into 3-space such that there exists a sphere congruence conformally enveloping both surfaces [10]. In this case, both surfaces f and $f^\sharp$ are isothermic, that is, they allow a conformal curvature line parametrisation. Algebraically, one obtains a classical Darboux transform of an isothermic surface by a solution to a Riccati equation which is given in terms of a dual isothermic surface and a real parameter, [17]. This directly links to integrability: the parameter can be considered as the spectral parameter of an integrable system [2]. Put differently, the Darboux transform is given in terms of a parallel section of an associated family of flat connections, e.g. [6,16]. In [3] the Darboux transformation has been generalised to arbitrary conformal immersions in the 4-sphere: geometrically, the enveloping

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K. Leschke (✉): Department of Mathematics, University of Leicester, University Road, Leicester LE1 7RH, UK. e-mail: k.leschke@le.ac.uk

K. Moriya: Department of Material Science, Graduate School of Material Science, University of Hyogo, 2167, Shosha, Himeji, Hyogo 671-2280, Japan.

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condition is weakened, algebraically, one uses quaternionic holomorphic sections instead of parallel sections. This way, one obtains a geometric interpretation of the spectral curve of a conformal torus as the normalisation of the set of all closed Darboux transforms: conformal tori can be constructed by spectral data. At the same time the Darboux transformation provides a tool to construct new surfaces from given, simpler ones while controlling the closing conditions.

In recent papers, it has been investigated how the classical spectral parameter corresponds to such generalised Darboux transforms. In case of a CMC surface, that is an immersion $f : M \to \mathbb{R}^3$ from a Riemann surface M into 3-space with non-vanishing constant mean curvature, the Gauss map N of f is harmonic by the Ruh–Vilms theorem. The introduction of a spectral parameter $\lambda \in \mathbb{C}_*$ gives rise to a $\mathbb{C}_*$-family of flat connections given by the harmonic Gauss map. A parallel section acts on N by conjugation and gives the associated family of harmonic maps when $\lambda \in S^1$, and thus CMC surfaces via the Sym–Bobenko formula. On the other hand, for fixed μ, a parallel section of the associated family of flat connections defines a quaternionic holomorphic section, and this way a special Darboux transform, a so-called *μ-Darboux transform* [9]. CMC surfaces are isothermic however a μ-Darboux transform of a CMC surface is only a classical Darboux transform in $\mathbb{R}^3$ if $\mu \in \mathbb{R}_* \cup S^1$. In general a μ-Darboux transform takes values in $\mathbb{R}^4$ but it can be shown to be a CMC surface in $\mathbb{R}^3$ up to a constant translation in $\mathbb{R}^4$. Moreover, the desingularisation of the set of closed μ-Darboux transforms of a CMC torus is biholomorphic to Hitchin's eigenline spectral curve [9,15].

Similar results hold for Willmore surfaces and Hamiltonian stationary Lagrangians [4,19,22]: again the associated harmonic map allows to define a family of flat connections and through those, μ-Darboux transforms which are again Willmore and Hamiltonian stationary respectively.

The case of minimal surfaces $f : M \to \mathbb{R}^3$ is surprisingly more complicated. Since a minimal surface has constant mean curvature its Gauss map is harmonic by the Ruh–Vilms theorem. In particular, there is an associated family of flat connections. However, the Gauss map does not determine a minimal surface uniquely so further information needs to be incorporated into the associated family of flat connections to encode a minimal surface via spectral data. To do so we consider in this paper a minimal surface as a Willmore surface: the conformal Gauss map of a minimal surface is harmonic and thus gives rise to a family of flat connections d_λ with $\lambda \in \mathbb{C}_*$. Parallel sections on the unit circle give again via conjugation new harmonic conformal Gauss maps and the associated surfaces for $\lambda = e^{i\theta} \in S^1$ give the classical associated family $f_\theta = \cos\theta f + \sin\theta f^*$ of isometric minimal surfaces where f^* is the conjugate minimal surface of f. This construction can be extended off the unit circle, and one obtains a generalised *left-and right-associated family* of minimal surfaces given by the $f^{p,q} = pf + qf^*$ and $f_{p,q} = fp + f^*q$ respectively where $p, q \in \mathbb{H}_*$ [21]. On the other hand, for fixed μ a parallel section of the flat connection d_μ in the associated family again gives rise to generalised Darboux transforms. As before, we call Darboux transforms which are given by parallel sections *μ-Darboux transforms*.

μ-Darboux transforms are not classical Darboux transforms but we show that a non-trivial μ-Darboux transform of a minimal surface $f : M \to \mathbb{R}^3$ is a twistor projection of a holomorphic curve in $\mathbb{CP}^3$. Therefore, a μ-Darboux transform f^μ is given by complex holomorphic data but we show that f^μ is not minimal if f is not a plane.

More precisely, recall [11,24] that a minimal surface has an *associated Willmore surface* which is the twistor projection of a holomorphic curve in complex projective 3-space: In case of a minimal surface $f : M \to \mathbb{R}^3$ the associated Willmore surface $f^\flat$ is the conformal immersion in 4-space which is given by

$$f^\flat = \begin{pmatrix} -< f, N > \\ f \times N - f^* \end{pmatrix},$$

where N is the Gauss map of f and f^* is a conjugate of f. We show that a μ-Darboux transform of f is the associated Willmore surfaces of an element of the right-associated family $f_{p,q}$ of f for $\mu \in \mathbb{C} \setminus \{0, 1\}$. At $\mu = -1$ we obtain the associated Willmore surface of the conjugate surface. For $\mu = 1$ all μ-Darboux transforms are constant, however, the limit of (appropriately scaled and rotated) Darboux transforms of f is the associated Willmore surface of f at $\mu = 1$.

Finally we show that the family of Willmore surfaces of μ-Darboux transforms, $\mu \in \mathbb{C}_*$, extends to a $\mathbb{CP}^1$ family of Willmore surfaces $f^\mu : M \to S^4$ where $\mu \in \mathbb{CP}^1$: the limits of μ-Darboux transforms at $\mu = 0, \infty$ are Darboux transforms $f^{0,\infty}$ in the 4-sphere of f but not μ-Darboux transforms since the associated family d_μ does not extend to $\mu = 0, \infty$. In fact, in an affine coordinate $f^{0,\infty}$ are minimal surfaces in $\mathbb{R}^4$ with an isolated set of ends.

2. Generalised Darboux transforms

We first recall some basic facts about conformal immersions in Euclidean space which will be needed in the following whilst setting up our notation. Although we are mostly interested in minimal surfaces in $\mathbb{R}^3$, some of our transforms will be surfaces in S^4. Therefore, we will study more generally conformal immersions in $\mathbb{R}^4$ and S^4. We recall basic facts in the quaternionic formalism, for details see [1].

2.1. Conformal immersions

We consider conformal immersions $f : M \to \mathbb{R}^4$ from a Riemann surface M into 4-space. In this paper, we model Euclidean 4-space by the quaternions $\mathbb{R}^4 = \mathbb{H}$, and the Euclidean 3-space in $\mathbb{R}^4$ by the imaginary quaternions $\mathbb{R}^3 = \operatorname{Im}\mathbb{H}$. Denote the complex structure of the Riemann surface M by J_{TM} and put

$$*\omega(X) = \omega(J_{TM}X)$$

for a 1-form $\omega \in \Omega^1(TM)$, $X \in TM$. Thus, $*$ is the negative Hodge star operator.

The conformality of an immersion $f : M \to \mathbb{R}^4$ gives [1, p. 10] the *left* and *right normal* $N, R : M \to S^2 = \{n \in \operatorname{Im}\mathbb{H} \mid n^2 = -1\}$ of f by

$$* df = N df = -df R, \tag{1}$$

Then the *Gauss map* of f is given by the map $(N, R) : M \to S^2 \times S^2 = \mathrm{Gr}_2(\mathbb{R}^4)$ and the *mean curvature vector* $\mathcal{H}$ of $f : M \to \mathbb{R}^4$ satisfies [1, p. 39]

$$\bar{\mathcal{H}} df = \frac{1}{2}(*dR + RdR), \quad \text{or, equivalently,} \quad df\bar{\mathcal{H}} = -\frac{1}{2}(*dN + NdN).$$

Since $\mathcal{H}$ is normal we have $N\mathcal{H} = \mathcal{H}R$. We put $H = -R\bar{\mathcal{H}}$ and denote by

$$(dR)' = \frac{1}{2}(dR - R * dR), \quad (dR)'' = \frac{1}{2}(dR + R * dR)$$

the $(1, 0)$ and $(0, 1)$-part of dR with respect to the complex structure R. Then the equation of the mean curvature vector becomes

$$Hdf = (dR)'. \tag{2}$$

Similarly, the equation for the mean curvature vector in terms of the left normal is

$$dfH = \frac{1}{2}(dN - N * dN) = (dN)'.$$

Note that if $f : M \to \mathbb{R}^3$ then H is the mean curvature of f.

If f is conformal then

$$d * df = dfH \wedge df, \tag{3}$$

where we used that $(dN)'' \wedge df = 0$ by type so that $dN \wedge df = (dN)' \wedge df$. For $f : M \to \mathbb{R}^3$ this is the well-known link between the Laplacian and the mean curvature, in this formulation see e.g. [20].

2.2. *General Darboux transformation*

Let us recall that two immersions $f, f^\sharp : M \to \mathbb{R}^4$ form a *classical Darboux pair* [10] if there exists a sphere congruence enveloping both f and $f^\sharp$. In this case, both f and $f^\sharp$ are *isothermic*, that is, they allow a conformal curvature line parametrisation. A classical Darboux transform $f^\sharp = f + T$ of f is given in terms of its *dual surface (or Christoffel surface)* f^d which is defined [8,16] by the property that

$$df^d \wedge df = df \wedge df^d = 0.$$

Then a classical Darboux tranform $f^\sharp = f + T$ is given, see [17], by a solution T of the Riccati equation

$$dT = -df + Tdf^d rT \tag{4}$$

where f^d is a dual surface of f and $r \in \mathbb{R}_*$.

By weakening the enveloping condition the notion of a classical Darboux transformation has been extended in [3] to general conformal immersions $f : M \to S^4$. In case of a conformal torus $f : T^2 \to S^4$, there exists at least a Riemann surface worth of Darboux transforms $f^\sharp : T^2 \to S^4$ of f. This way, one obtains a geometric interpretation of the spectral curve Σ of the conformal torus f as the normalisation of the set of closed Darboux transforms of f.

For the purposes of this paper, it is more useful to see generalised Darboux transforms as prolongations of holomorphic sections: Using the one-point compactification of $\mathbb{R}^4$ we consider a conformal immersion $f : M \to \mathbb{R}^4$ as a conformal immersion into the 4-sphere. We identify the 4-sphere $S^4 = \mathbb{HP}^1$ with the quaternionic projective line where the oriented Möbius transformations are given by GL(2, $\mathbb{H}$). In particular, a map $f : M \to \mathbb{HP}^1$ can be identified with a line subbundle $L \subset \underline{\mathbb{H}}^2 = M \times \mathbb{H}^2$ of the trivial $\mathbb{H}^2$ bundle over M whose fibers at $p \in M$ are given by

$$L_p = f(p).$$

For an immersion $f : M \to \mathbb{R}^4$ the line bundle L is given by

$$L = \psi\mathbb{H}, \quad \text{where} \quad \psi = \begin{pmatrix} f \\ 1 \end{pmatrix},$$

when choosing the point at infinity as $\infty = e\mathbb{H} \in \mathbb{HP}^1$ where

$$e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Given a conformal immersion $f : M \to \mathbb{R}^4$ with associated line bundle L, the left normal N of f induces a quaternionic holomorphic structure on the bundle $\underline{\mathbb{H}}^2/L$ via

$$D(e\alpha) = e\frac{1}{2}(d\alpha + N * d\alpha). \tag{5}$$

Here we identify $\underline{\mathbb{H}}^2/L = e\mathbb{H}$ via $(\pi_L)|_{e\mathbb{H}} : e\mathbb{H} \to \underline{\mathbb{H}}^2/L$ where $\pi_L : \underline{\mathbb{H}}^2 \to \underline{\mathbb{H}}^2/L$ is the canonical projection.

Denote by $\tilde{V}$ the bundle which is given by the pullback of a vector bundle V over M to the universal cover $\tilde{M}$ of M. Then D induces a holomorphic structure on $\widetilde{\underline{\mathbb{H}}^2/L}$ which we denote again, in abuse of notation, by D.

Definition 2.1. ([3,14]) Let $f : M \to \mathbb{R}^4$ be conformal with left normal N and L its associated line bundle.

A section $e\alpha \in \Gamma(\widetilde{e\mathbb{H}}) = \Gamma(\widetilde{\underline{\mathbb{H}}^2/L})$ is called *holomorphic* if $D(e\alpha) = 0$, or, equivalently, if $*d\alpha = Nd\alpha$.

Since $*df = Ndf$, for any holomorphic section $e\alpha$ there is $\beta : \tilde{M} \to \mathbb{H}$ with $d\alpha = -df\beta$. In particular, there exists a unique *prolongation* of the holomorphic section $e\alpha$, that is, a lift $\varphi = e\alpha + \psi\beta \in \Gamma(\widetilde{\underline{\mathbb{H}}^2})$ such that $d\varphi \in \Omega^1(\tilde{L})$.

Definition 2.2. ([3]) Let $f : M \to S^4$ be a conformal immersion. A *(generalised) Darboux transform* $f^\sharp : \tilde{M} \to S^4$ of f is given by the prolongation $\varphi \in \Gamma(\widetilde{\underline{\mathbb{H}}^2})$ of a holomorphic section of $\widetilde{\underline{\mathbb{H}}^2/L}$.

Away from the zeros of φ, the (singular) Darboux transform is given by the line bundle

$$L^\sharp = \varphi\mathbb{H}.$$

We obtain Darboux transforms of a conformal immersion f by finding non-trivial sections $\varphi \in \Gamma(\widetilde{\underline{\mathbb{H}}^2})$ with $d\varphi \in \Omega^1(\tilde{L})$: writing $\varphi = e\alpha + \psi\beta$ we see that $\alpha \neq 0$ since otherwise $0 = \pi_L d\varphi = edf\beta$ implies $\varphi = 0$. But then $\pi_L\varphi = e\alpha$ is a non-trivial holomorphic section since $*d\alpha = - * df\beta = Nd\alpha$ and φ is its prolongation.

If $f : M \to \mathbb{R}^4$ we write $T = \alpha\beta^{-1}$ and the Darboux transform is given as

$$f^\sharp = f + T$$

away of the zeros of β where T satisfies a generalisation of the Riccati equation (4)

$$dT = -df - Td\beta\alpha^{-1}T. \tag{6}$$

Here we used that $d\alpha = -df\beta$ since $d\varphi \in \Omega^1(\tilde{L})$. Moreover, $f^\sharp$ is a classical Darboux tranform if and only if $d\beta\alpha^{-1}$ is a closed 1-form. In this case, f is isothermic and a dual surface f^d of f is given by $df^d = d\beta\alpha^{-1}$, see [18].

Note that for α constant, we obtain the constant Darboux transform $f^\sharp = \infty$ given by the point $\infty = e\mathbb{H}$ at infinity.

2.3. *Willmore surfaces*

Willmore surfaces are critical points of the Willmore energy. This notion is conformally invariant hence it is useful to consider the conformal Gauss map rather than the Gauss map of a Willmore surface. Geometrically, the conformal Gauss map is a sphere congruence which is tangent at each point and has the same mean curvature vector as the surface at corresponding points. For our purposes it is convenient to model the conformal Gauss map by a complex structure:

Definition 2.3. ([1, p. 27]) The *conformal Gauss map* of a conformal immersion $f : M \to S^4$ is the unique complex structure S on $\underline{\mathbb{H}}^2$ such that S and dS stabilise the line bundle L of f and its Hopf field A is a 1-form with values in L.

Here, the *Hopf field* A of S is the 1-form given by

$$A = \frac{1}{4}(*dS + SdS) = \frac{1}{2}(*dS)'$$

where $(dS)' = \frac{1}{2}(dS - S * dS)$ is the $(1,0)$-part of the derivative of S with respect to the complex structure S.

In affine coordinates the conformal Gauss map is given by, see [1, p. 42],

$$S = \begin{pmatrix} N & fR - Nf \\ 0 & R \end{pmatrix} \tag{7}$$

where N, R are the left and right normals of f respectively and H is given by the mean curvature vector (2). Moreover, the Hopf field computes [1, Prop 12, p. 44] to

$$2 * A = \begin{pmatrix} f\omega & -f\omega f + f(dR)'' \\ \omega & -\omega f + (dR)'' \end{pmatrix} \tag{8}$$

where

$$(dR)'' = \frac{1}{2}(dR + R * dR)$$

and $\omega = \frac{1}{2}(dH + H * df H + R * dH - H * dN)$. Since $HN = RH$ and thus

$$\omega = \frac{1}{2}(dH + H * df H + *dHN - *dRH)$$

we see with (2) that ω satisfies

$$*\omega = -\omega N - (dR)'' H.$$

Theorem 2.4. ([1,13,25]) *Let $f : M \to S^4$ be a conformal immersion with conformal Gauss map S and Hopf field A. Then f is Willmore if and only if S is harmonic, that is, if and only if*

$$d * A = 0.$$

An important example are Willmore surfaces which are given by twistor projections of holomorphic curves:

Theorem 2.5. ([1, Thm 4, p. 47]) *Let $f : M \to \mathbb{R}^4$ be a conformal immersion with right normal R. Then f is the twistor projection of a holomorphic curve $F : M \to \mathbb{CP}^3$ if and only if $(dR)'' = \frac{1}{2}(dR + R * dR) = 0$.*

In this case, f is Willmore and the twistor lift of f is the holomorphic curve $F : M \to \mathbb{CP}^3$ which is given by the line subbundle $E \subset L$ via

$$F(p) = E_p,$$

where E is the $+i$ eigenspace of the conformal Gauss map of f restricted to the line bundle L of f. Moreover, the Hopf field A of f vanishes.

In particular, by [23] if $f : S^2 \to S^4$ is a Willmore sphere then f is either the stereographic projection of a minimal surface in $\mathbb{R}^4$ or a twistor projection of a holomorphic (or anti-holomorphic) curve in $\mathbb{CP}^3$.

2.4. μ-Darboux transforms

It is well-known [26] that an appropriate harmonic map gives rise to a family of flat connections. We now consider the associated family of flat connections of the conformal Gauss map S of a Willmore surface $f : M \to \mathbb{R}^4$, and use parallel sections to construct Darboux transforms of f.

We identify $\mathbb{C}^4 = (\mathbb{H}^2, I)$ where I is given by right multiplication by the unit quaternion i. If the conformal Gauss map of a conformal immersion $f : M \to S^4$ is harmonic, that is $d * A = 0$, the $\mathbb{C}_*$-family of connections

$$d_\lambda = d + (\lambda - 1)A^{(1,0)} + (\lambda^{-1} - 1)A^{(0,1)}, \quad \lambda \in \mathbb{C}_*, \tag{9}$$

is flat [19] on the trivial $\mathbb{C}^4$ bundle over M where A is the Hopf field of S and

$$A^{(1,0)} = \frac{1}{2}(A - I * A) \quad \text{and} \quad A^{(0,1)} = \frac{1}{2}(A + I * A)$$

denote the $(1, 0)$ and $(0, 1)$ parts of A with respect to I.

Since by definition of the Hopf field im $A \subset L$ we see that for fixed $\mu \in \mathbb{C}_*$ every d_μ-parallel section $\varphi \in \Gamma(\widetilde{\underline{\mathbb{H}}^2})$ has $d\varphi \in \Omega^1(\tilde{L})$, and thus $L^\sharp = \varphi\mathbb{H}$ is a Darboux transform of f on the universal cover $\tilde{M}$ of M.

Definition 2.6. Let $f : M \to S^4$ be a Willmore surface. Let $\mu \in \mathbb{C}_*$ and $\varphi \in \Gamma(\tilde{M} \times \mathbb{H}^2)$ be a non-trivial parallel section of the flat connection d_μ where $\tilde{M}$ is the universal cover of M. Then $f^\sharp = \varphi\mathbb{H} : \tilde{M} \to S^4$ is called a *μ-Darboux transform* of f.

Note that φ is a nowhere vanishing section since φ is d_μ-parallel. In particular, the μ-Darboux transform is well-defined on $\tilde{M}$.

3. Minimal surfaces

We first give some basic facts on minimal surfaces in Euclidean 4-space which will be needed in the following, in particular, we recall the extension of the associated family of minimal surfaces to allow quaternionic parameters, and characterise minimal surfaces by their Hopf fields, for details see e.g. [20,21].

3.1. Minimal surfaces in $\mathbb{R}^4$

Let $f : M \to \mathbb{R}^4$ be a conformal (branched) immersion from a Riemann surface M into 4-space. Then f is called *minimal* if its mean curvature vector is vanishing. In terms of the Gauss map of f we obtain with (2) that $f : M \to \mathbb{R}^4$ is minimal if and only if

$$(dR)' = 0, \quad \text{or, equivalently,} \quad (dN)' = 0.$$

In other words, if f is minimal then

$$*dR = -RdR = dRR \tag{10}$$

and $*dN = -NdN = dNN$ for the right and left normals of f respectively. Thus, both N and R are quaternionic holomorphic sections [14] with respect to the induced quaternionic holomorphic structures on the trivial $\mathbb{H}$ bundle $\underline{\mathbb{H}} = M \times \mathbb{H}$. Note also that a map $R : M \to S^2$ is harmonic if and only if

$$d(dR)' = 0 \quad \text{or, equivalently,} \quad d(dR)'' = 0. \tag{11}$$

In particular, both the left and right normal N and R of a minimal surface are conformal and harmonic.

On the other hand, (3) shows that f is minimal if and only if f is harmonic, i.e.,

$$d * df = 0.$$

In particular, $*df$ is closed if f is harmonic and there exists a *conjugate surface* f^* on the universal cover $\tilde{M}$ of M, given up to translation by

$$df^* = - * df.$$

Next, we observe that a conjugate surface f^* of a minimal surface f has the same left and right normal as f since

$$*df^* = - * (*df) = df = *df R = -df^* R,$$

and similarly $*df^* = N df^*$.

Note that f^* is minimal, and so is the *associated family* (when lifting f to the universal cover $\tilde{M}$), e.g. [12],

$$f_{\cos\theta,\sin\theta} = f\cos\theta + f^*\sin\theta, \quad \theta \in \mathbb{R}.$$

The associated family can be extended to a family depending on quaternionic parameters:

Definition 3.1. ([21]) Let $f : M \to \mathbb{R}^4$ be a minimal surface. The family of (branched) minimal immersions

$$f_{p,q} = fp + f^*q : \tilde{M} \to \mathbb{R}^4, \quad p, q \in \mathbb{H}, (p, q) \neq (0, 0), \tag{12}$$

where $f^* : \tilde{M} \to \mathbb{R}^4$ is a conjugate surface of f, is called the *right associated family* of f.

Note that for $p, q \in \mathbb{R}$, $(p, q) \neq (0, 0)$, we obtain the usual associated family of a minimal surface up to scaling. Moreover, $f_{pn,qn} = f_{p,q}n$ is given by a scaling of $f_{p,q}$ and an isometry on $\mathbb{R}^4$ for $n \in \mathbb{H}_*$.

Remark 3.2. Since $df_{p,q} = df(p+Rq)$ we see that $f_{p,q}$ has left normal $N_{p,q} = N$ and right normal $R_{p,q} = (p + Rq)^{-1}R(p + Rq)$.

In particular, for any immersion $f : M \to \mathbb{R}^3 = \operatorname{Im}\mathbb{H}$ in Euclidean 3-space, the left and right normal coincide and a surface $f_{p,q}$ in the right associated family has left normal $N_{p,q} = N$ and right normal $R_{p,q} = (p + Nq)^{-1}N(p + Nq)$. This shows that in general $N_{p,q} \neq R_{p,q}$ and thus, elements of the right associated families of a minimal surface $f : M \to \mathbb{R}^3$ are not necessarily minimal in 3-space but are minimal surface (Fig 1) in $\mathbb{R}^4$.

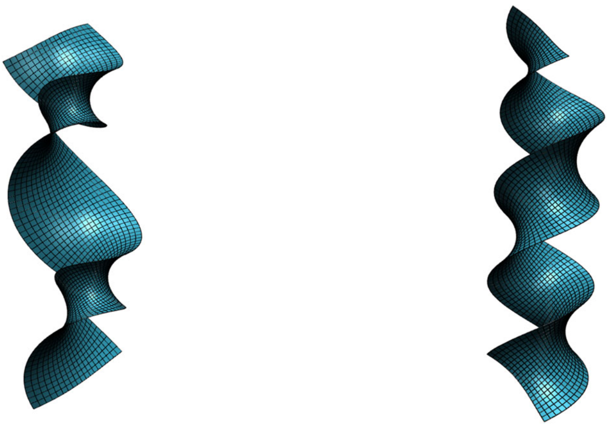

Fig. 1. Elements $f_{\frac{5}{8},\frac{1}{2}+\frac{3j}{8}}$, orthogonally projected into $\mathbb{R}^3$, and $f_{-\frac{1}{2},\frac{1}{2}}$ of the right associated family of the catenoid $f(x, y) = ix + j\cosh xe^{-iy}$

3.2. Hopf fields of minimal surfaces

We first discuss a characterisation of minimal surfaces by the shape of their Hopf field. This allows to see that every minimal surface is a Willmore surface and provides tools for later discussion of μ-Darboux transforms of minimal surfaces.

First note that surfaces which are both minimal and twistor projections are given by holomorphic maps into $\mathbb{C}^2$:

Proposition 3.3. *Let $f : M \to \mathbb{R}^4$ be a (branched) conformal immersion. Then the following statements are equivalent:*

(i) *There exists a constant complex structure* $\mathbf{i}$ *on* $\mathbb{R}^4$ *such that* $f : M \to \mathbb{C}^2 = (\mathbb{R}^4, \mathbf{i})$ *is complex holomorphic.*
(ii) *The right normal R of f is constant.*
(iii) *f is minimal and the twistor projection of a holomorphic curve in $\mathbb{CP}^3$.*

Proof. If f is complex holomorphic then $*df = df\mathbf{i}$ shows that the right normal of f is $R = -\mathbf{i}$, hence the right normal R is constant. But then Theorem 2.5 shows that f is the twistor projection of a holomorphic curve since $(dR)'' = dR = 0$. By (2) we see that $H = 0$ and f is minimal.

Finally, if f is minimal we have $*dR = -RdR$, thus $(dR)' = 0$. If f is also the twistor projection of a holomorphic curve then $(dR)'' = 0$. Thus, the right normal R is constant and the right multiplication by $-R$ gives a complex structure on $\mathbb{R}^4 = \mathbb{H}$. Then $*df = -df R$ shows that $f : M \to \mathbb{C}^2$ is complex holomorphic when identifying $\mathbb{C}^2 = (\mathbb{H}, -R)$. □

In the case when $f : M \to \mathbb{R}^3$ is minimal and the twistor projection of a holomorphic curve in $\mathbb{CP}^3$ then the previous Proposition implies that the Gauss map $N = R$ of f is constant. But then f is a plane. However, if we allow minimal surfaces in $\mathbb{R}^4$ we obtain non-trivial examples:

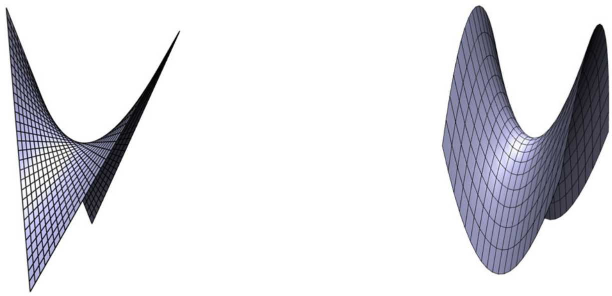

Fig. 2. $f(z) = z^2 + jz$, different orthogonal projections into $\mathbb{R}^3$

Consider the map $f : \mathbb{C} \to \mathbb{R}^4$ given by $f(z) = z^2 + jz$. Then f is a conformal immersion with right normal $R = -i$ and thus, R is conformal and harmonic: f is minimal and the (affine coordinate of the) twistor projection of the holomorphic curve

$$h(z) = \begin{pmatrix} z^2 \\ z \\ 1 \\ 0 \end{pmatrix} \mathbb{C}$$

to $\mathbb{HP}^1$ (Fig. 2).

We give a condition for minimality in terms of the Hopf field, see also [21].

Theorem 3.4. *A conformal immersion $f : M \to \mathbb{R}^4$, which is not complex holomorphic, is minimal if and only if the kernel of the Hopf field A is the point at ∞.*

Proof. Recalling (8) we have

$$2 * A = \begin{pmatrix} f\omega & -f\omega f + f(dR)'' \\ \omega & -\omega f + (dR)'' \end{pmatrix}$$

with $*\omega + \omega N = -(dR)''H$. Thus, if the kernel of A is $\infty = e\mathbb{H}$ then $\omega = 0$ and $(dR)'' \neq 0$. Therefore, $0 = *\omega + \omega N$ shows that f is a minimal surface.

Conversely, from (8) and (10) we see that the Hopf field A of a minimal immersion $f : M \to \mathbb{R}^4$ satisfies

$$2 * A = \begin{pmatrix} 0 & f dR \\ 0 & dR \end{pmatrix}, \tag{13}$$

where we used that $(dR)'' = dR$ by the conformality (10) of R and

$$\omega = \frac{1}{2}(dH + H * df H + *dHN - *dRH) = 0$$

since $H = 0$. By assumption R is not constant because f is not complex holomorphic. Therefore, $\ker A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbb{H} = \infty$.

□

From the expression (13) of the Hopf field A of a minimal surface $f : M \to \mathbb{R}^4$ we see that A is harmonic, that is, $d * A = 0$, since $*dR = -RdR$ and thus $df \wedge dR = 0$.

In particular, we obtain the well-known result:

Corollary 3.5. *Every minimal immersion in $\mathbb{R}^4$ is a Willmore surface in $\mathbb{R}^4$.*

4. The associated Willmore surface

We now discuss a correspondence between minimal surfaces and twistor projections of holomorphic curves in $\mathbb{CP}^3$.

If f is a minimal surface with constant right normal R then $df^* = - * df = dfR$ shows that the conjugate surface is by [7] a rigid motion $f^* = fR + c$ of f with $c \in \mathbb{H}$. For general minimal surfaces f the map $fR - f^*$ is a Willmore surface in $\mathbb{R}^4$: the following theorem is a special case of a more general statement for super-conformal maps, [11,24].

Theorem 4.1. *Let $f : M \to \mathbb{R}^4$ be a minimal surface with conjugate surface f^* on the simply connected Riemann surface M and assume that its right normal R is not constant. Then*

$$f^\flat = fR - f^* : M \to \mathbb{R}^4$$

is a twistor projection of a holomorphic curve in $\mathbb{CP}^3$ with right normal $R^\flat = -R$. Moreover, $f^\flat$ is branched if R is branched.

Proof. If f is minimal then its right normal R satisfies $*dR = -RdR = dRR$ and

$$d(fR - f^*) = fdR$$

shows that $f^\flat = fR - f^*$ is branched if R is branched. Moreover, the right normal of $f^\flat$ is $R^\flat = -R$ and

$$(dR^\flat)'' = \frac{1}{2}(dR^\flat + R^\flat * dR^\flat) = \frac{1}{2}(-dR + R * dR) = 0$$

shows that $f^\flat$ is a twistor projection of a holomorphic curve in $\mathbb{CP}^3$ by Theorem 2.5. □

This leads to the following definition:

Definition 4.2. If $f : M \to \mathbb{R}^4$ is a minimal surface with conjugate surface f^* and (non-constant) right normal R then we call

$$f^\flat = fR - f^*$$

an *associated Willmore surface* of f.

Note that $f^\flat$ is a Willmore surface but not a minimal surface by Proposition 3.3 since by assumption $R^\flat = -R$ is not constant.

Note that if we change the conjugate f^* by a constant, this results in a non-trivial change (Figs. 3, 4) of the associated Willmore surface $(f^* + c)^\flat = (f^*)^\flat + cR$.

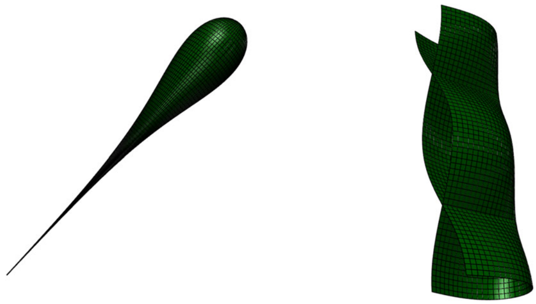

Fig. 3. The associated Willmore surface of the catenoid, various orthogonal projections into $\mathbb{R}^3$

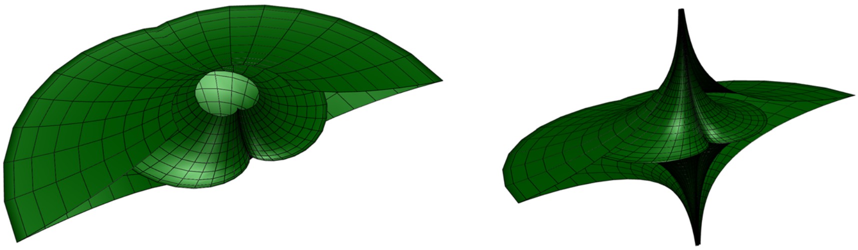

Fig. 4. The associated Willmore surface of the helicoid $f^*(x, y) = iy - k \sinh x e^{-iy}$ and the translated helicoid $f^* + i - 2j - k$, orthogonally projected into $\mathbb{R}^3$

5. μ-Darboux transforms

Previous results [4,5,9,22] seem to indicate that the Darboux transformation preserves a surface class which is given by a harmonicity condition as long as it is given by a parallel section of the associated family of flat connections of the harmonic map. In the case of a minimal surface $f : M \to \mathbb{R}^4$, the conformal Gauss map is harmonic and we will consider parallel sections of its associated family of flat connections d_μ. Then a μ-Darboux transform, that is, a Darboux transform which arises from parallel sections of d_μ, is indeed a Willmore surface but we will show that it is not minimal in $\mathbb{R}^4$. However, a μ-Darboux transform is still given by complex holomorphic data: it is the twistor projection of a holomorphic curve in $\mathbb{CP}^3$.

5.1. Parallel sections

For this paper to be self-contained, we briefly recall the results of [21] where the parallel sections of the family of flat connections of a minimal surface are computed. Let $f : M \to \mathbb{R}^4$ be minimal, and let as before

$$e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \psi = \begin{pmatrix} f \\ 1 \end{pmatrix}.$$

Let A be the Hopf field of the harmonic conformal Gauss map S of f. By the expression (13) of the Hopf field of a minimal surface we have

$$A\psi = -\frac{1}{2}\psi * dR, \quad \text{and} \quad Ae = 0.$$

Using $*dR = -RdR = dRR$ by the minimality of f we see that

$$A^{(1,0)}\psi = -\frac{1}{4}\psi dR(R+i), \quad A^{(0,1)}\psi = -\frac{1}{4}\psi dR(R-i).$$

We fix $\mu \in \mathbb{C}_*$ and compute all parallel sections of d_μ. If $\mu = 1$ then $d_\mu = d$ is trivial, and every constant section is parallel. Assume from now on that $\mu \neq 1$, and let $L = \psi\mathbb{H}$ the line bundle of f. Since $e\mathbb{H} \oplus L = \underline{\mathbb{H}}^2$ every d_μ-parallel section $\varphi \in \Gamma(\widetilde{\underline{\mathbb{H}}}^2)$ of the trivial $\mathbb{H}^2$ bundle over the universal cover $\tilde{M}$ of M can be written as $\varphi = e\alpha + \psi\beta$ with $\alpha, \beta : \tilde{M} \to \mathbb{H}$. Then

$$d_\mu\varphi = e(d\alpha + df\beta) + \psi\left(d\beta - \frac{1}{2}dR(R\beta(a-1) - \beta b)\right)$$

where $a = \frac{\mu+\mu^{-1}}{2}$, $b = i\frac{\mu^{-1}-\mu}{2}$. From this we see that φ is d_μ parallel if and only if

$$d\alpha = -df\beta, \quad \text{and} \quad d\beta = \frac{1}{2}dR(R\beta(a-1) - \beta b). \tag{14}$$

Let $\beta = Rm + m\frac{b}{a-1}$, $m \in \mathbb{H}$, and $\alpha = -fm\frac{b}{a-1} - f^*m$. Then

$$R\beta(a-1) - \beta b = 2m$$

where we used that $a^2 + b^2 = 1$. Therefore, β satisfies the second condition in (14) for $\varphi = e\alpha + \psi\beta$ to be d_μ-parallel since

$$d\beta = dRm = \frac{1}{2}dR(R\beta(a-1) - \beta b).$$

Moreover, using $df^* = - * df$ we have the first condition in (14) as

$$d\alpha = -df\left(Rm + m\frac{b}{a-1}\right) = -df\beta.$$

Note that $b = \frac{i(1-\mu^2)}{2\mu}$, $a - 1 = \frac{(1-\mu)^2}{2\mu}$ so that

$$\frac{b}{a-1} = \frac{i(1+\mu)}{1-\mu} \tag{15}$$

for $\mu \in \mathbb{C} \setminus \{0, 1\}$. Therefore, we see that

$$\varphi = -e\left(fm\frac{i(1+\mu)}{1-\mu} + f^*m\right) + \psi\left(Rm + m\frac{i(1+\mu)}{1-\mu}\right)$$

is a d_μ-parallel section. Indeed, every non-constant parallel section arises this way:

Proposition 5.1. ([21]) *Let $f : M \to \mathbb{R}^4$ be a minimal surface with conjugate surface f^* and d_λ the associated family of flat connections of the conformal Gauss map S of f. For $\mu \in \mathbb{C}\setminus\{0,1\}$ every d_μ-parallel section is either a constant section $\varphi = en$, $n \in \mathbb{H}$, or is given by $\varphi = e\alpha + \psi\beta \in \Gamma(\widetilde{\underline{\mathbb{H}}}^2)$ with*

$$\alpha = -f^*m - fm\rho, \quad \beta = Rm + m\rho, \quad m \in \mathbb{H}_* \tag{16}$$

where $\rho = \frac{i(1+\mu)}{1-\mu}$.

Remark 5.2. For $\mu \in \mathbb{C}\setminus\{0,1\}$ and $m \neq 0$, the section β is nowhere vanishing. This follows from the fact [21] that β is a parallel section of the associated family of flat connections of the right normal R of f but can also been shown directly: if $\beta(p) = 0$ then

$$R(p) = -m\frac{i(1+\mu)}{1-\mu}m^{-1}$$

which implies

$$-1 = m^{-1}R(p)^2 m = -\frac{(1+\mu)^2}{(1-\mu)^2}$$

contradicting $\mu \neq 0$.

5.2. μ-Darboux tranforms

If a minimal surface $f : M \to \mathbb{R}^4$ has constant right normal R then by (13) the Hopf field A vanishes and $d_\mu = d$ for all $\mu \in \mathbb{C}\setminus\{0,1\}$. That is, all μ-Darboux transforms of f are in this case the constant sections of $\Gamma(\underline{\mathbb{H}}^2)$. Therefore, from now on we will assume that f is not the twistor projection of a holomorphic curve in $\mathbb{CP}^3$.

With Proposition 5.1 at hand, we can again discuss all μ-Darboux transforms of the minimal surface f. If $\varphi = en$, $n \in \mathbb{H}_*$, is a constant parallel section then the corresponding μ-Darboux transform is the constant point $\infty = e\mathbb{H}$. On the other hand, every non-constant d_μ-parallel section $\varphi = e\alpha + \psi\beta \in \Gamma(\underline{\mathbb{H}}^2)$, $\mu \in \mathbb{C}\setminus\{0,1\}$, is given by (16) and the μ-Darboux transform is in this case given by

$$L^\mu = (e\alpha + \psi\beta)\mathbb{H},$$

where $\beta = Rm + m\frac{b}{a-1}$ is nowhere vanishing since $\mu \neq 0$, $m \in \mathbb{H}_*$. Therefore, the μ-Darboux transform is given by the affine coordinate $f^\mu = f + T : M \to \mathbb{R}^4$ with

$$T = \alpha\beta^{-1} = -\left(f^* + f\frac{\hat{b}}{\hat{a}-1}\right)\left(R + \frac{\hat{b}}{\hat{a}-1}\right)^{-1} \tag{17}$$

and $\hat{a} = mam^{-1}$, $\hat{b} = mbm^{-1}$. Note that although $\hat{b}$, $\hat{a} - 1 \in \mathbb{H}$, the fraction $\frac{\hat{b}}{\hat{a}-1}$ is well-defined since $(\hat{a} - 1)^{-1}\hat{b} = \hat{b}(\hat{a} - 1)^{-1} = m\frac{b}{a-1}m^{-1}$ and $\hat{a} \neq 1$. Moreover,

$$f^{\mu} = \left(f(R + \frac{\hat{b}}{\hat{a}-1}) - \left(f^* + f\frac{\hat{b}}{\hat{a}-1}\right)\right)\left(R + \frac{\hat{b}}{\hat{a}-1}\right)^{-1}$$
$$= \left(fR - f^*\right)\left(R + \frac{\hat{b}}{\hat{a}-1}\right)^{-1}$$

We summarise:

Theorem 5.3. *Let $f : M \to \mathbb{R}^4$ be a minimal surface in $\mathbb{R}^4$ on a simply connected Riemann surface M (and assume that f not holomorphic in some $\mathbb{C}^2$).*

Then every non-constant μ-Darboux transform of f is given by

$$f^{\mu} = (fR - f^*)(R + \hat{\rho})^{-1}, \tag{18}$$

where f^* is a conjugate surface of f and $\hat{\rho} = m\frac{i(1+\mu)}{1-\mu}m^{-1}$ with $\mu \in \mathbb{C} \setminus \{0, 1\}$, $m \in \mathbb{H}_*$.

Remark 5.4. (i) Whereas the associated Willmore surface is uniquely defined up to translation by the choice of the conjugate surface f^*, the μ-Darboux transformation depends non-trivially on f^*: a translation of f^* by $c \in \mathbb{H}$ results in an addition of $-c(R + \hat{\rho})^{-1}$ to f^{μ}.

(ii) The μ-Darboux transformation is independent of the choice of $m \in \mathbb{H}_*$ exactly when $\mu \in S^1$. This reflects the fact that the associated family of flat connections d_{μ} is a family of quaternionic connections on the unit circle.

(iii) If M is not simply connected we still obtain by (18) a μ-Darboux transform but on the universal cover $\tilde{M}$ of M. Note that f^{μ} is in this case globally defined on M only if the conjugate surface is defined on M. Put differently, the parallel section $\varphi = e\alpha + \psi\beta$ is only a section with multiplier [3] if f^* is globally defined. In this case, the multiplier is trivial.

We also observe that $\hat{\rho} = 0$ for $\mu = -1$ and thus:

Corollary 5.5. *The associated Willmore surface of a conjugate f^* of a minimal surface f is the μ-Darboux transform*

$$f^{\mu=-1}(x, y) = f^*R + f$$

of f for $\mu = -1$.

Therefore, the pictures in Fig. 4 show μ-Darboux transforms of the catenoid at $\mu = -1$.

Since the associated Willmore surface of a minimal surface is Willmore but not minimal, we see that μ-Darboux transforms are not minimal for $\mu = -1$. Indeed, this extends to all μ-Darboux transforms:

Theorem 5.6. *Let $f : M \to \mathbb{R}^4$ be minimal (and not holomorphic) on a simply connected Riemann surface M.*

Then every (non-constant) μ-Darboux transform $f^\mu : M \to \mathbb{R}^4$ of f is an associated Willmore surface of a minimal surface in the right associated family of f. In particular, f^μ is not minimal.

Proof. Consider the minimal surface

$$h = -\frac{1}{2}(f\hat{b} + f^*(\hat{a} - 1))$$

in the right associated family of f where $\hat{a} = m\frac{\mu+\mu^{-1}}{2}m^{-1}$, $\hat{b} = mi\frac{\mu^{-1}-\mu}{2}m^{-1}$ and $m \in \mathbb{H}_*$. From Remark 3.2 we see that the right normal of the minimal surface h is

$$R_h = (\hat{b} + R(\hat{a} - 1))^{-1}R(\hat{b} + R(\hat{a} - 1)). \tag{19}$$

Now let $\hat{\rho} = \frac{\hat{b}}{\hat{a}-1}$ then $\hat{\rho} = m\frac{i(1+\mu)}{1-\mu}m^{-1}$ by (15) and $1 + \hat{\rho}^2 = -2(\hat{a} - 1)^{-1}$. On the other hand

$$1 + \hat{\rho}^2 = 1 + ((R + \hat{\rho}) - R)^2 = (\hat{\rho} + R)^2 - (\hat{\rho} + R)R - R(\hat{\rho} + R).$$

so that

$$(\hat{\rho} + R)^{-1}(\hat{a} - 1)^{-1}(R + \hat{\rho})^{-1} = \frac{1}{2}(-1 + (\hat{\rho} + R)^{-1}R + R(\hat{\rho} + R)^{-1})$$

commutes with R, and so does its inverse $(\hat{\rho} + R)(\hat{a} - 1)(R + \hat{\rho})$. This shows by (19) that

$$R_h = (\hat{a} - 1)^{-1}(\hat{\rho} + R)^{-1}R(\hat{\rho} + R)(\hat{a} - 1) = (\hat{\rho} + R)R(\hat{\rho} + R)^{-1}. \tag{20}$$

We show next that R_h is not constant in our situation. Using $dRR = -RdR$ we obtain

$$dR_h = (\hat{\rho} - (\hat{\rho} + R)R(\hat{\rho} + R)^{-1})dR(\hat{\rho} + R)^{-1}$$

and since $dR \neq 0$, the right normal R_h is constant if and only if

$$\hat{\rho}(\hat{\rho} + R) = (\hat{\rho} + R)R.$$

But $\hat{\rho} = m\frac{i(1+\mu)}{1-\mu}m^{-1}$ with $\mu \neq 0$ so that $\hat{\rho}^2 \neq -1$. Thus, R_h is not constant and we can define, see Theorem 4.1, the associated Willmore surface $h^\flat$ of h: using the conjugate surface

$$h^* = \frac{1}{2}(f(\hat{a} - 1) - f^*\hat{b})$$

of the minimal surface $h = -\frac{1}{2}(f\hat{b} + f^*(\hat{a} - 1))$ a straight forward computation, using $\hat{a}^2 + \hat{b}^2 = 1$ and (20), gives

$$h^\flat = hR_h - h^* = (fR - f^*)(\hat{\rho} + R)^{-1}.$$

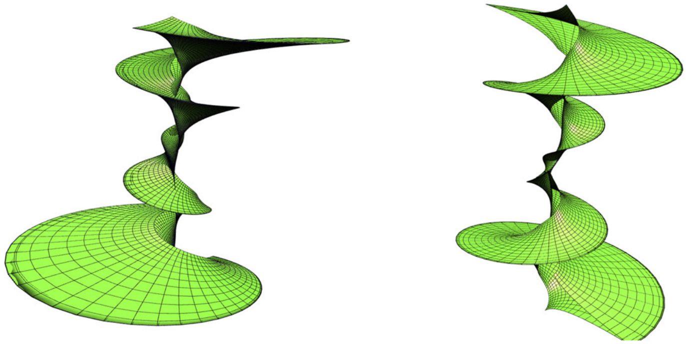

Fig. 5. μ-Darboux transforms of the catenoid with $\mu = -\frac{i}{2}$, $m = \frac{1}{2}(1+i-j-k)$ and $\mu = i$, $m = 1$, orthogonally projected into $\mathbb{R}^3$

In other words, $h^\flat$ is by (18) a μ-Darboux transform f^μ of f. By Theorem 5.3 every non-constant μ-Darboux transform arises this way. Moreover, since the right normal of the associated Willmore surface $h^\flat$ of h is given by Theorem 4.1 as $R_h^\flat = -R_h$ and R_h is not constant, the μ-Darboux transform f^μ is not minimal by Proposition 3.3. □

Since $f_{-\frac{\hat{b}}{2},-\frac{\hat{a}-1}{2}} = f_{\frac{5}{8},\frac{1}{2}+\frac{3j}{8}}$ for $\mu = -\frac{i}{2}$ and $m = \frac{1}{2}(1+i-j-k)$ and $f_{-\frac{\hat{b}}{2},-\frac{\hat{a}-1}{2}} = f_{-\frac{1}{2},\frac{1}{2}}$ for $\mu = i, m = 1$, the μ-Darboux transforms in Fig. 5 are also the associated Willmore surfaces of the minimal surfaces of the right associated family of the catenoid in Fig. 1.

Remark 5.7. Every μ-Darboux transform f^μ of a minimal surface, which is not holomorphic, is a twistor projection of a holomorphic curve with vanishing Hopf field, see Theorem 2.5. Therefore, the family of flat connections of f^μ is trivial and every μ-Darboux transform of f^μ is constant. This shows that the μ-Darboux transformation on minimal surfaces trivially satisfies Bianchi permutability.

By construction, all μ-Darboux transforms are (generalised) Darboux transforms. In particular, $T = \alpha\beta^{-1}$ satisfies the generalised Riccati equation (6). Since $d\beta = dRm$ by (16) the generalised Riccati equation becomes, away from the zeros of α,

$$dT = -df + TdR(-m\alpha^{-1})T$$

with $m\alpha^{-1}$ non-constant. In particular, if $f : M \to \mathbb{R}^3$ is minimal in $\mathbb{R}^3$ then the Gauss map N is the right normal of f, and the above equation generalizes the classical Ricatti equation (4) since $f^d = N$ is a dual surface of f. Note however that non-constant μ-Darboux transforms of a minimal surface $f : M \to \mathbb{R}^3$ are never classical: one can show that $-dNm\alpha^{-1}$ is not a closed 1-form.

5.3. $\mathbb{CP}^1$ *family*

We conclude this paper by investigating the limits of μ-Darboux transforms at $\mu = 1$ and $\mu = 0, \infty$. In the first case, the $d_{\mu=1} = d$ is the trivial connection and all parallel sections give rise to constant Darboux transforms. However, the limit of appropriately scaled and rotated μ-Darboux transforms is the associated Willmore surface of f (Fig. 6). In the second case, the family d_μ of flat connections does not extend to $\mu = 0, \infty$, however the pointwise limit of μ-Darboux transforms is still a Darboux transform (Fig. 7).

Theorem 5.8. *After rescaling and rigid motion, the pointwise limit of μ-Darboux transforms f^μ of f is the associated Willmore surface $f^\flat$, that is,*

$$\lim_{\mu \to 1} f^\mu \hat{\rho} = f^\flat.$$

Proof. First $f^\mu \hat{\rho}$ is a rigid motion of f^μ in $\mathbb{R}^4$ up to scaling [7] since $\rho \,\hat{\in}\, \mathbb{H}_*$. But then $\hat{\rho}^{-1} = mi \frac{\mu-1}{\mu+1} m^{-1}$ gives $\lim_{\mu \to 1} \hat{\rho}^{-1} = 0$ and we obtain

$$\lim_{\mu \to 1} (R + \hat{\rho})^{-1} \hat{\rho} = \lim_{\mu \to 1} (\widehat{\rho^{-1}} R + 1)^{-1} = 1.$$

This shows the claim since $f^\mu = (fR - f^*)(R + \hat{\rho})^{-1}$ and $f^\flat = fR - f^*$. $\square$

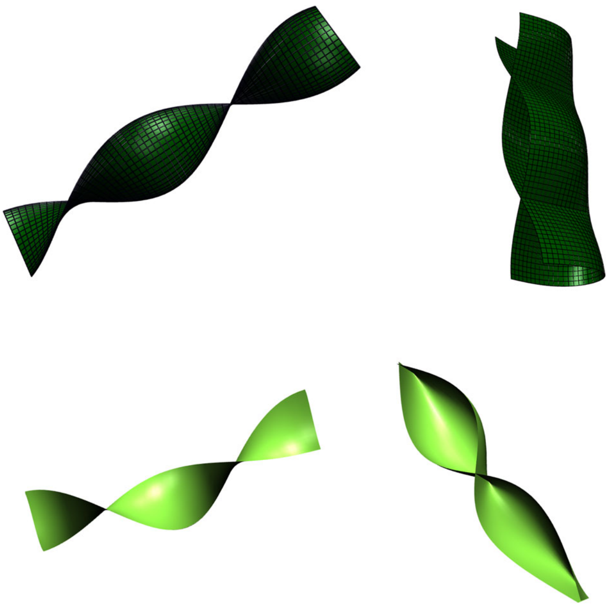

Fig. 6. Associated Willmore surface of the catenoid (first row), and μ-Darboux transforms of the catenoid with $\mu = \cos(0.3) + i \sin(0.3)$, $m = 1$ and $\mu = 1.3$, $m = \frac{1+j}{\sqrt{2}}$, orthogonally projected into $\mathbb{R}^3$

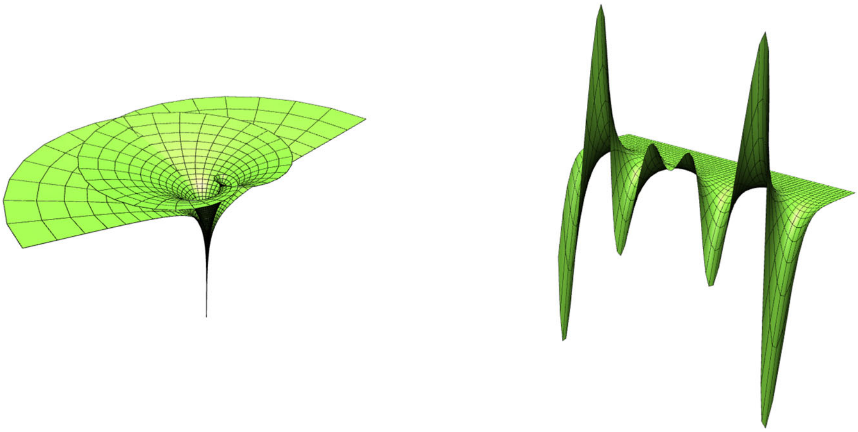

Fig. 7. f^0, $m = 1$, different orthogonal projections into $\mathbb{R}^3$

We recall that $f^\mu = (fR - f^*)(R + \hat{\rho})^{-1}$ for $m \in \mathbb{H}_*$, $\mu \in \mathbb{C} \setminus \{0, 1\}$. Since $\hat{\rho} = m\frac{i(1+\mu)}{1-\mu}$ has a well-define limit as μ approaches 0 and ∞ respectively, we can extend the μ-Darboux transformation to $\mu \in \mathbb{CP}^1$, at least away from the isolated zeros of $R \pm mim^{-1}$.

Theorem 5.9. *Let $f : M \to \mathbb{R}^4$ be minimal (and not holomorphic).*

- *The pointwise limit $f^0 = \lim_{\mu\to 0} f^\mu$ of the μ-Darboux tranforms of f is a Darboux transform of f. More precisely, $f^0 : M \to S^4$ is a Willmore surface and is given in affine coordinates by the minimal surface*

$$f^0 = (fR - f^*)(R + mim^{-1})^{-1}$$

which has its ends at the isolated zeros of $R + mim^{-1}$, $m \in \mathbb{H}_*$.

- *The pointwise limit $f^\infty = \lim_{\mu\to\infty} f^\mu$ of the μ-Darboux tranforms of f is a Darboux transform of f. More precisely, $f^\infty : M \to S^4$ is a Willmore surface and is given in affine coordinates by the minimal surface*

$$f^\infty = (fR - f^*)(R - mim^{-1})^{-1}$$

which has its ends at the isolated zeros of $R - mim^{-1}$, $m \in \mathbb{H}_*$.

Proof. We prove the statement for f^0, the second claim follows similarly.

Denote by $\hat{\rho}_0 = mim^{-1}$ and consider the section $\varphi = e\alpha + \psi\beta \in \Gamma(\underline{\mathbb{H}}^2)$ where

$$\alpha = -(f^*m + fmi), \quad \beta = Rm + mi.$$

Note that d_μ is not defined at $\mu = 0$ so φ is not a parallel section of our family of flat connections, but $e\alpha$ is still by (5) a holomorphic section with prolongation φ since as before $d\alpha = -df\beta$ and $d\varphi \in \Omega^1(L)$. Thus, φ defines a (generalised) Darboux transform $f^\sharp$ of f. Since R is conformal with $*dR = -RdR$ we see that $e(Rm + mi) = e\beta$ is a holomorphic section with respect to the quaternionic holomorphic structure $D(e\gamma) := \frac{1}{2}e(d\gamma - R * d\gamma)$. Thus, by [14] the zeros of β are isolated, and away from these zeros $f^\sharp$ has affine coordinate

$$f^\sharp = f + T = (fR - f^*)(R + \hat{\rho}_0)^{-1}$$

where $T = \alpha\beta^{-1} = -(f^* + f\hat{\rho}_0)(R + \hat{\rho}_0)^{-1}$. Thus, $f^0 = f^\sharp$ and f^0 is a Darboux transform of f. Since $\hat{\rho}_0 = \lim_{\mu\to 0} m\frac{i(1+\mu)}{1-\mu}m^{-1}$ we see that the Darboux transform f^0 is the pointwise limit $f^0 = \lim_{\mu\to 0} f^\mu$ of μ-Darboux transforms. It remains to show that f^0 is minimal. For this, we observe that

$$df^0 = -T dR(R + \hat{\rho}_0)^{-1}$$

so that the right normal of f^0 is

$$R^0 = -(R + \hat{\rho}_0)R(R + \hat{\rho}_0)^{-1}$$

since $*dR = dRR$. Since $\hat{\rho}_0^2 = -1$ we have $(R + \hat{\rho}_0)R = \hat{\rho}_0(\hat{\rho}_0 + R)$ and then $R^0 = -\hat{\rho}_0$ is constant. Therefore, by Proposition 3.3 the surface f^0 is both minimal and a twistor projection of a holomorphic curve in $\mathbb{CP}^3$. □

References

[1] Burstall, F., Ferus, D., Leschke, K., Pedit, F., Pinkall, U.: Conformal Geometry of Surfaces in S^4 and Quaternions. Lecture Notes in Mathematics. Springer, Berlin (2002)

[2] Burstall, F.E., Hertrich-Jeromin, U., Pedit, F., Pinkall, U.: Curved flats and isothermic surfaces. Math. Z. **225**, 199–209 (1997)

[3] Bohle, C., Leschke, K., Pedit, F., Pinkall, U.: Conformal maps from a 2-torus to the 4-sphere. J. Reine Angew. Math. **671**, 1–30 (2012)

[4] Bohle, C.: Constrained Willmore tori in the 4-sphere. J. Differ. Geom. **86**(1), 71–131 (2010)

[5] Burstall, F., Quintino, A.: Dressing transformations of constrained Willmore surfaces. Commun. Anal. Geom. **22**, 469–518 (2014)

[6] Burstall, F.E.: Isothermic surfaces: conformal geometry, Clifford algebras and integrable systems. In: Integrable Systems, Geometry, and Topology of AMS/IP Stud. Adv. Math., vol. 36, pp. 1–82. Am. Math. Soc., Providence (2006)

[7] Cayley, A.: On certain results relating to quaternions. Math. Pap. **1**, 123–126 (1889)

[8] Christoffel, E.: Ueber einige allgemeine Eigenschaften der Minimumsflächen. Crelle's J. **67**, 218–228 (1867)

[9] Carberry, E., Leschke, K., Pedit, F.: Darboux transforms and spectral curves of constant mean curvature surfaces revisited. Ann. Glob. Anal. Geom. **43**(4), 299–329 (2013)

[10] Darboux, G.: Sur les surfaces isothermiques. C. R. Acad. Sci. Paris **128**, 1299–1305 (1899)
[11] Dajczer, M., Tojeiro, R.: All superconformal surfaces in $\mathbb{R}^4$ in terms of minimal surfaces. Math. Z. **261**, 869–890 (2009)
[12] Eisenhart, L.: Minimal surfaces in Euclidean four-space. Am. J. Math. **34**(3), 215–236 (1912)
[13] Ejiri, N.: Willmore surfaces with a duality in $S^n(1)$. Proc. Lond. Math. Soc. **3–57**(2), 383–416 (1988)
[14] Ferus, D., Leschke, K., Pedit, F., Pinkall, U.: Quaternionic holomorphic geometry: Plücker formula, Dirac eigenvalue estimates and energy estimates of harmonic 2-tori. Invent. Math. **146**, 507–593 (2001)
[15] Hitchin, N.: Harmonic maps from a 2-torus to the 3-sphere. J. Differ. Geom. **31**(3), 627–710 (1990)
[16] Hertrich-Jeromin, U.: Models in Moebius Differential Geometry. Habilitations schrift (2002)
[17] Hertrich-Jeromin, U., Pedit, F.: Remarks on the Darboux transfoms of isothermic surfaces. Doc. Math. J. DMV **2**, 313–333 (1997)
[18] Leschke, K.: Transformations of isothermic surfaces. In preparation
[19] Leschke, K.: Harmonic map methods for Willmore surfaces. Cont. Math. **542**, 204–212 (2011)
[20] Leschke, K., Moriya, K.: Applications of quaternionic holomorphic geometry to minimal surfaces. Complex Manifolds **3**(1), 282–300 (2016)
[21] Leschke, K., Moriya, K.: Simple factor dressing and the Lopez-Ros deformation of minimal surfaces. Math. Z. **291**, 1015–1058 (2018)
[22] Leschke, K., Romon, P.: Spectral curve of Hamiltonian stationary tori. Calc. Var. Partial Differ. Equ. **38**(1), 45–74 (2010)
[23] Montiel, S.: Willmore two-spheres in the four sphere. Trans. Am. Math. Soc. **352**, 4449–4486 (2000)
[24] Moriya, K.: Super-conformal surfaces associated with null complex holomorphic curves. Bull. Lond. Math. Soc. **41**, 327–331 (2009)
[25] Rigoli, M.: The conformal Gauss map of submanifolds of the Moebius space. Ann. Glob. Anal. Geom **5**(2), 97–116 (1987)
[26] Uhlenbeck, K.: Harmonic maps into Lie groups (classical solutions of the chiral model). J. Diff. Geom. **30**, 1–50 (1989)