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Correction

Correction to: On the Quasi-Linear Elliptic PDE $-\nabla \cdot (\nabla u/\sqrt{1-|\nabla u|^2}) = 4\pi \sum_k a_k \delta_{s_k}$ in Physics and Geometry

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Denis Bonheure kindly informed me that [1], section 2.6 ("*The minimizer of $\mathcal{F}$ weakly satisfies the Euler–Lagrange equation*"), contains a logical gap on p. 516 where I wrote:

"The result $|\Omega_{\rm crit}| = 0$ means that $|\nabla v_\infty| < 1$ *a.e.*, and this already implies that the variation of $\mathcal{F}(v)$ about v_∞ to leading order (i.e. power 1) in ψ now reads

$$\mathcal{F}^{(1)}[v_\infty](\psi) = \int_{\mathbb{R}^3} \Bigg(\nabla\psi(s) \cdot \frac{\nabla v_\infty(s)}{\sqrt{1-|\nabla v_\infty(s)|^2}} - 4\pi\psi(s) \sum_{1\le n\le N} a_n \delta_{s_n}(s) \Bigg) \mathrm{d}^3 s. \tag{22}$$

Since $\mathcal{F}^{(1)}[v_\infty](\psi)$ is linear in ψ, v_∞ can minimize $\mathcal{F}$ over $\mathscr{A}$ only if $\mathcal{F}^{(1)}[v_\infty](\psi) = 0$ for all ψ, which is precisely (7). Thus the Euler-Lagrange equation (1) is satisfied by v_∞ in the weak sense, as claimed."

Bonheure's objection concerns the sentence: "Since $\mathcal{F}^{(1)}[v_\infty](\psi)$ is linear in ψ, ...", which alludes to the usual linearity-based argument (i.e., "Suppose $\mathcal{F}^{(1)}[v_\infty](\psi) \neq 0$ for some ψ; then either $\mathcal{F}^{(1)}[v_\infty](\psi) < 0$ or $\mathcal{F}^{(1)}[v_\infty](-\psi) < 0$; but this is impossible because v_∞ is the minimizer of $\mathcal{F}(v)$; hence, $\mathcal{F}^{(1)}[v_\infty](\psi) = 0$."). He notes that, although $|\Omega_{\rm crit}| = 0$ (cf. the proof on p. 515) implies that $\mathcal{F}^{(1)}[v_\infty](\psi)$ is given by (22), and although (22) does act linearly on the space of compactly supported $C^\infty(\mathbb{R}^3)$ test functions $C_c^\infty(\mathbb{R}^3)$, only a nonlinear subset of these supplies admissible perturbations of v_∞. More precisely, the restriction $v_\infty + \psi \in \mathscr{A}$ (the admissible set of v) rules out test functions ψ for which $\nabla v_\infty(s) \cdot \nabla\psi(s) \geq 0$ *a.e.* in some open ϵ-ball B_ϵ satisfying $\Omega_{\rm crit} \cap B_\epsilon \neq \emptyset$ while allowing those ψ for which $\nabla v_\infty(s) \cdot \nabla\psi(s) < 0$ *a.e.* in such B_ϵ.

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Thus one cannot invoke the linearity of $\mathcal{F}^{(1)}[v_\infty](\,\cdot\,) : C_c^\infty(\mathbb{R}^3) \to \mathbb{R}$ to conclude that $\mathcal{F}^{(1)}[v_\infty](\psi) = 0$ for all $\psi \in C_c^\infty(\mathbb{R}^3)$, which is equivalent to the Euler–Lagrange equation (1) (see (E28) below) in weak form (see (7) in [1]). Of course, the nonlinear set of admissible ψ contains the linear subset of $C_c^\infty(\mathbb{R}^3)$ test functions for which $\lim_{s\to\Omega_{\text{crit}}} |\nabla\psi(s)| = 0$ (incidentally, the larger linear subset of $C_c^\infty(\mathbb{R})$ for which $\lim_{s\to\Omega_{\text{crit}}} \nabla\psi(s)\cdot\nabla v_\infty(s) = 0$ contains inadmissible ψ, namely those for which $\lim_{s\to\Omega_{\text{crit}}} |\nabla\psi(s)| \neq 0$). For ψ in this linear subset we do have $\mathcal{F}^{(1)}[v_\infty](\psi) = 0$, by the familiar argument; however, Bonheure points out, knowing only that $|\Omega_{\text{crit}}| = 0$ does not allow us to conclude that the linear subset of test functions satisfying $\lim_{s\to\Omega_{\text{crit}}} |\nabla\psi(s)| = 0$ is dense in $C_c^\infty(\mathbb{R}^3)$.

In the following I respond to Bonheure's criticism (see also [2]) by showing that $\mathcal{F}^{(1)}[v_\infty](\psi) = 0$ not only for all $\psi \in C_c^\infty(\mathbb{R}^3)$ which satisfy $\lim_{s\to\Omega_{\text{crit}}} |\nabla\psi(s)| = 0$, but indeed for all $\psi \in C_c^\infty(\mathbb{R}^3)$. I pick up on Remark 3.4, see p. 518 in [1]; thus we play a variation of the convex duality theme on p. 517 of [1].

We work with the almost everywhere harmonic field (eq.(29) in [1])

$$V_h(s) = -\sum_{n=1}^{N} a_n \nabla \frac{1}{|s - s_n|}. \tag{E1}$$

We have (eq.(30) in [1])

$$\nabla \cdot V_h = 4\pi \sum_{n=1}^{N} a_n \delta_{s_n} \tag{E2}$$

in the sense of distributions. Moreover, recall that for any $w \in (\dot{W}_0^{1,p}(\mathbb{R}^3))^3$, $p \geq 1$,

$$\nabla \cdot \nabla \times w = 0 \tag{E3}$$

weakly; note that $\nabla \times w$ is well defined on $\mathbb{R}^3$ except on a set Ω_w with Lebesgue measure zero. More generally, linearity implies that $\nabla \cdot \nabla \times w = 0$ for any $w = \sum_p w_p$ with weak curls $\nabla \times w_p \in (L^p(\mathbb{R}^3))^3$, where $\sum_p$ sums over a countable (sub-)set of $p \geq 1$. We will work with $\sum_p (\dot{W}_0^{\nabla\times,p}(\mathbb{R}^3))^3$, $p \geq 1$, where $(\dot{W}_0^{\nabla\times,p}(\mathbb{R}^3))^3$ is defined as the closure of the set of divergence-free, compactly supported C^∞ vector fields w with respect to the norm $\||\nabla \times w|\|_{L^p(\mathbb{R}^3)}$.

Abbreviating $V_h + \nabla \times w =: V$, an integration by parts now yields

$$\begin{aligned}\mathcal{F}(v_\infty) &= \int_{\mathbb{R}^3} \Big(1 - \sqrt{1 - |\nabla v_\infty(s)|^2} - 4\pi v_\infty(s) \sum_{1\leq n\leq N} a_n \delta_{s_n}(s)\Big) \mathrm{d}^3 s \\ &= \int_{\mathbb{R}^3} \Big(1 - \sqrt{1 - |\nabla v_\infty(s)|^2} + V(s)\cdot\nabla v_\infty(s)\Big) \mathrm{d}^3 s\end{aligned} \tag{E4}$$

for any such $w \in \sum_p (\dot{W}_0^{\nabla\times,p}(\mathbb{R}^3))^3$, $p \geq 1$. Next observe that *pointwise*

$$\begin{aligned} V(s)\cdot\nabla v_\infty(s) - \sqrt{1 - |\nabla v_\infty(s)|^2} &\geq \min_{E(s)\in B_1 a.e.} \Big\{- V(s)\cdot E(s) - \sqrt{1 - |E(s)|^2}\Big\} \\ &= -\sqrt{1 + |V(s)|^2}, \quad s \in \mathbb{R}^3\backslash(\Omega_{\text{crit}} \cup \Omega_w),\end{aligned} \tag{E5}$$

where $B_1 \subset \mathbb{R}^3$ is the open unit ball. The unique minimizer $E_V(s)$ is given by

$$E_V(s) = \frac{V(s)}{\sqrt{1 + |V(s)|^2}}, \tag{E6}$$

defining a vector field on $\mathbb{R}^3$ *a.e.*, satisfying $|E_V(s)| < 1$, with $|E_V(s)| \to 1$ when $s \to \{s_n\}_{n=1}^N \subset \Omega_{\text{crit}}$ and possibly when $s \to \Omega_w$. Inverting (E6) yields

$$V(s) = \frac{E_V(s)}{\sqrt{1 - |E_V(s)|^2}}, \tag{E7}$$

and so, since $\nabla \cdot V = 4\pi \sum_{n=1}^N a_n \delta_{s_n}$ in the sense of distributions, we have

$$\nabla \cdot \frac{E_V(s)}{\sqrt{1 - |E_V(s)|^2}} = 4\pi \sum_{n=1}^N a_n \delta_{s_n} \tag{E8}$$

weakly, for *any* $V = V_h + \nabla \times w$. This almost is the Euler–Lagrange equation we seek to obtain, yet not quite: at this point we don't know whether $s \mapsto E_V(s)$ is a gradient field — indeed, for most w it's not! Also, the minimization w.r.t. E implies

$$\mathcal{F}(v_\infty) \geq \int_{\mathbb{R}^3} \left(1 - \sqrt{1 + |V(s)|^2}\right) \mathrm{d}^3 s \equiv -\mathcal{G}(V) \tag{E9}$$

for any $V = V_h + \nabla \times w$ with $w \in \sum_p (\dot{W}_0^{\nabla\times,p}(\mathbb{R}^3))^3$, $p \geq 1$. So we need to show that there does exist a $U = V_h + \nabla \times w_*$ such that for $a.e.\ s \in \mathbb{R}^3$ we have $E_U(s) = -\nabla v_*(s)$ for some $v_* \in \dot{W}_0^{1,\infty}$, and with $\mathcal{G}(U) = -\mathcal{F}(v_\infty)$. The existence and uniqueness of the minimizer $v_\infty(s)$ of $\mathcal{F}(v)$ (see [1]) then yields $v_*(s) = v_\infty(s)$. Clearly, $U = V_h + \nabla \times w_*$ will minimize $\mathcal{G}(V)$ among all $V = V_h + \nabla \times w$ with $w \in \sum_p (\dot{W}_0^{\nabla\times,p}(\mathbb{R}^3))^3$, $p \geq 1$. In fact, it suffices to minimize $\mathcal{G}(V_h + \nabla \times w)$ for $w \in \sum_p (\dot{W}_0^{\nabla\times,p}(\mathbb{R}^3))^3$, $p \in \{1, 2\}$. We remark that $\sum_{p\in\{1,2\}} (\dot{W}_0^{\nabla\times,p}(\mathbb{R}^3))^3$ is a Banach space with respect to the norm $\|w\| := \inf\{\| |\nabla \times w_1| \|_{L^1(\mathbb{R}^3)} + \| |\nabla \times w_2| \|_{L^2(\mathbb{R}^3)}\}$, where (given w) the infimum is over the set $\{w_1 + w_2 = w \mid w_p \in (\dot{W}_0^{\nabla\times,p}(\mathbb{R}^3))^3,\ p \in \{1, 2\}\}$; note that the splitting of w into a sum of w_1 and w_2 is not unique (we will take advantage of this to prove Lemma 0.7, and Theorem 0.1, below).

We now show that such a w_* exists.

In the special case $N = 1$ it is easily seen that $w_* \equiv 0$ is the minimizer. Indeed, V_h in this case is a spherically symmetric gradient field, and so is E_{V_h}; thus, taking the Gateaux derivative $\frac{d}{dt}\mathcal{G}(V_h + t\nabla \times w)$ at $t = 0$ with compactly supported $w \in C_c^\infty(\mathbb{R}^3)^3$ yields $\int E_{V_h} \cdot \nabla \times w\, d^3 s$, and integration by parts now shows that this integral does vanish because $\nabla \times E_{V_h} = 0$. Hence when $N = 1$ then V_h is a critical point of $\mathcal{G}(V)$, and the strict convexity of $\mathcal{G}(V)$ w.r.t. $\nabla \times w$ now establishes its minimality. Of course, this just re-expresses the long-ago solved $\mathcal{F}$ variational problem for $N = 1$ in terms of the $\mathcal{G}$ variational problem. Thus, in the following we assume $N > 1$.

When $N > 1$ then $w \equiv 0$ is not a minimizer. For suppose $w \equiv 0$ were a minimizer, then the Gateaux derivative $\frac{d}{dt}\mathcal{G}(V_h + t\nabla \times w)$ at $t = 0$ would have to vanish, yet it is easily seen that it doesn't vanish for all w because $\nabla \times E_{V_h} \not\equiv 0$ when $N > 1$.

We next prove

Theorem 0.1. $\exists! w_* \in (\dot{W}_0^{\nabla\times,1}(\mathbb{R}^3))^3 + (\dot{W}_0^{\nabla\times,2}(\mathbb{R}^3))^3$ *so that* $U = V_h + \nabla \times w_*$ *satisfies*

$$\mathcal{G}(U) = \inf \left\{ \mathcal{G}(V_h + \nabla \times w) \,\middle|\, w \in (\dot{W}_0^{\nabla\times,1}(\mathbb{R}^3))^3 + (\dot{W}_0^{\nabla\times,2}(\mathbb{R}^3))^3 \right\}. \tag{E10}$$

Proof of the Theorem. We begin by showing that $\mathcal{G}(V)$ is well defined on the stipulated set.

Proposition 0.2. *For $V = V_h + \nabla \times w$ with $w = w_1 + w_2$ as stipulated, the functional $\mathcal{G}(V)$ is well-defined and strongly continuous.*

Proof. Through telescoping $\mathcal{G}(V) = [\mathcal{G}(V_h + \nabla \times w_1 + \nabla \times w_2) - \mathcal{G}(V_h + \nabla \times w_1)] + [\mathcal{G}(V_h + \nabla \times w_1) - \mathcal{G}(V_h)] + \mathcal{G}(V_h)$, we right away note that $\mathcal{G}(V_h)$ is well-defined because $|V_h| \in L^1_{\mathrm{loc}}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3 \backslash B_R)$ for any open $B_R \supset \{s_n\}_{n=1}^N$, whereas the two difference terms (the [...] terms) are estimated as follows: We use the identity

$$\mathcal{G}(V_1 + V_2) - \mathcal{G}(V_1) = \int_0^1 \int_{\mathbb{R}^3} \frac{V_1(s) + \lambda V_2(s)}{\sqrt{1 + |V_1(s) + \lambda V_2(s)|^2}} \cdot V_2(s) \mathrm{d}^3 s \mathrm{d}\lambda, \tag{E11}$$

and note that $E_V = V\big/\sqrt{1 + |V|^2} \in ((L^2 \cap L^\infty)(\mathbb{R}^3))^3$, with $\||E_V|\|_{L^\infty(\mathbb{R}^3)} = 1$ and

$$\||E_V|\|_{L^2(\mathbb{R}^3)} \leq |\Omega|^{1/2} + \||V_h|\|_{L^2(\mathbb{R}^3 \backslash \Omega)} + \||\nabla \times w_1|\|_{L^1(\mathbb{R}^3)}^{1/2} + \||\nabla \times w_2|\|_{L^2(\mathbb{R}^3)}, \tag{E12}$$

where $\Omega = B_R \cup \{|\nabla \times w_1| > 1\} \cup \{|\nabla \times w_2| > 1\}$ for some convenient $B_R \supset \{s_n\}_{n=1}^N$ (see “Appendix A”); note that Ω is measurable but not necessarily open. Thus, and setting $\| \cdot \|_{L^p} := \| \cdot \|_{L^p(\mathbb{R}^3)}$ (below with $p = 1$ or 2),

(i) let $V_1 = V_h$ and $V_2 = \nabla \times w_1$, then Hölder's inequality applied to (E11) yields

$$|\mathcal{G}(V_h + \nabla \times w_1) - \mathcal{G}(V_h)| \leq \||\nabla \times w_1|\|_{L^1}; \tag{E13}$$

(ii) let $V_1 = V_h + \nabla \times w_1$ and $V_2 = \nabla \times w_2$ and apply Hölder to (E11) to get

$$|\mathcal{G}(V_h + \nabla \times (w_1 + w_2)) - \mathcal{G}(V_h + \nabla \times w_1)| \leq \sup_{0<\lambda<1} \left\| |E_{V_\lambda}| \right\|_{L^2} \||\nabla \times w_2|\|_{L^2}, \tag{E14}$$

where we have set $V_\lambda = V_h + \nabla \times w_1 + \lambda \nabla \times w_2$. The L^2 norm of E_{V_λ} is estimated by (E12) with w_2 replaced by λw_2, and the obvious estimate $\lambda < 1$. This establishes that $\mathcal{G}(V)$ is well-defined on the stipulated set.

This also proves that $\mathcal{G}(V)$ is strongly continuous at V_h, for $\left\| |\nabla \times w_1^{(n)}| \right\|_{L^1} \to 0$ and $\left\| |\nabla \times w_2^{(n)}| \right\|_{L^2} \to 0$ together imply $\mathcal{G}(V_h + \nabla \times (w_1^{(n)} + w_2^{(n)})) \to \mathcal{G}(V_h)$ as $n \to \infty$. The strong continuity of $\mathcal{G}(V)$ at any $V = V_h + \nabla \times w$ with $w \in (\dot{W}_0^{\nabla\times,1}(\mathbb{R}^3))^3 + (\dot{W}_0^{\nabla\times,2}(\mathbb{R}^3))^3$ follows essentially verbatim. □

The strong continuity of $\mathcal{G}(V)$ in concert with its strict convexity in $\nabla \times w$ implies:

Corollary 0.3. *The functional $\mathcal{G}(V)$ is weakly lower semi-continuous.*

Since $\mathcal{G}(V)$ is invariant under gauge transformations $w \to w + \nabla\gamma$ (because $\nabla \times w$, and thus V, are, also E_V is gauge invariant), the strict convexity of $\mathcal{G}(V)$ in $\nabla \times w$ does not automatically translate into strict convexity of $\mathcal{G}(V)$ in w. However, since we have stipulated w to be divergence-free, viz. $\nabla \cdot w = 0$, only gauge transformations $w \to w + \nabla\gamma$ with harmonic γ, i.e. with $\Delta\gamma = 0$, remain; but the only allowed harmonic γ are those which are constant at spatial ∞, which leaves the identity map as the only gauge transformation. Thus $\mathcal{G}(V)$ is strictly convex in w, and so we have

Corollary 0.4. *Any minimizer* $w_* \in (\dot{W}_0^{\nabla\times,1}(\mathbb{R}^3))^3 + (\dot{W}_0^{\nabla\times,2}(\mathbb{R}^3))^3$ *of* $\mathcal{G}(V_h + \nabla \times w)$ *is unique.*

We next show that it suffices to consider $\mathcal{G}(V_h + \nabla \times w)$ for $w \in (\dot{W}_0^{\nabla\times,1}(\mathbb{R}^3))^3 + (\dot{W}_0^{\nabla\times,2}(\mathbb{R}^3))^3$. For this we recall the chiral Helmholtz theorem of [3]:

Lemma 0.5. *Any vector field* $\mathbf{F}(s)$ *can be decomposed into* $\mathbf{F}(s) = \mathbf{f}(s) + \mathbf{g}(s)$*, where* $\mathbf{f}(s)$ *is divergence-free and* $\mathbf{g}(s)$ *is curl-free. The divergence-free part* $\mathbf{f}(s)$ *has the chiral Fourier representation*

$$\mathbf{f}(s) = \sum_{\eta\in\{\pm 1\}} \int_{\mathbb{R}^3} e^{i2\pi k\cdot s} \mathbf{Q}_\eta(k) f_\eta(k) \mathrm{d}^3 k \tag{E15}$$

with $k = (k_1, k_2, k_3)^T$ *and* $\mathbf{Q}_\eta(k) = -\frac{\eta}{\sqrt{2}}\left(\frac{k_1(k_1+i\eta k_2)}{|k|(|k|+k_3)} - 1, \frac{k_2(k_1+i\eta k_2)}{|k|(|k|+k_3)} - i\eta, \frac{k_1+i\eta k_2)}{|k|}\right)^T$. *So* $\mathbf{f}(s)$ *is uniquely characterized by two scalar functions,* $f_\eta(k)$*,* $\eta \in \{\pm 1\}$*, given by*

$$f_\eta(k) = \int_{\mathbb{R}^3} e^{-i2\pi k\cdot s} \mathbf{Q}^*_\eta(k) \cdot \mathbf{F}(s) \mathrm{d}^3 s. \tag{E16}$$

Note that $\mathbf{Q}_\eta(k)$ is a unit vector which depends on k only through $k/|k|$, and it can continuously be extended into the removable singularity at $k_3 = -|k|$. Note also that $k \cdot \mathbf{Q}_\eta(k) = 0$. We remark that this is not in violation of the "hairy ball theorem" because $\mathbf{Q}_\eta(k)$ is *complex*.

With the help of Lemma 0.5 we prove

Proposition 0.6. *Suppose* $\mathcal{G}(V_h + \nabla \times w) < \infty$*. Then, after at most a gauge transformation* $w \mapsto w + \nabla\gamma$*, we have* $w \in (\dot{W}_0^{\nabla\times,1}(\mathbb{R}^3))^3 + (\dot{W}_0^{\nabla\times,2}(\mathbb{R}^3))^3$.

Proof. For any $V = V_h + \nabla \times w$ with $\mathcal{G}(V) < \infty$ the subset $K_C \subset \mathbb{R}^3$ on which $|V| \geq C > 0$ *a.e.* has finite Lebesgue measure, for we have $\mathcal{G}(V) \geq (\sqrt{1+C^2} - 1)|K_C|$.

Partitioning $\mathbb{R}^3 = K_C \cup \mathbb{R}^3 \backslash K_C$ we thus estimate

$$\mathcal{G}(V) \geq \| |V| \|_{L^1(K_C)} - |K_C| + \tfrac{1}{1+\sqrt{1+C^2}} \| |V| \|^2_{L^2(\mathbb{R}^3\backslash K_C)}. \tag{E17}$$

Now let $C > 0$ be small enough so that $\{s_l\}_{l=1}^N \subset\subset K_C$. Then, by the triangle inequality, (E17) implies that, on the one hand,

$$\| |\nabla \times w| \|_{L^1(K_C)} \leq \mathcal{G}(V) + |K_C| + \| |V_h| \|_{L^1(K_C)}, \tag{E18}$$

and, on the other,

$$\| |\nabla \times w| \|_{L^2(\mathbb{R}^3\backslash K_C)} \leq \sqrt{(1 + \sqrt{1+C^2})(\mathcal{G}(V) + |K_C|)} + \| |V_h| \|_{L^2(\mathbb{R}^3\backslash K_C)}. \tag{E19}$$

Defining

$$f_{1;\eta}(k) := \int_{K_C} e^{-i2\pi k\cdot s} \mathbf{Q}^*_\eta(k) \cdot \nabla \times w(s) \mathrm{d}^3 s, \tag{E20}$$

$$f_{2;\eta}(k) := \int_{\mathbb{R}^3\backslash K_C} e^{-i2\pi k\cdot s} \mathbf{Q}^*_\eta(k) \cdot \nabla \times w(s) \mathrm{d}^3 s, \tag{E21}$$

and $f_\eta(k) := f_{1;\eta}(k)+f_{2;\eta}(k)$, we obtain a decomposition of $\nabla\times w$ into $\nabla\times w_1+\nabla\times w_2$ with $|\nabla\times w_1|\in L^1(\mathbb{R}^3)$ and $|\nabla\times w_2|\in L^2(\mathbb{R}^3)$. But then (see [3]),

$$w_l(s) := \sum_{\eta\in\{\pm 1\}}\eta\int_{\mathbb{R}^3} e^{i2\pi k\cdot s}\mathbf{Q}_\eta(k) f_{l;\eta}(k)\tfrac{1}{|k|}\mathrm{d}^3k \in \dot{W}_0^{\nabla\times,l}(\mathbb{R}^3)^3, \quad l\in\{1,2\}. \tag{E22}$$

□

Next we use that the splitting $w = w_1 + w_2$ is not unique to show:

Lemma 0.7. *Let $\{w^{(n)}\}_{n\in\mathbb{N}}\subset(\dot{W}_0^{\nabla\times,1}(\mathbb{R}^3))^3+(\dot{W}_0^{\nabla\times,2}(\mathbb{R}^3))^3$ be a minimizing sequence for $\mathcal{G}(V_h+\nabla\times w)$. Then without loss of generality we may assume that $\||\nabla\times w_2^{(n)}|\|_{L^2}\leq C_2$ for some convenient fixed $C_2>0$.*

Proof. Since $w\in(\dot{W}_0^{\nabla\times,1}(\mathbb{R}^3))^3+(\dot{W}_0^{\nabla\times,2}(\mathbb{R}^3))^3$ we have $f_\eta\in\widetilde{C}_0(\mathbb{R}^3)+L^2(\mathbb{R}^3)$, where $\widetilde{C}_0(\mathbb{R}^3)$, a subset of the continuous functions which vanish at spatial ∞, is the image of $L^1(\mathbb{R}^3)$ under Fourier transform.

Now, if $f = f_1 + f_2$ with $f_1\in\widetilde{C}_0(\mathbb{R}^3)$ and $f_2\in L^2(\mathbb{R}^3)$, then for $h\in S(\mathbb{R}^3)$ (Schwartz space) also $f=(f_1+h)+(f_2-h)$ with $f_1+h\in\widetilde{C}_0(\mathbb{R}^3)$ and $f_2-h\in L^2(\mathbb{R}^3)$. Since furthermore $S(\mathbb{R}^3)$ is dense in $L^2(\mathbb{R}^3)$ (and also in $C_0(\mathbb{R}^3)$, though that's not needed), if necessary after splitting $w=w_1+w_2$ corresponding to $f_\eta=f_{1,\eta}+f_{2,\eta}$, we can always find an h to "re-split" $w=w_1'+w_2'$ with $f_\eta=(f_{1,\eta}+h)+(f_{2,\eta}-h)$, such that $\||\nabla\times w_2'|\|_{L^2}\leq C_2$ for any fixed $C_2>0$. This establishes the Lemma. □

Now consider any minimizing sequence $\{V^{(n)} = V_h+\nabla\times w^{(n)}\}_{n\in\mathbb{N}}$ of $G(V)$ with $w^{(n)}\in(\dot{W}_0^{\nabla\times,1}(\mathbb{R}^3))^3+(\dot{W}_0^{\nabla\times,2}(\mathbb{R}^3))^3$ satisfying $\||\nabla\times w_2|\|_{L^2}\leq C_2$. The $(\dot{W}_0^{\nabla\times,2}(\mathbb{R}^3))^3$ norm bound on $w_2^{(n)}$ implies weak compactness, so from any minimizing sequence $\{V^{(n)}=V_h+\nabla\times(w_1^{(n)}+w_2^{(n)})\}_{n\in\mathbb{N}}$ we can extract a subsequence $\{V^{(n_j)}\}_{j\in\mathbb{N}}$ such that $(\dot{W}_0^{\nabla\times,2}(\mathbb{R}^3))^3$-weakly we have that $w_2^{(n_k)}\rightharpoonup w_2\in(\dot{W}_0^{\nabla\times,2}(\mathbb{R}^3))^3$ as $k\to\infty$; moreover, $\||\nabla\times w_2|\|_{L^2}\leq C_2$. This reduces the problem to proving weak compactness w.r.t. $(\dot{W}_0^{\nabla\times,1}(\mathbb{R}^3))^3$ of the sequence $\{w_1^{(n)}\}_{n_k\in\mathbb{N}}$ in a minimizing sequence $\{V^{(n)}\}_{n\in\mathbb{N}}$ given by $V^{(n)}=V_h+\nabla\times(w_1^{(n)}+w_2)$, with w_2 denoting a weak limit point of the sequence $\{w_2^{(n)}\}_{n_k\in\mathbb{N}}$ suitably chosen in the original minimizing sequence $\{V^{(n)}\}_{n\in\mathbb{N}}$. The fact that $\{V^{(n)}=V_h+\nabla\times(w_1^{(n)}+w_2)\}_{n_k\in\mathbb{N}}$ is a minimizing sequence follows from (E11) with $V_1=V_h+\nabla\times(w_1^{(n)}+w_2^{(n)})$ and $V_2=\nabla\times(w_2-w_2^{(n)})$, which yields the estimate $|\mathcal{G}(V_1+V_2)-\mathcal{G}(V_1)|\leq C\||\nabla\times(w_2^{(n)}-w_2)|\|_{L^2}$ (cf. (E14)).

We now prove weak compactness w.r.t. $(\dot{W}_0^{\nabla\times,1}(\mathbb{R}^3))^3$ of the sequence $\{w_1^{(n)}\}_{n_k\in\mathbb{N}}$.

First of all, since $w\equiv 0$ is not a minimizer if $N>1$, we have $\inf_V\mathcal{G}(V)<\mathcal{G}(V_h)$. Thus, without loss of generality we have $\mathcal{G}(V_h+\nabla\times w_2+\nabla\times w_1^{(n)})<\mathcal{G}(V_h)$ and $\nabla\times w_2+\nabla\times w_1^{(n)}\not\equiv 0$, for all $n\in\mathbb{N}$.

Second, recalling that for any $V=V_h+\nabla\times w$ with $w\in(\dot{W}_0^{\nabla\times,1}(\mathbb{R}^3))^3+(\dot{W}_0^{\nabla\times,2}(\mathbb{R}^3))^3$ the subset $K_C\subset\mathbb{R}^3$ on which $|V|\geq C>0$ *a.e.* has finite Lebesgue measure, since $\mathcal{G}(V)\geq(\sqrt{1+C^2}-1)|K_C|$, without loss of generality we now choose $C=\sqrt{3}$ and thus may assume that when $N>1$, then for a minimizing sequence

$\{V^{(n)}\}_{n\in\mathbb{N}}$ the size of the domains $K^{(n)}_{\sqrt{3}}$ on which $|V^{(n)}| \geq \sqrt{3}$ *a.e.* is bounded by $|K^{(n)}_{\sqrt{3}}| < \mathcal{G}(V_h)$, uniformly in n.

We also define $\Upsilon^{(n)} := K^{(n)}_{\sqrt{3}} \cup \overline{B_R}$, where $B_R \supset \{s_k\}_{l=1}^N$ is an open ball and $\overline{B_R}$ its closure; then $\mathrm{dist}(\partial B_R, \{s_l\}_{l=1}^N) > 0$. Note that $|\Upsilon^{(n)}| \leq |B_R| + |K^{(n)}_{\sqrt{3}}| < |B_R| + \mathcal{G}(V_h)$ uniformly in n.

Third, we establish the analog of the uniform upper norm bounds obtained in the proof of Proposition 0.6 for the partitioning $\mathbb{R}^3 = \Upsilon^{(n)} \cup \mathbb{R}^3\backslash\Upsilon^{(n)}$. We estimate

$$\mathcal{G}(V^{(n)}) \geq \Big\| |V^{(n)}| \Big\|_{L^1(\Upsilon^{(n)})} - |\Upsilon^{(n)}| + \tfrac{1}{3}\Big\| |V^{(n)}| \Big\|^2_{L^2(\mathbb{R}^3\backslash\Upsilon^{(n)})} \tag{E23}$$

and recall that $\mathcal{G}(V_h) > \mathcal{G}(V^{(n)})$ when $N > 1$.

On the one hand, (E23) implies $\big\||V^{(n)}|\big\|_{L^1(\Upsilon^{(n)})} < \mathcal{G}(V_h) + |\Upsilon^{(n)}|$. But then, since $\big\||V^{(n)}|\big\|_{L^1(\Upsilon^{(n)})} \geq \big\||V^{(n)}|\big\|_{L^1(B_R)} \geq \||\nabla\times w^{(n)}|\|_{L^1(B_R)} - \||V_h|\|_{L^1(B_R)}$, we obtain $\||\nabla\times w^{(n)}|\|_{L^1(B_R)} < \mathcal{G}(V_h) + |\Upsilon^{(n)}| + \||V_h|\|_{L^1(B_R)}$; and since the triangle inequality, followed by a radially-decreasing-rearrangement inequality, gives us $\||V_h|\|_{L^1(B_R)} \leq (4\pi)^{2/3}3^{1/3}(|B_R|)^{1/3}\sum_n |a_n|$, we conclude that (when $N > 1$) $\||\nabla\times w^{(n)}|\|_{L^1(B_R)}$ is bounded above uniformly in n. Also, since $\||\nabla\times w_2|\|_{L^1(B_R)} < \infty$ independently of n, for $w^{(n)} = w_1^{(n)} + w_2$ the triangle inequality now implies that $\||\nabla\times w_1^{(n)}|\|_{L^1(B_R)} < \mathcal{G}(V_h) + |\Upsilon^{(n)}| + \||V_h|\|_{L^1(\Upsilon^{(n)})} + \||\nabla\times w_2|\|_{L^1(B_R)} \leq \overline{C}_1(R)$ uniformly in n.

On the other hand, (E23) implies $\big\||V^{(n)}|\big\|^2_{L^2(\mathbb{R}^3\backslash\Upsilon^{(n)})} < 3\big(\mathcal{G}(V_h) + |\Upsilon^{(n)}|\big)$, which in concert with $\big\||V^{(n)}|\big\|_{L^2(\mathbb{R}^3\backslash\Upsilon^{(n)})} \geq \||\nabla\times w^{(n)}|\|_{L^2(\mathbb{R}^3\backslash\Upsilon^{(n)})} - \|V_h\|_{L^2(\mathbb{R}^3\backslash\Upsilon^{(n)})}$ yields the upper bound $\||\nabla\times w^{(n)}|\|_{L^2(\mathbb{R}^3\backslash\Upsilon^{(n)})} \leq \sqrt{3\big(\mathcal{G}(V_h) + |\Upsilon^{(n)}|\big)} + \|V_h\|_{L^2(\mathbb{R}^3\backslash\Upsilon^{(n)})}$. Since $\|V_h\|_{L^2(\mathbb{R}^3\backslash\Upsilon^{(n)})} \leq 4\pi\sum_k |a_k|/\mathrm{dist}(\partial B_R, \{s_j\}_{j=1}^N)$, and since $\||\nabla\times w_1^{(n)}|\|_{L^2(\mathbb{R}^3\backslash\Upsilon^{(n)})} \leq \||\nabla\times w^{(n)}|\|_{L^2(\mathbb{R}^3\backslash\Upsilon^{(n)})} + \||\nabla\times w_2|\|_{L^2(\mathbb{R}^3\backslash\Upsilon^{(n)})}$ by the triangle inequality, and since $\||\nabla\times w_2|\|_{L^2(\mathbb{R}^3\backslash\Upsilon^{(n)})} \leq C_2$ by Lemma 0.7, we conclude that for $N > 1$, we have $\||\nabla\times w_1^{(n)}|\|_{L^2(\mathbb{R}^3\backslash\Upsilon^{(n)})} \leq \overline{C}_2$ uniformly in n.

Fourth, by the $L^1(B_R)$ bounds a minimizing sequence $\{\big|\nabla\times w_1^{(n)}\big|\}_{n\in\mathbb{N}}$ has a weak* convergent subsequence on every $\overline{B_R}$; by the Lebesgue decomposition theorem the limit is the sum of an $L^1(B_R)$ function f and a measure μ which is singular w.r.t. Lebesgue measure, with $\|f\|_{L^1(B_R)} + \mu(\overline{B_R}) \leq \overline{C}(R)$. Now suppose that $\mu(\overline{B_R}) > 0$; note that $\mathrm{supp}(\mu)$ has Lebesgue measure zero. Then for any open neighborhood $\mathcal{N}_\epsilon$ of $\mathrm{supp}(\mu)$ of size ϵ we have

$$\int_{\mathcal{N}_\epsilon\cap B_R}\Big|V^{(n)}(s)\Big|\mathrm{d}^3s \geq \int_{\mathcal{N}_\epsilon\cap B_R}\Big(\sqrt{1+\big|V^{(n)}(s)\big|^2} - 1\Big)\mathrm{d}^3s \geq \int_{\mathcal{N}_\epsilon\cap B_R}\Big|V^{(n)}(s)\Big|\mathrm{d}^3s - \epsilon. \tag{E24}$$

Taking the limit $n_k \to \infty$ (along the convergent subsequence) and then $\epsilon \to 0$ reveals that the singular part makes an additive contribution $\mu(B_R) > 0$ to $G(V)$. Thus by subtracting the part of the $\nabla\times w_1^{(n)}$ subsequence which converges in absolute value to μ we can lower the value of $G(V)$; hence, $\{\nabla\times w_1^{(n)}\}_{n\in\mathbb{N}}$ was not a minimizing sequence—in contradiction to the hypothesis that it was. Therefore, after extracting a subsequence, we can assume that a minimizing sequence $\{\nabla \times w_1^{(n)}\}_{n\in\mathbb{N}}$ converges weakly in $L^1(B_R)$ to some $\nabla \times w_1$, for each B_R as stipulated. This means that $\{|\nabla \times w_1^{(n)}|\}_{n\in\mathbb{N}}$ converges weakly in $L^1_{\mathrm{loc}}(\mathbb{R}^3)$ to some $|\nabla \times w_1|$, and so $\{\nabla \times w_1^{(n)}\}_{n\in\mathbb{N}}$ converges weakly in $\dot{W}^{\nabla\times,1}_{\mathrm{loc}}(\mathbb{R}^3)^3$ to some $\nabla \times w_1$.

Summarizing so far: any minimizing sequence $\{w^{(n)}\}_{n\in\mathbb{N}}$ of $\mathcal{G}(V_h + \nabla \times w)$ with $w^{(n)} \in (\dot{W}^{\nabla\times,1}_0(\mathbb{R}^3))^3 + (\dot{W}^{\nabla\times,2}_0(\mathbb{R}^3))^3$ has a locally weakly convergent subsequence with limit $w_* \in (\dot{W}^{\nabla\times,1}_{\mathrm{loc}}(\mathbb{R}^3))^3 + (\dot{W}^{\nabla\times,2}_0(\mathbb{R}^3))^3$.

Fifth, by another variation of the reasoning in our proof of Proposition 0.6 we show that $w_* \in (\dot{W}^{\nabla\times,1}_0(\mathbb{R}^3))^3 + (\dot{W}^{\nabla\times,2}_0(\mathbb{R}^3))^3$. Namely, the subset $K^*_{\sqrt{3}} \subset \mathbb{R}^3$ on which $|V_h + \nabla \times w_*| \geq \sqrt{3}$ *a.e.* has finite Lebesgue measure, $|K^*_{\sqrt{3}}| < \mathcal{G}(V_h)$. We also define $\Upsilon^* := K^*_{\sqrt{3}} \cup \overline{B_R}$ and note that $|\Upsilon^*| \leq |B_R| + |K^*_{\sqrt{3}}| < |B_R| + \mathcal{G}(V_h)$. Essentially verbatim to our estimation of the sequence norms we have that $\||\nabla \times w_1|\|_{L^1(\Upsilon^*)} < \mathcal{G}(V_h) + |\Upsilon^*| + \||V_h|\|_{L^1(\Upsilon^*)} + \||\nabla \times w_2|\|_{L^1(\Upsilon^*)}$. By Hölder's inequality, $\||\nabla \times w_2|\|_{L^1(\Upsilon^*)} \leq |\Upsilon^*|^{1/2}\||\nabla \times w_2|\|^{1/2}_{L^2(\Upsilon^*)}$, and we have $\||\nabla \times w_2|\|_{L^2(\Upsilon^*)} \leq \||\nabla \times w_2|\|_{L^2(\mathbb{R}^3)}$. And so $\||\nabla \times w_1|\|_{L^1(\Upsilon^*)} \leq \overline{C}_1$.

On the other hand, we also know that $\||\nabla \times w_1^{(n)}|\|_{L^2(\mathbb{R}^3\setminus\Upsilon^{(n)})} \leq \overline{C}_2$ uniformly in n, and so $\||\nabla \times w_1|\|_{L^2(\mathbb{R}^3\setminus\Upsilon^*)} \leq \overline{C}_2$. Now suppose that $\||\nabla \times w_1|\|_{L^1(\mathbb{R}^3\setminus\Upsilon^*)} = \infty$. But then, since $|\nabla \times w_1| \in (L^2 \cap L^\infty)(\mathbb{R}^3\setminus\Upsilon^*)$, we can subtract the offending part from w_1 and add it to w_2, denoting the new decomposition by $\tilde{w}_1 + \tilde{w}_2$. Indeed, by the chiral Fourier representation we can find a $\nabla \times \tilde{w}$ with chiral Fourier components $\tilde{g}_\eta(k) \in (L^1 \cap L^2)(\mathbb{R}^3)$ such that $\||\nabla \times (w_1 - \tilde{w})|\|_{L^1(\mathbb{R}^3\setminus\Upsilon^*)} < \infty$; since $|\nabla \times \tilde{w}| \in (L^2 \cap L^\infty)(\mathbb{R}^3)$, it follows that $|\nabla \times \tilde{w}| \in L^1(\Upsilon^*)$, and so $\||\nabla \times (w_1 - \tilde{w})|\|_{L^1(\mathbb{R}^3)} < \infty$. Also we obviously have $|\nabla \times (w_2 - \tilde{w})| \in L^2(\mathbb{R}^3)$. Thus, setting $\tilde{w}_1 := w_1 - \tilde{w}$ and $\tilde{w}_2 := w_2 + \tilde{w}$ we have $w_1 + w_2 = \tilde{w}_1 + \tilde{w}$ with $\tilde{w}_1 \in \dot{W}^{\nabla\times,1}_0(\mathbb{R}^3))^3$ and $\tilde{w}_2 \in \dot{W}^{\nabla\times,2}_0(\mathbb{R}^3))^3$. Thus, $w_* \in \dot{W}^{\nabla\times,1}_0(\mathbb{R}^3))^3 + (\dot{W}^{\nabla\times,2}_0(\mathbb{R}^3))^3$, as claimed.

Lastly, by weak lower semi-continuity (Fatou's lemma),

$$U(s) := V_h(s) + \nabla \times w_*(s) \tag{E25}$$

is a minimizer of $\mathcal{G}(V)$. This completes the proof of Theorem 0.1. □

It now follows in the usual way that U given in (E25) is a critical point of $\mathcal{G}(V)$, satisfying the Euler–Lagrange equation

$$\nabla \times \frac{V_h + \nabla \times w_*}{\sqrt{1 + |V_h + \nabla \times w_*|^2}} = 0. \tag{E26}$$

Thus, and applying the Poincaré lemma, we find that *locally* in simply connected domains

$$E_U(s) = \frac{U(s)}{\sqrt{1 + |U(s)|^2}} \equiv -\nabla v_*(s) \tag{E27}$$

is a gradient field. Furthermore, as shown in (E8) for any E_V, we have that

$$-\nabla \cdot \frac{\nabla v_*(s)}{\sqrt{1 - |\nabla v_*(s)|^2}} = 4\pi \sum_{n=1}^{N} a_n \delta_{s_n} \tag{E28}$$

in the sense of distributions; thus, v_* exists globally $a.e.$ and satisfies $\mathcal{F}^{(1)}[v_*](\psi) = 0$ for all $\psi \in C_c^\infty(\mathbb{R}^3)$.

Finally, we show that $v_* = v_\infty$. Namely, we have

$$\mathcal{F}(v_*) \geq \mathcal{F}(v_\infty) \geq -\mathcal{G}(U) = \mathcal{F}(v_*). \tag{E29}$$

The first inequality expresses the fact that v_∞ is a minimizer of $\mathcal{F}(v)$; the second inequality follows from the fact that U is just a special V, and that $\mathcal{F}(v_\infty) \geq -\mathcal{G}(V)$ for all $V = V_h + \nabla \times w$, see (E9); lastly, the equality in (E29) follows from the fact that $U = -\nabla v_*(s)\big/\sqrt{1 - |\nabla v_*(s)|^2}$ satisfies (E28), equivalently v_* satisfies $\mathcal{F}^{(1)}[v_*](\psi) = 0$ for all $\psi \in C_c^\infty(\mathbb{R}^3)$ — more explicitly, inverting the above-stated algebraic relation between U and ∇v_*, we can rewrite $-\mathcal{G}(U)$ as follows (cf. (23)–(26) in [1]):

$$\int_{\mathbb{R}^3}\Big(1 - \sqrt{1 + |U(s)|^2}\Big)\mathrm{d}^3 s = \int_{\mathbb{R}^3}\Big(1 - \sqrt{1 - |\nabla v_*(s)|^2} + U(s)\cdot\nabla v_*(s)\Big)\mathrm{d}^3 s, \tag{E30}$$

and an integration by parts on the last term in the second integral, and using that $\nabla \cdot U = 4\pi\sum_{n=1}^N a_n\delta_{s_n}$ weakly, now yields $-\mathcal{G}(U) = \mathcal{F}(v_*)$. Thus, $\mathcal{F}(v_*) = \mathcal{F}(v_\infty)$, and by the uniqueness of the minimizer of $\mathcal{F}(v)$, we have $v_* = v_\infty$.

The proof that $\mathcal{F}^{(1)}[v_\infty](\psi) = 0$ for all $\psi \in C_c^\infty(\mathbb{R}^3)$ is complete. □

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Appendix A

Proof of (E12). Pick a ball $B_R \supset \{s_n\}_{n=1}^N$. Also let $\Omega_1 := \{|\nabla \times w_1| > 1\}$ and $\Omega_2 := \{|\nabla \times w_2| > 1\}$. Then $\infty > \||\nabla \times w_1|\|_{L^1} \geq \int_{\Omega_1}|\nabla \times w_1| d^3 s \geq |\Omega_1|$, and similarly $|\Omega_2| < \infty$. Let $\Omega = B_R \cup \Omega_1 \cup \Omega_2$; then $|\Omega| < \infty$. Let χ_S be the characteristic function of the set S. Then, by the triangle inequality, $\||E_V|\|_{L^2} \leq \big\||E_V|\chi_\Omega\big\|_{L^2} + \big\||E_V|\chi_{\mathbb{R}^3\setminus\Omega}\big\|_{L^2}$. Using $|E_V| \leq 1$ yields $\big\||E_V|\chi_\Omega\big\|_{L^2} \leq |\Omega|^{1/2}$, while using $|E_V| \leq |V|$ and again the triangle inequality yields

$$\||E_V|\|_{L^2(\mathbb{R}^3\setminus\Omega)} \leq \||V_h|\|_{L^2(\mathbb{R}^3\setminus\Omega)} + \||\nabla \times w_1|\|_{L^2(\mathbb{R}^3\setminus\Omega)} + \||\nabla \times w_2|\|_{L^2(\mathbb{R}^3\setminus\Omega)}. \tag{A1}$$

Next, $|\nabla \times w_1| \leq 1$ on $\mathbb{R}^3\setminus\Omega$ yields $\||\nabla \times w_1|\|_{L^2(\mathbb{R}^3\setminus\Omega)} \leq \||\nabla \times w_1|\|_{L^1(\mathbb{R}^3\setminus\Omega)}^{1/2} \leq \||\nabla \times w_1|\|_{L^1}^{1/2} < \infty$. Together with $\||\nabla \times w_2|\|_{L^2(\mathbb{R}^3\setminus\Omega)} \leq \||\nabla \times w_2|\|_{L^2} < \infty$ we obtain (E12). □

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