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# Invariants of simple algebras

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**Abstract.** We determine the group of invariants with values in Galois cohomology with coefficients $\mathbb{Z}/2\mathbb{Z}$ of central simple algebras of degree at most 8 and exponent dividing 2.

## 0. Introduction

Let $F$ be a field and let $A$ be an "algebraic structure" over field extensions of $F$. More precisely, $A$ is a functor from the category $\mathit{Fields}/F$ of field extensions over $F$ to the category $\mathit{Sets}$ of sets. For example, the values of $A$ can be the sets of isomorphism classes of central simple algebras of given degree $n$, quadratic forms of dimension $n$, étale algebras of rank $n$, etc. As defined in [7], an *invariant* of a functor $A$ with values in a cohomology theory $H$ (also viewed as a functor from $\mathit{Fields}/F$ to $\mathit{Sets}$) is a morphism of functors $A \to H$. All the invariants of $A$ with values in $H$ form a group $\operatorname{Inv}(A, H)$.

An interesting functor $\mathit{Tors}_G$ can be associated to an algebraic group $G$ defined over $F$ as follows. For a field extension $L/F$, $\mathit{Tors}_G(L)$ is the set of isomorphism classes of $G$-torsors over $\operatorname{Spec} L$. All examples of the functors $A$ listed above are isomorphic to the functors $\mathit{Tors}_G$ for certain groups $G$ (cf. [7, §3]). For example, $\mathit{Tors}_G(L)$ for the projective linear group $G = \mathbf{PGL}_n$ is naturally bijective to the set of isomorphism classes of central simple $L$-algebras of degree $n$.

The structure of the group $\operatorname{Inv}(A, H)$ was determined for various functors $A$ in [7]. The case $A = \mathit{Tors}_G$ for $G = \mathbf{PGL}_n$, i.e., the problem of classification of invariants of central simple algebras of degree $n$, is still wide open. In the present paper we determine the group of invariants with values in Galois cohomology with coefficients $\mathbb{Z}/2\mathbb{Z}$ of central simple algebras of degree at most 8 and exponent dividing 2, i.e., determine invariants of $\mathit{Tors}_G$ for $G = \mathbf{GL}_n/\boldsymbol{\mu}_2$ with $n$ dividing 8.

In the present paper, the word "variety" over a field $F$ means a separated integral scheme of finite type over $F$.

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## 1. Invariants

### *1.1. Cohomology theories, residues and values*

Let $F$ be a field and let $C$ be a Galois module for $F$ such that $nC = 0$ for some $n$ not divisible by char $F$. We define a graded *cohomology theory* $H$ *over* $F$ as follows. For any field extension $L/F$, we write

$$H(L) := \coprod_{r \geq 0} H^r\left(L, C(r)\right),$$

where $C(r)$ is the Tate twist of $C$ [7, 7.8]. Note that $H(L)$ is a (left) module over the cohomology ring

$$\coprod_{r \geq 0} H^r\left(L, (\mathbb{Z}/n\mathbb{Z})(r)\right)$$

with respect to the cup-product. We shall write $(x)$ for the element of

$$H^1\left(L, (\mathbb{Z}/n\mathbb{Z})(1)\right) = H^1(L, \mu_n) \simeq L^\times/L^{\times n}$$

corresponding to the coset $xL^{\times n}$.

Let $L$ be a field extension of $F$ with a discrete valuation $v$ trivial on $F$ and residue field $F(v)$. There is the *residue map* of degree $-1$ [7, §7.13]:

$$\partial_v : H^r(L) \to H^{r-1}\left(F(v)\right).$$

An element $h \in H^r(L)$ is called *unramified at* $v$ if $\partial_v(h) = 0$.

Let $\pi \in L$ be a prime element. The graded map

$$s_\pi : H^r(L) \to H^r\left(F(v)\right), \quad s_\pi(h) = \partial_v\left((-\pi) \cup h\right)$$

is called a *specialization map* [15, §1]. If $h \in H^r(L)$ is unramified at $v$, then the element $s_\pi(h)$ does not depend on the choice of $\pi$ and is called the *value of* $h$ *at* $v$, denoted $h(v)$.

### *1.2. The group* $A^0(X, H^r)$

Let $X$ be a variety over $F$ and let $H$ be a cohomology theory over $F$. Recall that for any point $x \in X$ of codimension 1 we have the *residue map*

$$\partial_x : H^r\left(F(X)\right) \to H^{r-1}\left(F(x)\right)$$

defined as follows [15, §2]:

$$\partial_x = \sum \mathrm{cor}_{F(v)/F(x)} \circ \partial_v,$$

where the sum is taken over all (finitely many) discrete valuations of $F(X)$ over $F$ dominating $x$, and $\partial_v : H^r\left(F(X)\right) \to H^{r-1}\left(F(v)\right)$ is the residue map for the discrete valuation $v$. We write

$$A^0(X, H^r) := \bigcap \mathrm{Ker}(\partial_x) \subset H^r\left(F(X)\right),$$

where the intersection is taken over all points $x \in X$ of codimension 1.

Let $K/F$ be a field extension, $p \in X(K)$ a point and $\alpha \in A^0(X, H^r)$ an arbitrary element. We say that $p$ is *nonsingular* if the image of $p : \mathrm{Spec}\, K \to X$ is a nonsingular point of $X$. If $p$ is nonsingular, the *value* $\alpha(p)$ *of* $\alpha$ *at* $p$ is the image of $\alpha$ under the pull-back map [15, §12]:

$$A^0(X, H^r) \to A^0(\operatorname{Spec} K, H^r) = H^r(K).$$

*1.3. Values of invariants*

We view the homogeneous components $H^r$ of the cohomology theory $H$ as functors from the category $\mathit{Fields}/F$ of field extensions over $F$ and field homomorphisms over $F$ to the category $\mathit{Sets}$ of sets. Let $S : \mathit{Fields}/F \to \mathit{Sets}$ be another functor. An $H$*-invariant of* $S$ *of degree* $r$ is a morphism of functors $q : S \to H^r$ [7, Def. 1.1]. We write $\operatorname{Inv}(S, H^r)$ for the group of $H$-invariant of $S$ of degree $r$ and $\operatorname{Inv}(S, H)$ for the graded group $\coprod_{r\geq 0} \operatorname{Inv}(S, H^r)$.

Let $G$ be an algebraic group defined over a field $F$. Let $\operatorname{Tors}_G : \mathit{Fields}/F \to \mathit{Sets}$ be the functor taking a field extension $K/F$ to the set of isomorphism classes of $G$-torsors over $\operatorname{Spec} K$. We have $\operatorname{Tors}_G(K) \simeq H^1(K, G)$ [11, Ch. VII]. We simply write $\operatorname{Inv}(G, H^r)$ for the group $\operatorname{Inv}(\operatorname{Tors}_G, H^r)$.

*Example 1.1.* Let $n > 0$ be an integer and $k > 0$ a divisor of $n$. We view the group $\boldsymbol{\mu}_k$ of $k$th roots of unity as a subgroup of $\mathbf{GL}_n$ via the embeddings $\boldsymbol{\mu}_k \subset \mathbf{G}_\mathrm{m} \subset \mathbf{GL}_n$ and set $G = \mathbf{GL}_n / \boldsymbol{\mu}_k$. By [11, Cor. 28.6], the exact sequence

$$1 \to \mathbf{G}_\mathrm{m} \xrightarrow{\alpha} G \xrightarrow{\beta} \mathbf{PGL}_n \to 1,$$

where $\alpha$ is the composition

$$\mathbf{G}_\mathrm{m} \xrightarrow{\sim} \mathbf{G}_\mathrm{m} / \boldsymbol{\mu}_k \to \mathbf{GL}_n / \boldsymbol{\mu}_k = G$$

and $\beta$ is the natural epimorphism, and Hilbert Theorem 90 yield a bijection between $H^1(F, G)$ and the kernel of the connecting map

$$\delta : H^1(F, \mathbf{PGL}_n) \to H^2(F, \mathbf{G}_\mathrm{m}) = \operatorname{Br}(F).$$

The set $H^1(F, \mathbf{PGL}_n)$ is bijective to the set of isomorphism classes of central simple $F$-algebras $A$ of degree $n$ and the map $\delta$ takes the class of $A$ to $k[A]$. Therefore, there is a natural bijection between $\operatorname{Tors}_G(F) = H^1(F, G)$ and the set of isomorphism classes of central simple $F$-algebras of degree $n$ and exponent dividing $k$.

We shall need the following statement:

**Proposition 1.2.** [7, Th. 11.7] *Let* $G$ *be an algebraic group over* $F$ *and* $q \in \operatorname{Inv}(G, H^r)$. *Let* $R$ *be a discrete valuation ring containing* $F$ *with quotient field* $L$ *and residue field* $K$. *Then for any* $G$*-torsor* $E$ *over* $\operatorname{Spec} R$, *we have*:

(1) *The residue of* $q(E_L)$ *at* $v$ *is zero, i.e.,* $q(E_L)$ *is unramified at* $v$.
(2) *The value* $q(E_L)(v)$ *of* $q(E_L)$ *at* $v$ *is* $q(E_K)$.

Let $X$ be a variety over $F$ and $E \to X$ a $G$-torsor. For a field extension $K/F$ and a point $p \in X(K)$, we write $E_p \to \operatorname{Spec} K$ for the pull-back of the torsor $E$ with respect to $p : \operatorname{Spec} K \to X$. Thus, we have a morphism of functors $X \to \operatorname{Tors}_G$

taking a point $p$ to $E_p$. We also write $E_{\mathrm{gen}}$ for the generic fiber of $E \to X$. It is a $G$-torsor over Spec $F(X)$.

**Theorem 1.3.** *Let $G$ be an algebraic group over $F$, $X$ a variety over $F$. Let $E \to X$ be a $G$-torsor and $q \in \mathrm{Inv}(G, H^r)$. Then*

(1) $q(E_{\mathrm{gen}}) \in A^0(X, H^r)$.
(2) *Let $K/F$ be a field extension and let $p \in X(K)$ be a nonsingular point. Then $q(E_p)$ is equal to the value of $q(E_{\mathrm{gen}})$ at $p$.*
(3) *Let $X$ be smooth and let $f : Y \to X$ be a morphism of varieties over $F$. Then*

$$f^*\left(q(E_{\mathrm{gen}})\right) = q(f^*(E)_{\mathrm{gen}})$$

*in $A^0(Y, H^r)$, where $f^* : A^0(X, H^r) \to A^0(Y, H^r)$ is the pull-back homomorphism.*

*Proof.* (1) and (2) follow from Proposition 1.2 and [15, Cor. 12.4].

(3): By (2), the pull-back homomorphism for the composition Spec $F(Y) \to Y \to X$ is equal to $q(f^*(E)_{\mathrm{gen}})$. The pull-back homomorphism for the first morphism Spec $F(Y) \to Y$ is the inclusion of $A^0(Y, H^r)$ into $H^r(F(Y))$. □

It follows from Theorem 1.3(1) that a $G$-torsor $E \to X$ gives rise to a group homomorphism

$$\varphi_E : \mathrm{Inv}(G, H^r) \to A^0(X, H^r), \quad q \mapsto q(E_{\mathrm{gen}}).$$

### *1.4. Classifying torsors*

A $G$-torsor $E \to X$ over $F$ is called *classifying* if $X$ is smooth and the corresponding morphism of functors $X \to \mathrm{Tors}_G$ is surjective, i.e., for any field extension $K/F$ and any $G$-torsor $E' \to$ Spec $K$, there is a point $p \in X(K)$ such that $E' \simeq E_p$.

*Remark 1.4.* We do not require the density condition as in [7, Def. 5.1].

**Theorem 1.5.** *Let $E \to X$ be a classifying $G$-torsor over $F$. Then the map $\varphi_E : \mathrm{Inv}(G, H^r) \to A^0(X, H^r)$ is injective.*

*Proof.* Let $q \in \mathrm{Ker}(\varphi_E)$, i.e., $q(E_{\mathrm{gen}}) = 0$. Let $K/F$ be a field extension and let $E' \to$ Spec $K$ be a $G$-torsor. Choose a point $p \in X(K)$ such that $E' \simeq E_p$. By Theorem 1.3(2), $q(E_p)$ is the value of $q(E_{\mathrm{gen}})$ at $p$. Hence $q(E') = 0$. □

## 2. Invariants of algebras of degree 8

In this section we assume that $\mathrm{char}(F) \neq 2$.

### *2.1. The functors $\mathsf{Alg}_n$ and $\mathsf{Dec}_n$*

For a commutative $F$-algebra $R$ and $a, b \in R^\times$ we write $(a, b) = (a, b)_R$ for the quaternion algebra $R \oplus Ri \oplus Rj \oplus Rk$ with the multiplication table $i^2 = a$, $j^2 = b, k = ij = -ji$. The class of $(a, b)_R$ in the Brauer group $Br(R)$ will be denoted by $[a, b] = [a, b]_R$. We write $\mathit{Quat}(R)$ for the set of isomorphism classes of quaternion algebras over $R$.

Let $a \in R^\times$ and $S = R[\sqrt{a}] := R[t]/(t^2 - a)$ the quadratic extension of $R$. We write $N_R(a)$ for the subgroup of $R^\times$ of all element of the form $x^2 - ay^2$ with $x, y \in R$, i.e., $N_R(a)$ is the image of the norm homomorphism $N_{S/R} : S^\times \to R^\times$. If $b \in N_R(a)$, then the quaternion algebra $(a, b)_R$ is isomorphic to the matrix algebra $M_2(R)$ by [10, Th. 6].

For every $n \geq 1$, $\mathit{Alg}_n(F)$ denotes the set of isomorphism classes of central simple $F$-algebras of degree $2^n$ and exponent dividing 2. We can identify $\mathit{Alg}_n(F)$ with the subset of $\mathrm{Br}(F)$ of classes of algebras of degree dividing $2^n$. In particular, we have that

$$\mathit{Alg}_1(F) \subset \mathit{Alg}_2(F) \subset \mathit{Alg}_3(F) \subset \cdots \subset \mathrm{Br}_2(F).$$

The isomorphism class of an algebra $A$ in $\mathit{Alg}_n(F)$ is called *decomposable* if $A$ is isomorphic to the tensor product of $n$ quaternion algebras over $F$. The subset of all decomposable classes in $\mathit{Alg}_n(F)$ is denoted by $\mathit{Dec}_n(F)$. The union of all $\mathit{Dec}_n(F)$ coincides with $\mathrm{Br}_2(F)$.

We view $\mathit{Alg}_n$ and $\mathit{Dec}_n$ as functors $\mathit{Fields}/F \to \mathit{Sets}$. By Example 1.1, the functor $\mathit{Alg}_n$ is isomorphic to the functor $\mathrm{Tors}_G$ for $G = \mathbf{GL}_{2^n}/\boldsymbol{\mu}_2$.

Obviously, $\mathit{Alg}_1(F) = \mathit{Dec}_1(F) = \mathit{Quat}(F)$. By Albert's theorem [12, Prop. 5.2], $\mathit{Alg}_2(F) = \mathit{Dec}_2(F)$.

The case $n = 3$ is more complicated. It is shown in [1] that $\mathit{Alg}_3(F) \neq \mathit{Dec}_3(F)$ in general. On the other hand, Tignol proved in [18] that $\mathit{Alg}_3(F) \subset \mathit{Dec}_4(F)$ as the subsets of $\mathrm{Br}_2(F)$.

### 2.2. *Tignol's construction*

We recall Tignol's argument given in [18]. Let $A$ be a central simple $F$-algebra in $\mathit{Alg}_3(F)$. By [16], there is a triquadratic splitting extension $F(\sqrt{a}, \sqrt{b}, \sqrt{c})/F$ of $A$ with $a, b, c \in F^\times$. Let $L = F(\sqrt{a})$. By Albert's Theorem, we have

$$[A]_L = [b, s] + [c, t] \tag{1}$$

in $\mathrm{Br}(L)$ for some $s, t \in L^\times$.

Taking the corestriction for the extension $L/F$ in (1), we get

$$0 = 2\,[A] = \big[b, N_{L/F}(s)\big] + \big[c, N_{L/F}(t)\big]$$

in $\mathrm{Br}(F)$, hence $\big[b, N_{L/F}(s)\big] = \big[c, N_{L/F}(t)\big]$. By the Common Slot Lemma [2, Lemma 1.7], we have

$$\big[b, N_{L/F}(s)\big] = \big[d, N_{L/F}(s)\big] = \big[d, N_{L/F}(t)\big] = \big[c, N_{L/F}(t)\big]$$

in $\mathrm{Br}(F)$ for some $d \in F^\times$. It follows that the classes $\big[bd, N_{L/F}(s)\big]$, $\big[cd, N_{L/F}(t)\big]$ and $\big[d, N_{L/F}(st)\big]$ are trivial. By [4, Lemma 2.3] (see also Lemma 2.2 below),

$$[bd, s] = [bd, k]\,,$$
$$[cd, t] = [cd, l]\,,$$
$$[d, st] = [d, m]\,.$$

in $\mathrm{Br}(L)$ for some $k, l, m \in F^\times$. It follows from (1) that

$$[A]_L = [bd, k]_L + [cd, l]_L + [d, m]_L$$

in $\mathrm{Br}(L)$. Hence

$$[A] = [a, e] + [bd, k] + [cd, l] + [d, m] = [a, e] + [b, k] + [c, l] + [d, klm] \quad (2)$$

in $\mathrm{Br}(F)$ for some $e \in F^\times$. We shall also need the following well known statements:

**Lemma 2.1.** *Let $K$ be a field and let $A$ be a central simple $K$-algebra such that $[A] \in \mathrm{Br}_2(K)$ and let $L/K$ be a quadratic field extension such that $[A]_L = [b, s] + [c, t]$ for some $b, c \in K^\times$ and $s, t \in L^\times$. Suppose that one of the classes $\big[b, N_{L/K}(s)\big]$ and $\big[c, N_{L/K}(t)\big]$ is zero in $\mathrm{Br}(K)$. Then $A \in \mathit{Dec}_3(K)$.*

*Proof.* Suppose that $\big[b, N_{L/K}(s)\big] = 0$. Taking the corestriction we get

$$0 = 2[A] = \big[b, N_{L/K}(s)\big] + \big[c, N_{L/K}(t)\big] = \big[c, N_{L/K}(t)\big].$$

By [4, Lemma 2.3], there are $u, v \in K^\times$ such that $[b, s] = [b, u]_L$ and $[c, t] = [c, v]_L$. It follows that the class $[A] - [b, u] - [c, v]$ is split by $L$, hence is the class of a quaternion algebra. Thus, $A \in \mathit{Dec}_3(K)$. ◻

**Lemma 2.2.** *Let $R$ be a commutative $F$-algebra, $a, b \in R^\times$, $T = R[\sqrt{a}]$ and $x + y\sqrt{a} \in T^\times$ such that $x^2 - ay^2 = u^2 - bv^2$ for some $u, v \in R$. If $x + u \in R^\times$, then $2(x + u)(x + y\sqrt{a}) \in N_T(b)$. In particular,*

$$\big[b, x + y\sqrt{a}\big]_T = [b, 2(x + u)]_T\,.$$

*Proof.* We have the equality

$$\begin{aligned}(x + y\sqrt{a} + u)^2 - bv^2 &= (x + y\sqrt{a})(x + y\sqrt{a} + 2u) + (u^2 - bv^2)\\ &= (x + y\sqrt{a})(x + y\sqrt{a} + 2u) + (x + y\sqrt{a})(x - y\sqrt{a})\\ &= (x + y\sqrt{a})(2x + 2u).\end{aligned}$$

◻

*2.3. The Azumaya algebra $\mathcal{A}$*

Consider the affine space $\mathbf{A}^8_F$ with coordinates $\mathbf{a}, \mathbf{e}, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}$ and define the rational functions:

$$\begin{aligned}\mathbf{f} &= \mathbf{xy} + \mathbf{az},\\ \mathbf{g} &= \mathbf{y} + \mathbf{xz},\\ \mathbf{d} &= \mathbf{w}^2 - \mathbf{f}^2 + \mathbf{ag}^2,\\ \mathbf{b} &= (\mathbf{u}^2 - \mathbf{x}^2 + \mathbf{a})\mathbf{d}^{-1},\\ \mathbf{c} &= (\mathbf{v}^2 - \mathbf{y}^2 + \mathbf{az}^2)\mathbf{d}^{-1},\\ \mathbf{p} &= (\mathbf{u} + \mathbf{x})(\mathbf{v} + \mathbf{y})(\mathbf{w} + \mathbf{f}).\end{aligned}$$

Let $X$ be the open subscheme of $\mathbf{A}^8_F$ given by

$$\mathbf{q} := \mathbf{adep}(\mathbf{u}^2 - \mathbf{x}^2 + \mathbf{a})(\mathbf{v}^2 - \mathbf{y}^2 + \mathbf{az}^2)(\mathbf{x}^2 - \mathbf{a})(\mathbf{y}^2 - \mathbf{az}^2)(\mathbf{f}^2 - \mathbf{ag}^2) \neq 0,$$

i.e., $X = \mathrm{Spec}(R)$ with $R = F[\mathbf{a}, \mathbf{e}, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{q}^{-1}]$. Let $S = R[\sqrt{\mathbf{a}}, \sqrt{\mathbf{b}}, \sqrt{\mathbf{c}}]$. Consider the Azumaya $R$-algebra

$$\mathcal{A}' = (\mathbf{a}, \mathbf{e})_R \otimes (\mathbf{b}, 2(\mathbf{u} + \mathbf{x}))_R \otimes (\mathbf{c}, 2(\mathbf{v} + \mathbf{y}))_R \otimes (\mathbf{d}, 2\mathbf{p})_R . \tag{3}$$

We view $S$ as a subring of $\mathcal{A}'$. Moreover, $(\mathbf{d}, 2\mathbf{p})_S := (\mathbf{d}, 2\mathbf{p}) \otimes_R S \subset \mathcal{A}'$.

Let $T = R[\sqrt{\mathbf{a}}]$. It follows from Lemma 2.2 that

$$2(\mathbf{u} + \mathbf{x})(\mathbf{x} + \sqrt{\mathbf{a}}) \in N_T(\mathbf{bd}) \subset N_S(\mathbf{d}),$$
$$2(\mathbf{v} + \mathbf{y})(\mathbf{y} + \mathbf{z}\sqrt{\mathbf{a}}) \in N_T(\mathbf{cd}) \subset N_S(\mathbf{d}),$$
$$2(\mathbf{w} + \mathbf{f})(\mathbf{x} + \sqrt{\mathbf{a}})(\mathbf{y} + \mathbf{z}\sqrt{\mathbf{a}}) \in N_T(\mathbf{d}) \subset N_S(\mathbf{d}).$$

It follows from (3) that

$$\left[\mathcal{A}'\right]_T = \left[\mathbf{b}, \mathbf{x} + \sqrt{\mathbf{a}}\right] + \left[\mathbf{c}, \mathbf{y} + \mathbf{z}\sqrt{\mathbf{a}}\right] \tag{4}$$

in $\mathrm{Br}(T)$.

Moreover, we have $2\mathbf{p} = 2(\mathbf{u}+\mathbf{x})(\mathbf{v}+\mathbf{y})(\mathbf{w}+\mathbf{f}) \in N_S(\mathbf{d})$, therefore, $(\mathbf{d}, 2\mathbf{p})_S$ is isomorphic to the matrix algebra $M_2(S)$. In particular,

$$M_2(R) \subset M_2(S) \simeq (\mathbf{d}, 2\mathbf{p})_S \subset \mathcal{A}'$$

and hence $\mathcal{A}' \simeq M_2(\mathcal{A})$ for the centralizer $\mathcal{A}$ of $M_2(R)$ in $\mathcal{A}'$ by the proof of [8, Th. 4.4.2]. Then $\mathcal{A}$ is an Azumaya $R$-algebra of degree 8 that is Brauer equivalent to $\mathcal{A}'$ by [17, Th. 3.10].

**Proposition 2.3.** *The Azumaya algebra $\mathcal{A}$ is classifying for $\mathsf{Alg}_3$, i.e, the corresponding $\mathbf{GL}_8/\boldsymbol{\mu}_2$-torsor over $X$ is classifying.*

*Proof.* Let $A \in \mathsf{Alg}_3(K)$, where $K$ is a field extension of $F$. We shall find a point $p \in X(K)$ such that $A \simeq \mathcal{A}(p)$.

We follow Tignol's construction. There is a triquadratic splitting extension $K(\sqrt{a}, \sqrt{b}, \sqrt{c})/K$ of $A$ with $a, b, c \in K^\times$. Let $L = K(\sqrt{a})$, so

$$[A]_L = [b, s] + [c, t]$$

in $\mathrm{Br}(L)$ for some $s = x + x'\sqrt{a}$, and $t = y + z\sqrt{a} \in L^\times$. Modifying $s$ by a norm for the extension $L(\sqrt{b})/L$, we may assume that $x' \neq 0$. Similarly, we may assume that $z \neq 0$. Moreover, replacing $a$ by $ax'^2$, we may assume that $x' = 1$.

We have

$$\left[b, x^2 - a\right] = \left[d, x^2 - a\right] = \left[d, y^2 - az^2\right] = \left[c, y^2 - az^2\right]$$

in $\mathrm{Br}(K)$ for some $d \in K^\times$, so the classes $\big[bd, x^2-a\big]$, $\big[cd, y^2-az^2\big]$ and $\big[d, (x^2-a)(y^2-az^2)\big]$ are trivial. Hence

$$\begin{aligned} bd &= u^2-(x^2-a)u'^2, \\ cd &= v^2-(y^2-az^2)v'^2, \\ d &= w^2-(x^2-a)(y^2-az^2)w'^2 \end{aligned}$$

for some $u, u', v, v', w, w'$ in $K$. Moreover, we may assume that $u' \neq 0$. Replacing $b$ and $u$ by $bu'^2$ and $uu'$ respectively, we may assume that $u' = 1$. Similarly, we may assume $v' = w' = 1$.

Replacing $u$ by $-u$ if necessary, we may assume that $u + x \neq 0$ and similarly $v + y \neq 0$ and $w + s \neq 0$, where $s = xy + az$. It follows from Lemma 2.2 that

$$\begin{aligned} \big[b, x+\sqrt{a}\big] &= [b, 2(u+x)]_L\,, \\ \big[c, y+z\sqrt{a}\big] &= [c, 2(v+y)]_L\,, \\ \big[d, (x+\sqrt{a})(y+z\sqrt{a})\big] &= [d, 2(w+s)]_L \end{aligned}$$

in $\mathrm{Br}(L)$. Hence

$$[A] = [a, e] + [b, 2(u+x)] + [c, 2(v+y)] + [d, 2(u+x)(v+y)(w+s)]$$

in $\mathrm{Br}(K)$ for some $e \in K^\times$.

Let $p$ be the point $(a, e, u, v, w, x, y, z)$ in $X(K)$. We have $[\mathcal{A}(p)] = [A]$ and hence $\mathcal{A}(p) \simeq A$ as $\mathcal{A}(p)$ and $A$ have the same dimension. □

**Proposition 2.4.** *Let $K$ be the quotient field of the ring $R = F[X]$. Let $\widehat{K}$ be the completion of $K$ with respect to the discrete valuation associated with one of the irreducible polynomials* $\mathbf{a}, \mathbf{u}^2-\mathbf{x}^2+\mathbf{a}, \mathbf{v}^2-\mathbf{y}^2+\mathbf{az}^2, \mathbf{d}, \mathbf{x}^2-\mathbf{a}, \mathbf{y}^2-\mathbf{az}^2, \mathbf{f}^2-\mathbf{ag}^2, \mathbf{u}+\mathbf{x}, \mathbf{v}+\mathbf{y}$ *and* $\mathbf{w}+\mathbf{f}$. *Then $\mathcal{A}_{\widehat{K}} \in Dec_3(\widehat{K})$.*

*Proof.* First assume that the valuation $v = v_{\mathbf{a}}$ is associated with $\mathbf{a}$. By Hensel's Lemma, $\mathbf{x}^2-\mathbf{a} \in \widehat{K}^{\times 2}$. It follows that $\big[\mathbf{b}, \mathbf{x}^2-\mathbf{a}\big]_{\widehat{K}} = 0$. By Lemma 2.1, applied to (4), $\mathcal{A}_{\widehat{K}} \in Dec_3(\widehat{K})$.

Let $v = v_{\mathbf{u}^2-\mathbf{x}^2+\mathbf{a}}$. In the residue field, $\bar{\mathbf{u}}^2-\bar{\mathbf{x}}^2+\bar{\mathbf{a}} = \bar{0}$, hence $\bar{\mathbf{x}}^2-\bar{\mathbf{a}}$ is a square. By Hensel's Lemma, $\mathbf{x}^2-\mathbf{a} \in \widehat{K}^{\times 2}$. Therefore, $\mathcal{A}_{\widehat{K}} \in Dec_3(\widehat{K})$ as in the previous case.

The case $v = v_{\mathbf{v}^2-\mathbf{y}^2+\mathbf{az}^2}$ is similar.

Let $v = v_{\mathbf{d}}$. In the residue field, $\bar{\mathbf{w}}^2-\bar{\mathbf{f}}^2+\bar{\mathbf{a}}\bar{\mathbf{g}}^2 = \bar{0}$, hence $\bar{\mathbf{f}}^2-\bar{\mathbf{a}}\bar{\mathbf{g}}^2$ is a square. By Hensel's Lemma, $\mathbf{f}^2-\mathbf{ag}^2 \in \widehat{K}^{\times 2}$, hence $\big[\mathbf{b}, \mathbf{f}^2-\mathbf{ag}^2\big]_{\widehat{K}} = 0$. It follows from (4) that

$$[\mathcal{A}]_T = \big[\mathbf{b}, \mathbf{x}+\sqrt{\mathbf{a}}\big] + \big[\mathbf{c}, \mathbf{y}+\mathbf{z}\sqrt{\mathbf{a}}\big] = \big[\mathbf{b}, \mathbf{f}+\mathbf{g}\sqrt{\mathbf{a}}\big] + \big[\mathbf{bc}, \mathbf{y}+\mathbf{z}\sqrt{\mathbf{a}}\big].$$

By Lemma 2.1, $\mathcal{A}_{\widehat{K}} \in Dec_3(\widehat{K})$.

Let $v = v_{\mathbf{x}^2-\mathbf{a}}$. In the residue field, $\bar{\mathbf{b}}\bar{\mathbf{d}} = \bar{\mathbf{u}}^2$ is a square. By Hensel's Lemma, $\mathbf{bd} \in \widehat{K}^{\times 2}$. It follows from (3) that $\mathcal{A}_{\widehat{K}} \in Dec_3(\widehat{K})$.

The cases $v = v_{\mathbf{y}^2-\mathbf{az}^2}$ and $v = v_{\mathbf{f}^2-\mathbf{ag}^2}$ are similar.

Let $v = v_{\mathbf{u}+\mathbf{x}}$. In the residue field, $\bar{\mathbf{b}}\bar{\mathbf{d}} = \bar{\mathbf{a}}$. By Hensel's Lemma, $\mathbf{abd} \in \widehat{K}^{\times 2}$. It follows again from (3) that $\mathcal{A}_{\widehat{K}} \in \mathit{Dec}_3(\widehat{K})$.

The cases $v = v_{\mathbf{v}+\mathbf{y}}$ and $v = v_{\mathbf{w}+\mathbf{f}}$ are similar. □

From now on we consider the cohomology theory with values in the Galois module $\mathbb{Z}/2\mathbb{Z}$, i.e., $H(L) = H(L, \mathbb{Z}/2\mathbb{Z})$ for any field extension of $F$. Note that $H(L)$ has structure of a commutative ring.

**Proposition 2.5.** *The restriction homomorphism*

$$\operatorname{Inv}(\mathit{Alg}_3, H^r) \to \operatorname{Inv}(\mathit{Dec}_3, H^r)$$

*is injective.*

*Proof.* Let $q$ be an invariant of $\mathit{Alg}_3$ of degree $r$ and let $K$ be the quotient field of the ring $R$, i.e., $K = F(X)$. By Theorem 1.3, we have $q(\mathcal{A}_K) \in A^0(X, H^r)$. Let $X'$ be the open subscheme of $\mathbf{A}^8_F$ given by $\mathbf{e} \neq 0$, so $X \subset X' \subset \mathbf{A}^8_F$ and $X' \simeq \mathbf{A}^7_F \times \mathbf{G}_{\mathrm{m}}$. Note that

$$A^0(X', H^r) = A^0(\mathbf{G}_{\mathrm{m}}, H^r) = H^r(F) \oplus (\mathbf{e}) \cup H^{r-1}(F)$$

by [15, Prop. 2.2 and Prop. 8.6].

Suppose that the restriction of $q$ on $\mathit{Dec}_3$ is zero. By Proposition 2.4, $\mathcal{A}_{\widehat{K}} \in \mathit{Dec}_3(\widehat{K})$, where $\widehat{K}$ is the completion of $K$ with respect to every divisor $x$ of $X'$ in $X' \backslash X$. Hence $q(\mathcal{A}_{\widehat{K}}) = 0$ for all such $\widehat{K}$. The residue homomorphism $\partial_x : H^r(K) \to H^{r-1}(F(x))$ factors through the group $H^r(\widehat{K})$. It follows that $\partial_x(q(\mathcal{A}_K)) = 0$ and therefore,

$$q(\mathcal{A}_K) \in A^0(X', H^r) = H^r(F) \oplus (\mathbf{e}) \cup H^{r-1}(F),$$

i.e., $q(\mathcal{A}_K) = h_K + (\mathbf{e}) \cup h'_K$ for some $h \in H^r(F)$ and $h' \in H^{r-1}(F)$. Consider a point $p \in X(E)$ with $E = F(\mathbf{e})$ such that $\mathbf{e}(p) = \mathbf{e}$ and $\mathbf{b}(p) = 1$. It follows from (3) that $\mathcal{A}(p) \in \mathit{Dec}_3(E)$. Hence by Theorem 1.3(2),

$$0 = q(\mathcal{A}(p)) = h_E + (\mathbf{e}) \cup h'_E,$$

therefore, $h = h' = 0$ and $q(\mathcal{A}_K) = 0$. By Proposition 2.3 and Theorem 1.5, $q = 0$. □

### *2.4. Invariants of $\mathit{Dec}_n$*

From now on we assume that $-1 \in F^{\times 2}$.

Let $K_*(F)$ denote the Milnor ring of a field $F$ and set $k_*(F) = K_*(F)/2K_*(F)$. For every $n \geq 0$, let $\gamma_n$ denote the *divided power* operation [9,19]:

$$k_2(F) \to k_{2m}(F)$$

defined by

$$\gamma_n\left(\sum_{i=1}^{r} \alpha_i\right) = \sum_{1 \leq i_1 \leq \cdots \leq i_m \leq n} \alpha_{i_1} \cdot \cdots \cdot \alpha_{i_m},$$

where the $\alpha_i$ are symbols. In particular, $\gamma_0 = 1 \in k_0(F) = \mathbb{Z}/2\mathbb{Z}$ and $\gamma_1$ is the identity.

We identify $k_2(F)$ with $\mathrm{Br}_2(F)$ via the norm residue isomorphism. Restricting $\gamma_m$ to $\mathit{Dec}_n$ and composing with the norm residue homomorphism $k_{2m}(F) \to H^{2m}(F)$, we can view the divided power operations (still denoted by $\gamma_m$) as invariants of $\mathit{Dec}_n$ with values in $H$, so $\gamma_m \in \mathrm{Inv}(\mathit{Dec}_n, H^{2m})$ for all $n$. Clearly, $\gamma_m = 0$ if $m > n$.

**Theorem 2.6.** *The $H(F)$-module* $\mathrm{Inv}(\mathit{Dec}_n, H)$ *is free with basis* $\{1 = \gamma_0, \gamma_1, \ldots, \gamma_n\}$.

*Proof.* The case $n = 1$, when $\mathit{Dec}_1 = \mathit{Quat}$ is proven in [7, Th. 18.1]. By [7, Ex. 16.5], the natural map

$$\mathrm{Inv}(\mathit{Quat}, H)^{\otimes n} \to \mathrm{Inv}(\mathit{Quat}^{\times n}, H)$$

is an isomorphism. It follows that $\mathrm{Inv}(\mathit{Quat}^{\times n}, H)$ is a free $H(F)$-module with basis of all monomials $\delta_1^{\varepsilon_1}\delta_2^{\varepsilon_2}\ldots\delta_n^{\varepsilon_n}$, where $\varepsilon_1 = 0$ or 1 and the invariant $\delta_i$ is defined by $\delta_i(\alpha_1, \ldots, \alpha_n) = \alpha_i$.

The natural morphism of functors

$$\mathit{Quat}^{\times n} \to \mathit{Dec}_n \tag{5}$$

given by the tensor product is surjective. It follows that the map

$$\mathrm{Inv}(\mathit{Dec}_n, H) \to \mathrm{Inv}(\mathit{Quat}^{\times n}, H)$$

is injective. The image of this map is element-wise invariant under the natural action of the symmetric group $S_n$ and hence is contained in the free $H(F)$-submodule generated by the standard symmetric functions $\gamma_m$ on the $\delta_1, \ldots, \delta_n$ that are precisely the divided powers. □

*Remark 2.7.* Vial has computed all invariants of $k_n$ in [19].

Restricting the divided powers on the subfunctors $\mathit{Alg}_n \subset \mathrm{Br}_2$ we view the $\gamma_m$ as invariants on $\mathit{Alg}_n$.

**Theorem 2.8.** *If $n \leq 3$, then the $H(F)$-module* $\mathrm{Inv}(\mathit{Alg}_n, H)$ *is free with basis* $\{1 = \gamma_0, \gamma_1, \ldots, \gamma_n\}$.

*Proof.* If $n \leq 2$, then $\mathit{Alg}_n = \mathit{Dec}_n$ and the statement follows from Theorem 2.6. The case $n = 3$ is implied by Proposition 2.5 and Theorem 2.6. □

### *2.5. Reduced trace form*

Let $A$ be a central simple algebra over a field $F$. Denote by $q_A$ the quadratic form on $A$ defined by $q_A(a) = \mathrm{Trd}_A(a^2)$ for $a \in A$, where $\mathrm{Trd}_A$ is the reduced trace form for $A$. If $A$ and $A'$ are two central simple algebras over $F$, then

$$q_{A\otimes A'} \simeq q_A \otimes q_{A'}.$$

*Example 2.9.* Let $A$ be a quaternion algebra over a field $F$. Then $q_A$ is the 2-fold Pfister form $\langle\langle a, b\rangle\rangle$, where $a, b \in F^\times$ such that $[A] = [a, b]$ in $\mathrm{Br}(F)$.

It follows from Example 2.9 that for any $A \in \mathit{Dec}_n(F)$ the form $q_A$ is a $2n$-fold Pfister form. Moreover, the invariant $e_{2n}(q_A)$ in $H^{2n}(F)$ (cf. [6, §16]) coincides with the divided power $\gamma_n(A)$.

**Theorem 2.10.** *If $n \le 3$, then for any $A \in \mathit{Alg}_n(F)$, the form $q_A$ is a $2n$-fold Pfister form such that $e_{2n}(q_A) = \gamma_n(A)$.*

*Proof.* If $n \le 2$, then $\mathit{Alg}_n = \mathit{Dec}_n$ and the statement follows.

Consider the case $n = 3$. Let $A \in \mathit{Alg}_3(F)$. Choose a splitting field $F(\sqrt{a}, \sqrt{b}, \sqrt{c})$ and set $L = F(\sqrt{a})$. We write $a \mapsto \bar{a}$ for the nontrivial automorphism of $L$ over $F$. Let $B$ be the centralizer of $L$ in $A$. By Skolem–Noether Theorem [11, Th. 1.4], there is an $s \in A$ such that $sxs^{-1} = \bar{x}$ for all $x$ in $L$. Note that $s^2$ commutes with all elements in $L$, hence $s^2 \in B$.

Let $\psi : B \to B$ be an automorphism defined by $y \mapsto sys^{-1}$. Then $A = B \oplus Bs$ with $sy = \psi(y)s$ for all $y \in B$. Since $\mathrm{Trd}_A(yzs) = \mathrm{Trd}_A(\sqrt{a}yzs(\sqrt{a})^{-1}) = -\mathrm{Trd}_A(yzs)$, we have $\mathrm{Trd}_A(yzs) = 0$ for any $y$ and $z$ in $B$. Moreover, $\mathrm{Trd}_A(y) = \mathrm{Tr}_{L/F}(\mathrm{Trd}_B(y))$ for any $y \in B$ by [5, §22, Cor. 5]. Therefore, for the trace forms we have

$$q_A = \mathrm{Tr}_{L/F}(q_B) \perp \mathrm{Tr}_{L/F}(q'_B),$$

where $q'_B(x) = \mathrm{Trd}_B\left((xs)^2\right)$.

Let $t \in F^\times$ and $A_t$ the $F$-algebra with presentation $A_t = B \oplus By$ and $yby^{-1} = sbs^{-1}$ for all $b \in B$ and $y^2 = ts^2$. By Proposition [11, Th. 13.41],

$$[A_t] = [a, t] + [A].$$

Moreover,

$$q_{A_t} = \mathrm{Tr}_{L/F}(q_B) \perp t\,\mathrm{Tr}_{L/F}(q'_B),$$

hence, by Lemma 2.11 below, in the Witt ring of $F$, we have

$$q_A - tq_{A_t} = \langle\langle t\rangle\rangle \cdot \mathrm{Tr}_{L/F}(q_B) \in I^6(F).$$

By (2), we can choose $t$ such that $A_t$ is decomposable, hence $q_{A_t} \in I^6(F)$ and therefore, $q_A \in I^6(F)$. As $\dim(q_A) = 64$, the form $q_A$ is a 6-fold Pfister form.

It follows that $e_6(q_A)$ is a well-defined invariant of $\mathit{Alg}_3$ that agrees with $\gamma_3$ on $\mathit{Dec}_3$. By Proposition 2.5, $e_6(q_A) = \gamma_3$ on $\mathit{Alg}_3$. □

**Lemma 2.11.** *In the notation above,* $\mathrm{Tr}_{L/F}(q_B) \in I^5(F)$.

*Proof.* In Tignol's construction (see (1) and (2)),

$$[A]_L = [b, s] + [c, t] = [a, e] + [b, k] + [c, l] + [d, klm]$$

in $\mathrm{Br}(L)$. Let

$$p := \langle\langle a, e\rangle\rangle + \langle\langle b, k\rangle\rangle + \langle\langle c, l\rangle\rangle + \langle\langle d, klm\rangle\rangle \in I^2(F). \tag{6}$$

It follows that

$$p_L \equiv \langle\langle b, s\rangle\rangle + \langle\langle c, t\rangle\rangle \mod I^3(L),$$

so $B \simeq (b, s) \otimes_L (c, t)$. We have in $W(L)$:

$$q_B = \langle\langle b, s\rangle\rangle \cdot \langle\langle c, t\rangle\rangle \equiv \langle\langle b, s\rangle\rangle \cdot p_L - \langle\langle b, s\rangle\rangle = \langle\langle b, s\rangle\rangle \cdot p_L \mod I^5(L)$$

since $\langle\langle b, b\rangle\rangle = 0$. By the projection formula and [6, Cor. 34.19],

$$\mathrm{Tr}_{L/F}(q_B) \equiv \mathrm{Tr}_{L/F}\left(\langle\langle b, s\rangle\rangle\right) \cdot p \equiv \langle\langle b, N_{L/F}(s)\rangle\rangle \cdot p \mod I^5(F). \quad (7)$$

We have $\langle\langle b, N_{L/F}(s)\rangle\rangle \simeq \langle\langle c, N_{L/F}(t)\rangle\rangle \simeq \langle\langle d, N_{L/F}(t)\rangle\rangle$. It follows that $\langle\langle b, N_{L/F}(s)\rangle\rangle$ annihilates all four summands in the right hand side of (6), hence $\langle\langle b, N_{L/F}(s)\rangle\rangle \cdot p = 0$. By (7), $\mathrm{Tr}_{L/F}(q_B) \in I^5(F)$. □

*2.6. Essential dimension of $\mathit{Dec}_n$ and $\mathit{Alg}_3$*

Let $S : \mathit{Fields}/F \to \mathit{Sets}$ be a functor, $E \in \mathit{Fields}/F$ and $K \subset E$ a subfield over $F$. An element $\alpha \in S(E)$ is said to be *defined over* $K$ (and $K$ is called a *field of definition of* $\alpha$) if there exists an element $\beta \in S(K)$ such that $\alpha$ is the image of $\beta$ under the map $S(K) \to S(E)$. The *essential dimension of* $\alpha$, denoted $\mathrm{ed}(\alpha)$, is the least transcendence degree $\mathrm{tr.deg}_F(K)$ over all fields of definition $K$ of $\alpha$. The *essential dimension of the functor* $S$ is

$$\mathrm{ed}(S) = \sup\{\mathrm{ed}(\alpha)\},$$

where the supremum is taken over fields $E \in \mathit{Fields}/F$ and all $\alpha \in S(E)$ (cf. [3, Def. 1.2]).

The highest invariant $\gamma_n$ of $\mathit{Alg}_n$ and $\mathit{Dec}_n$ of degree $2n$ is nontrivial, hence $\mathrm{ed}\left(\mathit{Alg}_n\right) \geq 2n$ and $\mathrm{ed}\left(\mathit{Dec}_n\right) \geq 2n$ by [3, Cor. 3.6]. On the other hand, using the surjection (5), we get

$$\mathrm{ed}\left(\mathit{Dec}_n\right) \leq \mathrm{ed}\left(\mathit{Quat}^{\times n}\right) \leq n \cdot \mathrm{ed}\left(\mathit{Quat}\right) = 2n.$$

Thus, $\mathrm{ed}\left(\mathit{Dec}_n\right) = 2n$.

It is proved in [13, Cor. 3.10] and [14, Th. 8.6] that $\mathrm{ed}\left(\mathit{Alg}_3\right) \leq 17$.

**Theorem 2.12.** $6 \leq \mathrm{ed}\left(\mathit{Alg}_3\right) \leq 8$.

*Proof.* By Proposition 2.3, there is a surjective morphism of functors $X \to \mathit{Alg}_3$, where $X$ is a variety defined in Sect. 2. By [3, Cor. 1.19], $\mathrm{ed}\left(\mathit{Alg}_3\right) \leq \dim(X) = 8$. □

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