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Singularities of generic characteristic polynomials and smooth finite splittings of Azumaya algebras over surfaces

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Abstract. Let k be an algebraically closed field. Let $P(X_{11}, \ldots, X_{nn}, T)$ be the characteristic polynomial of the generic matrix (X_{ij}) over k. We determine its singular locus as well as the singular locus of its Galois splitting. If X is a smooth quasi-projective surface over k and A an Azumaya algebra on X of degree n, using a method suggested by M. Artin, we construct finite smooth splittings for A of degree n over X whose Galois closures are smooth.

Introduction

Let k be an algebraically closed field and $A = k[X_{ij}, 1 \leq i, j \leq n]$ the polynomial ring in n^2 variables. Let $P(T) = T^n + a_1T^{n-1} + \cdots + a_n$ in $A[T]$ be the characteristic polynomial of the generic matrix (X_{ij}). We set

$$A_n = A[T]\big/(P(T)) \quad \text{and} \quad B_n = A[T_1, \ldots, T_n]\big/I$$

where I is the ideal of $A[T_1, \ldots, T_n]$ generated by the n polynomials $\sigma_i(T_1, \ldots, T_n) - (-1)^i a_i$, $1 \leq i \leq n$ where for each i, σ_i is the i-th elementary symmetric function. Let $Y_n = Spec(A_n)$ and $Z_n = Spec(B_n)$. In the first part of the paper we describe the singular loci of Y_n and Z_n and we prove that their codimension is equal to 3. Let X be a smooth quasi-projective surface over k. Let $\mathcal{A}$ be an Azumaya algebra of rank n^2 over X. There is a construction due to M. Artin of a degree n finite flat map $Y \to X$ with Y smooth which splits $\mathcal{A}$ (cf [8] for the case X projective and $\mathcal{A}$ generically a division ring). We use the method of proof in [8] to construct a degree n flat map $Y \to X$ which splits $\mathcal{A}$ where Y is smooth and has a smooth irreducible Galois closure.

1. The characteristic polynomial of the generic matrix

In this section we suppose that k is an algebraically closed field, of arbitrary characteristic. We denote by Sing(X) the singular locus of a given scheme X.

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Let

$$A_n = \frac{k[X_{11}, X_{12}, \ldots, X_{nn}][T]}{(P(T))}$$

where $P(T)$ is the characteristic polynomial of the generic matrix (X_{ij}) with $1 \leq i, j \leq n$. Let $Y_n = \mathrm{Spec}(A_n)$. We study the singular locus of Y_n.

Lemma 1.1. *Let* $\beta = \mathrm{diag}(\mathrm{B}_1, \ldots, \mathrm{B}_\mathrm{m})$ *be a matrix consisting of m cyclic Jordan blocks*

$$B_i = \begin{pmatrix} \lambda_i & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & \lambda_i & 1 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & \lambda_i & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \lambda_i & 1 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & \lambda_i \end{pmatrix}$$

with distinct eigenvalues λ_i*. Then, for any i, the scheme* Y_n *is smooth at* (β, λ_i)*.*

Proof. We denote by I_n the identity matrix of size n. Developing the determinant of $(X_{ij}) - T \cdot \mathrm{I}_n$ along the first column we get

$$\pm P(T) = (X_{11} - T)P_1(T) + X_{2,1}P_2(T) + \cdots + X_{n,1}P_n(T)$$

where the polynomials P_i are the cofactors of the first column. Let k_i be the size of B_i. We see that $P_{k_1}(T)(B, \lambda_1)$ is (up to sign) the determinant of a matrix of the form $\mathrm{diag}(\mathrm{I}_{\mathrm{k}_1-1}, \mathrm{B}_2 - \lambda_1\mathrm{I}_{\mathrm{k}_2}, \ldots, \mathrm{B}_\mathrm{m} - \lambda_1\mathrm{I}_{\mathrm{k}_\mathrm{m}})$, it being understood that the first block is missing if $k_1 = 1$. Since $\lambda_1 \neq \lambda_i$, this shows that $\partial P(T)/\partial X_{k_1,1} = P_{k_1}(T)$ is not zero at (B, λ_1). Thus Y_n is smooth at (β, λ_1) and the same clearly holds for any other λ_i. □

Lemma 1.2. *Every neighbourhood of a matrix* α *with an eigenvalue* $\lambda \neq 0$ *contains an invertible semisimple matrix with eigenvalue* λ*.*

Proof. We may assume that α is in Jordan form. The given neighbourhood of α contains an open set defined by the non-vanishing of a polynomial g in the coordinates of the generic matrix (X_{ij}). We may assume that the diagonal entries of α are $(\lambda, \lambda_2, \ldots, \lambda_n)$. Since $g(\alpha) \neq 0$ we may find values $\lambda'_2, \ldots, \lambda'_n$ all distinct and different from λ and different from 0, such that when we replace λ_i by λ'_i in α we obtain an α' for which $g(\alpha') \neq 0$. This new α' is in the given neighbourhood and is semisimple. □

Let Y_n be as before. The surjection $k[X_{11}, X_{12}, \ldots, X_{nn}][T] \to A_n$ induces a finite map $\pi : Y_n \to \mathbb{A}_k^{n^2}$. The projection $C = \pi(\mathrm{Sing}(Y_n))$ is a closed subscheme of $\mathbb{A}_k^{n^2}$ and is contained in the ramification locus of π, which is the closed subscheme of $\mathbb{A}_k^{n^2}$ whose closed points correspond to matrices with at least two equal eigenvalues.

Lemma 1.3. *Let $V \subset \mathbb{A}_k^{n^2}$ be the set of semisimple invertible matrices with at least two coincident eigenvalues. Then $V \subseteq C$.*

Proof. It suffices to check that any matrix of the form $\beta = \mathrm{diag}(\mu_1, \ldots, \mu_{\mathrm{n}-2}, \lambda, \lambda)$ is in C. We show that (β, λ) belongs to $\mathrm{Sing}(Y_n)$. Writing $X_{ii} = \mu_i + X_i$ for $i \leq n-2$, $X_{ii} = \lambda + X_i$ for $i \geq n-1$, $T = \lambda + t$ and $\nu_i = \mu_i - \lambda$ we see that $\pm P(T)$ is the determinant of the matrix

$$\begin{pmatrix} \nu_1 + X_1 - t & X_{12} & \cdots & X_{1n} \\ X_{2,1} & \nu_2 + X_2 - t & \cdots & X_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & X_{n-1} - t & X_{n-1,n} \\ \cdots & \cdots & X_{n,n-1} & X_n - t \end{pmatrix}$$

and it is clear that it does not contain any linear term in X_i,X_{ij} or T. Thus the variety it defines is singular at the origin, which corresponds to the point (β, λ) in the previous coordinates. □

Let P_n be the affine space of monic polynomials of degree n. Let $c : M_n \to P_n$ be the characteristic polynomial map associating to any $n \times n$-matrix its characteristic polynomial. We have the finite surjective map $\sigma : \mathbb{A}_k^n \to P_n$ sending $\xi = (\xi_1, \ldots, \xi_n)$ to the polynomial $T^n + \sigma_1(\xi)T^{n-1} + \cdots + \sigma_n(\xi)$, where, for $1 \leq i \leq n$, σ_i is the i-th elementary symmetric function. For a given positive integer $l \leq n$, the set of polynomials in P_n with at least l distinct eigenvalues is an open dense subscheme of P_n.

Lemma 1.4. *Let $W \subset M_n(k)$ be the set of all semisimple invertible matrices with at least $n-1$ distinct eigenvalues. Then W is open and dense in $M_n(k)$.*

Proof. The set M of all semisimple invertible matrices is open and dense in $M_n(k)$. The set P of all the polynomials in $P_n(k)$ which have at least $n-1$ distinct eigenvalues is open and dense. Hence $W = M \cap c^{-1}(P)$ is open and dense in $M_n(k)$. □

By 1.4 the set $U = W \cap C$ of all semisimple invertible matrices with exactly $n-1$ distinct eigenvalues is open in C.

Lemma 1.5. *The set U is dense in C.*

Proof. Let (β, λ) be a point of $\mathrm{Sing}(Y_n)$. By 1.1, β, which we may assume to be in Jordan canonical form, contains at least two cyclic Jordan blocks with the same eigenvalue. We write $\beta = \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_{\mathrm{r}})$ with the β_i's cyclic Jordan blocks of size s_i and β_1, β_2 having the same eigenvalue λ. Suppose that β is in the open set defined by $f \neq 0$ for some polynomial function f in the entries X_{ij} of the generic $n \times n$ matrix. Let $\widetilde{\beta} = \mathrm{diag}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \ldots, \widetilde{\beta}_{\mathrm{r}}\right)$ be a matrix where each $\widetilde{\beta}_i$ has the same size as β_i and the same off-diagonal entries. Suppose further that $\widetilde{\beta}$ has $n-1$ distinct eigenvalues, with $\widetilde{\beta}_1$ and $\widetilde{\beta}_2$ retaining the eigenvalue λ. Then $\widetilde{\beta}$ is semisimple and, for a general $\widetilde{\beta}$, $f(\widetilde{\beta}) \neq 0$.

For example, if

$$\beta = \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$$

then

$$\widetilde{\beta} = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda_3 & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$$

with $\lambda, \lambda_1, \lambda_2, \lambda_3$ distinct. □

Corollary 1.6. *The dimension of C is equal to the dimension of U.*

Lemma 1.7. *The dimension of U is $n^2 - 3$.*

Proof. Let $\Sigma_{n-1} \subset P_n$ be the subset of polynomials having $n-1$ distinct roots. Then Σ_{n-1}, being the image under σ of a closed subset of dimension $n-1$, has dimension $n-1$. The restriction of c to U yields a surjective map $c_U : U \to \Sigma_{n-1}$. The linear group $GL_n(k)$ acts by conjugation transitively on each fibre of c_U and the stabilizer of the matrix $\mathrm{diag}(\lambda, \lambda, \lambda_3, \dots, \lambda_n)$ is $GL_2(k) \times (k^*)^{n-2}$. Hence the dimension of U is $\dim(GL_n(k)) - \dim\left(GL_2(k) \times (k^*)^{n-2}\right) + \dim(\Sigma_{n-1}) = n^2 - (4 + n - 2) + n - 1 = n^2 - 3$. □

Corollary 1.8. *The closed set* $\mathrm{Sing}(Y_n)$ *is of codimension* 3.

Proof. The closure of U is $C = \pi(\mathrm{Sing}(Y_n))$ and π is a finite map. □

2. The generic Galois closure

Let X_{ij} with i, j running from 1 to n be indeterminates and write $P(T) = T^n + a_1 T^{n-1} + \cdots + a_n$ for the characteristic polynomial of the generic matrix (X_{ij}). Let A be the polynomial k-algebra in the X_{ij}. Consider another set $T_1, \dots, T_n$ of indeterminates and let

$$B_n = A[T_1, \dots, T_n]\big/ I$$

where I is the ideal generated by all the polynomials $\sigma_i(T_1, \dots, T_n) - (-1)^i a_i$ for $1 \le i \le n$. Let $Z_n = \mathrm{Spec}(B_n)$. We want to determine $\mathrm{Sing}(Z_n)$.
A k-point of Z_n is a pair (α, t) with the characteristic polynomial of α,

$$P(\alpha)(T) = T^n + a_1(\alpha) T^{n-1} + \cdots + a_n(\alpha)$$

satisfying $a_i(\alpha) = \sigma_i(t)$, $1 \le i \le n$.
Let $\pi : Z_n \to \operatorname{Spec}(A)$ be the first projection and let $S = \pi(\operatorname{Sing}(Z_n))$. We want to compute the dimension of S.
Let (α, t) be a singularity of Z_n. Since no $\sigma_i(T_1, \dots, T_n)$ involves the X_{ij} and no a_j involves the T_i, if we order the X_{ij} lexicographically, the Jacobian matrix of the equations $\sigma_i(T_1, \dots, T_n) - (-1)^i a_i = 0$ is of size $(n^2 + n) \times n$ and looks as follows:

$$J = \begin{pmatrix} \frac{\partial \sigma_1}{\partial T_1} & \cdots & \frac{\partial \sigma_n}{\partial T_1} \\ \vdots & & \vdots \\ \frac{\partial \sigma_1}{\partial T_n} & \cdots & \frac{\partial \sigma_n}{\partial T_n} \\ \frac{\partial a_1}{\partial X_{11}} & \cdots & \frac{(-1)^{n-1}\partial a_n}{\partial X_{11}} \\ \vdots & & \vdots \\ \frac{\partial a_1}{\partial X_{nn}} & \cdots & \frac{(-1)^{n-1}\partial a_n}{\partial X_{nn}} \end{pmatrix}.$$

Since π is a finite map, the dimension of Z_n is n^2. The point (α, t) being a singularity of Z_n, the Jacobian criterion implies that the rank of J at (α, t) is at most $n - 1$. Thus, in particular, the determinant δ of the top $n \times n$ block of J must vanish at (α, t). It is well-known that $\delta = \pm \prod_{i<j}(T_i - T_j)$. This shows that α has at least two equal eigenvalues. In other words, denoting by $V(–)$ the vanishing locus of a given set of polynomials, (α, t) belongs to the vanishing locus $V(\delta^2)$ of the discriminant δ^2 of $P(T)$.
Consider now $\operatorname{Sing}(Z_n) \cap V(a_1, \dots, a_n)$. Since $\operatorname{Sing}(Z_n) \subset V(\delta^2)$ we have

$$\operatorname{Sing}(Z_n \cap V(a_1, \dots, a_n)) = \operatorname{Sing}(Z_n \cap V(\delta^2, a_1, \dots, a_n)).$$

But the vanishing of $a_1, \dots, a_{n-1}$ and δ^2 already implies the vanishing of a_n; in fact, if $T^n - a_n$ has a multiple root, then $a_n = 0$ (we are in characteristic 0). Thus

$$\operatorname{Sing}(Z_n) \cap V(a_1, \dots, a_{n-1}) = \operatorname{Sing}(Z_n) \cap V(a_1, \dots, a_n)$$

and therefore

$$\dim(\operatorname{Sing}(Z_n)) \le \dim(\operatorname{Sing}(Z_n) \cap V(a_1, \dots, a_n)) + n - 1.$$

The set $V(a_1, \dots, a_n)$ is the set $\mathcal{N}$ of nilpotent matrices. On the other hand, the bottom block of the Jacobian matrix must have rank at most $n - 1$, which means that α is a singular point of $\mathcal{N}$. This shows that $\operatorname{Sing}(Z_n) \cap \mathcal{N} \subseteq \operatorname{Sing}(\mathcal{N})$ and from the previous inequality we obtain the next result.

Lemma 2.4. *The dimension of* $\operatorname{Sing}(Z_n)$ *is at most* $\dim(\operatorname{Sing}(\mathcal{N})) + n - 1$.

We now compute the dimension of $\operatorname{Sing}(\mathcal{N})$. As pointed out by George McNinch, our computation could be deduced from results already in the literature (see for instance [7], Sect. 7) but we prefer to be as self-contained as possible. We begin with the computation of the dimension of $\mathcal{N}$.

Proposition 2.5. *Let $\mathcal{N} \subset M_n$ denote the variety of nilpotent matrices. Then the dimension of $\mathcal{N}$ is $n^2 - n$.*

Proof. Since $\mathcal{N}$ is defined by the ideal $(a_1, \ldots, a_n)$ of $A = k[X_{11}, X_{12}, \ldots, X_{nn}]$, it suffices to show that this ideal has height n. Let I be the ideal generated by

$$(a_1, \ldots a_n, X_{ij} \mid i \neq j).$$

We claim that this ideal has height n^2. The ring A/I is isomorphic to

$$k[X_{11}, X_{2,2}, \ldots, X_{nn}]\big/ J$$

where J is the ideal generated by the elementary symmetric functions $\sigma_1, \ldots, \sigma_n$ in $X_{11}, X_{2,2}, \ldots, X_{nn}$. Since $k[X_{11}, \ldots, X_{nn}]$ is finite over $k[\sigma_1, \ldots, \sigma_n]$, the ideal J has height n in $k[X_{11}, \ldots, X_{nn}]$. Hence I is supported only at closed points. Since the a_i are homogeneous, it follows that the ideal $(a_1, \ldots, a_n)$ has height n. □

Lemma 2.6. *A nilpotent matrix α whose Jordan form consists of only one cyclic block is not a singularity of $\mathcal{N}$. More precisely, the determinant of $\left(\frac{\partial a_i}{\partial X_{j1}}\right)$ is not zero at α.*

Proof. Let A be as before and $P(T) = T^n + a_1 T^{n-1} + \cdots + a_n$ the characteristic polynomial of the generic matrix (X_{ij}). The variety of nilpotent matrices is $\mathcal{N} = V(a_1, \ldots, a_n)$. We show that at

$$\alpha = \begin{pmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \end{pmatrix}$$

the jacobian matrix $\left(\frac{\partial a_i}{\partial X_{jk}}\right)$ has rank n. We compute the $n \times n$ matrix $\left(\frac{\partial a_i}{\partial X_{j1}}\right)$. The derivative of a_i by X_{j1} is the coefficient of T^{n-i} in $\frac{\partial P(T)}{\partial X_{j1}}$. Developing the determinant of $(X_{ij}) - T\mathrm{I}_n$ along the first column we find

$$\pm P(T) = (X_{11} - T)P_1(T) + X_{2,1}P_2(T) + \cdots + X_{n,1}P_n(T)$$

where $P_i(T)$ is the determinant of an $(n-1) \times (n-1)$ matrix M_i. At $(X_{ij}) = \alpha$ we find

$$M_i(\alpha) = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$$

with

$$B_1 = \begin{pmatrix} 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ -T & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & -T & 1 & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & -T & 1 \end{pmatrix}$$

of size $j-1$ and

$$B_2 = \begin{pmatrix} -T & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & -T & 1 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & -T & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & -T & 1 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & -T \end{pmatrix}$$

of size $n-j$. Thus $P_j(T) = \pm T^{n-j}$ and $\frac{\partial a_i}{\partial X_{j1}}(\alpha)$ is ± 1 for $j = i$ and zero otherwise. This proves the lemma. $\square$

Lemma 2.7. *The set $\mathcal{N}_2$ of nilpotent matrices whose Jordan form has exactly two cyclic blocks are dense in the set of nilpotent matrices whose Jordan form has two or more blocks.*

Proof. Let $\alpha = \operatorname{diag}(B_1, B_2, \ldots, B_m)$ be a nilpotent matrix which we can assume to be in Jordan form with blocks B_1, ...,B_m, $m \geq 3$. Let $g \neq 0$ with $g \in A$ define a neighbourhood of α. We can find constants $\epsilon_2, \ldots, \epsilon_{m-1}$ such that replacing the zeros between the superdiagonals of B_2 and B_3, between the superdiagonals B_3 and B_4 and so on, by the ϵ_i we obtain a matrix α' such that $g(\alpha') \neq 0$. Clearly α' has two cyclic blocks. $\square$

Lemma 2.8. *If $\alpha \in \mathcal{N}$ has a Jordan form with two or more cyclic blocks, then α is a singularity of $\mathcal{N}$.*

Proof. We may assume that α is in Jordan form and can be written as

$$\operatorname{diag}(B_1, B_2, \ldots, B_m)$$

where $m \geq 2$, each B_i is a cyclic Jordan block, B_1 is of size p and B_2 of size q. We can write the generic matrix as $(X_{ij}) = (\alpha + Y_{ij})$. Then $\frac{\partial a_i}{\partial X_{ij}}(\alpha) = \frac{\partial a_i}{\partial Y_{ij}}(0)$. But in the matrix $\alpha + (Y_{ij})$ the p-th line and the $(p+q)$-th line are linear homogeneous in the Y_{ij}, hence developing the determinant of $\alpha + (Y_{ij})$ along these two lines we see that $a_n(Y_{ij} \mid 1 \leq i, j \leq n)$ has no constant and no linear term. This shows that all the derivatives $\frac{\partial a_n}{\partial Y_{ij}}$ vanish at the origin and therefore the Jacobian matrix $\frac{\partial a_i}{\partial Y_{ij}}$ cannot be of rank n. $\square$

Corollary 2.9. *The set $\mathcal{N}_2$ is dense in* $\operatorname{Sing}(\mathcal{N})$.

The set $\mathcal{N}_2$ is the union of the $GL_n(k)$-orbits $S_{p,q}$ of all the matrices of the form $\beta = \operatorname{diag}(B_p, B_q)$ where B_p is the nilpotent cyclic Jordan block of size p and B_q the nilpotent cyclic Jordan block of size $q = n - p$. In particular, it is the finite union of the constructible sets $S_{p,q}$. The dimension of $S_{p,q}$ is $n^2 - s$ where s is the dimension of the isotropy group of β.

Lemma 2.10. *The dimension of the isotropy group of* $\operatorname{diag}(B_p, B_q)$ *is*

$$p + q + 2\min(p, q).$$

In particular it is always at least $p + q + 2$.

Proof. Let $\Gamma \subset GL_n(K)$ be the isotropy group of $\beta = \mathrm{diag}(B_p, B_q)$. Let

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be an element of Γ, written with blocks A, B, C, D of suitable sizes. The condition $\gamma\beta\gamma^{-1} = \beta$ is equivalent to the conditions

$$AB_p = B_pA, \ DB_q = B_qD, \ BB_q = B_pB, \ CB_p = B_qC.$$

We compute the dimension of the linear subspace Γ_0 of $M_{p+q}(K)$ consisting of matrices that satisfy the four conditions above.
An explicit matrix computation shows that the first condition gives

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \cdot & \cdot & \cdot & a_{p-1} & a_p \\ 0 & a_1 & a_2 & \cdot & \cdot & \cdot & a_{p-2} & a_{p-1} \\ 0 & 0 & a_1 & \cdot & \cdot & \cdot & a_{p-3} & a_{p-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & a_1 & a_2 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & a_1 \end{pmatrix}$$

A similar result holds for D, hence the matrices $\mathrm{diag}(A, D)$ in Γ_0 span a linear space of dimension $p + q$.
Assume now that $p \leq q$. An explicit computation shows that the third condition gives

$$B = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & 0 & b_1 & b_2 & b_3 & \cdot & \cdot & \cdot & b_{p-1} & b_p \\ 0 & \cdot & \cdot & \cdot & 0 & 0 & b_1 & b_2 & \cdot & \cdot & \cdot & b_{p-2} & b_{p-1} \\ 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 & b_1 & \cdot & \cdot & \cdot & b_{p-3} & b_{p-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_1 & b_2 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & b_1 \end{pmatrix}$$

A similar result holds for C, hence, when $p \leq q$ the dimension of Γ_0 is $p + q + p + p = p + q + 2\min(p, q)$ and clearly this is also the dimension (as a variety) of Γ. □

Proposition 2.11. *For $n \geq 3$ the dimension of* $\mathrm{Sing}(\mathcal{N})$ *is $n^2 - n - 2$.*

Proof. By 2.9 and 2.10, $\dim(\mathrm{Sing}(\mathcal{N})) = \dim(\mathcal{N}_2) = n^2 - \min_{p,q}(\dim(S_{p,q}))$. The isotropy group of minimal dimension is $S_{1,n-1}$ which has dimension $n + 2$. Thus $\dim(\mathcal{N}_2) = n^2 - (n + 2)$. □

Theorem 2.12. *For $n \geq 3$ the dimension of* $\mathrm{Sing}(Z_n)$ *is at most $n^2 - 3$.*

Proof. This immediately follows from 2.4 and 2.11. □

3. Finite splitting of Azumaya algebras

Let X be a smooth quasi-projective irreducible surface over an algebraically closed field k, $K = k(X)$ the field of rational functions of X and A a central simple algebra of degree n over K. Let $\mathcal{A}$ be a maximal order in A defined over X. We do not assume that A is a division ring.

Lemma 3.1. *There exists an element σ in A whose characteristic polynomial is irreducible, separable and has Galois group $\mathcal{S}_n$.*

Proof. Let $\sigma_1, \ldots, \sigma_m$ be a K-basis of A (m being equal to n^2). Let $K \subset L$ be a separable finite extension of K such that $A \otimes_K L = M_n(L)$. Let $X_1, \ldots, X_m$ be indeterminates and $\widetilde{\sigma} = X_1\sigma_1 + \cdots + X_m\sigma_m$. After an L-linear change of variables the characteristic polynomial $P_{\widetilde{\sigma}}(T)$ of $\widetilde{\sigma}$ is the characteristic polynomial of the generic matrix, hence it is irreducible and separable over $L(X_1, \ldots, X_m)$, and has Galois group $\mathcal{S}_n$. Since it is defined over $K(X_1, \ldots, X_m)$ it has the same properties over this smaller field. By Hilbert's irreducibility theorem (see for instance [4], Proposition 16.1.5) there exist $\xi_1, \ldots, \xi_m$ in K such that the characteristic polynomial of $\sigma = \xi_1\sigma_1 + \cdots + \xi_m\sigma_m$ is irreducible, separable, with Galois group $\mathcal{S}_n$. □

We fix a smooth embedding of X in a projective space. If d is sufficiently large, the twisted sheaf $\mathcal{A}(d)$ is generated by global sections $s_1, \ldots s_N$. For σ as in Lemma 1 and a suitable global section g of $\mathcal{O}_X(d)$, σg is a global section of $\mathcal{A}(d)$ and we may assume that $s_N = \sigma g$. Such a set of global sections will be called *admissible*. We set $\mathcal{L} = \mathcal{O}_X(d)$.
Let s be any global section of $\mathcal{A}(d) = \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{L}$. Choose an arbitrary affine non-empty open set $U \subset X$ over which $\mathcal{L}$ is principal: $\mathcal{L}_{|U} = \mathcal{O}_U f$ for some $f \in \mathcal{L}(U)$. Then $sf^{-1} \in \mathcal{A}(U)$, which is a maximal order over $\mathcal{O}_X(U)$. Let

$$P_{f,U}(T) = T^n + b_1 T^{n-1} + \cdots + b_n$$

with $b_1, \ldots, b_n \in k[U]$ be the characteristic polynomial of sf^{-1}. We define $J_{f,U}$ as the ideal of

$$Sym\left(\mathcal{L}^{-1}\big|_U\right) = \mathcal{O}_U \oplus \mathcal{L}^{-1}\big|_U \oplus \mathcal{L}^{-2}\big|_U \oplus \cdots = \mathcal{O}_U \oplus \mathcal{O}_U f^{-1} \oplus \mathcal{O}_U f^{-2} \oplus \cdots$$

generated by $f^{-n} \oplus b_1 f^{-(n-1)} \oplus \cdots \oplus b_n$.

Lemma 3.2. *Let Λ be a central simple algebra of rank n^2 over a field K. For any $\alpha \in \Lambda$ and any $c \in K$, the characteristic polynomial $P_\alpha(T)$ of α satisfies the relation $c^n P_\alpha(T) = P_{c\alpha}(cT)$.*

Proof. It immediately follows from the split case $\Lambda = M_n(K)$. □

Lemma 3.3. *The ideal $J_{f,U}$ does not depend on the choice of f.*

Proof. We apply 3.2 with $f = ug$ for some other generator g of $\mathcal{L}|_U$ and u invertible on U. (We note that the suffixes f or g stand for the elements s/f, s/g in the algebra). We have

$$P_{g,U}(T) = P_{u^{-1}f,U}(T) = u^n P_{f,U}(u^{-1}T) = T^n + ub_1T^{n-1} + \cdots + u^n b_n.$$

Thus the ideal $J_{g,U}$ is generated by

$$g^{-n} \oplus b_1ug^{-(n-1)} \oplus \cdots \oplus u^n b_n = u^n(f^{-n} \oplus b_1 f^{-(n-1)} \oplus \cdots \oplus b_n).$$

and coincides therefore with $J_{f,U}$.
Patching the ideals $J_{f,U}$ over a suitable affine covering of X yields a global ideal J_s of $Sym(\mathcal{L}^{-1})$ that only depends on the section s. We call J_s *the characteristic ideal of* s. □

The ideal J_s defines a closed subscheme Y_s of Spec $\left(Sym(\mathcal{L}^{-1})\right)$ which is clearly finite and flat over X.
To simplify notation, if $s = \lambda_1 s_1 + \cdots + \lambda_N s_N$ we put $\lambda = (\lambda_1, \ldots, \lambda_N) \in k^N$, $J_s = J_\lambda$ and $Y_s = Y_\lambda$. We denote by $\pi_\lambda : Y_\lambda \to X$ the natural map.

Theorem 3.4. *Let X be a smooth quasi-projective irreducible surface over an algebraically closed field k, $K = k(X)$ the field of rational functions of X and A a central simple algebra of degree n over K. Let $\mathcal{A}$ be a maximal order in A defined over X. Let $s_1, \ldots, s_N$ be an admissible set of sections of $\mathcal{A}(d)$ and for any $\lambda \in k^N$, let Y_λ be as above. There exists a nonempty open set $U \subset k^N$ such that, for any $\lambda \in U$, Y_λ is an irreducible quasi-projective surface.*

Before proving this theorem we recall, without proof, two easy lemmas.

Lemma 3.5. *Let $\pi : Y \to X$ be a flat dominant morphism, with X integral. Then Y is reduced if and only if the generic fibre of π is reduced.*

Lemma 3.6. *Let $\pi : Y \to X$ be a flat dominant morphism, with X integral. Then Y is irreducible if and only if the generic fibre of π is irreducible.*

Proof of Theorem 3.4. We set $\mathbb{A}_k^N = \text{Spec}\,(k[t_1, \ldots, t_N])$ and extend the base to $\widetilde{X} = X \times \mathbb{A}_k^N$. Let $\widetilde{A}$ and $\widetilde{\mathcal{L}}$ be the inverse images of A and $\mathcal{L}$ under the projection $\pi : \widetilde{X} \to X$. Put $\widetilde{s} = t_1 s_1 + \cdots + t_N s_N$ and let $\widetilde{J}_t(T)$ be the characteristic ideal of $\widetilde{s}$ and $\widetilde{Y}$ the closed subscheme of Spec $\left(\text{Sym}(\widetilde{\mathcal{L}}^{-1})\right)$ defined by $\widetilde{J}_t(T)$. Look at the diagram

$$\begin{tikzcd} & \widetilde{Y} \arrow[dl, "p"'] \arrow[d, "\pi"] \arrow[dr, "q"] & \\ X & \widetilde{X} \arrow[l] \arrow[r] & \mathbb{A}_k^N \end{tikzcd}$$

The map π is clearly finite and flat and the two projections from $X \times \mathbb{A}_k^N$ are flat, hence p and q are flat. We set $\widetilde{Y}_K = \widetilde{Y} \times_X \text{Spec}(K)$ and $q_K : \widetilde{Y}_K \to \mathbb{A}_K^N$ the restriction of q to $\widetilde{Y}_K$. We first note that, by the choice of s_N made above, the fibre $q_K^{-1}(0,\dots,0,1)$ is integral. By Theorem 9.7.7 of [5], to prove the theorem it suffices to show that the geometric generic fibre of q is integral. Let Ω be an algebraic closure of $k(t_1,\dots,t_N)$, $\widetilde{Y}_\Omega = \widetilde{Y} \times_{\mathbb{A}_k^N} \mathrm{Spec}(\Omega)$ the generic fibre of q, $\widetilde{X}_\Omega = X \times_k \Omega$ and $\pi_\Omega : \widetilde{Y}_\Omega \to \widetilde{X}_\Omega$ the extension of π. Let S be the integral closure of $k[t_1,\dots,t_N]$ in Ω and $\Lambda = K \otimes_k S$. We set $\widetilde{Y}_\Lambda = \widetilde{Y} \times_{\widetilde{X}} \mathrm{Spec}(\Lambda)$, $\widetilde{X}_\Lambda = \mathrm{Spec}(\Lambda)$ and $\pi_\Lambda : \widetilde{Y}_\Lambda \to \widetilde{X}_\Lambda$ the extension of π. Assume that $\widetilde{Y}_\Omega$ is not integral. Since π_Ω is flat, by 3.5 and 3.6 the generic fibre of π_Ω is not integral. But π_Λ is also flat and has the same generic fibre as π_Ω, hence, again by 3.5 and 3.5, $\widetilde{Y}_\Lambda$ is not integral. The characteristic polynomial $P_{\widetilde{s}/f}(T) \in K[t_1,\dots,t_N]$ that generates $\widetilde{J}_t(T)$ over a suitable open set of X is clearly separable over $K(t_1,\dots,t_N)$, hence $\widetilde{Y}_\Lambda$ is reduced by Lemma 3.5. If $\widetilde{Y}_\Lambda$ is not integral, being reduced it has more than one component and since π_Λ is finite and flat, each component maps surjectively onto $\widetilde{X}_\Lambda$ and hence no fibre is integral. Let z be a point of $\widetilde{X}_\Lambda$ over the point $(0,\dots,0,1)$ of $\mathbb{A}_K^N$. Specializing at z we get a contradiction with the irreducibility of $\pi_\Lambda^{-1}(0,\dots,0,1) = \mathrm{Spec}(K) \times_X Y_{(0,\dots,0,1)}$. □

Corollary 3.7. *Let U be as in 3.4. For any $\lambda \in W$ the field $k(Y_\lambda)$ splits A.*

Proof. By construction the field $k(Y_\lambda)$ is a maximal subfield of A. □

We now assume that $\mathcal{A}$ is an Azumaya algebra over X and show how to construct a smooth splitting, dealing first with the quasiprojective case in characteristic zero.

Proposition 3.8. *Assume that $\mathcal{A}$ is an Azumaya algebra over X. The dimension of* $\mathrm{Sing}(\widetilde{Y})$ *is at most $N-1$.*

Proof. We try to determine the singularities of $\widetilde{Y}$ using the following lemma. □

Lemma 3.9. *Let $f : Z \to X$ be a flat map of schemes. Suppose that X is regular. If $z \in Z$ is a singular point of Z, then z is a singularity of its fibre $f^{-1}(f(z))$.*

Proof. Let C be the local ring of Z at z and A be the local ring of $f(z)$. By assumption the maximal ideal of A is generated by a regular sequence $(x_1,\dots,x_m)$. Since f is flat, C is faithfully flat over A and this sequence is still regular as a sequence in C. If z is not a singular point of its fibre, then $C/(x_1,\dots,x_m)$ is regular and hence its maximal ideal is generated by a regular sequence $(\overline{y}_1,\dots,\overline{y}_r)$. This implies that the maximal ideal of C is generated by the regular sequence $(x_1,\dots,x_m,y_1,\dots,y_r)$, hence C is regular. □

By 3.9 the singularities of $\widetilde{Y}$ are contained in the union of the singularities of the fibres of p.

Lemma 3.10. *For any $x \in X$ the singular locus of the fibre $p^{-1}(x)$ of p has codimension* 3 *in $p^{-1}(x)$.*

Proof. Let $k(x)$ be the residue field of $x \in X$, Ω its algebraic closure and F_x the fibre of p at x. The geometric fibre $\mathcal{A}(\overline{x})$ of $\mathcal{A}$ at x is a matrix algebra $M_n(\Omega)$ and

$$F_{\overline{x}} = \mathrm{Spec}\,(\Omega[t_1,\dots,t_N][T]/(P_x(T))),$$

where $P_x(T)$ is the characteristic polynomial of $\overline{s} = (t_1 s_1(x) + \cdots + t_N s_N(x))\big/ f(x)$ for some generator f of $\mathcal{L}|_U$, U a neighbourhood of x. Since the sections $s_i(x)/f(x)$ generate $M_n(\Omega)$ over Ω, by a linear change of coordinates we may assume that $\overline{s} = t_1 e_1 + \cdots + t_m e_m$ where $m = n^2$ and $\{e_1, \ldots, e_m\}$ form a basis of $M_n(\Omega)$. Then

$$F_{\overline{x}} = Y_n \times \operatorname{Spec}(\Omega[t_{m+1}, \ldots, t_N]).$$

We proved that $\operatorname{Sing}(Y_n)$ has codimension 3, hence the same holds for $\operatorname{Sing}(F_{\overline{x}})$ and for $\operatorname{Sing}(F_x)$. □

Theorem 3.11. *The dimension of* $\operatorname{Sing}(\widetilde{Y})$ *is at most* $N-1$.

Proof. For every $x \in X$ the fibre F_x of p is a finite cover of $\mathbb{A}_k^N$ and hence the dimension of F_x is N. Let $\operatorname{Sing}(\widetilde{Y})$ be the singular locus of $\widetilde{Y}$. By 3.9, for every $x \in X$, the fibre at x of $p|_{\operatorname{Sing}(\widetilde{Y})} : \operatorname{Sing}(\widetilde{Y}) \to X$ is contained in the singular locus of F_x and has therefore dimension at most $N-3$. Since X is 2-dimensional, the dimension of $\operatorname{Sing}(\widetilde{Y})$ is at most $N-1$. □

4. Smooth splitting in characteristic zero

Theorem 4.1. *Let k be an algebraically closed field of characteristic* 0*, X a smooth quasi-projective irreducible surface over k, $K = k(X)$ the field of rational functions of X. Let $\mathcal{A}$ be an Azumaya algebra over X and $s_1, \ldots, s_N$ an admissible set of sections of $\mathcal{A}(d)$ as defined in Sect. 3. For any $\lambda \in k^N$ let Y_λ be the surface associated to the section $\lambda_1 s_1 + \cdots + \lambda_N s_N$. There exists a nonempty open set $V \subset k^N$ such that for any $\lambda \in V$, Y_λ is a smooth integral quasi-projective surface. Further, the pull-back $\pi_\lambda^* \mathcal{A}$ is trivial in* $\operatorname{Br}(Y_\lambda)$.

Proof. Look at $q : \widetilde{Y} \to \mathbb{A}_k^N$. Since by 3.11 $\operatorname{Sing}(\widetilde{Y})$ is at most $(N-1)$-dimensional, its image $q(\operatorname{Sing}(\widetilde{Y}))$ is contained in a proper closed subset of $\mathbb{A}_k^N$. Choose an open set $W \subset \mathbb{A}_k^N$ which does not intersect $q(\operatorname{Sing}(\widetilde{Y}))$ and let $\widetilde{W} = q^{-1}(W) \cap \widetilde{Y}$. We now have a map $q : \widetilde{W} \to W$ of smooth varieties. This map is clearly flat and surjective and therefore, if k is of characteristic zero, it is generically smooth (see [6], Chap. III, Corollary 10.7). By definition of generic smoothness there exists a dense open set $U' \subset \mathbb{A}_k^N$ such that $q^{-1}(U') \cap \widetilde{Y} \to U'$ is smooth. Thus for any $\lambda \in U'$ the fibre $Y_\lambda = q^{-1}(\lambda) \cap \widetilde{Y}$ is smooth. By 3.4, if $\lambda \in U$ then Y_λ is integral, hence for any $\lambda \in V = U \cap U'$ the surface Y_λ is smooth and integral. By 3.7 the field $k(Y_\lambda)$ splits A. But Y_λ being smooth, the canonical map $\operatorname{Br}(Y_\lambda) \to \operatorname{Br}(k(Y_\lambda))$ is injective and thus $\pi_\lambda^* \mathcal{A}$ is trivial in $\operatorname{Br}(Y_\lambda)$. □

Remark. In positive characteristic Theorem 4.1 is not true for arbitrary sets of admissible sections. Let for instance X be the affine plane $X = \operatorname{Spec}(k[u, v])$ (the affine line would also suffice) over a field of odd characteristic p and $\mathcal{A}$ the trivial Azumaya algebra $M_2(\mathcal{O}_X)$ over X. Then $\mathcal{A}$ is generated by its global sections

$$s_1 = \begin{pmatrix} 1 & u^p \\ 0 & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad s_4 = \begin{pmatrix} 1 & u^p \\ 1 & 1 \end{pmatrix},$$

and the generic splitting that we denoted $\widetilde{Y}$ is the spectrum of

$$S = k[u, v, t_1, t_2, t_3, t_4][T]/(P(T))$$

where the determinant $P(T)$ of $T \cdot \mathrm{I}_2 - (t_1 s_1 + t_2 s_2 + t_3 s_3 + t_4 s_4)$ is

$$T^2 - (t_1 + 2t_4)T + t_4(t_1 + t_4) - (t_3 + t_4)(t_2 + t_4 u^p).$$

The algebra S is smooth over k if and only if P, P', $\partial P/\partial u$ and $\partial P/\partial v$ have no common zero over the algebraic closure of $k(t_1, t_2, t_3, t_4)$. But in fact, they are easily seen to be solvable with respect to u provided $(t_3 + t_4)t_4 \neq 0$.
Still, the theorem is true in any characteristic if we choose more accurately the sections $s_1, \dots, s_N$.

5. Smooth splitting in arbitrary characteristic

Lemma 5.1. *Let $X \subset \mathbb{P}^n_k$ be a quasiprojective variety and let $\mathcal{F}$ be a coherent sheaf on X generated by global sections $s_1, \dots, s_N$. Let $V = H^0(X, \mathcal{O}_X(1)) = kx_0 + \cdots + kx_n$ where $x_0, \dots, x_n$ are the projective coordinates on X. Let $W \subseteq H^0(X, \mathcal{F})$ be the k-space generated by $s_1, \dots, s_N$. We denote by m_x the maximal ideal of the local ring of any closed point x of X.*

(a) *For any $x \in X(k)$ the canonical map*

$$V \to H^0\left(X, \mathcal{O}_X(1) \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x}/m_x^2\right)$$

is surjective.

(b) *For any $x \in X(k)$ the canonical map*

$$V \otimes_k W \to H^0\left(X, \mathcal{F}(1) \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x}/m_x^2\right)$$

is surjective.

Proof. The second assertion immediately follows from the first one. As to the first one, let $x \in \mathbb{P}^n_k$ be any closed point of X. It will be defined by the vanishing of n linear forms, which we may assume to be $x_1, \dots, x_n$. Then m_x is the ideal of $\mathcal{O}_{X,x}$ generated by $x_1/x_0, \dots, x_n/x_0$ and

$$\mathcal{O}_{X,x}/m_x^2 = k + k\overline{(x_1/x_0)} + \cdots + k\overline{(x_n/x_0)}$$

where the bar denotes the class modulo m_x^2. We thus have

$$H^0\left(\mathcal{O}_X(1) \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x}/m_x^2\right) = k\overline{x}_0 + \cdots + k\overline{x}_n$$

which proves the assertion. □

Let X be an irreducible quasiprojective smooth surface over k and $\mathcal{A}$ an Azumaya algebra of degree n over X. We assume here that, by the lemma we just proved, we have chosen the line bundle $\mathcal{L}$ such that the global sections $s_1, \dots, s_N$ generate

$$H^0\left(X, \mathcal{A} \otimes_{\mathcal{O}_x} \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x}/m_x^2\right)$$

as a vector space over k for every closed point $x \in X(k)$.
We still assume that $s_N = \sigma g$ with $g \neq 0$ a section of $\mathcal{L}$ and σ as in Lemma 3.1. Let $p : \widetilde{Y} \to X$ and $\widetilde{Y} \to \mathbb{A}_k^N$ be as above. We study under which conditions the fibre of $Y_\lambda \to X$ at $x \in X(k)$ is singular. We fix an x in $X(k)$ and set $R = \mathcal{O}_{X,x}$, $m = m_x$ and $\overline{R} = R/m^2$. Reduction modulo m^2 will systematically be denoted by a bar. Let ξ, η be generators of m. Then, $\overline{R} = k[\xi, \eta]$ with $\xi^2 = \xi\eta = \eta^2 = 0$. We choose an isomorphism $\mathcal{A}(\mathrm{Spec}(R)) \otimes_R \overline{R} \simeq M_n(\overline{R})$, and a local section $f \neq 0$ of $\mathcal{L}$ defining an isomorphism $\mathcal{L}(\mathrm{Spec}(R)) \to R$. Consider the composition of k-linear maps

$$\begin{aligned}\varphi : k^N \to H^0\left(X, \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{L}\right) \to \mathcal{A}(\mathrm{Spec}(R)) \otimes_R \mathcal{L}(\mathrm{Spec}(R)) \to \mathcal{A}(\mathrm{Spec}(R))\\ \to M_n(\overline{R})\end{aligned}$$

mapping λ to the image of s_λ/f.
We write every element $\overline{a}$ of $M_n(\overline{R})$ as $\overline{a} = \alpha + \beta\xi + \gamma\eta$ with α, β and $\gamma \in M_n(k)$. Suppose now that $s_\lambda/f = a \in \mathcal{A}(R)$ is the local section corresponding to $\lambda \in \mathbb{A}_k^N$ and $\overline{a}$ its image in $M_n(\overline{R})$. The reduction modulo m^2 of the local affine algebra of $\widetilde{Y}$ at (x, λ) is

$$\overline{R}[T]\big/\overline{P}_\lambda(T)$$

where

$$P(T) = T^n + a_1 T^{n-1} + \cdots + a_{n-1}T + a_n$$

is the characteristic polynomial of a. We denote its reduction modulo m by $\overline{\overline{P}}(T)$. We introduce the set of matrices

$$S(x) = \{\overline{a} \in M_n(\overline{R}) \bigm| \exists\, \lambda \in k^N \text{ s.t. } \varphi(\lambda) = \overline{a} \text{ and } Y_\lambda \text{ is singular}\}$$

and set $\widetilde{S}(x) = \varphi^{-1}(S(x))$. Observe that $\widetilde{S}(x)$ does not depend on the choice of the local section f because if $\overline{a} \in S(x)$ then $\overline{a}u \in S(x)$ for any unit u of $\overline{R}$.

Proposition 5.2. *The codimension of $S(x)$ in $M_n(\overline{R})$ is as least* 3.

Proof. We consider more cases than what is really necessary because we want to prepare the way for the Galois splitting in the next section. □

Fix a point $y = (x, \mu) \in Y_\lambda$ in the fibre of x, where μ is a root of $\overline{\overline{P}}(T) \in k[T]$. The fibre of $p : Y_\lambda \to X$ at x is singular at y if and only if the derivatives $\frac{\partial \overline{P}}{\partial T}, \frac{\partial \overline{P}}{\partial \xi}, \frac{\partial \overline{P}}{\partial \eta}$ vanish at $y = (x, \mu)$. To see what this means we write $\overline{a} = \alpha + \xi\beta + \eta\gamma$ with α, β and γ in $M_n(k)$. If μ is a simple root, then $\frac{\partial \overline{P}}{\partial T} \neq 0$ at (x, μ) and (x, μ) is a smooth point of Y_λ. Assume therefore that α has at least two identical eigenvalues. The set of all matrices $\alpha \in M_n(k)$ with at most $n-3$ different eigenvalue has codimension 3, so we only have to deal with the cases in which α has $n-1$ or $n-2$ distinct eigenvalues. This is the same as saying that α is conjugated to a matrix

$$\begin{pmatrix} J_i & 0 \\ 0 & D \end{pmatrix}$$

where D is a diagonal matrix with distinct eigenvalues, different from μ for $1 \leq i \leq 5$ and distinct from μ and ν for $6 \leq i \leq 8$ and $\mu \neq \nu$ and J_i is one of the following matrices

$$J_1 = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}, J_2 = \begin{pmatrix} \mu & 1 \\ 0 & \mu \end{pmatrix},$$

$$J_3 = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}, \ J_4 = \begin{pmatrix} \mu & 1 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}, \ J_5 = \begin{pmatrix} \mu & 1 & 0 \\ 0 & \mu & 1 \\ 0 & 0 & \mu \end{pmatrix},$$

$$J_6 = \begin{pmatrix} \mu & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \nu & 0 \\ 0 & 0 & 0 & \nu \end{pmatrix}, \ J_7 = \begin{pmatrix} \mu & 1 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \nu & 0 \\ 0 & 0 & 0 & \nu \end{pmatrix}, \ J_8 = \begin{pmatrix} \mu & 1 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \nu & 1 \\ 0 & 0 & 0 & \nu \end{pmatrix}.$$

For $1 \leq i \leq 8$ let M_n^i be the set of all matrices $\overline{a} \in M_n(\overline{R})$ for which α is of the form $\mathrm{diag}(J_i, D)$ and β and γ are arbitrary matrices in $M_n(k)$. These sets are open subsets of affine spaces, in particular they are irreducible. We denote by $\widehat{M}_n^i$ the $Gl_n(k)$-orbit of M_n^i and by G_i the stabilizer of M_n^i in $Gl_n(k)$. Since $Gl_n(k)$ is irreducible, all $\widehat{M}_n^i$'s are irreducible. From the formula

$$\dim(\widehat{M}_n^i) \leq \dim(M_n^i) + \dim(Gl_n(K)) - \dim(G_i)$$

we first compute an upper bound for the dimension of each $\widehat{M}_n^i$.
Using that if $M \in M_m(k)$ is either a Jordan block or a diagonal matrix with distinct eigenvalues, then its stabilizer in $Gl_m(k)$ has dimension m, together with a direct computation for G_4 we find $\dim(G_1) \geq n+2$, $\dim(G_2) \geq n$, $\dim(G_3) \geq n+6$, $\dim(G_4) \geq n+2$, $\dim(G_5) \geq n$, $\dim(G_6) \geq n+4$, $\dim(G_7) \geq n+2$, $\dim(G_8) \geq n+2$.
On the other hand, $\dim(M_n^i) = 2n^2 + n - 1$ for $i = 1, 2$ and $2n^2 + n - 2$ for $3 \leq i \leq 8$. Thus the codimension of $\widehat{M}_n^2$ is 1, that of $\widehat{M}_n^5$, $\widehat{M}_n^8$ is 2 and the remaining ones have codimension ≥ 3. hence we only have to consider the singularities arising from $\widehat{M}_n^2$, $\widehat{M}_n^5$, and $\widehat{M}_n^8$.
We shall show that if $\overline{a} = \alpha + \xi\beta + \eta\gamma$ is in $S(x) \cap \widehat{M}_n^2$, then β and γ must both belong to certain proper closed subsets of $M_n(k)$.
The point (x, μ) is singular if and only if both $\frac{\partial \overline{P}}{\partial \xi}$ and $\frac{\partial \overline{P}}{\partial \eta}$ vanish at $T = \mu$. To compute $\overline{P}(T)$ we can use the following lemma. □

Lemma 5.3. *Let A be a commutative ring, $I \subset A$ an ideal such that $I^2 = (0)$, and $M \in M_n(A)$ a matrix of the form*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with a, d square blocks and b, c having entries in I. The characteristic polynomial of M is $P_M(T) = P_a(T)P_d(T)$ where P_a and P_d are the characteristic polynomials of a and d respectively.

Proof. Since $P_a(T)$ is not a zero divisor, we can embed A into $A[T, 1/P_a(T)]$ and compute in this overring, using the fact that $M_n(A[T, 1/P_a(T)])$ contains $(T-a)^{-1}$. We have

$$\det\begin{pmatrix} T-a & -b \\ -c & T-d \end{pmatrix} = \det\begin{pmatrix} 1 & 0 \\ c(T-a)^{-1} & 1 \end{pmatrix} \det\begin{pmatrix} T-a & -b \\ -c & T-d \end{pmatrix}$$

$$= \det\begin{pmatrix} T-a & -b \\ -0 & -c(T-a)^{-1}b + T - d \end{pmatrix} = \det(\mathrm{T_a})\det(\mathrm{T_d})$$

because $c(T-a)^{-1}b = 0$. □

We now complete the proof of 5.2. Using 5.3 we see that, if $\overline{a}$ is in M_n^2, $\beta = (\beta_{i,j})$ and $\gamma = (\gamma_{i,j})$, then

$$\left(\frac{\partial \overline{P}}{\partial \xi}, \frac{\partial \overline{P}}{\partial \eta}\right)\Bigg|_{\substack{T=\mu \\ (\xi,\eta)=(0,0)}} = (-\beta_{2,1}, -\gamma_{2,1})P_D(\mu)$$

where $P_D(T)$—the characteristic polynomial of D—does not vanish at μ. Hence, the point (x, μ) is singular if and only if

$$\beta_{2,1} = 0 \quad \text{and} \quad \gamma_{2,1} = 0.$$

This shows that $S(x) \cap M_n^2$ is of codimension 2 in M_n^2, hence of codimension at least 3 in $M_n(\overline{R})$. Since G_2 also stabilizes $S(x) \cap M_n^2$, the codimension of its orbit $S(x) \cap \widehat{M}_n^2$ is at least 3.
In the remaining two cases the codimension of $\widehat{M}_n^i$ is 2 and, as we have seen, the set $\widehat{M}_n^i$ is irreducible. Since the set of matrices $\overline{a} \in M_n(\overline{R})$ for which (x, μ) is a smooth point is an open set, to show that $S(x) \cap \widehat{M}_n^i$ is of codimension ≥ 3 it suffices to show that $\widehat{M}_n^i$ contains a matrix for which the fibre of x consists of smooth points.
A direct computation shows that if

$$A = \begin{pmatrix} 0 & 1 & 0 \\ \xi & 0 & 1 \\ \eta & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \xi & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \eta & 1 \end{pmatrix},$$

then for a diagonal with distinct eigenvalues different from 0 and 1, $\mathrm{diag}(A, D) \in \widehat{M}_n^5 \backslash S(x)$ and $\mathrm{diag}(B, D) \in \widehat{M}_n^8 \backslash S(x)$.
This finishes the proof of 5.2. □

We now show the existence of smooth splittings.

Theorem 5.4. *Let X be an irreducible quasiprojective smooth surface over k and $\mathcal{A}$ an Azumaya algebra of degree n over X. Assume (5.1) that we have chosen the line bundle $\mathcal{L}$ such that the global sections $s_1, \ldots, s_N$ generate*

$$H^0\left(X, \mathcal{A} \otimes_{\mathcal{O}_x} \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x}/m_x^2\right)$$

for every closed point $x \in X(k)$. Assume also that $s_N = \sigma g$ with $g \neq 0$ a section of $\mathcal{L}$ and σ are as in Lemma 3.1. Then there exists an open dense set $U \subset k^N$ such that, for any $\lambda \in U$ the surface Y_λ is a smooth irreducible finite cover of X and splits $\mathcal{A}$.

Proof. It only remains to prove smoothness for λ varying in a suitable open set U. Since, by the choice of $s_1, \ldots, s_N$, the linear map φ is surjective, $\widetilde{S}(x)$ is a closed set of codimension ≥ 3 in k^N. Let $\widetilde{S}$ be the union of all $\widetilde{S}(x)$ for x running over $X(k)$.
Let now $\Sigma \subset \widetilde{Y}(k)$ be the closed set of points of $\widetilde{Y}(k)$ at which the map $q : \widetilde{Y} \to \mathbb{A}_k^N$ is not smooth. Since q is flat, being smooth is the same as having smooth fibres and therefore its image $q(\Sigma)$ in k^N is $\widetilde{S}$, which is closed because q is a projective map. We want to show that $\widetilde{S}$ is a proper closed subset of k^N. For any $x \in X(k)$ the closed set $\Sigma(x) := \pi^{-1}(x \times k^N) \cap \Sigma$ is mapped by q onto $\widetilde{S}(x)$, which has codimension ≥ 3 in k^N. Since q is a flat surjective map, $\Sigma(x)$ has codimension ≥ 3 in $\pi^{-1}(x \times k^N)$, hence dimension at most $N-3$. Since X is two-dimensional the dimension of Σ is at most $N-1$. This shows that its image $\widetilde{S}$ in k^N is a proper closed subset of k^N. From this we conclude that for a general $\lambda \in k^N$ the surface Y_λ is smooth. □

6. Smooth finite Galois splitting of Azumaya algebras

We now construct, for any $\lambda \in k^N$, a Galois covering Z_λ of X with group $G = \mathcal{S}_n$, such that $X = Z_\lambda/G$. Notice that, in general, even if Y_λ is smooth its Galois closure may be singular. Therefore, in order to have Y and Z smooth we must construct both at the same time. We achieve this by globalizing the construction of the universal splitting algebra of a monic polynomial, which we now recall.
Let R be a commutative ring and $P(T) = T^n + b_1 T^{n-1} + \cdots + b_n$ a monic polynomial with coefficients in R. For $1 \leq i \leq n$ let σ_i be the i-th elementary symmetric function in the n variables $T_1, \ldots, T_n$. The universal splitting algebra of $P(T)$ is the quotient S of the polynomial algebra $R[T_1, \ldots, T_n]$ by the ideal I generated by the elements

$$\sigma_i(T_1, \ldots, T_n) - (-1)^i b_i, \quad 1 \leq i \leq n.$$

We denote by $\tau_1, \ldots, \tau_n$ the classes modulo I of $T_1, \ldots, T_n$. We clearly have

$$P(T) = (T - \tau_1) \cdots (T - \tau_n).$$

The symmetric group $\mathcal{S}_n$ operates on S by permuting $\tau_1, \ldots, \tau_n$.
We will use the following properties of S. (For more details and proofs see [1] or [3]).

$P1$. The construction of S commutes with scalar extensions ([3], 1.9).
$P2$. As an R-module S is free of rank $n!$ ([3], 1.10).
$P3$. For any commutative R-algebra A and any n-tuple $(a_1, \dots, a_n)$ of elements of A such that $p(T) = (T - a_1) \cdots (T - a_n)$ in $A[T]$ there is a unique R-homomorphism $\varphi : S \to A$ such that $\varphi(\tau_i) = a_i$ ([3], 1.3).
$P4$. The subalgebra $R[\tau_n]$ of S is isomorphic to $R[T]\big/(P(T))$ and S is the universal splitting algebra of $P(T)\big/(T - \tau_n)$ over $R[\tau_n]$ ([3], 1.8).
$P5$. If the discriminant of $P(T)$ is a regular element of R, then $S^{\mathcal{S}_n} = R$ ([3], 2.2).
$P6$. If R is a field and $P(T)$ is separable with Galois group $\mathcal{S}_n$, then S is a Galois extension of R with Galois group $\mathcal{S}_n$.

We now construct Z_λ. Let $\mathcal{L}$ be a very ample line bundle such that $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{L}$ is generated by global sections $s_1, \dots, s_N$ and assume that $s_N = \sigma g$ with $g \neq 0$ a global section of $\mathcal{L}$ and σ as in Lemma 3.1. Let $U \subset X$ be an affine open set for which $\mathcal{L}|_U$ is isomorphic to $\mathcal{O}_U f$ for some section f on U. We set, as in Sect. 3, $s = \lambda_1 s_1 + \cdots \lambda_N s_N$. Let $P_{f,U}(T) = T^n + b_1 T^{n-1} + \cdots + b_n$ be the characteristic polynomial of $s/f \in \mathcal{A}(U)$. We choose n isomorphic copies $\mathcal{L}_1, \dots, \mathcal{L}_n$ of $\mathcal{L}$ and for each i, $f_i = f$ the generator of $\mathcal{L}_i|_U$. Consider

$$\mathcal{T} = Sym\left(\mathcal{L}_1^{-1} \oplus \cdots \oplus \mathcal{L}_n^{-1}\right).$$

Writing $f_i^{-1} f_j^{-1}$ instead of $f_i^{-1} \otimes_{\mathcal{O}_U} f_j^{-1}$ we shall write the restriction of $\mathcal{T}$ to U simply as

$$\bigoplus \mathcal{O}_U f_1^{-i_1} \cdots f_n^{-i_n}.$$

Note that $\mathcal{O}_U[T_1, \dots, T_n]$ is isomorphic to $\mathcal{T}|_U$ under $T_i \mapsto f_i^{-1}$.
We define $\mathcal{J}_{f,U} \subset \mathcal{T}|_U$ as the ideal generated by

$$\sigma_i\left(f_1^{-1}, \dots, f_n^{-1}\right) - (-1)^i b_i, \ 1 \leq i \leq n.$$

It corresponds in the polynomial algebra to the ideal generated by

$$F_i = \sigma_i(T_1, \dots, T_n) - (-1)^i b_i, \ 1 \leq i \leq n$$

which defines the universal splitting algebra of $P_{f,U}(T)$. As in the preceding section, it is easy to check that these ideals do not depend on the choice of f and can therefore be patched over the various U's to obtain a global ideal $\mathcal{J}_\lambda \subset \mathcal{T}$.
Let Z_λ be the closed subscheme of $\mathrm{Spec}(\mathcal{T})$ defined by $\mathcal{J}_\lambda$.

Proposition 6.1. *Assume that $\lambda \in k^N$ has been chosen such that $P_{f,U}(T) = P(T)$ is separable and irreducible over K. The symmetric group $\mathcal{S}_n$ acts on Z_λ via its obvious action on $\mathcal{T}$. The quotient $Z_\lambda/\mathcal{S}_n$ coincides with X and Y_λ coincides with the quotient $Z_\lambda/\mathcal{S}_{n-1}$, where $\mathcal{S}_{n-1}$ is the isotropy group of* 1.

Proof. It suffices to deal with the affine case, when S is the universal splitting algebra of $P(T)$ over $R = k[U]$ and show that $S^{\mathcal{S}_n} = R$ and $S^{\mathcal{S}_{n-1}} = R[T]\big/(P(T))$. Since $P(T)$ is separable over K the first assertion follows from property P6 and the second from properties P3 and P6. □

Theorem 6.2. *There exists a nonempty open set $U \subset k^N$ such that, for any $\lambda \in U$, Z_λ is an irreducible quasi-projective surface. The natural map $\pi_\lambda : Z_\lambda \to X$ is a ramified Galois cover with group $\mathcal{S}_n$ and splits $\mathcal{A}$.*

Proof. The splitting property follows from Proposition 6.1 because $Z_\lambda/\mathcal{S}_{n-1} = Y_\lambda$ which splits $\mathcal{A}$. It remains to prove that for a general λ the fibre Z_λ is irreducible. We extend the base to $\widetilde{X} = X \times \mathbb{A}_k^N$ where $\mathbb{A}_k^N = \operatorname{Spec}(k[t_1, \ldots, t_N])$ and define $\widetilde{\mathcal{A}}$, $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{L}}_i$ for $1 \leq i \leq n$ as the inverse images of $\mathcal{A}$, $\mathcal{L}$ and the $\mathcal{L}_i$'s under the projection $\pi : \widetilde{X} \to X$. Repeating the construction of $\mathcal{J}_\lambda$ we obtain an ideal $\mathcal{J}_t$, where $t = (t_1, \ldots, t_N)$, which specializes to $\mathcal{J}_\lambda$ when we specialize t to λ. The scheme $\widetilde{Z}$ is the closed subscheme of

$$\operatorname{Spec}\left(\widetilde{\mathcal{T}}\right) = \operatorname{Spec}\left(Sym\left(\widetilde{\mathcal{L}_1}^{-1} \oplus \cdots \oplus \widetilde{\mathcal{L}_n}^{-1}\right)\right)$$

defined by $\mathcal{J}_t$.
Look at the diagram

$$\begin{tikzcd}
 & \widetilde{Z} \arrow[dl, "p"'] \arrow[d, "\pi"] \arrow[dr, "q"] & \\
X & X \times \mathbb{A}_k^N \arrow[l] \arrow[r] & \mathbb{A}_k^N
\end{tikzcd}$$

The map π is clearly finite and flat and the two projections from $X \times \mathbb{A}_k^N$ are flat, hence p and q are flat. As in the previous section we set $\widetilde{Z}_K = \widetilde{Z} \times_X \operatorname{Spec}(K)$ and $q_K : \widetilde{Z}_K \to \mathbb{A}_K^N$ the restriction of q to $\widetilde{Z}_K$. We first note that, by the choice of s_N made above, the fibre $q_K^{-1}(0, \ldots, 0, 1)$ is integral. In fact, by construction, its coordinate algebra is the universal splitting algebra of the characteristic polynomial $P_{s_N/f}(T)$ of s_N/f. Since the Galois group of $P_{s_N/f}(T)$ is $\mathcal{S}_n$, its universal splitting algebra, by property P6, is a field. We can now complete the proof exactly as we did in the proof of Theorem 3.4. By Theorem 9.7.7 of [5], it suffices to show that the geometric generic fibre of q is integral. Let Ω, S, Λ and $\widetilde{X}_\Lambda$ be as in Sect. 3 and define $\widetilde{Z}_\Omega$, $\widetilde{Z}_\Lambda$, π_Ω and π_Λ as we did there for $\widetilde{Y}_\Omega$ and so on. The proof given in Sect. 3 goes through once we remark that the universal splitting algebra $\widetilde{Z}_\Lambda$ is reduced. This is a special case of the following lemma. $\square$

Lemma 6.3. *Let R be a domain, K its field of fractions and $P(T) \in R[T]$ a monic polynomial. Assume that $P(T)$ is separable over K. Then the universal splitting algebra of $P(T)$ over R is reduced.*

Proof. Let S be the universal splitting algebra of $P(T)$ over R. It is a free R-algebra of degree $n!$. The construction of the universal splitting algebra commutes with scalar extensions (property P1), hence $S \otimes_R K$ is the splitting algebra of $P(T)$ over K. Since $P(T)$ is separable over K, it follows immediately from property P4 that $S \otimes_R K$ is étale over K, in particular reduced. By Lemma 3.5 S is reduced too. $\square$

7. Smooth Galois splitting in characteristic zero

Theorem 7.1. *Assume that k is of characteristic zero. There exists a nonempty open set $U \subset k^N$ such that, for any $\lambda \in U$, Z_λ is a quasi-projective irreducible smooth Galois covering of X with Galois group $\mathcal{S}_n$ which splits $\mathcal{A}$.*

Proof. If $n = 2$ then $U = k^N$ and for any $\lambda \in k^N$, $Z_\lambda = Y_\lambda$. We therefore assume that $n \geq 3$. In this case the proof is on similar lines as the proof of Theorem 3.11. By 2.12 the singularities of $\widetilde{Z}$ are contained in the union of the singularities of the fibers of p. Since, by Theorem 4.1, the singularities of the closed fibres of p are at worst in codimension 3, we can argue exactly as in the proof of Theorem 3.12 and conclude that q is generically smooth. The other assertion are given by Theorem 6.2. □

8. Smooth Galois splitting in arbitrary characteristic

Theorem 8.1. *Let X be an irreducible quasiprojective smooth surface over k and $\mathcal{A}$ an Azumaya algebra of degree n over X. Assume (5.1) that we have chosen the line bundle $\mathcal{L}$ such that the global sections $s_1, \dots, s_N$ generate*

$$H^0\left(X, \mathcal{A} \otimes_{\mathcal{O}_x} \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x} / m_x^2\right)$$

for every closed point $x \in X(k)$. Assume also that $s_N = \sigma g$ with $f \neq 0$ a section of $\mathcal{L}$ and σ are as in Lemma 3.1. Then there exists an open dense set $U \subset k^N$ such that, for any $\lambda \in U$ the surface Z_λ is a smooth irreducible finite Galois cover of X with Galois group $\mathcal{S}_n$, and splits $\mathcal{A}$.

Only the smoothness of a general fibre needs to be proved.
Let x be closed point of X, $\lambda \in k^N$, and

$$\overline{P}(T) = T^n + \overline{a}_1 T^{n-1} + \cdots + \overline{a}_n$$

the characteristic polynomial of $\varphi(\lambda) \in M_n(\overline{R})$. We defined $F_i = \sigma_i(T_1, \dots, T_n) - (-1)^i \overline{a}_i$ where σ_i is the i-th elementary symmetric function. We define $\sigma'_{i,j}$ as the i-th elementary symmetric function in $T_1, \dots, T_{j-1}, T_{j+1}, \dots, T_n$ and set $\sigma'_{0,j} = 1$. Note that $\partial F_i / \partial T_j = \sigma'_{i-1,j}$. Let $(\mu_1, \dots, \mu_n)$ be the roots of $\overline{\overline{P}}(T)$ in some chosen order. Then $z = (x, \mu_1, \dots, \mu_n)$ is a point of Z_λ. It is smooth if and only if the jacobian matrix

$$J(z) = \frac{\partial(F_1, \dots, F_n)}{\partial(T_1, \dots, T_n, \xi, \eta)} = \begin{pmatrix} 1 & \cdots & 1 & -\frac{\partial a_1}{\partial \xi} & -\frac{\partial a_1}{\partial \eta} \\ \sigma'_{1,1} & \cdots & \sigma'_{1,n} & \frac{\partial a_2}{\partial \xi} & \frac{\partial a_2}{\partial \eta} \\ \vdots & & \vdots & \vdots & \vdots \\ \sigma'_{n-1,1} & \cdots & \sigma'_{n-1,n} & (-1)^n \frac{\partial a_n}{\partial \xi} & (-1)^n \frac{\partial a_n}{\partial \eta} \end{pmatrix}$$

evaluated at z (we denote it by $J(z)$) has rank n. In this section $S(x)$ will denote the set of $\overline{a} = \alpha + \xi\beta + \eta\gamma \in M_n(\overline{R})$ for which the fibre of x contains a singular point of Z_λ, which is the same as saying that the corresponding Jacobian matrix has rank less than n.

Proposition 8.2. *The codimension of $S(x)$ in $M_n(\overline{R})$ is at least* 3.

Proof. If $\mu_1, \ldots, \mu_n$ are all distinct, then the Jacobian $\left(\partial\sigma_i \big/ \partial T_j\right)$ evaluated at the point $(\mu_1, \ldots, \mu_n)$ is invertible and hence $J(z)$ has rank n. Suppose now that α has a multiple eigenvalue. As in Sect. 3 we only have to consider matrices in $\widehat{M}_n^2$, $\widehat{M}_n^5$ and $\widehat{M}_n^8$.
Suppose first that $\overline{a}$ is in M_n^2. In this case α has two equal eigenvalues $\mu_1 = \mu_2 = \mu$. Consider the $(n-1) \times (n-1)$ submatrix $T = (\sigma'_{i-1,j})$ of $J(z)$, with $1 \leq i \leq n-1$ and $2 \leq j \leq n$, evaluated at z

By multiplying the first row of $J(z)$ by μ and substracting it from the second, then multiplying the second by μ and substracting it from the third, and so on, we transform T into $T' = (\partial s_i \big/ \partial T_j)$, $1 \leq i \leq n-1$, $2 \leq j \leq n$, evaluated at $(\mu, \mu_3, \ldots, \mu_n)$ where s_i is the i-th elementary symmetric function in the $n-1$ variables $T_2, \ldots, T_n$. Since $\mu, \mu_3, \ldots, \mu_n$ are all distinct T', is invertible. This proves that the columns of $J(z)$ from the second to the n-th are independent. By these row operations the last row of $J(z)$ becomes

$$\left(0, 0, \ldots, 0, (-1)^{n-1}\frac{\partial\overline{P}}{\partial\xi}(\mu), (-1)^{n-1}\frac{\partial\overline{P}}{\partial\eta}(\mu)\right)$$

and therefore the rank of $J(z)$ will be n if and only if

$$\left(\frac{\partial\overline{P}}{\partial\xi}(\mu), \frac{\partial\overline{P}}{\partial\eta}(\mu)\right) \neq (0, 0).$$

We already computed $\overline{P}(T)$ in 3 and found that its derivatives with respect to ξ and η both vanish for $\xi = \eta = 0$ and $T = \mu$ if and only if

$$\beta_{2,1} = 0 \quad \text{and} \quad \gamma_{2,1} = 0.$$

These two conditions show that the codimension of $\widehat{M}_n^2 \cap S(x)$ is ≥ 3.
The case $n = 4$ will illustrate what we said. The matrix $J(z)$ is

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \frac{\partial\overline{a}_1}{\partial\xi} & \frac{\partial\overline{a}_1}{\partial\eta} \\ \mu+\mu_3+\mu_4 & \mu+\mu_3+\mu_4 & \mu+\mu+\mu_4 & \mu+\mu+\mu_3 & -\frac{\partial\overline{a}_2}{\partial\xi} & -\frac{\partial\overline{a}_2}{\partial\eta} \\ \mu\mu_3+\mu\mu_4+\mu_3\mu_4 & \mu\mu_3+\mu\mu_4+\mu_3\mu_4 & \mu\mu+\mu\mu_4+\mu\mu_4 & \mu\mu+\mu\mu_3+\mu\mu_3 & \frac{\partial\overline{a}_3}{\partial\xi} & \frac{\partial\overline{a}_3}{\partial\eta} \\ \mu\mu_3\mu_4 & \mu\mu_3\mu_4 & \mu\mu\mu_4 & \mu\mu\mu_3 & -\frac{\partial\overline{a}_4}{\partial\xi} & -\frac{\partial\overline{a}_4}{\partial\eta} \end{pmatrix}$$

and the row operations transform it into

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \frac{\partial\overline{a}_1}{\partial\xi} & \frac{\partial\overline{a}_1}{\partial\eta} \\ \mu_3+\mu_4 & \mu_3+\mu_4 & \mu+\mu_4 & \mu+\mu_3 & \star & \star \\ \mu_3\mu_4 & \mu_3\mu_4 & \mu\mu_4 & \mu\mu_3 & \star & \star \\ 0 & 0 & 0 & 0 & \frac{\partial\overline{P}}{\partial\xi} & \frac{\partial\overline{P}}{\partial\eta} \end{pmatrix}.$$

For the remaining two cases, the same examples as in 3 and essentially the same computations as for M_n^2 show that the codimension of $\widehat{M}^5 \cap S(z)$ and $\widehat{M}^8 \cap S(z)$

is ≥ 3 as well. Let us consider for example the case of $\widehat{M}_n^8$. We choose $\overline{a} = \alpha + \xi\beta + \eta\gamma \in M_n^8$ with $\alpha = \mathrm{diag}(B, D)$ with

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mu & 1 \\ 0 & 0 & 0 & \mu \end{pmatrix},$$

β, γ arbitrary matrices in $M_n(k)$ and $D = \mathrm{diag}(\mu_5, \ldots, \mu_n)$ where all the entries are distinct and different from 0 and μ. We want to find the conditions for $z = (x, 0, 0, \mu, \mu, \mu_5, \ldots, \mu_n)$ to be smooth. The first n entries of the last row of $J(z)$ vanish and in the last but one row the entries from the 3d to the n-th also vanish. Consider the $(n-2) \times (n-2)$ submatrix T of $J(z)$ formed by the first $n-2$ rows and the 2, 4, 5, …, nth column. By multiplying the first row of $J(z)$ by μ and substractig it from the second, then multiplying the second by μ and substracting it from the third, and so on, we transform T into $T' = (\partial s_i \big/ \partial T_j)$, $1 \leq i \leq n-2$, $j = 2, 4, 5, \ldots, n$, evaluated at $(0, \mu, \mu_5, \ldots, \mu_n)$ where s_i is the i-th elementary symmetric function in the $n-2$ variables $T_2, T_4, T_5, \ldots, T_n$. Since $0, \mu, \mu_5, \ldots, \mu_n$ are all distinct, T' is invertible. This proves that the 2, 4, …, nth columns of $J(z)$ are independent. In the process, the first n entries of the last two rows have become zero. To show that the last two rows are independent from the other ones it suffices now to show that the 2×2 determinant in the right bottom square does not vanish.

Let us compute the four entries of this determinant. We already saw, in the case of $\widehat{M}_n^2$ that the last two entries of the last row are $(-1)^{n-1}\frac{\partial \overline{P}}{\partial \xi}(\mu)$ and $(-1)^{n-1}\frac{\partial \overline{P}}{\partial \eta}(\mu)$. The last two entries of the last but one row are, up to sign,

$$\frac{\partial \overline{a}_{n-1}}{\partial \xi} + \frac{\partial \overline{a}_{n-2}}{\partial \xi}\mu + \cdots + \frac{\partial \overline{a}_1}{\partial \xi}\mu^{n-1} \quad \text{and} \quad \frac{\partial \overline{a}_{n-1}}{\partial \eta} + \frac{\partial \overline{a}_{n-2}}{\partial \eta}\mu + \cdots + \frac{\partial \overline{a}_1}{\partial \eta}\mu^{n-1}$$

which can be computed as

$$\frac{\frac{\partial \overline{P}}{\partial \xi}(\mu) - \frac{\partial \overline{a}_n}{\partial \xi}}{\mu} \quad \text{and} \quad \frac{\frac{\partial \overline{P}}{\partial \eta}(\mu) - \frac{\partial \overline{a}_n}{\partial \eta}}{\mu}$$

Hence, up to a nonzero factor, the determinant we want is

$$\det \begin{pmatrix} \dfrac{\frac{\partial \overline{P}}{\partial \xi}(\mu) - \frac{\partial \overline{a}_n}{\partial \xi}}{\mu} & \dfrac{\frac{\partial \overline{P}}{\partial \eta}(\mu) - \frac{\partial \overline{a}_n}{\partial \eta}}{\mu} \\ \dfrac{\partial \overline{P}}{\partial \xi}(\mu) & \dfrac{\partial \overline{P}}{\partial \eta}(\mu) \end{pmatrix} = -\frac{1}{\mu}\det \begin{pmatrix} \dfrac{\partial \overline{a}_n}{\partial \xi} & \dfrac{\partial \overline{a}_n}{\partial \eta} \\ \dfrac{\partial \overline{P}}{\partial \xi}(\mu) & \dfrac{\partial \overline{P}}{\partial \eta}(\mu) \end{pmatrix} \tag{$\dagger$}$$

We can now compute $\overline{P}$. Using Lemma 5.3 and writing $\overline{a} \in M_n(\overline{R})$ as

$$\mathrm{diag}(J_8, \mu_5, \ldots, \mu_n) + (\overline{a}_{i,j})$$

we find that $\overline{P}(T)$ is

$$\begin{aligned} &\left(T^2 - (\overline{a}_{1,1} + \overline{a}_{2,2})T - \overline{a}_{2,1}\right)\left(T^2 - (2\mu + \overline{a}_{3,3} + \overline{a}_{4,4})T + \mu^2 \right. \\ &\quad \left. + \mu(\overline{a}_{3,3} + \overline{a}_{4,4}) - \overline{a}_{4,3}\right) P_D(T) \end{aligned}$$

where P_D is the characteristic polynomial of $\operatorname{diag}(\mu_5, \ldots, \mu_n)$. Denoting by c the constant term of $P_D(T)$, we can compute the entries of the determinant above. Since

$$\overline{a}_n = (-\overline{a}_{2,1})(\mu^2 + \mu(\overline{a}_{3,3} + \overline{a}_{4,4}) - \overline{a}_{4,3})c = -\overline{a}_{2,1}\mu^2 c$$

and

$$\overline{P}(\mu) = \left(\mu^2 - (\overline{a}_{1,1} + \overline{a}_{2,2})\mu - \overline{a}_{2,1}\right)\left(-a_{4,3}\right)\overline{P}(\mu) = -\mu^2\overline{a}_{4,3}\overline{P}(\mu)$$

the determinant in (†) is, up to a constant nonzero factor,

$$\begin{pmatrix} \beta_{2,1} & \gamma_{2,1} \\ \beta_{4,3} & \gamma_{4,3} \end{pmatrix}$$

and in the example given this determinant is $\neq 0$. □

The rest of the proof of Theorem 8.1 is exactly the same as in Sect. 3.

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References

[1] Bourbaki, N.: Algèbre, Chapitre IV, Polynômes et fractions rationnelles. Masson, Paris (1981)

[2] Colliot-Thélène, J.-L.: Algèbres simples centrales sur les corps de fonctions de deux variables (d'après A. J. de Jong), Séminaire Bourbaki, juin 2005, Exp. No. 949. Astérisque **307**, 379–413 (2006)

[3] Ekedahl, T., Laksov, D.: Splitting algebras, symmetric functions and Galois theory. J. Algebra Appl. **4**, 59–75 (2004)

[4] Fried, M.D., Jarden, M.: Field Arithmetic, Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer, New York (2005)

[5] Grothendieck, A.: Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné) : IV. Étude locale des schémas et des morphismes de schémas, Troisième partie. Publications Mathématiques de l'IHÉS **28**, 5–255 (1966)

[6] Hartshorne, R.: Algebraic Geometry, Graduate Texts in Mathematics, vol. 52. Springer-Verlag, New York (1977)

[7] Jantzen, J.C.: Nilpotent orbits in representation theory. In: Ørsted, B., Anker, J.-Ph. (eds.) Lie Theory: Lie Algebras and Representations, pp. 1–211. Birkhäuser, Boston (2004)

[8] Van den Bergh, M.: Notes on de Jong's period=index theorem for central simple algebras over fields of transcendence degree two. Preprint (2007). http://archiv.org/abs/0807.1403