Commun. Math. Phys. 362, 609–657 (2018)
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-018-3186-9

Communications in Mathematical Physics

On Ergodicity of Foliations on $\mathbb{Z}^d$-Covers of Half-Translation Surfaces and Some Applications to Periodic Systems of Eaton Lenses

Krzysztof Frączek[1], **Martin Schmoll**[2]

[1] Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, ul. Chopina 12/18, 87-100 Toruń, Poland. E-mail: fraczek@mat.umk.pl
[2] Department of Mathematical Sciences, Clemson University, Clemson SC, 29634, USA. E-mail: schmoll@clemson.edu

Received: 23 September 2017 / Accepted: 2 June 2018
Published online: 2 July 2018 –

Abstract: We consider the geodesic flow defined by periodic Eaton lens patterns in the plane and discover ergodic ones among those. The ergodicity result on Eaton lenses is derived from a result for quadratic differentials on the plane that are pull backs of quadratic differentials on tori. Ergodicity itself is concluded for $\mathbb{Z}^d$-covers of quadratic differentials on compact surfaces with vanishing Lyapunov exponents.

1. Introduction

1.1. Periodic Eaton lens distributions in the plane. An *Eaton lens* is a circular lens on the plane $\mathbb{R}^2$ which acts as a perfect retroreflector, i.e. so that each ray of light after passing through the Eaton lens is directed back toward its source, see Fig. 1. More precisely, if an Eaton lens is of radius $R > 0$, then the refractive index (RI for short) inside the lens depends only on the distance from the center r and is given by the formula $n(x, y) = n(r) = \sqrt{2R/r - 1}$. The refractive index $n(x, y)$ is constant and equals 1 outside the lens.

In this paper we consider dynamics of light rays in periodic Eaton lens distributions in the plane $\mathbb{R}^2 \cong \mathbb{C}$. As a simple example take a lattice $\Lambda \subset \mathbb{R}^2$ and consider an Eaton lens of radius $R > 0$ centered at each lattice point of Λ. This configuration of lenses will be denoted by $L(\Lambda, R)$

Let us call an Eaton lens distribution, say $\mathcal{L}$, in $\mathbb{R}^2$ admissible, if no pair of lenses intersects. For every admissible Eaton lens configuration $\mathcal{L}$, the dynamics of the light rays can be considered as a geodesic flow $(\mathfrak{g}_t^{\mathcal{L}})_{t\in\mathbb{R}}$ on the unit tangent bundle of $\mathbb{R}^2$ with lens centers removed, see Section A for details. The Riemannian metric inducing the flow is given by $g_{(x,y)} = n(x, y) \cdot (dx \otimes dx + dy \otimes dy)$, where $n(x, y)$ is the refractive index at (x, y).

K. Frączek: The first author was partially supported by the Narodowe Centrum Nauki Grant 2014/13/B/ST1/03153.

M. Schmoll: The second author was partially supported by Simons Collaboration Grant 318898.

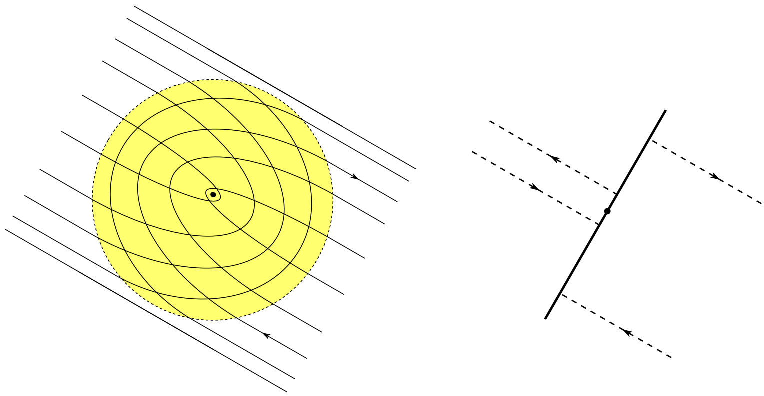

Fig. 1. Light rays passing through an Eaton lens and its flat counterpart

Since each Eaton lens in $\mathcal{L}$ acts as a perfect retroreflector, for any given slope $\theta \in \mathbb{R}/\pi\mathbb{Z}$ there is an invariant set $\mathscr{P}_{\mathcal{L},\theta}$ in the unit tangent bundle, such that all trajectories on $\mathscr{P}_{\mathcal{L},\theta}$ have direction θ or $\theta+\pi$ outside the lenses. The restriction of the geodesic flow $(\mathfrak{g}_t^{\mathcal{L}})_{t\in\mathbb{R}}$ to $\mathscr{P}_{\mathcal{L},\theta}$ will be denoted by $(\mathfrak{g}_t^{\mathcal{L},\theta})_{t\in\mathbb{R}}$. Moreover, $(\mathfrak{g}_t^{\mathcal{L},\theta})_{t\in\mathbb{R}}$ possesses a natural invariant infinite measure $\mu_{\mathcal{L},\theta}$ equivalent to the Lebesgue measure on $\mathscr{P}_{\mathcal{L},\theta}$, see Appendix A for details. With respect to this setting we consider measure-theoretic questions. Denote by $\pi_{\mathcal{L},\theta} : \mathscr{P}_{\mathcal{L},\theta} \to \mathbb{C}$ the map associating to a unit tangent vector in $\mathscr{P}_{S,\theta}$ its footpoint (in $\mathbb{C}$).

For example, in [17], the authors have shown, that simple periodic Eaton lens configurations, for example $L(\Lambda, R)$, have the opposite behavior of *ergodicity*. More precisely, a light ray in an Eaton lens configuration is called *trapped*, if the ray never leaves a strip parallel to a line in $\mathbb{R}^2$. The trapping phenomenon observed in [17] was extended in [16] to the following result:

Theorem 1.1. *If $L(\Lambda, R)$ is an admissible configuration then for a.e. direction $\theta \in \mathbb{R}/\pi\mathbb{Z}$ there exist constants $C = C(\Lambda, R, \theta) > 0$ and $v = v(\Lambda, R, \theta) \in \mathbb{R}/\pi\mathbb{Z}$, such that the $\pi_{L(\Lambda,R),\theta}$-image of every geodesic orbit in $\mathscr{P}_{L(\Lambda,R),\theta}$ is trapped in an infinite band of width $C > 0$ in direction v.*

We say that a flow $(\mathfrak{g}_t^{\mathcal{L},\theta})_{t\in\mathbb{R}}$ is trapped if there exist constants $C = C(\mathcal{L}, \theta) > 0$ and $v = v(\mathcal{L}, \theta) \in \mathbb{R}/\pi\mathbb{Z}$ such that the $\pi_{\mathcal{L},\theta}$-image of every orbit is trapped in an infinite band of width $C > 0$ in direction v.

Knieper and Glasmachers [18,19] have trapping results for geodesic flows on Riemannian planes. Among other things, Theorem 2.4 in [19] says, that for all Riemann metrics on the plane that are pull backs of Riemann metrics on a torus with vanishing topological entropy, the geodesics are trapped. Nevertheless, the trapping phenomena obtained in [16–19] have different flavors. The former is transient whereas the latter is recurrent.

Let us further mention that Artigiani describes a set of exceptional triples (Λ, R, θ) for which the flow $(\mathfrak{g}_t^{L(\Lambda,R),\theta})_{t\in\mathbb{R}}$ is ergodic in [2].

In this paper we investigate ergodicity and trapping for more complicated periodic Eaton lens distributions. In fact, given a lattice $\Lambda \subset \mathbb{C}$ let us denote a Λ-periodic distribution of k Eaton lenses with center $c_i \in \mathbb{C}$ and radius $r_i \geq 0$ for $i = 1, \ldots, k$ by

$L(\Lambda, c_1, \ldots, c_k, r_1, \ldots, r_k)$. Of course, we will only consider admissible configurations. If the list of Eaton lenses has centrally symmetric pairs, we write $\pm c_i$ for their centers and list their common radius only once. We adopt the convention that if the radius of a lens is zero then this lens disappears.

For a random choice of admissible parameters in this family of configurations in Sect. 5 we prove trapping.

Theorem 1.2. *For every lattice $\Lambda \subset \mathbb{C}$, every vector of centers $\overline{c} \in \mathbb{C}^k$ and almost every $\overline{r} \in \mathbb{R}^k_{>0}$ such that $L(\Lambda, \overline{c}, \overline{r})$ is admissible the geodesic flow $(\mathfrak{g}^{L(\Lambda,\overline{c},\overline{r}),\theta})_{t\in\mathbb{R}}$ is trapped for a.e. $\theta \in \mathbb{R}/\pi\mathbb{Z}$.*

An admissible ergodic Eaton lens configuration in the plane. As a consequence we have that the set of parameters $(\Lambda, \overline{c}, \overline{r}, \theta)$ for which $(\mathfrak{g}_t^{L(\Lambda,\overline{c},\overline{r}),\theta})_{t\in\mathbb{R}}$ is ergodic is very rare. Despite this, in this paper, we find exceptional one-dimensional ergodic sets (piecewise smooth curves) of parameters such that a random choice inside such a curve provides an ergodic behavior of light rays. In fact the configurations we found are curves

$$\theta \longmapsto L(\Lambda_\theta, c_1(\theta), \ldots, c_k(\theta), r_1(\theta), \ldots, r_k(\theta))$$

parameterized with the angle $\theta \in \mathbb{R}/\pi\mathbb{Z}$ such that $\theta \mapsto \Lambda_\theta$ is constant or varies in a diagonal way, i.e. $\Lambda_\theta = diag(a(\theta), b(\theta))\Lambda$. We should stress that such ergodic curves do not exist when $k = 1$. More precisely, from results of [16] one can conclude:

Theorem 1.3. *Suppose that $\theta \longmapsto L(\Lambda_\theta, c(\theta), r(\theta))$ is a C^2-curve such that $\Lambda_\theta = diag(a(\theta), b(\theta))\Lambda$ for a lattice Λ and two positive C^2-maps a and b. Then for a.e. θ the geodesic flow $(\mathfrak{g}^{L(\Lambda_\theta,c(\theta),r(\theta)),\theta})_{t\in\mathbb{R}}$ is trapped.*

We now describe the simplest curve with ergodic directions. It is a loop defined for every angle $\theta \in [0, \pi]$. To start, we take the function

$$l(\theta) := 2 - |\cot\theta|(1 - |\cot\theta|)$$

and consider the curve of lattices

$$\Lambda_\theta = \mathbb{Z}(0, 4) \oplus \mathbb{Z}(4, 2) \text{ for } \theta \bmod \pi \in [-\pi/4, \pi/4]$$

continued by

$$\Lambda_\theta = \mathbb{Z}(0, 4) \oplus \mathbb{Z}(2l(\theta), 2) \text{ for } \theta \bmod \pi \in [\pi/4, 3\pi/4].$$

Both families of lattices agree on the respective boundaries of their defining intervals and so we obtain a continuous loop of lattices since $\Lambda_\pi = \Lambda_0$. Next define the curve $\theta \mapsto \gamma_W(\theta)$ of admissible Eaton lens configurations for every $\theta \in \mathbb{R}/\pi\mathbb{Z}$ as follows:

$$\gamma_W(\theta) = \begin{cases} L\left(\Lambda_\theta, (0,0), \pm(1, 1+\tan\theta), 2\sin\theta, \cos\theta\right) & \text{if } \theta \bmod \pi \in [0, \pi/4] \\ L(\Lambda_\theta, (0,0), \pm(\cot\theta, 2), l(\theta)\sin\theta, \cos\theta) & \text{if } \theta \bmod \pi \in [\pi/4, \pi/2] \\ L(\Lambda_\theta, (0,0), \pm(\cot\theta, 2), l(\theta)\sin\theta, -\cos\theta) & \text{if } \theta \bmod \pi \in [\pi/2, 3\pi/4] \\ L\left(\Lambda_\theta, (0,0), \pm(-1, 1-\tan\theta), 2\sin\theta, -\cos\theta\right) & \text{if } \theta \bmod \pi \in [3\pi/4, \pi] \end{cases}$$

We want to assume, that two Eaton lens configurations in the plane are the same, if they differ by a translation. After all, that is equivalent to a translation of the origin, preserving dynamical properties. Then the curve of Eaton lens distribution closes, since

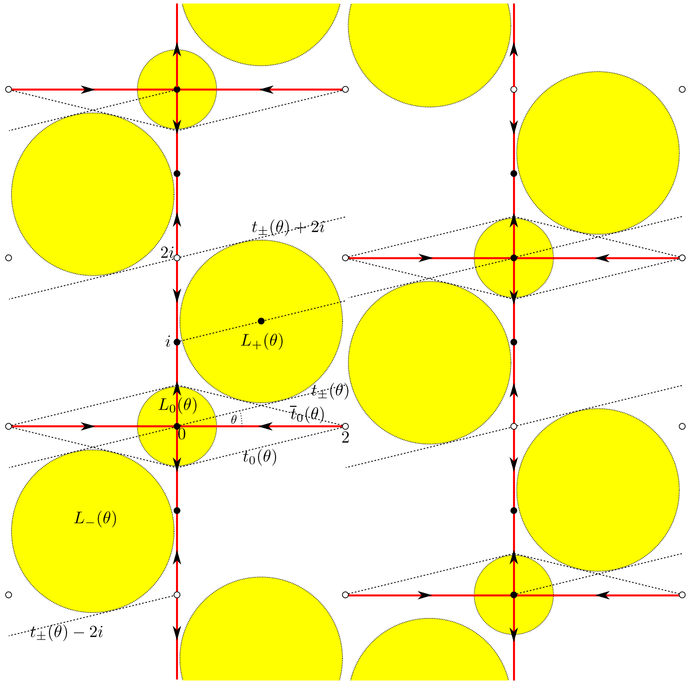

Fig. 2. Ergodic curve for angles $|\theta| \leq \pi/4$ and $|\theta - \pi| \leq \pi/4$

$\gamma_W(0) = \gamma_W(\pi) + (0, 2)$. The admissibility of all Eaton lens configurations in the image of γ_W is shown in Sect. 2.1. To give a geometric outline of the lens configurations we add a cartoon showing the configurations at representative angles in the interval $[0, \pi/4]$ (Fig. 2) and $[\pi/4, \pi/2]$ (Fig. 3).

In "Appendix A" we show that each Eaton lens can be replaced by a flat lens (slit-fold) whose ends are located on straight lines in direction θ tangents to the lens. Note that $\gamma_W(\theta)$ arises as Λ_θ-lattice of three lenses: one is central and other two are symmetric with respect to the center. After the replacement of every central lens by its flat horizontal counterpart (with the same center), and the replacement of the symmetric lens couples by their vertical flat counterparts hooked in the center, we obtain a Λ_θ-periodic system of slit-folds. Moreover, after a horizontal rescaling, for $\theta \in (\pi/4, 3\pi/4)$, we have the same (independent of θ) $\Lambda = \mathbb{Z}(0, 4) \oplus \mathbb{Z}(4, 2)$-periodic system of slit-folds on $\mathbb{C}$, denoted in the next sections by $\widehat{X}_3$, which can be treated as a Λ-periodic quadratic differential on $\mathbb{C}$. Then for every $\theta \in \mathbb{R}/\pi\mathbb{Z}$ the study of the behavior of geodesic orbits on $\mathscr{P}_{\gamma_W(\theta),\theta}$ comes down to the study of a directional measured foliation on $\widehat{X}_3$. Passing to the quotient quadratic differential $X_3 := \widehat{X}_3/\Lambda$ we have a half-translation torus whose orientation cover is the famous translation surface called *Eierlegende Wollmilchsau*. Since the Eierlegende Wollmilchsau is a square tiled surface, its directional flow in every direction with rational slope is not ergodic. Coming back to systems of lenses, it follows that there exists a dense set of directions θ in $\mathbb{R}/\pi\mathbb{Z}$, such that the geodesic flow $(\mathfrak{g}_t^{\gamma_W(\theta),\theta})_{t\in\mathbb{R}}$ is not ergodic.

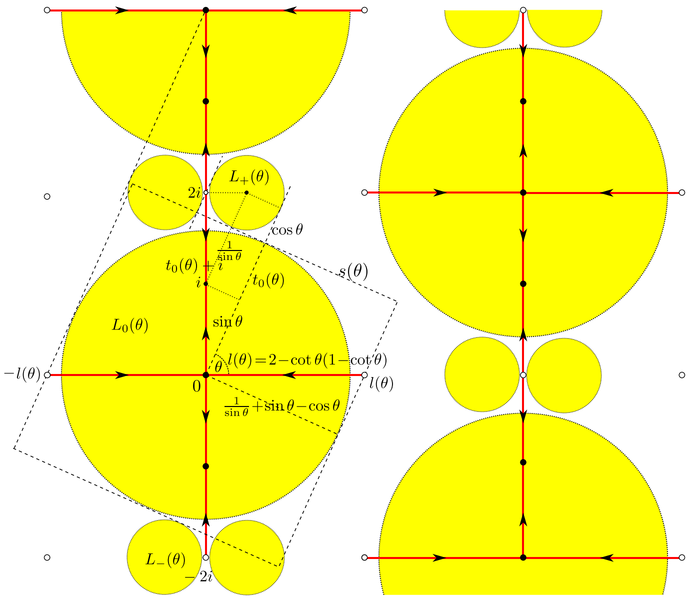

Fig. 3. Ergodic curve for the angles $|\theta \pm \pi/2| \leq \pi/4$

Theorem 1.4. *For almost every $\theta \in \mathbb{R}/\pi\mathbb{Z}$ the geodesic flow $(\mathfrak{g}_t^{\gamma_W(\theta),\theta})_{t\in\mathbb{R}}$ is ergodic.*

Part of the paper shows several curves of ergodic Eaton lens configurations in the plane, see Figs. 28, 29 and 30. For some of those curves we describe admissible Eaton lens configurations only for an interval of slopes in $\mathbb{R}/\pi\mathbb{Z}$.

Reduction to quadratic differentials and cyclic pillow case covers. The dynamical results for periodic Eaton lens distributions in the plane rely on the equivalence of the Eaton dynamics in a fixed direction, say θ, to the (dynamics on a) direction foliation $\mathcal{F}_\theta(q)$ of a quadratic differential (half-translation structure) q on the plane. Starting from a (slit-fold) quadratic differential, the connection is made by replacing a *slit-fold*, as shown in Fig. 4, by an Eaton lens. For a given direction the dynamical equivalence of a slit-fold and an Eaton lens is motivated by Fig. 1. This equivalence is described in detail in Appendix A. We distinguish two objects: A *flat lens* is a one-dimensional replacement of an Eaton lens perpendicular to the light direction, that does not change the future and the past of the light in the complement of the Eaton lens that is replaced, see Fig. 1. A *slit-fold* on the other hand is a flat lens in the language of quadratic differentials. In fact, a slit-fold is constructed by removing a line segment, say $[a, b]$ with $a, b \in \mathbb{C}$, from the plane (or any flat surface), then a closure is taken so that the removed segment is replaced by two parallel and disjoint segments. Then for each segment one identifies those pairs of points, that have equal distance from the segments center point. Once this is done we obtain a slit-fold that we denote by $\rangle a, b\langle$ on the given surface, see Fig. 4. The single slit-fold $\rangle a, b\langle$ defines a quadratic differential on the plane with two singular points located on the (doubled) centers of the segment and a zero at its (identified) endpoints. Alternatively that quadratic differential on the plane is obtained as quotient of the abelian differential defined by gluing two copies of the slit plane $\mathbb{C}\backslash[a, b]$ crosswise

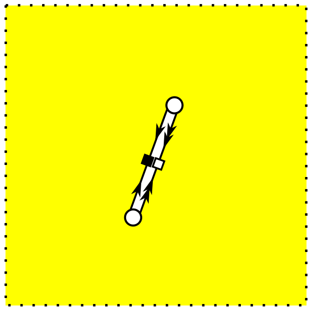

Fig. 4. A slit-fold

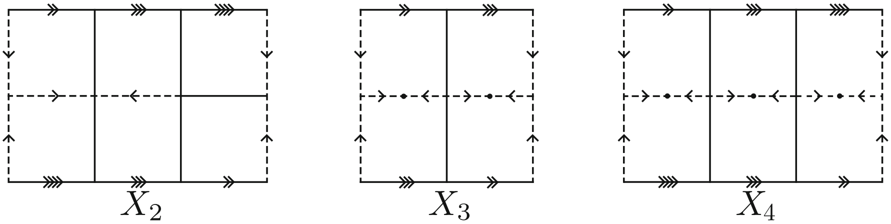

Fig. 5. Torus quadratic differentials X_2, X_3 and X_4; slit-folds are marked with a dashed line

along its strands. Then a quotient is taken with respect to the sheet exchange map that lifts the rotation by π around the center point of $[a, b]$. By adding slit-folds we can construct a variety of quadratic differentials starting from any flat surface.

For fixed $k \in \mathbb{N}$ the set $\mathcal{S}_k$ of quadratic differentials made of k disjoint slit-folds is a subset of $\mathcal{Q}((-1)^{2k}, 2^k)$, the vector space of genus one quadratic differentials that have $2k$ singular points and k cone points of order 2. Disjointness of slit-folds means, cone points of different slit-folds do not fall together. We need the superset $\overline{\mathcal{S}}_k \supset \mathcal{S}_k$ of those quadratic differentials that are made of exactly k slit-folds, including the ones with merged cone points. Let us consider the three particular half-translation surfaces $X_2 \in \overline{\mathcal{S}}_2$, $X_3 \in \overline{\mathcal{S}}_3$ and $X_4 \in \overline{\mathcal{S}}_4$ drawn on Fig. 5.

Theorem 1.5. *Let $X = X_k$ for $k = 2, 3, 4$ and denote by $\widetilde{X}$ its universal cover, a quadratic differential on the plane. Then for almost every $\theta \in \mathbb{R}/\pi\mathbb{Z}$ the foliation in direction θ on $\widetilde{X}$ is ergodic.*

Those ergodic foliations on the plane can be converted into ergodic curves of admissible Eaton lens distributions.

The ergodicity of universal covers of quadratic surfaces in S_k on the other hand is rather exceptional. If $X \in S_k$ satisfies a separation condition on slit-folds (which is an open condition) then the foliation in direction θ on $\widetilde{X}$ is trapped for a.e. $\theta \in \mathbb{R}/\pi\mathbb{Z}$, see Corollary 5.6 for details.

The following more general ergodicity result supplies the key to the proof of Theorems 1.5 and 1.4.

Theorem 1.6. *Let (X, q) be a quadratic differential on a compact, connected surface such that all Lyapunov exponents of the Kontsevich–Zorich cocycle of (X, q) are zero. Then for every connected unbranched $\mathbb{Z}^d$-cover $(\widetilde{X}, \widetilde{q})$, almost every directional foliation on $(\widetilde{X}, \widetilde{q})$ is ergodic.*

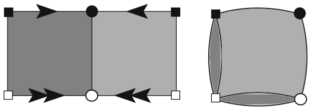

Fig. 6. The pillowcase quadratic differential in polygonal representation

Note, that a $\mathbb{Z}^d$-cover is an abelian cover and so $d \le 2g_X$, g_X being the genus of X. This result is in fact a consequence of the more general Theorem 4.6 that provides a criterion on ergodicity for translation flows on $\mathbb{Z}^d$-covers of compact translation surfaces. We would like to mention that a similar result was obtained independently by Avila, Delecroix, Hubert and Matheus but it was never published (communicated by Pascal Hubert). Some related research was also recently done by Hooper, Hubert and Weiss who studied ergodicity of directional flows on translation surfaces with infinite area, see e.g. [22] and [23].

2. Ergodic Slit-Fold Configurations on Planes by Cyclic Pillowcase Covers

In this section we outline the strategy to construct the ergodic quadratic differentials on the plane assuming the validity of Theorem 1.6. Theorem 1.6 reduces the problem of ergodicity from cyclic quadratic differentials in the plane to quadratic differentials $(\mathcal{T}, q)$ on the torus $\mathcal{T}$ whose all Lyapunov exponents are zero. A recent criterion of Grivaux and Hubert [20] implies that a cyclic cover of the *pillowcase* has all Lyapunov exponents zero, if it is branched at (exactly) three singular points. Now it turns out that there is a only a short list of those branched cyclic covers $\mathcal{T} \to \mathcal{P}$. Recall, the *pillowcase* $\mathcal{P}$ is a quadratic differential $q_{\mathcal{P}}$ on the sphere S^2. To characterize it, consider the quadratic differential dz^2 on $\mathbb{R}^2 \cong \mathbb{C}$. This differential is invariant under translations and the central reflection $-id : \mathbb{R}^2 \to \mathbb{R}^2$. Thus it descends to the torus $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ defining a quadratic differential invariant under the hyperelliptic involution $\varphi : \mathbb{T}^2 \to \mathbb{T}^2$ induced by the central reflection of $\mathbb{R}^2$. So it further descends to a quadratic differential $q_{\mathcal{P}}$ on the quotient sphere $S^2 = \mathbb{T}^2/\varphi$. The pillowcase is the pair $\mathcal{P} = (S^2, q_{\mathcal{P}})$, see Fig. 6. Putting the result from [20] on cyclic pillowcase covers and Theorem 1.6 together, one has:

Corollary 2.1. *Let $\pi : X \to \mathcal{P}$ be a finite cyclic cover branched over three of the singular points of $\mathcal{P}$ and let $q = \pi^* q_{\mathcal{P}}$ be the pull back quadratic differential to X. If $(\widetilde{X}, \widetilde{q}) \to (X, q)$ is a connected $\mathbb{Z}^d$-cover with $d \le 2g_X$, then almost every directional foliation on $(\widetilde{X}, \widetilde{q})$ is ergodic.*

We now give a list of relevant pillow-case covers:

Proposition 2.2. *Up to the action of $SL_2(\mathbb{Z})$ on covers and up to isomorphism of covers, there are three cyclic covers $(\mathbb{T}, q) \to \mathcal{P}$ that are branched over exactly three cone points of $q_{\mathcal{P}}$. The degree of each such cover is* 3, 4 *or* 6.

The proof of Proposition 2.2 is the content of Sects. 3.2, 3.3 and 3.4.

Figure 7 shows polygonal one strip representations of one cyclic pillowcase cover in each degree. We note that the quadratic differential on the degree 3 cover has the *Ornithorynque* (see [11] for the description of the surface) as its orientation cover and the quadratic differential on the degree 4 cover has the *Eierlegende Wollmilchsau* (see

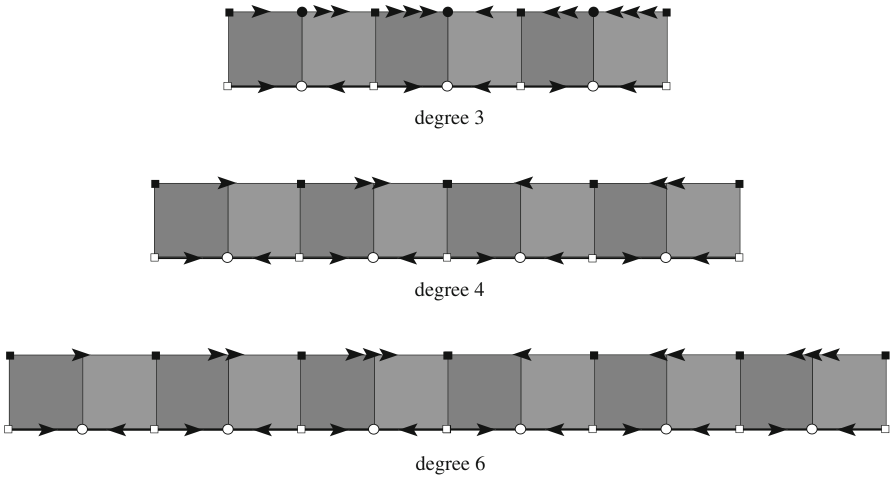

Fig. 7. Torus differentials with zero Lyapunov exponents

also [11]) as its orientation cover. There are questions particular to the conversion of a quadratic differential into an admissible Eaton lens distribution in the plane. For example to convert the torus differentials from Proposition 2.2 to Eaton lens distributions one needs a cover that is a slit-fold differential in the plane. We do this below for the Eaton curve presented in the introduction. The construction of some other curves need more sophisticated geometric arguments that can be found in "Appendix B".

Eaton differentials and skeletons. For a fixed direction the (long term) Eaton lens dynamics on the plane or on a torus is equivalent to the dynamics on a particular slit-fold, so we call a quadratic differential that is given by a union of slit-folds a *pre-Eaton differential*. Since the radius of an Eaton lens replacing a slit-fold depends on the angle between the light ray and the slit-fold, a light direction needs to be specified. Recall that a configuration of Eaton lenses is *admissible*, if no pair of Eaton lenses intersects. A pre-Eaton differential q is called an *Eaton differential*, if there is a nonempty open interval $I \subset \mathbb{R}$ such that for every (light) direction $\theta \in I \mod \pi$ the translation flow on the orientation cover of q in direction θ is measure equivalent (here flows are treated as measured foliations) to the geodesic flow of an admissible Eaton lens configuration, whose lens centers and radii depend continuously on $\theta \in I$. We further call an Eaton differential *maximal*, if $I \to \mathbb{R}/\pi\mathbb{Z}$, $x \mapsto x \mod \pi$ is onto. Finally let us call a (pre-)Eaton differential *ergodic*, if its direction foliations are ergodic in almost every direction. Note, that a pre-Eaton differential must be located on a torus, or a plane, since it has no singular points besides the ones of its slit-folds. So it is enough to present a pre-Eaton differential by a union of slit-folds, that we will call *skeleton*. Below we introduce and use geometric as well as algebraic presentations of skeletons.

Note that, in view of the celebrated Kerckhoff–Masur–Smillie [24] result, every pre-Eaton differential on a torus is ergodic, even uniquely ergodic. The ergodicity problem is more subtle when we pass to Λ-periodic (Λ a lattice) pre-Eaton differentials on the plane. Then ergodicity depends on the values of Lyapunov exponents of the quadratic differential on the quotient torus $\mathbb{C}/\Lambda$. Theorem 1.6 and Proposition 5.3 show that the

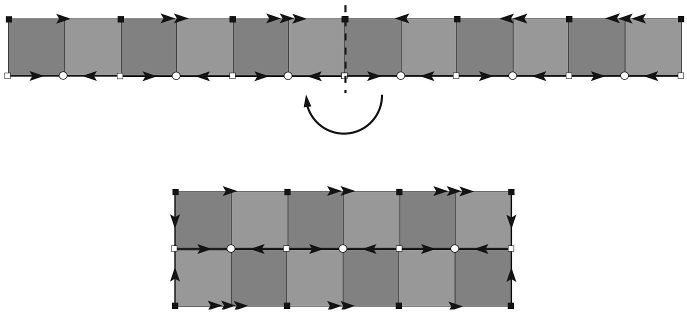

Fig. 8. Cutting and turning polygonal pieces gives a pure slit-fold representation modulo absolute homology

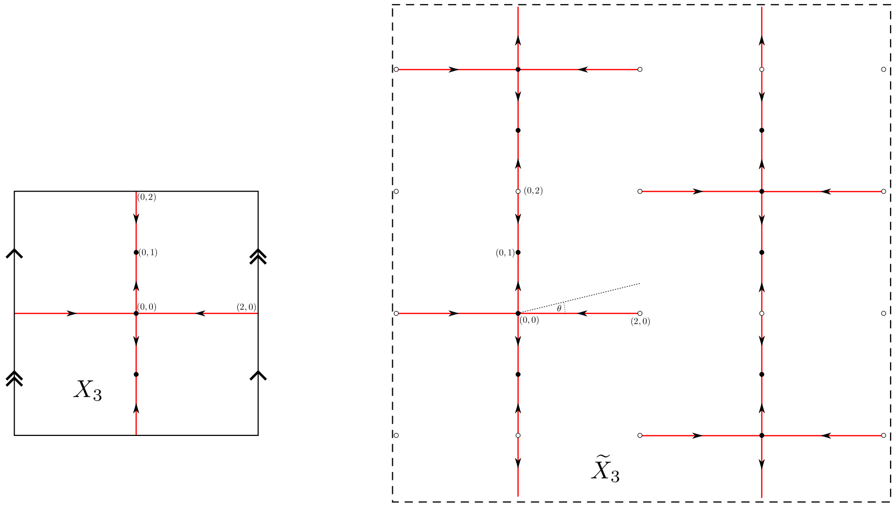

Fig. 9. The quadratic surface X_3 (rotated by $\pi/2$) and its universal cover $\widetilde{X}_3$ (Wollmilchsau differential)

pre-Eaton differential is ergodic if and only if the top Lyapunov exponent of the quotient quadratic differential is zero.

Proof of Theorem 1.5. Pre-Eaton differentials are obtained from all three torus differentials in Fig. 7, by first cutting vertically through their center and then rotating either one of the halves underneath the other, as in Fig. 8. Up to rescaling the resulting pre-Eaton differentials are X_2 (from the degree 3 cover), X_3 (from the degree 4 cover) and X_4 (from the degree 6 cover) as shown in Fig. 5. It follows, that X_2, X_3 and X_4 are cyclic covers of the pillowcase and branched over exactly three singularities of $\mathcal{P}$. Passing to their universal covers we obtain three pre-Eaton differentials $\widetilde{X}_2$, $\widetilde{X}_3$, $\widetilde{X}_4$ on the plane. In view of Corollary 2.1 almost every directional foliation for every such differential is ergodic. □

Below we call the quadratic differential $\widetilde{X}_3$ on the complex plane obtained from the degree 4 pillowcase cover X_3 the *Wollmilchsau differential*, see Fig. 9.

Theorem 2.3. *The Wollmilchsau differential is an ergodic, maximal Eaton differential.*

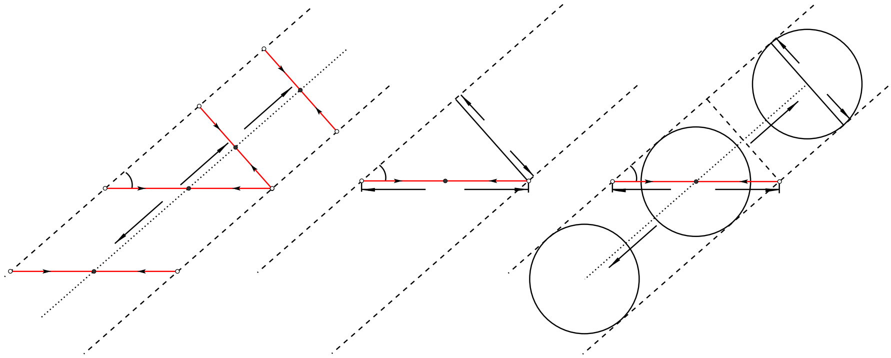

Fig. 10. Railed moves of slit-folds and Eaton lenses in direction θ

Ergodicity follows because Theorem 1.5 applies. To show the other statements of the Theorem we need to describe an Eaton lens configuration depending continuously on $\theta \in \mathbb{R}/\pi\mathbb{Z}$ and show that it is admissible. This is done in Proposition 2.4, see the comment after that.

Eaton lenses may overlap when placed at slit-fold centers. To resolve this problem we deform the measured foliation tangential to its direction $\theta \in \mathbb{R}/\pi\mathbb{Z}$ to a measure equivalent foliation by moving slit-folds parallel to θ. More precisely take a direction foliation $\mathcal{F}_\theta(q)$ of a quadratic differential q that contains a slit-fold. Then changing the location of the slit-fold while keeping its endpoints (and therefore its center points) on the same leaves of $\mathcal{F}_\theta(q)$ is called a *railed motion*. Changing a slit-fold skeleton using railed motions is called a *railed deformation*. In terms of Teichmüller Theory railed deformations are isotopies, or *Whitehead moves* that preserve the transverse measure of a measured foliation. In particular, two measured foliations that differ by railed deformations are Whitehead equivalent. A Whitehead move is a deformation of a foliated surface that collapses a leaf connecting two singular points, or it is the inverse of such a deformation, see [27, page 116]. Figure 11 shows railed deformations deforming skeletons into disjoint slit-folds. Each of those consists of several Whitehead moves. Some railed motions are shown in Fig. 10 to the left. After performing a railed deformation, appropriately sized Eaton lenses are placed at the slit-fold centers.

2.1. The Eaton lens configurations along γ_W are admissible. The following result together with Theorem 1.5 gives the proof of Theorem 1.4.

Proposition 2.4. *The Eaton lens configurations defined by $\gamma_W(\theta)$ are admissible and for all $\theta \in [0,\pi]$ the ergodicity of the geodesic flow $(\mathfrak{g}_t^{\gamma_W(\theta),\theta})_{t\in\mathbb{R}}$ is equivalent to the ergodicity of the directional foliation generated by the Wollmilchsau differential $\widetilde{X}_3$ in direction θ' such that*

$$\theta' = \begin{cases} \theta & \textit{if } \theta \bmod \pi \in [-\pi/4, \pi/4] \\ \operatorname{arccot}\Big(\cot\theta\,\Big(1 - \frac{|\cot\theta|(1-|\cot(\theta)|)}{2}\Big)\Big) & \textit{if } \theta \bmod \pi \in [\pi/4, 3\pi/4]. \end{cases}$$

Remark 2.5. Proposition 2.4 says that $\widetilde{X}_3$ is a maximal Eaton differential. We believe that every periodic pre-Eaton differential on the plane is an Eaton differential for all intervals of a finite partition of $\mathbb{R}/\pi\mathbb{Z}$. A possible general construction of the corresponding Eaton systems is probably very technical and difficult to describe. Instead, in "Appendix B" we present (in drawings) examples of such construction for the pre-Eaton differentials $\widetilde{X}_2$ and $\widetilde{X}_4$.

Proof of Proposition 2.4. For this proof we will use complex coordinates on the plane. Let us consider the situation for light directions $\theta \in [0, \pi/4]$ first. For those angles the Eaton lens configurations are periodic with respect to the lattice $\Lambda := \mathbb{Z}4i \oplus \mathbb{Z}(4 + 2i)$. Therefore it is enough to show that Eaton lenses centered inside the strip $S = \{z \in \mathbb{C}; |\Re z| \leq 2\}$ are pairwise disjoint and do not leave the strip, i.e. do not cross the boundary of the strip.

Modulo the action of Λ there are three Eaton lenses on $\gamma_W(\theta)$. The first one $L_0(\theta)$ has radius $r_0(\theta) = 2\sin\theta$ and is centered at the origin. Then there is a pair of lenses denoted by $L_\pm(\theta)$ centered at $c_\pm(\theta) = \pm(1+i(1+\tan\theta))$, both of radius $r_\pm(\theta) = \cos\theta$, see Fig. 2. Since the radius of the Eaton lenses $L_\pm(\theta)$ is less then 1 and the radius of $L_0(\theta)$ is bounded by 2, the lenses in the Λ orbit of any one of those three Eaton lenses are pairwise disjoint. For the same reason the $\mathbb{Z}4i$ orbit of all three Eaton lenses lies in the strip S.

The line in direction θ through the point i contains the center of $L_+(\theta)$ since its slope is $\tan\theta$. The distance of that line to its parallel through the origin, denoted by $t_\pm(\theta)$, is $\cos\theta$, equaling the radius of $L_+(\theta)$. So the lines $t_\pm(\theta)$ and $t_\pm(\theta) + 2i$ are tangent to $L_+(\theta)$. Then by central symmetry the lines $t_\pm(\theta)$ and $t_\pm(\theta) - 2i$ are tangents to $L_-(\theta)$. It follows that $L_+(\theta) + 4ni$ lies between the lines $t_\pm(\theta) + 4ni$ and $t_\pm(\theta) + (4n + 2)i$ and $L_-(\theta) + 4ni$ lies between the lines $t_\pm(\theta) + (4n - 2)i$ and $t_\pm(\theta) + 4ni$ for every $n \in \mathbb{Z}$. Therefore, no pair of Eaton lenses in the $\mathbb{Z}4i$ orbits of $L_\pm(\theta)$ intersect. Since the $\mathbb{Z}(4 + 2i)$ translates of S cover the whole plane, intersecting only in their boundary lines, we conclude that no pair of Eaton lenses in the Λ orbits of $L_\pm(\theta)$ intersect.

Since $L_0(\theta)$, the lens in the origin, has radius $2\sin\theta$ the line in direction θ through 2, denoted by $t_0(\theta)$, is tangent to it. By reflection symmetry with respect to the horizontal axis, the line through 2 in direction $\pi - \theta$ is also a tangent to $L_0(\theta)$. Let us denote this (tangent-)line by $\bar{t}_0(\theta)$, we shall see it is also tangent to $L_+(\theta)$. Indeed, the reflection of $\bar{t}_0(\theta)$ with respect to the vertical through the center of $L_+(\theta)$ is the tangent $t_\pm(\theta)$. Since the centers of $L_+(\theta)$ and $L_0(\theta)$ lie on different sides of their common tangent $\bar{t}_0(\theta)$ these lenses do not intersect. By central symmetry the same is true for $L_-(\theta)$ and $L_0(\theta)$. Since all three lenses $L_\pm(\theta)$ and $L_0(\theta)$ in the parallelogram in S bounded by $t_\pm(\theta) \pm 2i$ are disjoint and these parallelograms have a (modulo boundary) disjoint Λ orbit, we conclude that the lens distribution given by $\gamma_W(\theta)$ is disjoint for all $\theta \in [0, \pi/4]$.

For the same interval of angles the geodesic flow $(\mathfrak{g}_t^{\gamma_W(\theta),\theta})_{t\in\mathbb{R}}$ is measure equivalent to direction θ dynamics defined by the surface $\widetilde{X}_3$. First the results of "Appendix A" imply, that for given $\theta \in [0, \pi/4]$ the ergodicity of the geodesic flow is equivalent to the ergodicity of the measured foliation defined by the slit-fold distribution obtained from the flat lens representation of Eaton lenses. That is, for given $\theta \in [0, \pi/4]$ we replace every Eaton lens by a slit-fold centered at the lens' center, perpendicular to θ and with length equal to the diameter of the lens. In fact modulo Λ we obtain the slit-folds

$$S_\pm(\theta) := \pm\rangle 1 + \cos\theta\sin\theta + i(1 + \tan\theta - \cos^2\theta),$$
$$1 - \cos\theta\sin\theta + i(1 + \tan\theta + \cos^2\theta)\langle$$

through the centers of $L_\pm(\theta)$ and

$$S_0(\theta) := \rangle -2\sin^2\theta + 2i\sin\theta\cos\theta, 2\sin^2\theta - 2i\sin\theta\cos\theta\langle$$

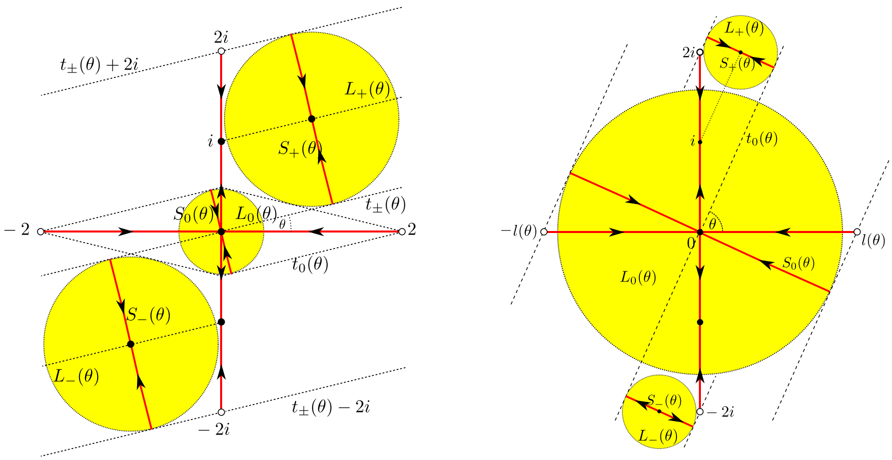

Fig. 11. Transitions from Eaton lenses to the Wollmilchsau skeleton

through the origin, see Fig. 11. The endpoints of the slit-fold $S_+(\theta)$ lie on the lines (and direction θ foliation leaves) $t_\pm(\theta)$ and $t_\pm(\theta)+2i$. That means we can perform a railed deformation of $S_+(\theta)$ along those leaves terminating in the slit-fold $\rangle 0, 2i\langle$. By central symmetry there is a railed deformation of $S_-(\theta)$ to the slit-fold $\rangle 0, -2i\langle$. The end points of the slit-fold $S_0(\theta)$ are located on direction θ foliation leaves through the point ± 2, so $S_0(\theta)$ has a railed deformation to the slit-fold $\rangle -2, 2\langle$. But that means the skeleton $S_+ \cup S_- \cup S_0$ is Whitehead equivalent to the skeleton $\rangle -2, 2\langle \cup \rangle 0, 2i\langle \cup \rangle 0, -2i\langle$. The Λ orbit of the latter is the Wollmilchsau skeleton in the plane, showing the claim on equivalence of ergodicity for angles $\theta \in [0, \pi/4]$.

The strategy we have just used to replace an Eaton lens with a slit-fold is the same for every angle. Let us describe this process for the slit-folds in the Wollmilchsau skeleton: For a fixed direction $\theta \in \mathbb{R}/\pi\mathbb{Z}$ a slit-fold, say S, replaces an Eaton lens, say L, if the two lines in direction θ through the endpoints of S are tangent to L. Step by step, the flat lens equivalent to L is in the quadratic differential interpretation the slit-fold S_L perpendicular to the direction θ with diameter and center matching those of L. In that case, the endpoints of S_L lie on the two said tangents to L and therefore there is a railed deformation of S_L to S. If, as in our case, more than one slit-fold is involved it must be checked that the tangent segments between S and S_L do not cross another slit-fold. This is illustrated in Fig. 11 for an angle $\theta \in [0, \pi/4]$ (left) and for an angle $\theta \in [\pi/4, \pi/2]$ (right). This same strategy is applied for the angles $\theta \in [\pi/4, \pi/2]$ below. The tangent lines necessary to show equivalence to the Wollmilchsau skeleton are also needed to show admissibility.

For the angles $\theta \in [\pi/4, \pi/2]$ the lattice of translation depends on the angle. In fact $\Lambda_\theta := \mathbb{Z}4i + \mathbb{Z}(2l(\theta) + 2i)$, where $l(\theta) = 2 - \cot\theta(1 - \cot\theta)$. While $L_0(\theta)$ is still centered at the origin, now with radius $r_0(\theta) = l(\theta)\sin\theta = \frac{1}{\sin\theta} + \sin\theta - \cos\theta$ the other two lenses $L_\pm(\theta)$ as before of radius $\cos\theta$ are now centered at $c_\pm(\theta) = \pm(\cot\theta + 2i)$, see Fig. 3. In particular the radii of the lenses $L_\pm(\theta)$ are bounded by $1 < l(\theta) \leq 2$ and the radius of the lens $L_0(\theta)$ is bounded by $l(\theta) \leq 2$. Because the generators of the lattice Λ_θ move each lens by at least twice their diameter there are no pairwise intersections possible among the lenses in one Λ_θ orbit. Moreover the $\mathbb{Z}4i$ orbit of $L_-(\theta)$ lies on the left of the vertical through the origin while the $\mathbb{Z}4i$ orbit of $L_+(\theta)$ lies on the right of that line. As $0 \leq \cot\theta \leq 1$ we have

$$\cot\theta + \cos\theta \leq 2\cot\theta \leq 2 - \cot\theta(1 - \cot\theta) = l(\theta).$$

Moreover, $r_0(\theta) = l(\theta)\sin\theta \leq l(\theta)$. It follows that the $\mathbb{Z}4i$ orbits of $L_\pm(\theta)$ and $L_0(\theta)$ are contained in the strip $S = \{z \in \mathbb{C}; |\Re z| \leq l(\theta)\}$. Since the $\mathbb{Z}(2l(\theta) + 2i)$ translates of S cover the whole plane, intersecting only in their boundary lines, we conclude that no pair of Eaton lenses in the Λ_θ orbits of $L_\pm(\theta)$ intersect.

Restricted to the $\mathbb{Z}4i$ orbit the lens configuration have for all $\theta \in [\pi/4, \pi/2]$ reflection symmetries around the coordinate axes. More precisely the $\mathbb{Z}4i$ orbit of each lens is invariant under the reflection at the horizontal while the $\mathbb{Z}4i$ orbits of $L_\pm(\theta)$ are interchanged by reflection at the vertical. Given these symmetries, all that remains to be seen is that $L_0(\theta)$ does not intersect with $L_+(\theta)$. To do this we find a common tangent to $L_0(\theta)$ and $L_+(\theta)$ that separates them. Let us consider the tangent line $s(\theta)$ to $L_0(\theta)$ at the intersection point of its boundary with the half-line $t_0(\theta)$ in direction θ through the origin. The direction of $s(\theta)$ is $\pi/2 - \theta$. The half-line $t_0(\theta) + i$ in direction θ through the point i intersects perpendicularly $s(\theta)$ and goes through the center of $L_+(\theta)$. By elementary geometry, see also Fig. 3, the distance from i to the center of $L_+(\theta)$ is $(\sin\theta)^{-1}$. The leg of the right triangle with hypothenuse the segment from 0 to i lying on $t_0(\theta)$ has length $\sin\theta$. So the intersection point of $s(\theta)$ with $t_0(\theta) + i$ must be at distance $r_0(\theta) - \sin\theta = \frac{1}{\sin\theta} - \cos\theta$ from i. But then it has distance $\cos\theta$ from the center of $L_+(\theta)$ and so the tangent $s(\theta)$ to $L_0(\theta)$ is also tangent to $L_+(\theta)$.

To show admissibility for one of the remaining angles, say $\theta \in [\pi/2, \pi]$, notice that $L_\pm(\theta)$ are the lenses $L_\pm(\pi - \theta)$ reflected at the vertical through the origin. We also have $L_0(\theta) = L_0(\pi - \theta)$ and the lattice of translations has the same symmetry $\Lambda_\theta = \Lambda_{\pi-\theta}$. So the Λ_θ orbits of these (reflected) lenses match the distribution given in the introduction. Since for $\theta = \pi/2$ the lenses $L_\pm(\theta)$ are located on the vertical coordinate axis, this continuation of γ_W is continuous at $\pi/2$. Moreover globally the lens distribution for $\theta \in [\pi/2, \pi]$ equals the one for $\pi - \theta$ reflected at the vertical coordinate axis. Since a reflection is an isometry, it preserves admissibility of lens distributions. Finally the Eaton lens configuration at $\theta = \pi$ matches that at $\theta = 0$, since $\gamma_W(\pi) + 2i = \gamma_W(0)$. □

In particular the proof of Proposition 2.4 shows that the Wollmilchsau differential is a maximal Eaton differential. This, together with the fact that the Wollmilchsau differential appears as a cyclic pillow case cover branched over exactly three points, shows Theorem 2.3.

3. Quadratic Differentials on Tori in the Determinant Locus

3.1. Quadratic and Abelian differentials. In this article quadratic differentials are the fundamental objects. They appear in various presentations, analytical, polygonal and geometrical. All of those play important roles in different parts of our text.

Consider a Riemann surface X, i.e. a one dimensional complex manifold, not necessarily compact, and a quadratic differential q on X with poles of order at most one. A quadratic differential is a tensor that can locally be written as $f(z)\,dz^2 = f(z)\,dz \otimes dz$, where f is a meromorphic function with poles of order at most one. Away from the poles and zeros of f one may use q to define *natural coordinates* on X

$$\zeta = \int_{z_0}^{z} \sqrt{f(z)}\,dz = \int_{z_0}^{z} \sqrt{q}.$$

If ζ_1 and ζ_2 are local coordinates, then $d\zeta_1 = \sqrt{f(z)}\,dz = \pm d\zeta_2$ in the intersection of the coordinate patches, so $\zeta_1 = \pm\zeta_2 + c$ for some $c \in \mathbb{C}$. That way the pair (X, q) defines a maximal atlas made of natural coordinates and is therefore called *half-translation surface*. The maximal atlas is also called *half-translation structure*. The coordinate changes for any two charts from a half-translation structure are translations combined with half-turns (180 degree rotations) and this motivates the name half-translation surface. Similarly to a quadratic differential it is possible to consider an Abelian differential (holomorphic 1-form) ω on X. If $\Sigma \subset X$ denotes the set of zeros of ω, as for quadratic differentials, away from Σ Abelian differential defines *natural coordinates* on X

$$\zeta = \int_{z_0}^{z} \omega.$$

If ζ_1 and ζ_2 are local coordinates and their coordinate patches intersect then $\zeta_1 = \zeta_2 + c$ for some $c \in \mathbb{C}$. So the pair (X, ω) defines a maximal atlas made of natural coordinates and is called *translation surface*. Here the maximal atlas is called *translation structure*.

Objects on the plane that are invariant under *translations* pull back via natural charts to X and glue together to give global objects on the translation surface (X, ω). Among those objects are the euclidean metric, the differential dz, and constant vector fields in any given direction. In fact, the pull back of the differential dz recovers ω on the translation surface (X, ω). Similarly objects on the plane that are invariant under *translations* and *half-turns* define global objects on the half-translation surface (X, q). Here objects of interest are again the euclidean metric, the quadratic differential dz^2 (recovering q), and any direction foliation by (non-oriented) parallel lines. Since there is one line foliation on $\mathbb{C}$ for each angle $\theta \in \mathbb{R}/\pi\mathbb{Z}$ that is tangent to $\pm \exp i\theta$, we denote its pullback to X by $\mathcal{F}_\theta(q)$, or $\mathcal{F}_\theta$ if there is no confusion about the quadratic differential. For a translation surface, say (X, ω), the constant unit vector field on $\mathbb{C}$ in direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ defines a directional unit vector field $V_\theta = V_\theta^\omega$ on $X \setminus \Sigma$. Then the corresponding directional flow $(\varphi_t^\theta)_{t\in\mathbb{R}} = (\varphi_t^{\omega,\theta})_{t\in\mathbb{R}}$ (also known as *translation flow*) on $X \setminus \Sigma$ preserves the area measure μ_ω given by $\mu_\omega(A) = \frac{1}{2}|\int_A \omega \wedge \overline{\omega}|$. If the surface X is compact then the measure μ_ω is finite. We will use the notation $(\varphi_t^v)_{t\in\mathbb{R}}$ for the *vertical flow* (corresponding to $\theta = \frac{\pi}{2}$) and $(\varphi_t^h)_{t\in\mathbb{R}}$ for the *horizontal flow* respectively ($\theta = 0$).

For every half-translation surface (X, q) there exists a unique double cover $\pi_o : (\widehat{X}, \widehat{q}) \to (X, q)$, the *orientation cover*, characterized by the property that it is branched precisely over all singular points with odd order. The pull-back $\widehat{q} = \pi_o^* q$ is the square $\widehat{q} = \omega^2$ of an abelian differential $\omega \in \Omega(X)$. If $M = \widehat{X}$ then the translation surface (M, ω) is called also the orientation cover of the half-translation surface (X, q). The pull-back $\widehat{\mathcal{F}_\theta}$ of any direction foliation $\mathcal{F}_\theta$ is orientable. This foliation coincides with the foliations determined by the directional flows $(\varphi_t^\theta)_{t\in\mathbb{R}}$ and $(\varphi_t^{\theta+\pi})_{t\in\mathbb{R}}$ on (M, ω).

Suppose that q is not the square of an abelian differential on X. Then its orientation cover (M, ω) is connected. We call the foliation $\mathcal{F}_\theta$ on (X, q) *ergodic* if the translation flow $(\varphi_t^\theta)_{t\in\mathbb{R}}$ on M is ergodic with respect to the measure μ_ω.

Particular representations of half-translation structures. The quadratic differential $(dz)^2$ on $\mathbb{C}$ is invariant under translations and rotations of 180 degrees, that group generated by those isometries are in the group of half-translations. Invariance of $(dz)^2$ under that group results in a variety of possible constructions of quadratic differentials, or equivalently half-translation surfaces.

Most notably a (compact) polygon in $\mathbb{C}$ all of whose edges appear in parallel pairs, together with an prescribed identification of edge pairs by half-translations. It is known, that any quadratic differential on a compact surface can be represented by such a polygon. A second way is to take suitable quotients of $\mathbb{C}$ under certain discrete groups of half-translations. Here any torus $\mathbb{C}/\Lambda$ with a lattice Λ of translations is an example. Our way to built quadratic differentials in the plane $\mathbb{C} \cong \mathbb{R}^2$ and on a torus is by successively adding (non-intersecting) slit-folds. Since the identifications of the edges of a slit-fold are half-translations it defines a canonical new quadratic differential on the surface with the slit-fold. One important property of slit-folds is, that they do not change the genus of the half-translation surface to which they are added. There are other types of "folds" with this property. Examples are shown in "Appendix B". They are helpful in the construction of other ergodic curves.

3.2. Cyclic covers of pillowcases. In this section we classify those quadratic differentials on tori that arise as pullbacks of the *pillowcase* along a covering map (cyclic covers) which is unbranched over one point. Two of those examples are quotients of the well known *Ornithorynque* and *Eierlegende Wollmilchsau* under an involution.

Given a Riemann surface X and a finite subset $\Sigma^* \subset X$ it is well known that the elements of $\xi \in H_1(X, \Sigma^*; G)$, G an abelian group, define a regular cover $\pi : X_\xi \to X$ over $X \setminus \Sigma^*$ branched over $\Sigma^* \subset X$ with deck transformation group G. To describe this cover formally first denote by $\langle \cdot, \cdot \rangle : H_1(X, \Sigma^*; G) \times H_1(X \setminus \Sigma^*; G) \to G$ the algebraic intersection form. If $\sigma : [t_0, t_1] \to X$ is a closed curve in X and $\sigma_\xi : [t_0, t_1] \to X_\xi$ is any of its lifts to X_ξ then $\sigma_\xi(t_1) = \langle \xi, [\sigma] \rangle \cdot \sigma_\xi(t_0)$, where $\cdot$ denotes the deck group action of G on X_ξ.

Let us look at the pillowcase $\mathcal{P}$ with underlying space $X = \mathbb{CP}^1$ and take $\Sigma \subset \mathbb{CP}^1$ to be the pillowcases four singular points. We are looking for pillowcase covers with at most three branch points. That means such a cover is unbranched over at least one singular point of the pillowcase. Then the result of Hubert and Grivaux [20] implies that the cover is in the *determinant locus*. We now construct those covers.

3.3. Differentials in the determinant locus. Take the pillowcase $X = \mathcal{P}$ with named singular points $p_1, p_2, p_3, p_4 = p_F \in \mathcal{P}$ put in clockwise order starting from the upper left. We assume the point p_F is fixed under all automorphisms (and affine maps) of $\mathcal{P}$. We further assume all branching of covers is restricted to the set $\Sigma^* = \{p_1, p_2, p_3\}$.

Let γ_{12}, γ_{23} be generators in $H_1(\mathcal{P}, \{p_1, p_2, p_3\}; \mathbb{Z}/d\mathbb{Z})$ so that γ_{12} is the class of the oriented horizontal path joining p_1 and p_2 and γ_{23} is the class of the oriented vertical path joining p_2 and p_3. Let γ_h, γ_v be generators in $H_1(\mathcal{P} \setminus \{p_1, p_2, p_3\}; \mathbb{Z}/d\mathbb{Z})$ such that γ_h is the class the horizontal (right oriented) simple loop and γ_v is the class of the simple loop around p_1 with counterclockwise orientation. Then

$$\langle \gamma_{12}, \gamma_v \rangle = \langle \gamma_{23}, \gamma_h \rangle = 1 \text{ and } \langle \gamma_{12}, \gamma_h \rangle = \langle \gamma_{23}, \gamma_v \rangle = 0.$$

Let us consider any cyclic degree d cover $\mathcal{P}_\xi$ of $\mathcal{P}$ branched over Σ^* which is defined by a homology class $\xi = w_h \gamma_{12} + w_v \gamma_{23} \in H_1(X, \Sigma^*; \mathbb{Z}/d\mathbb{Z})$. Here

$$w_h = \langle \xi, \gamma_v \rangle \in \mathbb{Z}/d\mathbb{Z} \text{ and } w_v = \langle \xi, \gamma_h \rangle \in \mathbb{Z}/d\mathbb{Z}$$

are called weights of the cover $\mathcal{P}_\xi \to \mathcal{P}$. Therefore the cover is determined by the triple $(d, w_h, w_v) \in \mathbb{N} \times \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$ and we will denote it by $X_d(w_h, w_v) \to \mathcal{P}$. The cover $X_d(w_h, w_v)$ is connected iff $\gcd(d, w_h, w_v) = 1$. The cover defined by those data has a straightforward geometric realization. Namely, cut the pillowcase along the three line segments joining: p_1 with p_2, p_2 with p_3 and p_3 with p_F. The resulting surface is isometric to a rectangle of width 2 and height 1 in the complex plane. Let us denote this polygonal presentation of X with cuts by X^c and take d labeled copies $X^c \times \{1, \ldots, d\} = X_1^c \sqcup \cdots \sqcup X_d^c$. Now identify the vertical right edge of X_i^c with the vertical left edge $X^c_{i+w_v \bmod d}$ by a translation. Then identify the right half of the upper horizontal edge of X_i^c with the left half of the upper horizontal edge of $X^c_{i+w_h \bmod d}$ using a half turn and identify the right half of the lower horizontal edge of X_i^c with the left half of the lower horizontal edge of X_i^c using a half turn. This determines $X_d(w_h, w_v)$ because of the covers cyclic nature. By eventually renaming the decks we may assume that $w_v = \gcd(w_v, d)$ divides d. Indeed, if $A : \mathbb{Z}/d\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}$ is a group automorphism then using A to rename the decks we obtain $X_d(w_h, w_v) \cong X_d(Aw_h, Aw_v)$. If $w_v = \gcd(w_v, d)l$ then let A be the multiplication by l on $\mathbb{Z}/d$. Since $\gcd(l, d) = 1$, A is an automorphism for which $A(\gcd(w_v, d)) = w_v$. Then after renaming the decks using the inverse automorphism A^{-1} we obtain $X_d(w_h, w_v) \cong X_d(A^{-1}w_h, \gcd(w_v, d))$. See [5,12] for a more background and applications of cyclic covers.

We now determine those cyclic covers that are torus differentials, i.e. have genus 1. To calculate the genus of $X_d(w_h, w_v)$ we note, that the covering has $\gcd(w_h, d)$ preimages over p_1, $\gcd(|w_h - w_v|, d)$ preimages over p_2 and $w_v = \gcd(w_v, d)$ preimages over p_3 because it is cyclic. It follows that the respective branching orders are $o_1 = d/\gcd(w_h, d)$ at p_1, $o_2 = d/\gcd(|w_h - w_v|, d)$ at p_2 and $o_3 = d/\gcd(w_v, d) = d/w_v$ at p_3. That means we have an angle excess of $(o_i - 2)\pi$ around any preimage of p_i for $i = 1, 2, 3$.

Proposition 3.1. *The genus g_{d,w_h,w_v} of $X_d(w_h, w_v)$ is given by*

$$\begin{aligned} g_{d,w_h,w_v} - 1 &= (d - \gcd(w_h, d) - \gcd(w_v, d) - \gcd(|w_h - w_v|, d))/2 \\ &= (d - w_v - \gcd(w_h, d) - \gcd(|w_h - w_v|, d))/2. \end{aligned}$$

Proof. Write down the standard formula expressing the Euler characteristic of quadratic differentials in terms of total angle deficit for singular points and total angle excess for cone points:

$$\begin{aligned} &2\chi(X_d(w_h, w_v)) = d + \gcd(w_h, d)(2 - d/\gcd(w_h, d)) \\ &\quad + \gcd(w_v, d)(2 - d/\gcd(w_v, d)) + \gcd(|w_h - w_v|, d)(2 - d/\gcd(|w_h - w_v|, d)) \\ &\quad = 2(-d + \gcd(w_h, d) + \gcd(w_v, d) + \gcd(|w_h - w_v|, d)). \end{aligned}$$

The result follows since $\chi(X_d(w_1, w_2)) = 2(1 - g_{d,w_h,w_v})$. □

By definition the degree of the pillowcase cover $\pi_d(w_h, w_v) : X_d(w_h, w_v) \to \mathcal{P}$ is d.

Proposition 3.2. *If $X_d(w_h, w_v)$ has genus* 1*, then $d \in \{3, 4, 6\}$.*

Proof. A torus has vanishing Euler characteristic, thus from Proposition 3.1 we directly derive the condition

$$d = \gcd(w_h, d) + \gcd(w_v, d) + \gcd(|w_h - w_v|, d).$$

Dividing by d, we see that a torus presents a positive integer solution of the problem

$$1 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c},$$

where a, b, c represent the natural numbers $d/\gcd(w_h, d), d/\gcd(w_v, d), d/\gcd(|w_h - w_v|, d)$. Without restriction of generality we may assume that any solution fulfills $c \geq b \geq a > 0$. It follows that $2 \leq a \leq 3$.

If $a = 2$ then $1/b + 1/c = 1/2$ which gives $b \leq 4$. Therefore we obtain two possibilities $(b, c) = (3, 6)$ or $(4, 4)$.

If $a = 3$ then $1/b + 1/c = 2/3$ with $c \geq b \geq 3$. It leads to $(b, c) = (3, 3)$. It follows that we get only $(3, 3, 3)$, $(2, 4, 4)$, $(2, 3, 6)$ as solutions. Since

$$\gcd\left(\gcd(w_h, d), \gcd(w_v, d), \gcd(|w_h - w_v|, d)\right) = 1,$$

we obtain $\operatorname{lcm}(a, b, c) = d$. It follows that $d = 3, 4, 6$ respectively. □

3.4. Branched pillow case covers that are torus differentials. In spite of Proposition 3.2 all we need to do to exhaust the list of possible torus covers is to go over a short list of possible cases. Because p_F is assumed to be fixed the pillowcase has no automorphisms. For $d = 3, 4$ and 6 we need to find the weights $1 \leq w_v, w_h < d$ with $\gcd(w_h, w_v, d) = 1$ satisfying the condition

$$w_v | d \quad \text{and} \quad d = \gcd(w_h, d) + w_v + \gcd(|w_h - w_v|, d).$$

The weights cannot be 0 or d, because the cover must be branched over all three points p_1, p_2 and p_3 to give a surface of genus larger than zero, the genus of the pillowcase. Thus without loss of generality we can pick the weights w_h, w_v from $\{1, \ldots, d - 1\}$. For $d = 6$ we obtain the following weight pairs fulfilling the conditions:

$$(w_h, w_v) \in \{(1, 3), (3, 1), (3, 2), (2, 3), (4, 1), (4, 3), (5, 2), (5, 3)\}$$

The weights tell us the number of deck changes that occur when we (positively) cross over either homology class (generator). By renaming the decks so that deck k becomes deck $d - k$ we obtain the cover $X_d(d - w_h, d - w_v)$ from $X_d(w_h, w_v)$. Thus those are isomorphic, in particular for $d = 6$ we have $X_6(1, 3) \cong X_6(5, 3)$ and $X_6(2, 3) \cong X_6(4, 3)$. For $d = 3$ and $d = 4$ the same line of arguments applies and leads to the following list of covers:

Torus differentials of degree $d = 3, 4$ and 6

Degree d	w_h	w_v	$\# \pi^{-1}(p_1)$	$\# \pi^{-1}(p_2)$	$\# \pi^{-1}(p_3)$	Surface
3	2	1	1	1	1	$X_3(2, 1)$
4	2	1	2	1	1	$X_4(2, 1)$
	3	1	1	2	1	$X_4(3, 1)$
	3	2	1	1	2	$X_4(3, 2)$
6	3	1	3	2	1	$X_6(3, 1)$
	3	2	3	1	2	$X_6(3, 2)$
	4	1	2	3	1	$X_6(4, 1)$
	4	3	2	1	3	$X_6(4, 3)$
	5	2	1	3	2	$X_6(5, 2)$
	5	3	1	2	3	$X_6(5, 3)$

The group $SL_2(\mathbb{R})$ acts real linearly on the plane and defines a map on half-translation surfaces by post composition with local coordinates. Alternatively one may take a polygon representation of the surface and apply a matrix $A \in SL_2(\mathbb{R})$, viewed as linear map of $\mathbb{R}^2$, to it. The edges of the polygon are then identified exactly as before the deformation. That defines an action of $\mathrm{SL}_2(\mathbb{R})$ on surfaces with quadratic differential. Denote by $A \cdot X$ the deformation of X by $A \in \mathrm{SL}_2(\mathbb{R})$.

Let $X_\xi \to X$ be the G-cover with branching in $\Sigma^* \subset X$ determined by $\xi \in H_1(X, \Sigma^*; G)$. Then the deformation $A \cdot X_\xi$ is a branched cover determined by $A_*\xi \in H_1(A \cdot X, \Sigma^*; G)$.

The pillowcase is stabilized by all elements of $\mathrm{SL}_2(\mathbb{Z})$, as one can easily check on the two (parabolic) generators $P_h := \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z})$ and $P_v := \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z})$. Stabilized means the original pillowcase can be obtained from the deformed pillowcase by successively cutting off polygons, translating and if needed rotating them to another boundary in tune with the edge identification rules of the pillowcase.

Let us consider any cover $X_d(w_h, w_v) = \mathcal{P}_\xi$ (with $\xi = w_h\gamma_{12} + w_v\gamma_{23}$) and $A \in SL_2(\mathbb{Z})$. Since $A \cdot \mathcal{P} = \mathcal{P}$, we have

$$A \cdot X_d(w_h, w_v) = A \cdot \left(\mathcal{P}_\xi\right) = \left(A \cdot \mathcal{P}\right)_{A_*\xi} = \mathcal{P}_{A_*\xi} = X_d(\langle A_*\xi, \gamma_v\rangle, \langle A_*\xi, \gamma_h\rangle)$$

and $\langle A_*\xi, \gamma_h\rangle = \langle \xi, A_*^{-1}\gamma_h\rangle$, $\langle A_*\xi, \gamma_v\rangle = \langle \xi, A_*^{-1}\gamma_v\rangle$. Moreover for the parabolic generators P_h and P_v we have

$$(P_h^{-1})_*\gamma_h = \gamma_h, \quad (P_h^{-1})_*\gamma_v = \gamma_h - \gamma_v, \quad (P_v^{-1})_*\gamma_v = \gamma_v, \quad (P_v^{-1})_*\gamma_h = \gamma_v - \gamma_h,$$

and hence

$$\begin{aligned}
\langle \xi, (P_h^{-1})_*\gamma_v\rangle &= \langle w_h\gamma_{12} + w_v\gamma_{23}, \gamma_h - \gamma_v\rangle = w_v - w_h,\\
\langle \xi, (P_h^{-1})_*\gamma_h\rangle &= \langle w_h\gamma_{12} + w_v\gamma_{23}, \gamma_h\rangle = w_v,\\
\langle \xi, (P_v^{-1})_*\gamma_v\rangle &= \langle w_h\gamma_{12} + w_v\gamma_{23}, \gamma_v\rangle = w_h,\\
\langle \xi, (P_v^{-1})_*\gamma_h\rangle &= \langle w_h\gamma_{12} + w_v\gamma_{23}, \gamma_v - \gamma_h\rangle = w_h - w_v.
\end{aligned}$$

This yields the action of parabolic matrices on degree d pillowcase covers:

$$P_h \cdot X_d(w_h, w_v) = X_d(w_v - w_h, w_v) \text{ and } P_v \cdot X_d(w_h, w_v) = X_d(w_h, w_h - w_v).$$

Since the group of maps generated by two involutions $(x, y) \mapsto (x, y - x)$ and $(x, y) \mapsto (y - x, y)$ has exactly 6 elements, so we obtain the following:

Proposition 3.3. *The* $\mathrm{SL}_2(\mathbb{Z})$ *orbit of a pillowcase cover is given by*

$$\begin{aligned}
\mathrm{SL}_2(\mathbb{Z}) \cdot X_d(w_h, w_v) = \big\{&X_d(w_h, w_v), X_d(w_h, w_h - w_v), X_d(w_v - w_h, w_v),\\
&X_d(-w_v, w_h - w_v), X_d(w_v - w_h, -w_h), X_d(-w_v, -w_h)\big\}.
\end{aligned}$$

Note, that for low degree this orbit is even smaller: The orbits of degree three and four covers contain less than six tori. As can be easily seen from the proposition, compare the table of surfaces, that the relevant torus differentials of fixed degree lie on one $\mathrm{SL}_2(\mathbb{Z})$ orbit.

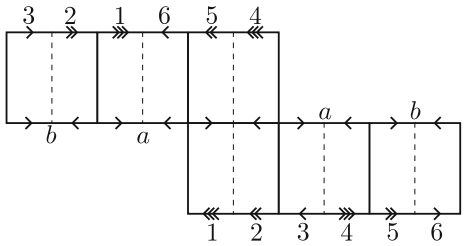

Fig. 12. The Ornithorynque as orientation cover of $X_3(2, 1)$

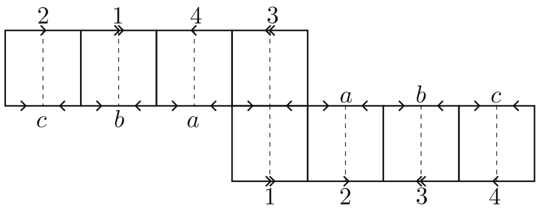

Fig. 13. The Eierlegende Wollmilchsau as orientation cover of $X_4(2, 1)$

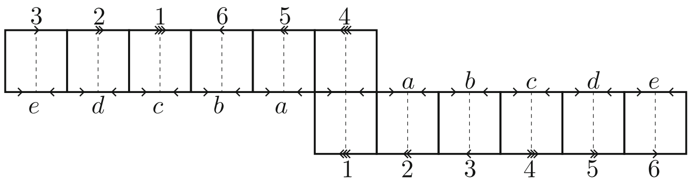

Fig. 14. The orientation cover of $X_6(3, 1)$

Orientation covers of some pillow case covers. We consider the orientation covers $X_d(2, 1)$, for $d = 3, 4$ and $X_d(3, 1)$ for $d = 6$ drawn on Fig. 7. Recall that the orientation cover $(\widehat{X}, \omega^2) \to (X, q)$ of a quadratic differential (X, q) is uniquely characterized as the degree two cover, branched precisely over the cone points having an odd total angle (in multiples of π). There is a sheet exchanging involution ρ on $\widehat{X}$ that has the preimages of the odd cone points as fixed-points. The involution is locally a rotation by π, eventually followed by a translation. Using this one may construct orientation covers given a polygonal representation. One considers two copies of the polygon and whenever two edges were identified by a rotation on the original polygon, one identifies any of those two edges as before but now to the corresponding edge of the other copy. Turning any one copy by 180 degrees the new identifications become translations and we have a translation surface. For the surfaces at hand this procedure is reflected in the following Figs. 12, 13 and 14. The first two are splendid specimens in the zoo of square tiled surfaces. If the notation square tiled did not immediately give it away, a look at the figures will explain the idea of a square tiled surface. In fact, $\widehat{X}_3(2, 1) \cong \widehat{X}_2$ is the is known as the *Ornithorynque* and $\widehat{X}_4(2, 1) \cong \widehat{X}_3$ is the *Eierlegende Wollmilchsau* already mentioned in the introduction. Both names reflect that the surfaces carry an abundance of rather exceptional properties, for one all of them have all vanishing Lyapunov exponents. To our best knowledge the orientation cover of $X_6(3, 1)$ is not a well studied square tiled surface. On the other hand, we are not able to provide a direct reason to motivate such research.

4. Ergodicity of Translation Flows and Measured Foliations on Infinite Covers

In this section we prove a useful criterion on ergodicity for translation flows on $\mathbb{Z}^d$-covers (see Theorem 4.6). The key Theorem 1.6 follows directly from this criterion.

For relevant background material concerning IETs and their relations to translation surfaces, we refer the reader to [11,26,29–31].

4.1. $\mathbb{Z}^d$ covers. Let $\widetilde{X}$ be a labeled unbranched $\mathbb{Z}^d$-cover of a compact connected surface X and let $p : \widetilde{X} \to X$ be the covering map, i.e. there exists a properly discontinuous $\mathbb{Z}^d$-action on $\widetilde{X}$ such that $\widetilde{X}/\mathbb{Z}^d$ is homeomorphic to X. Then $p : \widetilde{X} \to X$ is the composition of the natural projection $\widetilde{X} \to \widetilde{X}/\mathbb{Z}^d$ and the homeomorphism. Denote by $\langle\,\cdot\,,\,\cdot\,\rangle : H_1(X,\mathbb{Z}) \times H_1(X,\mathbb{Z}) \to \mathbb{Z}$ the algebraic intersection form. Then any $\mathbb{Z}^d$-cover $\widetilde{X}$ is determined by a d-tuple $\gamma = (\gamma_1,\ldots,\gamma_d) \in H_1(X,\mathbb{Z})^d$ of independent homology classes, so that if $\sigma : [t_0,t_1] \to X$ is a closed curve in X and $\widetilde{\sigma} : [t_0,t_1] \to \widetilde{X}$ is any its lift to $\widetilde{X}$ then

$$\widetilde{\sigma}(t_1) = \langle\gamma,[\sigma]\rangle \cdot \widetilde{\sigma}(t_0),$$

where

$$\langle\gamma,[\sigma]\rangle = (\langle\gamma_1,[\sigma]\rangle,\ldots,\langle\gamma_d,[\sigma]\rangle) \in \mathbb{Z}^d \qquad ([\sigma] \in H_1(X,\mathbb{Z}))$$

and $\cdot$ denotes the action of $\mathbb{Z}^d$ on $\widetilde{X}$. The $\mathbb{Z}^d$-cover corresponding to γ will be denoted by $\widetilde{X}_\gamma$.

Remark 4.1. Note that the surface $\widetilde{X}_\gamma$ is connected if and only if the group homomorphism $H_1(X,\mathbb{Z}) \ni \xi \mapsto \langle\gamma,\xi\rangle \in \mathbb{Z}^d$ is surjective.

If q is a quadratic differential on X then the pull-back $p^*(q)$ of q by p is also a quadratic differential on $\widetilde{X}_\gamma$ and will be denoted by $\widetilde{q}_\gamma$. For any $\theta \in \mathbb{R}/\pi\mathbb{Z}$ we denote by $\widetilde{\mathcal{F}}_\theta = \widetilde{\mathcal{F}}_\theta^\gamma$ the corresponding measurable foliation on $(\widetilde{X}_\gamma,\widetilde{q}_\gamma)$.

If (M,ω) is a compact translation surface and $\gamma \in H_1(M,\mathbb{Z})^d$ is a d-tuple then the translation flow on the $\mathbb{Z}^d$-cover $(\widetilde{M}_\gamma,\widetilde{\omega}_\gamma)$ in direction θ is denoted by $(\widetilde{\varphi}_t^\theta)_{t\in\mathbb{R}}$.

Let (X,q) be a connected half-translation surface and denote by (M,ω) its orientation cover which is a translation surface. Then there exist a branched covering map $\pi : M \to X$ such that $\pi^*(q) = \omega^2$ and an idempotent $\sigma : X \to X$ such that $\pi \circ \sigma = \pi$ and $\sigma^*(\omega) = -\omega$.

The space $H_1(M,\mathbb{R})$ has an orthogonal (symplectic) splitting into spaces $H_1^+(M,\mathbb{R})$ and $H_1^-(M,\mathbb{R})$ of σ_*-invariant and σ_*-anti-invariant homology classes, respectively. Moreover, the subspace $H_1^+(M,\mathbb{R})$ is canonically isomorphic to $H_1(X,\mathbb{R})$ via the map $\pi_* : H_1^+(M,\mathbb{R}) \to H_1(X,\mathbb{R})$, so we identify both spaces.

Remark 4.2. Suppose that (X,q) is a compact connected half-translation surface and q is not the square of an abelian differential. Let $\gamma \in (H_1(X,\mathbb{Z}))^d$ be a d-tuple such that the $\mathbb{Z}^d$-cover $\widetilde{X}_\gamma$ is connected. Because the cover is unbranched, the lifted quadratic differential $\widetilde{q}_\gamma$ is not the square of an abelian differential as well.

Since $H_1^+(M,\mathbb{Z})$ and $H_1(X,\mathbb{Z})$ are identified, we can treat γ as a d-tuple in $(H_1^+(M,\mathbb{Z}))^d$. Let us consider the corresponding $\mathbb{Z}^d$-cover $\widetilde{M}_\gamma$. Then the maps $\pi : M \to X$ and $\sigma : M \to M$ can be lifted to a branched covering map $\widetilde{\pi} : \widetilde{M}_\gamma \to \widetilde{X}_\gamma$ and an involution $\widetilde{\sigma} : \widetilde{M}_\gamma \to \widetilde{M}_\gamma$ so that $\widetilde{\pi} \circ \widetilde{\sigma} = \widetilde{\pi}$. Then $\widetilde{\pi}$ establishes an orientation cover $(\widetilde{M}_\gamma, \widetilde{\omega}_\gamma)$ of the half-translation surface $(\widetilde{X}_\gamma, \widetilde{q}_\gamma)$. Since the quadratic differential $\widetilde{q}_\gamma$ is not the square of an abelian differential, for every $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ the ergodicity of the measured foliation $\widetilde{\mathcal{F}}_\theta$ of $(\widetilde{X}_\gamma, \widetilde{q}_\gamma)$ is equivalent to the ergodicity of the translation flow $(\widetilde{\varphi}_t^\theta)_{t\in\mathbb{R}}$ on $(\widetilde{M}_\gamma, \widetilde{\omega}_\gamma)$. Note, that the measure $\mu_{\widetilde{\omega}_\gamma}$ is an infinite Radon measure.

4.2. The Teichmüller flow and the Kontsevich–Zorich cocycle. Given a connected compact oriented surface M of genus g, denote by $\mathrm{Diff}^+(M)$ the group of orientation-preserving homeomorphisms of M. Denote by $\mathrm{Diff}_0^+(M)$ the subgroup of elements $\mathrm{Diff}^+(M)$ which are isotopic to the identity. Let us denote by $\Gamma(M) := \mathrm{Diff}^+(M)/\mathrm{Diff}_0^+(M)$ the *mapping-class* group. We will denote by $\mathcal{T}(M)$ (respectively $\mathcal{T}_1(M)$) the *Teichmüller space of Abelian differentials* (respectively of unit area Abelian differentials), that is the space of orbits of the natural action of $\mathrm{Diff}_0^+(M)$ on the space of all Abelian differentials on M (respectively, the ones with total area $\mu_\omega(M) = 1$). We will denote by $\mathcal{M}(M)$ ($\mathcal{M}_1(M)$) the *moduli space of (unit area) Abelian differentials*, that is the space of orbits of the natural action of $\mathrm{Diff}^+(M)$ on the space of (unit area) Abelian differentials on M. Thus $\mathcal{M}(M) = \mathcal{T}(M)/\Gamma(M)$ and $\mathcal{M}_1(M) = \mathcal{T}_1(M)/\Gamma(M)$.

The moduli space $\mathcal{M}(M)$ is stratified according to the number and multiplicity of the holomorphic one-forms zeros and the $SL(2,\mathbb{R})$-action respects this stratification. Define the stratum $\mathcal{M}(\kappa_1, \ldots, \kappa_s)$ as the collection of translations surfaces (M, ω) such ω has s zeros and the multiplicity of the zeros of ω is given by $(\kappa_1, \ldots, \kappa_s)$. Then $\kappa_1 + \cdots + \kappa_s = 2g - 2$.

Denote by $\mathcal{Q}(X)$ the moduli space of half-translation surfaces which is also naturally stratified by the number and the types of singularities. We denote by $\mathcal{Q}(\kappa_1, \ldots, \kappa_s)$ the stratum of quadratic differentials (X, q) which are not the squares of Abelian differentials, and which have s singularities and their orders are $(\kappa_1, \ldots, \kappa_s)$, where $\kappa_i \geq -1$. Then $\kappa_1 + \cdots + \kappa_s = 4g_X - 4$, where g_X is the genus of X.

The group $SL(2,\mathbb{R})$ acts naturally on $\mathcal{T}_1(M)$ and $\mathcal{M}_1(M)$ as follows. Given a translation structure ω, consider the charts given by local primitives of the holomorphic 1-form. The new charts defined by postcomposition of these charts with an element of $SL(2,\mathbb{R})$ yield a new complex structure and a new differential that is Abelian with respect to this new complex structure, thus a new translation structure. We denote by $g \cdot \omega$ the translation structure on M obtained acting by $g \in SL(2,\mathbb{R})$ on a translation structure ω on M.

The *Teichmüller flow* $(g_t)_{t\in\mathbb{R}}$ is the restriction of this action to the diagonal subgroup $(\mathrm{diag}(e^t, e^{-t}))_{t\in\mathbb{R}}$ of $SL(2,\mathbb{R})$ on $\mathcal{T}_1(M)$ and $\mathcal{M}_1(M)$. We will deal also with the rotations $(r_\theta)_{\theta\in\mathbb{R}/2\pi\mathbb{Z}}$ that acts on $\mathcal{T}_1(M)$ and $\mathcal{M}_1(M)$ by $r_\theta\omega = e^{i\theta}\omega$.

Theorem 4.3 (Theorem 2 in [24]). *For every Abelian differential ω on a compact connected surface M for almost all directions $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ the vertical and horizontal flows on $(M, r_\theta\omega)$ are uniquely ergodic.*

Let us call a $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ for which the assertion of the theorem holds *Masur generic*.

The *Kontsevich–Zorich (KZ) cocycle* $(G_t^{KZ})_{t\in\mathbb{R}}$ is the quotient of the trivial cocycle

$$g_t \times \mathrm{Id} : \mathcal{T}_1(M) \times H_1(M,\mathbb{R}) \to \mathcal{T}_1(M) \times H_1(M,\mathbb{R})$$

by the action of the mapping-class group $\Gamma(M)$. The mapping class group acts on the fiber $H_1(M,\mathbb{R})$ by induced maps. The cocycle $(G_t^{KZ})_{t\in\mathbb{R}}$ acts on the homology vector bundle

$$\mathcal{H}_1(M,\mathbb{R}) = (\mathcal{T}_1(M) \times H_1(M,\mathbb{R}))/\Gamma(M)$$

over the Teichmüller flow $(g_t)_{t\in\mathbb{R}}$ on the moduli space $\mathcal{M}_1(M)$.

Clearly the fibers of the bundle $\mathcal{H}_1(M,\mathbb{R})$ can be identified with $H_1(M,\mathbb{R})$. The space $H_1(M,\mathbb{R})$ is endowed with the symplectic form given by the algebraic intersection number. This symplectic structure is preserved by the action of the mapping-class group and hence is invariant under the action of $SL(2,\mathbb{R})$.

The standard definition of KZ-cocycle uses the cohomological bundle. The identification of the homological and cohomological bundle and the corresponding KZ-cocycles is established by the Poincaré duality $\mathcal{P}: H_1(M,\mathbb{R}) \to H^1(M,\mathbb{R})$. This correspondence allows us to define the so called Hodge norm (see [9] for cohomological bundle) on each fiber of the bundle $\mathcal{H}_1(M,\mathbb{R})$. The norm on the fiber $H_1(M,\mathbb{R})$ over $\omega \in \mathcal{M}_1(M)$ will be denoted by $\|\cdot\|_\omega$.

Let $\omega \in \mathcal{M}_1(M)$ and denote by $\mathcal{M} = \overline{SL(2,\mathbb{R})\omega}$ the closure of the $SL(2,\mathbb{R})$-orbit of ω in $\mathcal{M}_1(M)$. The celebrated result of Eskin, Mirzakhani and Mohammadi, proved in [6,7], says that $\mathcal{M} \subset \mathcal{M}_1(M)$ is an affine (i.e. an affine subspace in periodic coordinates) $SL(2,\mathbb{R})$-invariant immersed submanifold and there is an affine (i.e. the affine area measure in periodic coordinates) $SL(2,\mathbb{R})$-invariant probability measure $\nu_{\mathcal{M}}$ whose support is $\mathcal{M}$. The above results say in addition, that the measure $\nu_{\mathcal{M}}$ is ergodic under the action of the Teichmüller flow. It follows, that $\nu_{\mathcal{M}}$-almost every element of $\mathcal{M}$ is Birkhoff generic, i.e. the pointwise ergodic theorem holds for the Teichmüller flow and every continuous integrable function on $\mathcal{M}$. The following recent result is more refined and yields Birkhoff generic elements among $r_\theta\omega$ for $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.

Theorem 4.4 (Theorem 1.1 in [3]). *For almost all $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ we have*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \phi(g_t r_\theta \omega)\, dt = \int_{\mathcal{M}} \phi\, d\nu_{\mathcal{M}} \ \text{for every} \ \phi \in C_c(\mathcal{M}_1(M)).$$

All directions $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ for which the assertion of the theorem holds are called *Birkhoff generic*.

By Masur's celebrated criterion [25], Birkhoff genericity implies Masur genericity. Nevertheless, for the sake of clarity we will treat both separately.

Let $\mathcal{V} \to \mathcal{M}$ be an $SL(2,\mathbb{R})$-invariant subbundle of $\mathcal{H}_1(M,\mathbb{R})$ which is defined and continuous over $\mathcal{M}$. For every $\omega \in \mathcal{M}$ we denote by $\mathcal{V}_\omega$ its fiber over ω.

Let us consider the KZ-cocycle $(G_t^{\mathcal{V}})_{t\in\mathbb{R}}$ restricted to $\mathcal{V}$. By Oseledets' theorem, there exists Lyapunov exponents of $(G_t^{\mathcal{V}})_{t\in\mathbb{R}}$ with respect to the measure $\nu_{\mathcal{M}}$. If additionally, the subbundle $\mathcal{V}$ is symplectic, its Lyapunov exponents with respect to the measure $\nu_{\mathcal{M}}$ are:

$$\lambda_1^{\mathcal{V}} \geq \lambda_2^{\mathcal{V}} \geq \cdots \geq \lambda_d^{\mathcal{V}} \geq -\lambda_d^{\mathcal{V}} \geq \cdots \geq -\lambda_2^{\mathcal{V}} \geq -\lambda_1^{\mathcal{V}}.$$

Theorem 4.5 (Theorem 1.4 in [3]). *Let $\lambda_1^{\mathcal{V}} = \overline{\lambda}_1 > \overline{\lambda}_2 > \cdots > \overline{\lambda}_{s-1} > \overline{\lambda}_s = -\lambda_1^{\mathcal{V}}$ be distinct Lyapunov exponents of $(G_t^{\mathcal{V}})_{t\in\mathbb{R}}$ with respect to $\nu_{\mathcal{M}}$. Then for a.e. $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ there exists a direct splitting of the fiber $\mathcal{V}_{r_\theta\omega} = \bigoplus_{i=1}^s \mathcal{U}^i_{r_\theta\omega}$ such that for every $\xi \in \mathcal{U}^i_{r_\theta\omega}$ we have*

$$\lim_{t\to\infty} \frac{1}{t}\log \|\xi\|_{g_t r_\theta \omega} = \overline{\lambda}_i. \tag{4.1}$$

Each $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ for which the assertion of the theorem holds is called *Oseledets generic*. Then $\mathcal{V}_{r_\theta\omega}$ has a direct splitting

$$\mathcal{V}_{r_\theta\omega} = E^+_{r_\theta\omega} \oplus E^0_{r_\theta\omega} \oplus E^-_{r_\theta\omega}$$

into unstable, central and stable subspaces

$$E^+_{r_\theta\omega} = \left\{\xi \in \mathcal{V}_{r_\theta\omega} : \lim_{t\to+\infty} \frac{1}{t} \log \|\xi\|_{g_{-t}r_\theta\omega} < 0\right\},$$
$$E^0_{r_\theta\omega} = \left\{\xi \in \mathcal{V}_{r_\theta\omega} : \lim_{t\to\infty} \frac{1}{t} \log \|\xi\|_{g_t r_\theta\omega} = 0\right\},$$
$$E^-_{r_\theta\omega} = \left\{\xi \in \mathcal{V}_{r_\theta\omega} : \lim_{t\to+\infty} \frac{1}{t} \log \|\xi\|_{g_t r_\theta\omega} < 0\right\}.$$

The dimensions of $E^+_{r_\theta\omega}$ and $E^-_{r_\theta\omega}$ are equal to the number of positive Lyapunov exponents of $(G^{\mathcal{V}}_t)_{t\in\mathbb{R}}$.

One of the main objectives of this paper is to prove (in Sect. 4.5) the following criterion on ergodicity for translation flows on $\mathbb{Z}^d$-covers.

Theorem 4.6. *Let (M, ω) be a compact connected translation surface and let $\mathcal{M} = \overline{SL(2,\mathbb{R})\omega}$. Suppose that $\mathcal{V} \to \mathcal{M}$ is a continuous $SL(2,\mathbb{R})$-invariant subbundle of $\mathcal{H}_1(M,\mathbb{R})$ such that all Lyapunov exponents of the KZ-cocycle $(G^{\mathcal{V}}_t)_{t\in\mathbb{R}}$ vanish. Then for every connected $\mathbb{Z}^d$-cover $(\widetilde{M}_\gamma, \widetilde{\omega}_\gamma)$ given by a d-tuple $\gamma = (\gamma_1, \ldots, \gamma_d) \in (\mathcal{V}_\omega \cap H_1(M,\mathbb{Z}))^d$ the directional flow in direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ on the translation surface $(\widetilde{M}_\gamma, \widetilde{\omega}_\gamma)$ is ergodic for a.e. θ.*

The following result is a key step of the proof of Theorem 4.6.

Theorem 4.7. *Let $\mathcal{V} \to \mathcal{M}$ be a continuous $SL(2,\mathbb{R})$-invariant subbundle of $\mathcal{H}_1(M,\mathbb{R})$. If all Lyapunov exponents of the KZ-cocycle $(G^{\mathcal{V}}_t)_{t\in\mathbb{R}}$ vanish then $\|\xi\|_{g\omega} = \|\xi\|_\omega$ for all $\xi \in \mathcal{V}_\omega$ and $g \in SL(2,\mathbb{R})$.*

Proof. The proof follows from two results in [13]. By assumption, $\mathcal{V}$ is an $SL(2,\mathbb{R})$-invariant subbundle of the central Oseledets bundle E^0 (defined $\nu_{\mathcal{M}}$-a.e. for the bundle $\mathcal{H}_1(M,\mathbb{R}) \to \mathcal{M}$). Therefore, in view of Theorem 3 in [13], $\mathcal{V}$ is a subbundle of the annihilator $\mathrm{Ann}(B^{\mathbb{R}})$ of the bilinear form $B^{\mathbb{R}}$. Since the bilinear form $B^{\mathbb{R}}$ vanishes on $\mathcal{V}$, by Lemma 4.1 in [13], the Kontsevich–Zorich cocycle is isometric on $\mathcal{V}$, which completes the proof. □

Suppose that (M, ω) is an orientation cover of a compact half-translation surface (X, q). Then the $SL(2,\mathbb{R})$-invariant symplectic subspace $H_1^+(M,\mathbb{R})$ determines an $SL(2,\mathbb{R})$-invariant symplectic subbundle $\mathcal{H}_1^+$ which is defined and continuous over $\mathcal{M}$. The fibers of this bundle can be identified with the space $H_1^+(M,\mathbb{R}) = H_1(X,\mathbb{R})$ so the dimension of each fiber is $2g_X$, where g_X is the genus of X. The Lyapunov exponents of the bundle $\mathcal{H}_1^+$ are called the Lyapunov exponents of the half-translation surface (X, q). We denote by $\lambda_{top}(q)$ the largest exponent.

Proof of Theorem 1.6. Theorem 4.6 applied to the subbundle $\mathcal{H}_1^+$ together with Remark 4.2 completes the proof. □

4.3. Skew product representation. Let $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ be a direction such that the flow $(\varphi^\theta_t)_{t\in\mathbb{R}}$ on (M, ω) is ergodic and has no saddle connections. Let $I \subset M \setminus \Sigma$ be an interval transversal to the direction θ with no self-intersections. Then the Poincaré return map $T : I \to I$ is an ergodic interval exchange transformation (IET) which satisfies the Keane property. Denote by $(I_\alpha)_{\alpha\in\mathcal{A}}$ the family of exchanged intervals. For every $\alpha \in \mathcal{A}$

we will denote by $\xi_\alpha = \xi_\alpha(\omega, I) \in H_1(M, \mathbb{Z})$ the homology class of any loop formed by the segment of orbit for $(\varphi_t^\theta)_{t\in\mathbb{R}}$ starting at any $x \in \mathrm{Int}\, I_\alpha$ and ending at Tx together with the segment of I that joins Tx and x, that we will denote by $[Tx, x]$.

Proposition 4.8 (see Lemma 2.1 in [15] for $d = 1$). *For every $\gamma \in H_1(M, \mathbb{Z})^d$ the directional flow $(\widetilde{\varphi}_t^\theta)_{t\in\mathbb{R}}$ on the $\mathbb{Z}^d$-cover $(\widetilde{M}_\gamma, \widetilde{\omega}_\gamma)$ has a special representation over the skew product $T_{\psi_\gamma} : I \times \mathbb{Z}^d \to I \times \mathbb{Z}^d$ of the form $T_{\psi_\gamma}(x, n) = (Tx, n + \psi_\gamma(x))$, where $\psi_\gamma : I \to \mathbb{Z}^d$ is a piecewise constant function given by*

$$\psi_\gamma(x) = \langle \gamma, \xi_\alpha \rangle = \big(\langle \gamma_1, \xi_\alpha \rangle, \ldots, \langle \gamma_d, \xi_\alpha \rangle\big) \quad \text{if} \quad x \in I_\alpha \quad \text{for } \alpha \in \mathcal{A}. \tag{4.2}$$

In particular, the ergodicity of the flow $(\widetilde{\varphi}_t^\theta)_{t\in\mathbb{R}}$ on $(\widetilde{M}_\gamma, \widetilde{\omega}_\gamma)$ is equivalent to the ergodicity of the skew product $T_{\psi_\gamma} : I \times \mathbb{Z}^d \to I \times \mathbb{Z}^d$.

Since the ergodicity of the flow $(\widetilde{\varphi}_t^\theta)_{t\in\mathbb{R}}$ is equivalent to the ergodicity of T_{ψ_γ}, this will allow us to apply the theory of essential values of cocycles to prove Theorem 4.6 in Sect. 4.5.

4.4. Ergodicity of skew products. In this subsection we recall some general facts about cocycles. For relevant background material concerning skew products and infinite measure-preserving dynamical systems, we refer the reader to [1,28].

Let G be a locally compact abelian second countable group. We denote by 0 its identity element, by $\mathcal{B}_G$ its σ-algebra of Borel sets and by m_G its Haar measure. Recall that, for each ergodic automorphism $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ of a standard Borel probability space, each measurable function $\psi : X \to G$ defines a *skew product* automorphism T_ψ which preserves the σ-finite measure $\mu \times m_G$:

$$T_\psi : (X \times G, \mathcal{B} \times \mathcal{B}_G, \mu \times m_G) \to (X \times G, \mathcal{B} \times \mathcal{B}_G, \mu \times m_G),$$
$$T_\psi(x, g) = (Tx, g + \psi(x)),$$

Here we use $G = \mathbb{Z}^d$. The function $\psi : X \to G$ determines also a *cocycle* $\psi^{(\cdot)} : \mathbb{Z} \times X \to G$ for the automorphism T by the formula

$$\psi^{(n)}(x) = \begin{cases} \sum_{0\le j<n} \psi(T^j x) & \text{if } n \ge 0 \\ -\sum_{n\le j<0} \psi(T^j x) & \text{if } n < 0. \end{cases}$$

Then $T_\psi^n(x, g) = (T^n x, g + \psi^{(n)}(x))$ for every $n \in \mathbb{Z}$.

An element $g \in G$ is said to be an *essential value* of ψ, if for every open neighbourhood V_g of g in G and any set $B \in \mathcal{B}$, $\mu(B) > 0$, there exists $n \in \mathbb{Z}$ such that

$$\mu(B \cap T^{-n} B \cap \{x \in X : \psi^{(n)}(x) \in V_g\}) > 0.$$

The set of essential values of ψ is denoted by $E(\psi)$.

Proposition 4.9 (see Lemma 3.3 and Corollary 5.4 in [28]). *The set of essential values $E(\psi)$ is a closed subgroup of G and the skew product T_ψ is ergodic if and only if $E(\psi) = G$.*

Proposition 4.10 (see Corollary 2.8 in [4]). *Let (X, d) be a compact metric space, $\mathcal{B}$ the σ–algebra of Borel sets and μ be a probability Borel measure on X. Suppose that $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is an ergodic measure–preserving automorphism and there exists an increasing sequence of natural numbers $(h_n)_{n\geq 1}$ and a sequence of Borel sets $(C_n)_{n\geq 1}$ such that*

$$\mu(C_n) \to \alpha > 0, \quad \mu(C_n \triangle T^{-1}C_n) \to 0 \quad \textit{and} \quad \sup_{x\in C_n} d(x, T^{h_n}x) \to 0.$$

If $\psi : X \to G$ is a measurable cocycle such that $\psi^{(h_n)}(x) = g$ for all $x \in C_n$, then $g \in E(\psi)$.

4.5. Proof of Theorem 4.6. In this section we prove the following result. In view of Theorems 4.3, 4.4 and 4.7, it proves Theorem 4.6.

Theorem 4.11. *Let (M, ω) be a compact connected translation surface and let $\gamma = (\gamma_1, \ldots, \gamma_d) \in H_1(M, \mathbb{Z})^d$ be a d-tuple such that the $\mathbb{Z}^d$-cover $\widetilde{M}_\gamma$ is connected and $\|\gamma_i\|_{g\omega} = \|\gamma_i\|_\omega$ for all $1 \leq i \leq d$ and $g \in SL(2, \mathbb{R})$. If a direction $\pi/2 - \theta \in \mathbb{R}/2\pi\mathbb{Z}$ is Birkhoff and Masur generic for ω then the directional flow in direction θ on $(\widetilde{M}_\gamma, \widetilde{\omega}_\gamma)$ is ergodic.*

Suppose that the directional flow $(\varphi_t^\theta)_{t\in\mathbb{R}}$ on (M, ω) in a direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ is ergodic and minimal. Let $I \subset M \setminus \Sigma$ (Σ is the set of zeros of ω) be an interval transversal to the direction θ with no self-intersections. The Poincaré return map $T : I \to I$ is a minimal ergodic IET. Denoted by $I_\alpha, \alpha \in \mathcal{A}$ the intervals exchanged by T. Let $\lambda_\alpha(\omega, I)$ stand for the length of the interval I_α.

Denote by $\tau : I \to \mathbb{R}_+$ the map giving the first return time to I of the flow $(\varphi_t^\theta)_{t\in\mathbb{R}}$. Then τ is constant on each I_α and we denote by $\tau_\alpha = \tau_\alpha(\omega, I) > 0$ its value on I_α for all $\alpha \in \mathcal{A}$. Let us further denote by $\delta(\omega, I) > 0$ the maximal number $\Delta > 0$ for which the set $\{\varphi_t^\theta x : t \in [0, \Delta), x \in I\}$ does not contain any singular point (from Σ).

Denote by $(\widetilde{\varphi}_t^\theta)_{t\in\mathbb{R}}$ the directional flow for a $\mathbb{Z}^d$-cover $(\widetilde{M}_\gamma, \widetilde{\omega}_\gamma)$ of (M, ω).

Recall that for every transversal interval $I \subset M$ the Poincaré return map $T : I \to I$ of the flow $(\varphi_t^\theta)_{t\in\mathbb{R}}$ is an IET exchanging subintervals $(I_\alpha)_{\alpha\in\mathcal{A}}$. Moreover, for every $\alpha \in \mathcal{A}$ we denote by $\xi_\alpha(I) = \xi_\alpha(\omega, I) \in H_1(M, \mathbb{Z})$ the homology class of a trajectory starting from the interval I_α until the first return to I and closed up within the interval. By Proposition 4.8, the Poincaré return map $\widetilde{T}$ of the flow $(\widetilde{\varphi}_t^\theta)_{t\in\mathbb{R}}$ to $p^{-1}(I)$ ($p : \widetilde{M}_\gamma \to M$ the covering map) is isomorphic to the skew product $T_\psi : I \times \mathbb{Z}^d \to I \times \mathbb{Z}^d$ of the form $T_\psi(x, n) = (Tx, n + \psi(x))$, where $\psi = \psi_{\gamma,I} : I \to \mathbb{Z}^d$ is a piecewise constant function given by

$$\psi_{\gamma,I}(x) = \langle\gamma, \xi_\alpha(I)\rangle = \big(\langle\gamma_1, \xi_\alpha(I)\rangle, \ldots, \langle\gamma_d, \xi_\alpha(I)\rangle\big) \text{ if } x \in I_\alpha \text{ for } \alpha \in \mathcal{A}. \tag{4.3}$$

Suppose that $J \subset I$ is a subinterval. Denote by $S : J \to J$ the Poincaré return map to J for the flow $(\varphi_t^\theta)_{t\in\mathbb{R}}$. Then S is also an IET and suppose it exchanges intervals $(J_\alpha)_{\alpha\in\mathcal{A}'}$ (the set $\mathcal{A}'$ may differ from $\mathcal{A}$ by at most two elements). The IET S is the induced transformation for T on J. Moreover, all elements of J_α have the same first return time to J for the transformation T. Let us denote this return time by $h_\alpha \geq 0$ for all $\alpha \in \mathcal{A}'$. Then I is the union of disjoint towers $\{T^j J_\alpha : 0 \leq j < h_\alpha\}$ for $\alpha \in \mathcal{A}'$.

Lemma 4.12. *Suppose that* $0 \leq h \leq \min\{h_\alpha : \alpha \in \mathcal{A}'\}$ *is a number such that for every* $0 \leq j < h$ *the transformation* T^j *restricted to* J *is continuous. Then for every* $\alpha \in \mathcal{A}'$ *we have*

$$\psi^{(h_\alpha)}_{\gamma,I}(x) = \langle \gamma, \xi_\alpha(J)\rangle \text{ and } |T^{h_\alpha}x - x| \leq |J| \tag{4.4}$$

for every $x \in C_\alpha := \bigcup_{0\leq j\leq h} T^j J_\alpha$.

Proof. The proof goes in two steps. First we show (4.4) for the bases of the towers, i.e. if $x \in J_\alpha$ for some $\alpha \in \mathcal{A}'$. Since h_α is the first return time of $x \in J_\alpha$ to J, both x and $T^{h_\alpha}x$ belong to J, so $|T^{h_\alpha}x - x| \leq |J|$. As for $0 \leq j < h$ the transformation T^j acts on J as a translation, the inequality $|T^{h_\alpha}x - x| \leq |J|$ extends to the whole tower $C_\alpha = \bigcup_{0\leq j\leq h} T^j J_\alpha$.

Let $\psi_{\gamma,J} : J \to \mathbb{Z}^d$ be the cocycle associated to the interval J. Then $\psi_{\gamma,J}$ arises from $\psi_{\gamma,I}$ by inducing on the interval $J \subset I$. Therefore, for every $x \in J$ the value $\psi_{\gamma,J}(x)$ is the sum of the values of $\psi_{\gamma,I}$ along the T-orbit of x until the first return to J. Hence

$$\psi_{\gamma,J}(x) = \sum_{0\leq j<h_\alpha} \psi_{\gamma,I}(T^j x) = \psi^{(h_\alpha)}_{\gamma,I}(x) \quad if \quad x \in J_\alpha.$$

In view of (4.3) applied to the interval J, we have $\psi_{\gamma,J}(x) = \langle \gamma, \xi_\alpha(J)\rangle$ for $x \in J_\alpha$, so $\psi^{(h_\alpha)}_{\gamma,I} = \langle \gamma, \xi_\alpha(J)\rangle$ on J_α.

If $x \in C_\alpha$ then $x = T^j x_0$ with $x_0 \in J_\alpha$ and $0 \leq j \leq h$. Moreover,

$$\begin{aligned}\psi^{(h_\alpha)}_{\gamma,I}(x) - \psi^{(h_\alpha)}_{\gamma,I}(x_0) &= \psi^{(h_\alpha)}_{\gamma,I}(T^j x_0) - \psi^{(h_\alpha)}_{\gamma,I}(x_0)\\ &= \sum_{i=0}^{j-1}(\psi_{\gamma,I}(T^i T^{h_\alpha} x_0) - \psi_{\gamma,I}(T^i x_0)).\end{aligned}$$

Since x_0 and $T^{h_\alpha}x_0 = Sx_0$ belong to J, by assumption, for all $0 \leq i < h$ the points $T^i T^{h_\alpha} x_0$ and $T^i x_0$ belong to the interval $T^i J \subset I_\beta$ for some $\beta \in \mathcal{A}$. Therefore,

$$\psi_{\gamma,I}(T^i T^{h_\alpha} x_0) = \psi_{\gamma,I}(T^i x_0) \quad \text{for every} \quad 0 \leq i < j.$$

It follows that $\psi^{(h_\alpha)}_{\gamma,I}(x) = \psi^{(h_\alpha)}_{\gamma,I}(x_0) = \langle \gamma, \xi_\alpha(J)\rangle$. □

Lemma 4.13. *Let* $\Delta > 0$ *be so, that the set* $\{\varphi^\theta_t x : t \in [0, \Delta), x \in J\}$ *does not contain any singular point. Let* $h = [\Delta/|\tau|]$, *where* $|\tau| = \max\{\tau_\alpha : \alpha \in \mathcal{A}\}$. *Then for every* $0 \leq j < h$ *the transformation* T^j *restricted to* J *is continuous.*

Proof. Suppose, contrary to our claim, that $T^j J$ contains an end x of some interval I_β. Then $x = \varphi^\theta_{\tau^{(j)}(x_0)}(x_0)$ for some $x_0 \in J$ and there is $0 \leq s < \tau(x)$ such that $\varphi^\theta_s x$ is a singular point. Therefore, $\varphi^\theta_{\tau^{(j)}(x_0)+s} x_0$ is a singular point and $\tau^{(j)}(x_0)+s < (j+1)|\tau| \leq h|\tau| \leq \Delta$, contrary to the assumption. □

The following result follows directly from Lemmas A.3 and A.4 in [14].

Lemma 4.14. *For every (M,ω) there exist positive constants $A, C, c > 0$ such that if $0 \in \mathbb{R}/2\pi\mathbb{Z}$ is Birkhoff and Masur generic then there exists a sequence of nested horizontal intervals $(I_k)_{k\geq 0}$ in (M,ω) and an increasing divergent sequence of real numbers $(t_k)_{k\geq 0}$ such that $t_0 = 0$ and for every $k \geq 0$ the corresponding subintervals of I_k are parameterized with the same set $\mathcal{A}$ and*

$$\frac{1}{c}\|\xi\|_{g_{t_k}\omega} \leq \max_{\alpha} \left|\langle \xi_\alpha(g_{t_k}\omega, I_k), \xi\rangle\right| \leq c\|\xi\|_{g_{t_k}\omega} \quad \textit{for every} \quad \xi \in H_1(M,\mathbb{R}), \tag{4.5}$$

$$\lambda_\alpha(g_{t_k}\omega, I_k)\,\delta(g_{t_k}\omega, I_k) \geq A \textit{ and } \frac{1}{C} \leq \tau_\alpha(g_{t_k}\omega, I_k) \leq C \textit{ for any } \alpha \in \mathcal{A}. \tag{4.6}$$

Proof of Theorem 4.11. Assume that the total area of (M,ω) is 1. Taking $\omega_0 = r_{\pi/2-\theta}\omega$ we have $0 \in \mathbb{R}/2\pi\mathbb{Z}$ is Birkhoff and Masur generic for ω_0. Since the flow $(\widetilde{\varphi}^\theta_t)_{t\in\mathbb{R}}$ on $(\widetilde{M}_\gamma, \widetilde{\omega}_\gamma)$ coincides with the vertical flow on $(\widetilde{M}_\gamma, \widetilde{(\omega_0)}_\gamma)$, we need to prove the ergodicity of the latter flow.

By Lemma 4.14, there exists a sequence of nested horizontal intervals $(I_k)_{k\geq 0}$ in (M,ω_0) and an increasing divergent sequence of real numbers $(t_k)_{k\geq 0}$ such that (4.5) and (4.6) hold for $k \geq 0$ and $t_0 = 0$.

Let $I := I_0$ and for the flow $(\widetilde{\varphi}^v_t)_{t\in\mathbb{R}}$ on $(\widetilde{M}_\gamma, \widetilde{(\omega_0)}_\gamma)$ denote by $T : I \to I$ and $\psi : I \to \mathbb{Z}^d$ the corresponding IET and cocycle respectively. For every $k \geq 1$ the Poincaré first return map $T_k : I_k \to I_k$ to I_k for the vertical flow $(\varphi^v_t)_{t\in\mathbb{R}}$ on (M,ω_0) is an IET exchanging intervals $(I_k)_\alpha$, $\alpha \in \mathcal{A}$ whose lengths in (M,ω_0) are equal to $e^{-t_k}\lambda_\alpha(g_{t_k}\omega_0, I_k)$, $\alpha \in \mathcal{A}$, resp. In view of (4.6), the length of I_k in (M,ω_0) is

$$|I_k| = \sum_{\alpha\in\mathcal{A}} e^{-t_k}\lambda_\alpha(g_{t_k}\omega_0, I_k) \leq Ce^{-t_k}\sum_{\alpha\in\mathcal{A}}\lambda_\alpha(g_{t_k}\omega_0, I_k)\tau_\alpha(g_{t_k}\omega_0, I_k) = Ce^{-t_k}.$$

Moreover, by the definition of δ, the set

$$\left\{\varphi^v_t(x) : t \in \left[0, e^{t_k}\delta(g_{t_k}\omega_0, I_k)\right), x \in I_k\right\}$$

does not contain any singular points.

Denote by $h^k_\alpha \geq 0$ the first return time of the interval $(I_k)_\alpha$ to I_k for the IET T. Let

$$h_k := \left[e^{t_k}\delta(g_{t_k}\omega_0, I_k)/|\tau(\omega_0, I)|\right] \text{ and } C^k_\alpha := \bigcup_{0\leq j\leq h_k} T^j(I_k)_\alpha.$$

Now Lemmas 4.12 and 4.13 applied to $J = I_k$ and $\Delta = e^{t_k}\delta(g_{t_k}\omega_0)$ give

$$\psi^{(h^k_\alpha)}(x) = \langle\gamma, \xi_\alpha(g_{t_k}\omega_0, I_k)\rangle \text{ and } |T^{h^k_\alpha}x - x| \leq |I_k| \leq Ce^{-t_k} \quad \text{for } x \in C^k_\alpha \tag{4.7}$$

for every $k \geq 1$ and $\alpha \in \mathcal{A}$. Moreover, by (4.6),

$$Leb(C^k_\alpha) = (h_k+1)|(I_k)_\alpha| \geq \frac{e^{t_k}\delta(g_{t_k}\omega_0, I_k)}{|\tau(\omega_0, I)|}e^{-t_k}\lambda_\alpha(g_{t_k}\omega_0, I_k) \geq \frac{A}{|\tau(\omega_0, I)|}. \tag{4.8}$$

By assumption, in view of (4.5), we have

$$c^{-1}\|\gamma_i\|_{g_{t_k}\omega_0} \leq \max_{\alpha\in\mathcal{A}}\|\langle\gamma_i, \xi_\alpha(g_{t_k}\omega_0, I_k)\rangle\| \leq c\|\gamma_i\|_{g_{t_k}\omega_0} = c\|\gamma_i\|_{\omega_0} \text{ for } 1 \leq i \leq d.$$

Therefore for every $\alpha \in \mathcal{A}$ the sequence $\{\langle\gamma, \xi_\alpha(g_{t_k}\omega_0, I_k)\rangle\}_{k\geq 1}$ in $\mathbb{Z}^d$ is bounded. Passing to a subsequence, if necessary, we can assume the sequence is constant. In view of (4.7) and (4.8), Proposition 4.10 gives $\langle\gamma, \xi_\alpha(g_{t_k}\omega_0, I_k)\rangle \in E(\psi)$ for every $\alpha \in \mathcal{A}$ and $k \geq 1$. Recall that for every $k \geq 1$ the homology classes $\xi_\alpha(g_{t_k}\omega_0, I_k)$, $\alpha \in \mathcal{A}$ generate $H_1(M, \mathbb{Z})$. As $\widetilde{M}_\gamma$ is connected, the homomorphism $H_1(M, \mathbb{Z}) \ni \xi \mapsto \langle\gamma, \xi\rangle \in \mathbb{Z}^d$ is surjective. Therefore, for every $k \geq 1$ the vectors $\langle\gamma, \xi_\alpha(g_{t_k}\omega_0, I_k)\rangle$, $\alpha \in \mathcal{A}$ generate $\mathbb{Z}^d$. Since $E(\psi)$ is a group and contains all these vectors, we obtain $E(\psi) = \mathbb{Z}^d$, so the skew product T_ψ is ergodic. In view of Proposition 4.8, the vertical flow on $(\widetilde{M}_\gamma, \widetilde{(\omega_0)_\gamma})$ is ergodic, which completes the proof. □

4.6. Some comments on Theorem 4.6. Let $\omega \in \mathcal{M}_1(M)$ and denote by $\mathcal{M} = \overline{SL(2,\mathbb{R})\omega}$ the closure of the $SL(2,\mathbb{R})$-orbit of ω in $\mathcal{M}_1(M)$. Denote by $\nu_{\mathcal{M}}$ the corresponding affine $SL(2,\mathbb{R})$-invariant ergodic probability measure supported on $\mathcal{M}$. In view of [6,8], for any $SL(2,\mathbb{R})$-invariant symplectic subbundle $\mathcal{V}$ defined over $\mathcal{M}$ there exists an $SL(2,\mathbb{R})$-invariant continuous direct decomposition

$$\mathcal{V} = \mathcal{V}^1 \oplus \mathcal{V}^2 \oplus \ldots \oplus \mathcal{V}^m$$

such that each subbundle $\mathcal{V}^i$ is strongly irreducible. Denote by $\lambda_{top}^{\mathcal{V}^i}$ the maximal Lyapunov exponent of the reduced Kontsevich–Zorich cocycle $(G_t^{\mathcal{V}^i})_{t\in\mathbb{R}}$ with respect to the measure $\nu_{\mathcal{M}}$. As a step of the proof of Theorem 1.4 in [3] the authors showed also the following result:

Theorem 4.15. *If $\xi \in \mathcal{V}^i_\omega$ is non-zero then for a.e. $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ we have*

$$\lim_{t\to\infty} \frac{1}{|t|} \log \|\xi\|_{g_t r_\theta \omega} = \lambda_{top}^{\mathcal{V}^i}.$$

A consequence of this result is the following:

Theorem 4.16. *For every $\omega \in \mathcal{M}_1(M)$ and $\xi \in H_1(M,\mathbb{R})$ there exists $\lambda(\omega,\xi) \geq 0$ such that*

$$\lim_{t\to\infty} \frac{1}{|t|} \log \|\xi\|_{g_t r_\theta \omega} = \lambda(\omega,\xi) \textit{ for a.e. } \theta \in \mathbb{R}/2\pi\mathbb{Z}.$$

Proof. Let us consider the bundle $\mathcal{H}_1(M,\mathbb{R})$ defined over $\mathcal{M}$. Then there exists a continuous $SL_2(\mathbb{R})$-invariant splitting

$$\mathcal{H}_1(M,\mathbb{R}) = \mathcal{V}^1 \oplus \mathcal{V}^2 \oplus \ldots \oplus \mathcal{V}^m \tag{4.9}$$

such that each subbundle $\mathcal{V}^i$ is strongly irreducible. Then $\xi = \sum_{i=1}^m \xi_i$ where $\xi_i \in \mathcal{V}^i_\omega$. Therefore, by Theorem 4.15, for a.e. θ we have

$$\lim_{t\to\infty} \frac{1}{|t|} \log \|\xi\|_{g_t r_\theta \omega} = \max\{\lambda_{top}^{\mathcal{V}^i} : 1 \leq i \leq m,\ \xi_i \neq 0\}$$

which completes the proof. □

The following result is a direct consequence of Theorem 4.6 and yields some relationship between the value of the Lyapunov exponent $\lambda(\omega,\gamma)$ for $\gamma \in H_1(M,\mathbb{Z})$ and the ergodic properties of translation flows on the $\mathbb{Z}^d$-cover $(\widetilde{M}_\gamma, \widetilde{\omega}_\gamma)$.

Theorem 4.17. *Let (M, ω) be a compact translation surface and let $\gamma \in H_1(M, \mathbb{Z})^d$ be so, that $\widetilde{M}_\gamma$ is connected and $\lambda(\omega, \gamma_i) = 0$ for $1 \le i \le d$. Then $(\widetilde{\varphi}_t^\theta)_{t\in\mathbb{R}}$ is ergodic for almost every $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.*

Proof. We present the arguments of the proof only for $d = 1$. In the higher dimensional case, the proof runs along similar lines.

Let us consider the $SL_2(\mathbb{R})$-invariant splitting (4.9) into strongly irreducible subbundles and let $\gamma = \sum_{i=1}^m \gamma_i$ with $\gamma_i \in \mathcal{V}_\omega^i$. Since $\lambda(\omega, \gamma) = 0$, by Theorem 4.15, $\gamma_i \neq 0$ implies $\lambda_{top}^{\mathcal{V}^i} = 0$. Let

$$\mathcal{V}^\gamma := \bigoplus_{\{1\le i\le m:\ \gamma_i\neq 0\}} \mathcal{V}^i.$$

Then $\mathcal{V}^\gamma$ is a non-zero $SL_2(\mathbb{R})$-invariant subbundle so that $\gamma \in \mathcal{V}_\omega^\gamma$ and all Lyapunov exponents of the restricted KZ-cocycle $(G_t^{\mathcal{V}^\gamma})_{t\in\mathbb{R}}$ with respect to the measure $\nu_{\mathcal{M}}$ vanish. Then Theorem 4.6 provides the final argument. □

Finally, we can formulate a conjecture which was stated so far informally in the translation surface community. It completely describes the relationship between the value of the Lyapunov exponent and the ergodic properties of translation flows on the $\mathbb{Z}$-covers on compact surfaces.

Conjecture. *Let (M, ω) be a compact translation surface and let $(\widetilde{M}_\gamma, \widetilde{\omega}_\gamma)$ be its connected $\mathbb{Z}$-cover given by $\gamma \in H_1(M, \mathbb{Z})$. Then*

(i) *if $\lambda(\omega, \gamma) = 0$ then $(\widetilde{\varphi}_t^\theta)_{t\in\mathbb{R}}$ is ergodic for almost every $\theta \in \mathbb{R}/2\pi\mathbb{Z}$;*
(ii) *if $\lambda(\omega, \gamma) > 0$ then $(\widetilde{\varphi}_t^\theta)_{t\in\mathbb{R}}$ is non-ergodic for almost every $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.*

The claim (i) is confirmed by Theorem 4.17. The truth of claim (ii) is suggested only by a much weaker result proved in [15].

Remark 4.18. One can formulate the above conjecture also in the framework of branched $\mathbb{Z}$-covers, with branching over the zero set Σ of $\omega \in \Omega^1(M)$. Each such cover is determined by a relative homology element $\gamma \in H_1(M, \Sigma, \mathbb{Z})$ and so the corresponding Lyapunov exponent $\lambda(\omega, \gamma)$ is well defined. However, in this framework even the claim (i) is not clear. For the Kontsevich–Zorich cocycle acting on the relative homological bundle $\mathcal{H}_1(M, \Sigma, \mathbb{R})$ the key Theorem 4.7 does not hold. More precisely, the K-Z cocycle can have infinite growth for SL(2, $\mathbb{R}$)-invariant subbundles having zero Lyapunov exponents. This fact obstructs the applicability of the method to prove ergodicity presented here.

5. Non-ergodicity and Trapping for Typical Choice of Periodic System of Eaton Lenses

In this section we present the proof of Theorem 1.2.

Let $\Lambda \subset \mathbb{C}$ be a lattice. For any quadratic differential q on the torus $X := \mathbb{C}/\Lambda$ we denote by $\widetilde{q}$ the pullback of q by the projection map $p : \mathbb{C} \to \mathbb{C}/\Lambda$. Denote by $\mathcal{F}_\theta$ and $\widetilde{\mathcal{F}}_\theta$ the measured foliations in a direction $\theta \in \mathbb{R}/\pi\mathbb{Z}$ derived from (X, q) and $(\mathbb{C}, \widetilde{q})$ respectively. We call the foliation $\widetilde{\mathcal{F}}_\theta$ *trapped*, if there exists a vector $v \in S^1 \subset \mathbb{C}$ and a constant C such that every leaf of $\widetilde{\mathcal{F}}_\theta$ is trapped in an infinite strip of width C parallel to v. Of course, every trapped foliation is highly non-ergodic.

Let (M, ω) be the orientation cover of the half-translation torus (X, q) and let $\pi : M \to X$ be the corresponding branched covering map. Then the space $H_1^+(M, \mathbb{R}) \simeq H_1(X, \mathbb{R})$ of vectors invariant under the deck exchange map on homology is a two dimensional real space. Denote by $\gamma_1, \gamma_2 \in H_1(X, \mathbb{Z}) \simeq H_1^+(M, \mathbb{Z})$ two homology elements determining the $\mathbb{Z}^2$-covering $p : \mathbb{C} \to X$. Since γ_1, γ_2 are linearly independent, they span the space $H_1(X, \mathbb{R}) \simeq H_1^+(M, \mathbb{R})$. Let $(\widetilde{M}, \widetilde{\omega})$ be the $\mathbb{Z}^2$-cover of (M, ω) given by the pair $(\gamma_1, \gamma_2) \in H_1^+(M, \mathbb{Z})^2$. Denote by $p_M : \widetilde{M} \to M$ the covering map. For every $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ let M_θ^+ be the set of points $x \in M$ such that the positive semi-orbit $(\varphi_t^\theta(x))_{t \geq 0}$ on (M, ω) is well defined.

Let $D \subset \widetilde{M}$ be a bounded fundamental domain of the $\mathbb{Z}^2$-cover such that the interior of D is path-connected and the boundary of D is a finite union of intervals. For every $x \in M_\theta^+$ and $t > 0$ define the element $\sigma_t^\theta(x) \in H_1(M, \mathbb{Z})$ as the homology class of the loop formed by the segment of the orbit of x from x to $\varphi_t^\theta(x)$ closed up by the shortest curve joining $\varphi_t^\theta(x)$ with x that does not cross $p_M(\partial D)$.

The following result is a more general version of Theorem 3.2 in [17]. Since its proof runs essentially as in [17], we omit it.

Proposition 5.1. *Assume that for a direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ there is a non-zero homology class $\xi \in H_1^+(M, \mathbb{R}) = \mathbb{R}\gamma_1 + \mathbb{R}\gamma_2$ and $C > 0$ such that*

$$|\langle \sigma_t^\theta(x), \xi \rangle| \leq C \text{ for every } x \in M_\theta^+ \text{ and } t > 0.$$

If the foliation $\mathcal{F}_\theta$ has no vertical saddle connection the lifted foliation $\widetilde{\mathcal{F}}_\theta$ is trapped.

Let $\mathcal{M}$ be the closure of the $SL(2, \mathbb{R})$-orbit of (M, ω) and denote by $\nu_{\mathcal{M}}$ the affine probability measure on $\mathcal{M}$. Let us consider the restriction $(G_t^{\mathcal{H}_1^+})_{t \in \mathbb{R}}$ of the Kontsevich–Zorich cocycle to the subbundle $\mathcal{H}_1^+(M, \mathbb{R}) \to \mathcal{M}$. Recall that a.e. $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ is Oseledets generic for the subbundle. This implies the existence of the stable subspace $E^-_{r_\theta \omega} \subset H_1^+(M, \mathbb{R})$ whose dimension is equal to the number of positive Lyapunov exponents of $(G_t^{\mathcal{H}_1^+})_{t \in \mathbb{R}}$. Moreover, by Theorem 4.4 in [14] we have.

Proposition 5.2. *Suppose that $\pi/2 - \theta \in \mathbb{R}/2\pi\mathbb{Z}$ is a Birkhoff, Oseledets and Masur (BOM) generic direction for (M, ω). Then for every $\xi \in E^-_{r_{\pi/2-\theta}\omega}$ there exists $C > 0$ such that $|\langle \sigma_t^\theta(x), \xi \rangle| \leq C$ for all $x \in M_\theta^+$ and $t > 0$.*

Since almost every direction is BOM generic, the previous two results yield the following criterion.

Proposition 5.3. *Suppose that the Lyapunov exponent $\lambda_{top}(q)$ of $(\mathbb{C}/\Lambda, q)$ is positive. Then for a.e. $\theta \in \mathbb{R}/\pi\mathbb{Z}$ the measured foliation $\widetilde{\mathcal{F}}_\theta$ on $(\mathbb{C}, \widetilde{q})$ is trapped.*

To show the positivity of the Lyapunov exponents we will use Forni's criterion:

Proposition 5.4 (Theorem 1.6 in [10]). *Let (M, ω) be a translation surface of genus g. Let $\mathcal{M}$ be the closure of the $SL(2, \mathbb{R})$-orbit of (M, ω) and denote by $\nu_{\mathcal{M}}$ the affine probability measure on $\mathcal{M}$. Suppose that all vertical regular orbits on (M, ω) are periodic and there are g different periodic orbits $\mathcal{O}_1, \ldots, \mathcal{O}_g$ such that $M \setminus \{\mathcal{O}_1, \ldots, \mathcal{O}_g\}$ is homeomorphic to the $2g$-holed sphere. Then all Lyapunov exponents of the Kontsevich–Zorich cocycle with respect to the measure $\nu_{\mathcal{M}}$ are positive.*

Recall that every $SL(2,\mathbb{R})$-invariant affine measure has a *local product structure* in the sense of Definition 1.3 in [10]. By assumption, the vertical foliation on (M,ω) is *completely periodic* in the sense of Definition 1.4 in [10]. The existence of periodic vertical orbits $\mathscr{O}_1,\ldots,\mathscr{O}_g$ such that $M\setminus\{\mathscr{O}_1,\ldots,\mathscr{O}_g\}$ is homeomorphic to the $2g$-holed sphere implies that the vertical foliation is also Lagrangian. Recall Definition 1.4 in [10]: If (M,ω) admits a completely periodic foliation it is called *Lagrangian*, if the subspace of $H_1(M,\mathbb{R})$ generated by the regular leaves of the foliation is Lagrangian with respect to the symplectic structure on $H_1(M,\mathbb{R})$ given by the intersection form. Since (M,ω) belongs to the support of $\nu_{\mathcal{M}}$ (i.e. $\mathcal{M}$), it follows that the measure $\nu_{\mathcal{M}}$ is *cuspidal* and *Lagrangian* in the sense of Definition 1.5 in [10]. Therefore, Proposition 5.4 indeed directly follows from Theorem 1.6 in [10].

Let $\Lambda\subset\mathbb{C}$ be a lattice and $w\in\Lambda$ a non-zero vector. Let us fix a unit vector $v\in S^1\subset\mathbb{C}$, a k-tuple $\overline{c}=(c_1,\ldots,c_k)$ of different points on the torus $\mathbb{C}/\Lambda$ and a k-tuple $\overline{r}=(r_1,\ldots,r_k)$ of positive numbers. Denote by $q_{v,\overline{c},\overline{r}}$ the quadratic differential on the torus $\mathbb{C}/\Lambda$ arising from the k slit-folds parallel to v, centered at points $c_1,\ldots,c_k\in\mathbb{C}/\Lambda$ with respective radii $r_1,\ldots,r_k$. If all slit-folds are pairwise disjoint, then $q_{v,\overline{c},\overline{r}}\in\mathcal{Q}((-1)^{2k},2^k)$.

For every $1\le j\le k$ denote by $S_j(w)\subset\mathbb{C}/\Lambda$ the shadow of the j-th slit in direction w, i.e. $S_j(w)=\{c_j+sv+tw : s\in[-r_j,r_j], t\in[0,1]\}$. A quadratic differential $q_{v,\overline{c},\overline{r}}$ is called *separated by the vector* $w\in\Lambda$, if each shadow $S_j(w)$ is a proper cylinder (not the whole torus or a linear loop) and any two different shadows $S_j(w)$, $S_{j'}(w)$ are either pairwise disjoint or the centers c_j, $c_{j'}$ lie on the same linear loop parallel to the vector $w\in\Lambda$.

Lemma 5.5. *If $q_{v,\overline{c},\overline{r}}$ is a quadratic differential on $\mathbb{C}/\Lambda$ which is separated by a non-zero vector $w\in\Lambda$ then the Lyapunov exponent $\lambda_{top}(q_{v,\overline{c},\overline{r}})$ is positive.*

In fact, we show the stronger result that all Lyapunov exponents of the orientation cover of the quadratic differential $q_{v,\overline{c},\overline{r}}$ are positive.

Proof. Without loss of generality we may assume $\Lambda=\mathbb{Z}^2$, so $w=(0,1)$ and $v=(1,0)$. This assumption simplifies the argument. Let us divide the slit centers into N cliques ($1\le N\le k$). Centers that lie on the same vertical linear loop are in a clique. Denote by $x_1,\ldots,x_N\in\mathbb{R}/\mathbb{Z}$ the horizontal coordinates of the cliques so that $x_1<x_2<\cdots<x_N<x_1+1$. We will also need cliques of the corresponding slit-folds; two slit-folds are in the same clique, if and only if their shadows in the vertical direction intersect, see Fig. 15.

Suppose that the j-th clique contains $m_j\ge 1$ slit-folds centered at $c_{j,l}:=(x_j,y_{j,l})\in\mathbb{C}/\Lambda$ for $1\le l\le m_j$ so that $y_{j,1}<y_{j,2}<\cdots<y_{j,m_j}<y_{j,1}+1$. Then $\sum_{j=1}^{N}m_j=k$.

Since the quadratic differential is separated by the vertical direction, there are exactly N vertical linear loops that separate the cliques of slit-folds. For $1\le j\le N$, denote by s_j a vertical upward-oriented linear loop separating the j-th and $(j+1)$-th cliques of slit-folds, see Fig. 15. We adopt throughout the periodicity convention that the $(N+1)$-th clique is the first one, i.e. $x_{N+1}=x_1$.

Let $(M_{v,\overline{c},\overline{r}},\omega_{v,\overline{c},\overline{r}})$ be the orientation cover of $(\mathbb{C}/\Lambda,q_{v,\overline{c},\overline{r}})$. Using Forni's criterion we will show that all Lyapunov exponents of $\omega_{v,\overline{c},\overline{r}}$ are positive. This implies the positivity of $\lambda_{top}(q_{v,\overline{c},\overline{r}})$. Let $\pi:(M_{v,\overline{c},\overline{r}},\omega_{v,\overline{c},\overline{r}})\to(\mathbb{C}/\Lambda,q_{v,\overline{c},\overline{r}})$ be the natural projection. Then the holomorphic one form $\omega_{v,\overline{c},\overline{r}}$ lies in $\mathcal{M}(1^{2k})$ and the genus of $M_{v,\overline{c},\overline{r}}$ is $k+1$. More geometrically, $M_{v,\overline{c},\overline{r}}$ is the translation surface made of two copies of a slitted torus $\mathbb{C}/\Lambda$ (denoted by $\mathbb{T}_+$ – left; and $\mathbb{T}_-$ – right), where the slits replace the slit-folds

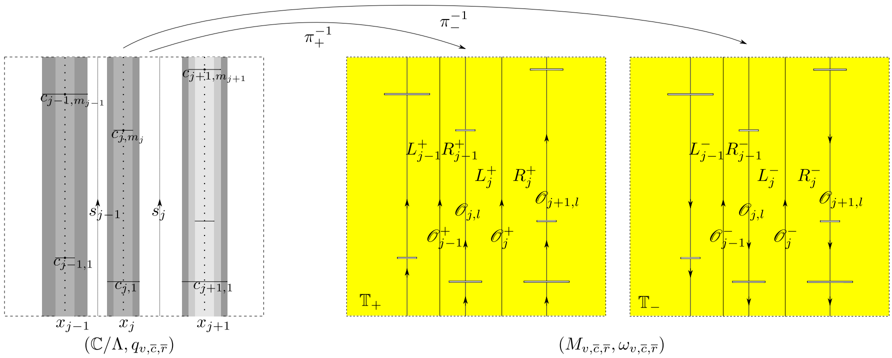

Fig. 15. The half-translation surface $(\mathbb{C}/\Lambda, q_{v,\bar{c},\bar{r}})$ and its orientation cover $(M_{v,\bar{c},\bar{r}}, \omega_{v,\bar{c},\bar{r}})$

on $(\mathbb{C}/\Lambda, q_{v,\bar{c},\bar{r}})$, see Fig. 15. Let $\sigma : M_{v,\bar{c},\bar{r}} \to M_{v,\bar{c},\bar{r}}$ be the involution that exchanges the slitted tori $\mathbb{T}_+$ and $\mathbb{T}_-$ by translation. Finally, each side of any slit on $\mathbb{T}_+$ and $\mathbb{T}_-$ is glued to its σ-image by a 180 degree rotation. Denote by $\pi_\pm^{-1} : \mathbb{C}/\Lambda \to \mathbb{T}_\pm$ the two branches of the inverse of π.

Note that all regular vertical orbits on $(M_{v,\bar{c},\bar{r}}, \omega_{v,\bar{c},\bar{r}})$ are periodic. We distinguish $k + 2N$ such orbits:

- for every $1 \le j \le N$ let $\mathscr{O}_j^\pm = \pi_\pm^{-1}(s_j)$;
- for every $1 \le j \le N$ and $1 \le l \le m_j$ the orbit $\mathscr{O}_{j,l}$ is made of two vertical segments: the first one joins $\pi_+^{-1}(c_{j,l})$ and $\pi_+^{-1}(c_{j,l+1})$ inside $\mathbb{T}_+$ and the second one joins $\pi_-^{-1}(c_{j,l+1})$ and $\pi_-^{-1}(c_{j,l})$ inside $\mathbb{T}_-$ (we adopt the convention that $c_{j,m_j+1} = c_{j,1}$).

Since $\pi_+^{-1}(c_{j,l}) = \pi_-^{-1}(c_{j,l})$ in $M_{v,\bar{c},\bar{r}}$, the above two segments together yield a periodic orbit $\mathscr{O}_{j,l}$.

From these $k + 2N$ periodic orbits we choose $k + 1$, so that the surface obtained after removing the distinguished $k + 1$ orbits from $M_{v,\bar{c},\bar{r}}$ is homeomorphic to the $2(k + 1)$-punctured sphere. The choice of the periodic orbits depends on the parity of N. At first let us look at the surface

$$\underline{M} := M_{v,\bar{c},\bar{r}} \setminus \Big(\bigcup_{j=1}^{N} \mathscr{O}_j^+ \cup \bigcup_{j=1}^{N} \mathscr{O}_j^- \cup \bigcup_{j=1}^{N} \bigcup_{l=1}^{m_j} \mathscr{O}_{j,l} \Big).$$

For every $1 \le j \le N$ let $R_j^\pm$ be the region of $\mathbb{T}_\pm$ that is bounded by the orbit $\mathscr{O}_j^\pm$ and the union $\bigcup_{l=1}^{m_{j+1}} \mathscr{O}_{j+1,l}$, see Fig. 15. Similarly, $L_j^\pm$ is the region of $\mathbb{T}_\pm$ bounded by the orbit $\mathscr{O}_j^\pm$ and the union $\bigcup_{l=1}^{m_j} \mathscr{O}_{j,l}$. Then $\underline{M}$ is the union of $2N$ connected components and each such component $A_j^\pm$ is the union of $L_j^\pm \cup R_{j-1}^\mp$ for $1 \le j \le N$; where we adopt the convention that $R_0^\pm = R_N^\pm$. The component $A_j^\pm$ is homeomorphic to the m_j-punctured annulus ($(m_j + 2)$–punctured sphere) and its boundary consists of orbits $\mathscr{O}_j^\pm$, $\mathscr{O}_{j-1}^\mp$ and $\mathscr{O}_{j,l}$ for $1 \le l \le m_j$, see Fig. 16.

Odd case If N is odd then we take: $\mathscr{O}_1^+$ and $\mathscr{O}_{j,l}$ for $1 \le j \le N$ and $1 \le l \le m_j$. Since $\sum_{j=1}^N m_j = k$, this yields a family of $k + 1$ vertical periodic orbits. Then the

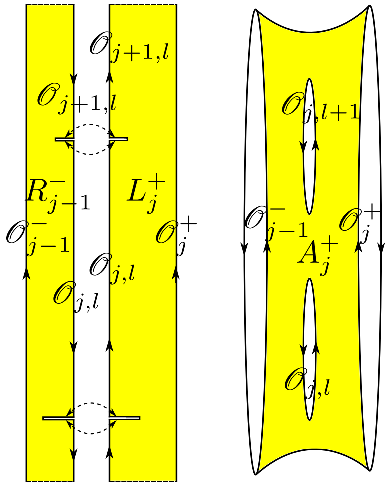

Fig. 16. The annulus A_j^+

surface

$$\underline{M}_1 := M_{v,\bar{c},\bar{r}} \setminus \left(\mathscr{O}_1^+ \cup \bigcup_{j=1}^{N} \bigcup_{l=1}^{m_j} \mathscr{O}_{j,l} \right)$$

is made of the punctured annuli A_j^+, A_j^-, $1 \le j \le N$ glued along the loops $\mathscr{O}_j^+$ for $2 \le j \le N$ and $\mathscr{O}_j^-$ for $1 \le j \le N$. Each such gluing yields a pattern $A_j^+, \mathscr{O}_j^+, A_{j+1}^-$ or $A_j^-, \mathscr{O}_j^-, A_{j+1}^+$; we adopt the convention that $A_{N+1}^\pm = A_1^\pm$. Since N is odd, all such junctures taken together are arranged in the following pattern:

$$A_2^-, \mathscr{O}_2^-, A_3^+, \ldots, A_{N-1}^-, \mathscr{O}_{N-1}^-, A_N^+, \mathscr{O}_N^+, A_1^-, \mathscr{O}_1^-, A_2^+, \ldots, A_{N-1}^+, \mathscr{O}_{N-1}^+, A_N^-, \mathscr{O}_N^-, A_1^+.$$

Since each annulus $A_j^\pm$ has m_j punctures and appears in the above sequence exactly once, it follows that $\underline{M}_1$ is an annulus with $2\sum_{j=1}^N m_j = 2k$ punctures. Therefore, $\underline{M}_1$ is homeomorphic to the $2(k+1)$-punctured sphere.

Even case If N is even then we take $k+1$ vertical periodic orbits: $\mathscr{O}_1^+$, $\mathscr{O}_1^-$, $\mathscr{O}_{1,l}$ for $2 \le l \le m_j$ and $\mathscr{O}_{j,l}$ for $2 \le j \le N$ and $1 \le l \le m_j$. Then the surface

$$\underline{M}_2 := M_{v,\bar{c},\bar{r}} \setminus \left(\mathscr{O}_1^+ \cup \mathscr{O}_1^- \cup \bigcup_{l=2}^{m_1} \mathscr{O}_{1,l} \cup \bigcup_{j=2}^{N} \bigcup_{l=1}^{m_j} \mathscr{O}_{j,l} \right)$$

is made of the punctured annuli A_j^+, A_j^-, $1 \le j \le N$ glued along the loops $\mathscr{O}_j^+$, $\mathscr{O}_j^-$ for $2 \le j \le N$ and $\mathscr{O}_{1,1}$. Each such gluing yields a pattern $A_j^+, \mathscr{O}_j^+, A_{j+1}^-$ or $A_j^-, \mathscr{O}_j^-, A_{j+1}^+$ or $A_1^+, \mathscr{O}_{1,1}, A_1^-$. Since N is even, all such junctures together are arranged in the following pattern:

$$A_2^-, \mathscr{O}_2^-, A_3^+, \ldots, A_{N-1}^+, \mathscr{O}_{N-1}^+, A_N^-, \mathscr{O}_N^-, A_1^+, \mathscr{O}_{1,1}, A_1^-, \mathscr{O}_N^+, A_N^+, \mathscr{O}_{N-1}^-, A_{N-1}^+, \ldots$$
$$\ldots, A_3^-, \mathscr{O}_2^+, A_2^+.$$

Since each annulus $A_j^{\pm}$ has m_j punctures and appears in the above sequence exactly once, it follows that $\underline{M}_2$ is an annulus with $2\sum_{j=1}^N m_j = 2k$ punctures. Therefore, $\underline{M}_1$ is homeomorphic to the $2(k+1)$-punctured sphere.

Applying Proposition 5.4 to the translation surface $(M_{v,\overline{c},\overline{r}}, \omega_{v,\overline{c},\overline{r}})$ then yields the positivity of all Lyapunov exponents of $\omega_{v,\overline{c},\overline{r}}$, and finally the positivity of $\lambda_{top}(q_{v,\overline{c},\overline{r}})$. □

Lemma 5.5 combined with Proposition 5.3 leads to a trapping criterion for slit-folds systems $\widetilde{q}_{v,\overline{c},\overline{r}}$. Recall that $\widetilde{q}_{v,\overline{c},\overline{r}}$ is the half-translation structure on $\mathbb{C}$ given by the system of slit-folds parallel to the vector v, centered at $\{c_1, \ldots, c_k\} + \Lambda$ and radii $r_1, \ldots, r_k$ respectively.

Corollary 5.6. *If $q_{v,\overline{c},\overline{r}}$ is a quadratic differential on $\mathbb{C}/\Lambda$ which is separated by a non-zero vector $w \in \Lambda$ then the measured foliation $\widetilde{\mathcal{F}}_\theta$ of $(\mathbb{C}, \widetilde{q}_{v,\overline{c},\overline{r}})$ is trapped for almost every $\theta \in \mathbb{R}/\pi\mathbb{Z}$.*

Let S be an infinite system of Eaton lenses on $\mathbb{C}$ and let $\theta \in \mathbb{R}/\pi\mathbb{Z}$. Then $\mathscr{P}_{S,\theta}$ is an invariant set for the geodesic flow consisting of four copies of each lens and two copies of the complement of the lenses with planar geometry. This gives a natural projection $\pi_{S,\theta} : \mathscr{P}_{S,\theta} \to \mathbb{C}$ associating the footpoint (in $\mathbb{C}$) to any unit tangent vector in $\mathscr{P}_{S,\theta}$. We call the geodesic flow on $\mathscr{P}_{S,\theta}$ trapped, if

$$\exists_{C>0}\ \exists_{u\in\mathbb{C},|u|=1}\ \forall_{t\in\mathbb{R}}\ \forall_{x\in\mathscr{P}_{S,\theta}}\quad |\langle \pi_{S,\theta}(\mathfrak{g}_t^{S,\theta}x) - \pi_{S,\theta}(x), u\rangle| \le C.$$

Remark 5.7. Note, that the geodesic flow on $\mathscr{P}_{S,\theta}$ is trapped, if and only if

$$\exists_{0<C\in\mathbb{Q}}\ \forall_{N\in\mathbb{N}}\ \exists_{u_N\in\mathbb{Q}\times\mathbb{Q},1\le|u_N|\le 2}\ \forall_{t\in\mathbb{Q}\cap[-N,N]}\ \forall_{y\in\mathbb{Q}\times\mathbb{Q}}$$
$$(\pi_{S,\theta}(x) = y) \implies |\langle \pi_{S,\theta}(\mathfrak{g}_t^{S,\theta}x) - y, u_N\rangle| \le C.$$

Moreover, the geodesic flow on $\mathscr{P}_{S,\theta}$ is trapped, if and only if the direction θ foliation on the corresponding slit-fold plane is trapped.

Let Λ be a lattice on $\mathbb{C}$ and let $\overline{c} = (c_1, c_2, \ldots, c_k) \in \mathbb{C}^k$ be a vector such that the points $c_j + w$ are pairwise distinct for $1 \le j \le k$ and $w \in \Lambda$. Each such vector is called *proper*. A vector of radii $\overline{r} = (r_1, r_2, \ldots, r_k) \in \mathbb{R}^k_{>0}$ is called $(\Lambda, \overline{c})$-*admissible* if $\mathrm{dist}(c_i + \Lambda, c_j + \Lambda) > r_i + r_j$ for $i \ne j$. Admissibility guarantees that Eaton lenses of radius r_j centered at $c_j + \Lambda$ for $1 \le j \le k$ do not intersect. Recall, that such a Λ-periodic system of Eaton lenses is denoted by $L(\Lambda, \overline{c}, \overline{r})$. Of course, the set of $(\Lambda, \overline{c})$-admissible vectors is open in $\mathbb{R}^k$.

Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ be a partition of $\{1, \ldots, k\}$. Then for every $\overline{r} \in \mathbb{R}^k$ and $\overline{x} \in \mathbb{R}^m$ denote by $\overline{r}^{\mathcal{A}}(\overline{x}) = \overline{r}(\overline{x})$ the vector in $\mathbb{R}^k$ defined by $\overline{r}(\overline{x})_j = x_l r_j$ whenever $j \in A_l$. In particular, taking $\overline{x} = \overline{1} = (1, \ldots, 1) \in \mathbb{R}^k$ gives $\overline{r}(\overline{1}) = \overline{r}$.

Denote by $\mathrm{Adm}_{\Lambda,\overline{c},\mathcal{A}} \subset \mathbb{R}^m_{>0}$ the set of all $\overline{x} \in \mathbb{R}^m_{>0}$ such that the vector $\overline{1}^{\mathcal{A}}(\overline{x})$ is $(\Lambda, \overline{c})$-admissible. This is a non-empty open subset.

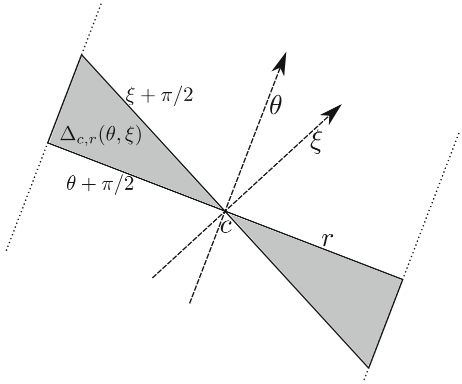

Fig. 17. The set $\Delta_{c,r}(\theta, \xi)$

Theorem 5.8. *Let $\mathcal{A}$ be an m-element partition of $\{1, \ldots, k\}$. Suppose that a vector $\overline{r}_0 \in \mathbb{R}^k_{>0}$ is $(\Lambda, \overline{c})$-admissible. Then for every $\theta_0 \in \mathbb{R}/\pi\mathbb{Z}$ there exists an open neighborhood U of $(\overline{1}, \theta_0)$ in $\mathbb{R}^m_{>0} \times \mathbb{R}/\pi\mathbb{Z}$ such that for almost every $(\overline{x}, \theta) \in U$ the vector $\overline{r}_0^{\mathcal{A}}(\overline{x})$ is $(\Lambda, \overline{c})$-admissible and the geodesic flow on $\mathscr{P}_{L(\Lambda,\overline{c},\overline{r}_0(\overline{x})),\theta}$ is trapped, and hence non-ergodic.*

Proof. First we pass to the flat version of any admissible system $L(\Lambda, \overline{c}, \overline{r})$ and its geodesic flow in direction $\theta \in \mathbb{R}/\pi\mathbb{Z}$. The resulting object is the quadratic differential $\widetilde{q}_{e^{i(\theta+\pi/2)},\overline{c},\overline{r}}$ on $\mathbb{C}$ and its foliation $\widetilde{\mathcal{F}}_\theta$. The $\pi_{L(\Lambda,\overline{c},\overline{r}),\theta}$-projection of geodesic orbits on $\mathscr{P}_{L(\Lambda,\overline{c},\overline{r}),\theta}$ coincide with the leaves of the foliation $\widetilde{\mathcal{F}}_\theta$ outside the lenses.

For every $c \in \mathbb{C}$, $r > 0$, $\theta \in \mathbb{R}/\pi\mathbb{Z}$ and $\xi \in \mathbb{R}/\pi\mathbb{Z} \setminus \{\theta \pm \pi/2\}$ let

$$\Delta_{c,r}(\theta, \xi) = \left\{c + rte^{i\theta}(s\tan(\theta - \xi) + i) : t \in [-1, 1], s \in [0, 1]\right\} \text{(Fig. 17)}.$$

Since $\overline{r}_0$ is $(\Lambda, \overline{c})$-admissible, the line segments $w + \Delta_{c_j,(\overline{r}_0)_j}(\theta_0, \theta_0)$ are pairwise disjoint for $1 \le j \le k$ and $w \in \Lambda$. Therefore we can choose $\epsilon > 0$ such that for all $\overline{r} \in \overline{r}_0\big((0, (1+\epsilon)\sec\epsilon)^m\big)$ and $\theta, \xi \in (\theta_0-\epsilon, \theta_0+\epsilon)$ the sets $w+\Delta_{c_j,r_j}(\theta, \xi)$ are pairwise disjoint for $1 \le j \le k$ and $w \in \Lambda$. Then $\widetilde{q}_{e^{i(\xi+\pi/2)},\overline{c},\sec(\theta-\xi)\overline{r}}$ is a railed deformation of $\widetilde{q}_{e^{i(\theta+\pi/2)},\overline{c},\overline{r}}$ along the direction θ and so their foliations in direction θ are Whitehead equivalent.

Since the set of directions arising from vectors in the lattice Λ is dense, there is a vector $w \in \Lambda$ such that $w/|w| = ie^{i\theta_1}$ with $|\theta_1 - \theta_0| < \epsilon$. Then all slit-folds of $\widetilde{q}_{ie^{i\theta_1},\overline{c},\sec(\theta_0-\theta_1)(1+\epsilon)\overline{r}_0}$ are pairwise disjoint and parallel to the vector $w \in \Lambda$. Since all shadows $S_j(w)$ for $\widetilde{q}_{ie^{i\theta_1},\overline{c},\sec(\theta_0-\theta_1)(1+\epsilon)\overline{r}_0}$ are linear loops such that $S_j(w)$, $S_{j'}(w)$ are either pairwise disjoint or $S_j(w) = S_{j'}(w)$ if the centers c_j, $c_{j'}$ lie on the same linear loop parallel to $w \in \Lambda$, by continuity, there exists $\delta > 0$ such that if $0 < |\theta - \theta_1| < \delta$ then $\widetilde{q}_{ie^{i\theta},\overline{c},\sec(\theta_0-\theta)(1+\epsilon)\overline{r}_0}$ is separated by $w \in \Lambda$. Next, choose a direction $\theta_2 \neq \theta_1$ so that $|\theta_2 - \theta_0| < \epsilon$ and $|\theta_2 - \theta_1| < \delta$. Then $\widetilde{q}_{ie^{i\theta_2},\overline{c},\sec(\theta_0-\theta_2)(1+\epsilon)\overline{r}_0}$ is separated by $w \in \Lambda$. It follows that $\widetilde{q}_{ie^{i\theta_2},\overline{c},\sec(\theta_0-\theta_2)\overline{r}_0(\overline{x})}$ is separated by $w \in \Lambda$ for every $\overline{x} \in (1-\epsilon, 1+\epsilon)^m$. Therefore, by Corollary 5.6, for every $\overline{x} \in (1-\epsilon, 1+\epsilon)^m$ and for a.e. $\theta \in \mathbb{R}/\pi\mathbb{Z}$ (in particular, for a.e. $\theta \in (\theta_0 - \epsilon, \theta_0 + \epsilon)$) the foliation $\widetilde{\mathcal{F}}_\theta$ on $\mathbb{C}$ derived from $\widetilde{q}_{ie^{i\theta_2},\overline{c},\sec(\theta_0-\theta_2)\overline{r}_0(\overline{x})}$ is trapped.

On the other hand for every $\overline{x} \in (1-\epsilon, 1+\epsilon)^m$ and $\theta \in (\theta_0 - \epsilon, \theta_0 + \epsilon)$ we have

$$\cos(\theta - \theta_2)\sec(\theta_0 - \theta_2)\overline{x} \in (0, (1+\epsilon)\sec\epsilon)^m \text{ and } \theta, \theta_2 \in (\theta_0 - \epsilon, \theta_0 + \epsilon).$$

Hence the quadratic differential $\widetilde{q}_{ie^{i\theta_2},\overline{c},\sec(\theta_0-\theta_2)\overline{r}_0(\overline{x})}$ is a railed deformation of $\widetilde{q}_{ie^{i\theta},\overline{c},\cos(\theta-\theta_2)\sec(\theta_0-\theta_2)\overline{r}_0(\overline{x})}$ along the direction θ. It follows that for every $\overline{x} \in (1-\epsilon, 1+\epsilon)^m$ and for a.e. $\theta \in (\theta_0-\epsilon, \theta_0+\epsilon)$ the foliation $\widetilde{\mathcal{F}}_\theta$ on $\mathbb{C}$ derived from $\widetilde{q}_{ie^{i\theta},\overline{c},\cos(\theta-\theta_2)\sec(\theta_0-\theta_2)\overline{r}_0(\overline{x})}$ is trapped, and hence the geodesic flow restricted to $\mathscr{P}_{L(\Lambda,\overline{c},\cos(\theta-\theta_2)\sec(\theta_0-\theta_2)\overline{r}_0(\overline{x})),\theta}$ is also trapped.

By Remark 5.7, trapping is a measurable condition. Then a Fubini argument shows, that the geodesic flow on $\mathscr{P}_{L(\Lambda,\overline{c},\overline{r}_0(\cos(\theta-\theta_2)\sec(\theta_0-\theta_2)\overline{x})),\theta}$ is trapped for a.e. $(\overline{x},\theta) \in (1-\epsilon, 1+\epsilon)^m \times (\theta_0-\epsilon, \theta_0+\epsilon)$. Moreover, the map

$$(\overline{x},\theta) \mapsto (\cos(\theta-\theta_2)\sec(\theta_0-\theta_2)\overline{x}, \theta)$$

on $(1-\epsilon, 1+\epsilon)^m \times (\theta_0-\epsilon, \theta_0+\epsilon)$ is a C^∞ diffeomorphism. Denote by U its image which is an open neighborhood of $(\overline{1},\theta_0)$. It follows that $\mathscr{P}_{L(\Lambda,\overline{c},\overline{r}_0(\overline{x})),\theta}$ is trapped for a.e. $(\overline{x},\theta) \in U$, which completes the proof. □

As a corollary we obtain the following more general version of Theorem 1.2.

Corollary 5.9. *For every lattice* $\Lambda \subset \mathbb{C}$, *every proper vector of centers* $\overline{c} \in \mathbb{C}^k$ *and every partition* $\mathcal{A}$ *of* $\{1,\ldots,k\}$ *the geodesic flow on* $\mathscr{P}_{L(\Lambda,\overline{c},\overline{1}(\overline{r})),\theta}$ *is trapped for a.e.* $(\overline{r},\theta) \in Adm_{\Lambda,\overline{c},\mathcal{A}} \times \mathbb{R}/\pi\mathbb{Z}$.

Example 1. Let $\Lambda := \mathbb{Z}(0,4) \oplus \mathbb{Z}(4,2)$. For every $\theta \in [0,\pi/4)$ let us consider the Λ-periodic pattern of lenses

$$\mathcal{L}_\theta = L\big(\Lambda, (0, \pm(1+i(1+\tan\theta)), (2\sin\theta, \cos\theta)\big).$$

This is the pattern of lenses drawn on Fig. 2. By Theorem 1.4, for a.e. $\theta_0 \in [0,\pi/4)$ the geodesic flow on $\mathscr{P}_{\mathcal{L}_{\theta_0},\theta_0}$ is ergodic. On the other hand each pair $(\mathcal{L}_{\theta_0},\theta_0)$ satisfies the assumption of Theorem 5.8. Let us consider the partition $\mathcal{A} = \{\{1\},\{2,3\}\}$. Then, by Theorem 5.8, after almost every small perturbation of the direction θ_0, the radius of the central lens and the radii of the pair of symmetrically placed lenses, the ergodic properties of the geodesic flow change dramatically to a highly non-ergodic trapped flow.

Let us now consider the partitions $\{\{1\},\{2\},\{3\}\}$ and $\{\{1,2,3\}\}$. By applying Theorem 5.8 to those, we obtain another type of results saying, that almost every small perturbation of $(\mathcal{L}_{\theta_0},\theta_0)$ leads to a trapped geodesic flow. In the first case all radii are perturbed independently whereas in the second case all radii are perturbed simultaneously.

In summary, the curves of ergodic lens distributions described in the paper are very exceptional. They are surrounded by highly non-ergodic systems. We have shown this phenomenon only for a particular "ergodic" curve, but for the other "ergodic" curves it can be shown along the same lines.

Moreover, we conjecture that the trapping property is measurably typical along many curves transversal to the ergodic curves described in the paper. An interesting and highly involved result of that type was proved in [16], where the authors consider curves arising from fixed systems of lenses for which the direction θ varies.

Acknowledgements. We would like to thank the anonymous referees for suggestions to improve the clarity of the paper.

Appendix A. Eaton Lens Dynamics

To precisely describe the dynamics of light rays passing through an Eaton lens, we denote the lens of radius $R > 0$ and centered at $(0, 0)$ by $\overline{B}_R$. The refractive index (RI for short) in $\overline{B}_R$ depends only on the distance from the center $r := \sqrt{x^2 + y^2} \in (0, R]$ and is given by the formula $n(x, y) = n(r) = \sqrt{2R/r - 1}$; at the center we put $n(0, 0) = +\infty$. Suppose, for simplicity, that the refractive index $n(x, y)$ is constant and equals 1 outside $\overline{B}_R$. Recall that the dynamics of light rays can be described as the geodesic flow on $\mathbb{R}^2 \setminus (0, 0)$ equipped with the Riemannian metric $g = n \cdot (\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y)$. Of course, the geodesics are straight lines or semi-lines outside $\overline{B}_R$. The dynamics of the geodesic flow inside $\overline{B}_R$ was described for example in [21]. After passing to polar coordinates (r, θ) we use the Euler-Lagrange equation to see that any geodesic inside $\overline{B}_R$ satisfies

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \pm\frac{r\sqrt{n(r)^2r^2 - n(r_0)^2r_0^2}}{n(r_0)r_0} = \pm\frac{r\sqrt{r(2R-r) - r_0(2R-r_0)}}{\sqrt{r_0(2R-r_0)}}, \tag{A.1}$$

where (r_0, θ_0) is a point of the geodesic minimizing the distance to the center. It follows that for any point (r, θ) of the geodesic in $\overline{B}_R$ we have

$$\begin{aligned}\pm(\theta - \theta_0) &= \int_{r_0}^{r} \frac{\sqrt{r_0(2R-r_0)}}{u\sqrt{u(2R-u) - r_0(2R-r_0)}}\,\mathrm{d}u \\ &= \Big[\arcsin\frac{Ru - r_0(2R-r_0)}{u(R-r_0)}\Big]_{r_0}^{r} = \arcsin\frac{Rr - r_0(2R-r_0)}{r(R-r_0)} + \frac{\pi}{2}.\end{aligned}$$

Consequently

$$-\cos(\theta - \theta_0) = \frac{Rr - r_0(2R-r_0)}{r(R-r_0)}, \tag{A.2}$$

and hence

$$\Big(\frac{r\cos(\theta-\theta_0) + (R-r_0)}{R}\Big)^2 + \frac{(r\sin(\theta-\theta_0))^2}{R^2 - (R-r_0)^2} = 1.$$

In particular inside of $\overline{B}_R$ the geodesic is an arc of an ellipse. Let $s := \sqrt{R^2 - (R-r_0)^2}$ and rotate the geodesic by $-\theta_0$. Then the equation of the ellipse becomes

$$\Big(\frac{x + \sqrt{R^2 - s^2}}{R}\Big)^2 + \Big(\frac{y}{s}\Big)^2 = 1.$$

Since the ellipse is centered at $(-\sqrt{R^2 - s^2}, 0)$ and $(-\sqrt{R^2 - s^2}, \pm s)$ are its intersection points with the boundary of $\overline{B}_R$, the geodesic has horizontal tangents at these intersecting points. Rotating everything back to the original position we see, that the direction of any geodesic is reversed after passing through $\overline{B}_R$. The only exception is the trajectory that hits the center of the lens. For this trajectory we adopt the convention, that at the center it turns and continues its motion backwards.

Now for every $\theta \in \mathbb{R}/\pi\mathbb{Z}$ consider the restriction of the geodesic flow $(\mathfrak{g}_t^\theta)_{t\in\mathbb{R}}$ to its invariant subset of the unit tangent bundle of $\mathbb{R}^2$ consisting of all trajectories assuming direction θ or $\pi + \theta$ outside $\overline{B}_R$. Denote by $\mathscr{P}_\theta$ the phase space of that flow. Since all flows $(\mathfrak{g}_t^\theta)_{t\in\mathbb{R}}$ are isomorphic by rotations, we restrict our considerations to the horizontal flow $(\mathfrak{g}_t)_{t\in\mathbb{R}} = (\mathfrak{g}_t^0)_{t\in\mathbb{R}}$.

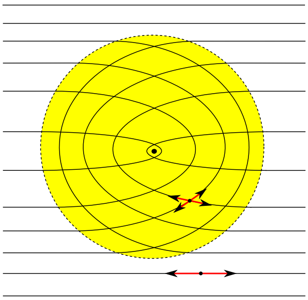

Fig. 18. Flow directions inside and outside of an Eaton lens

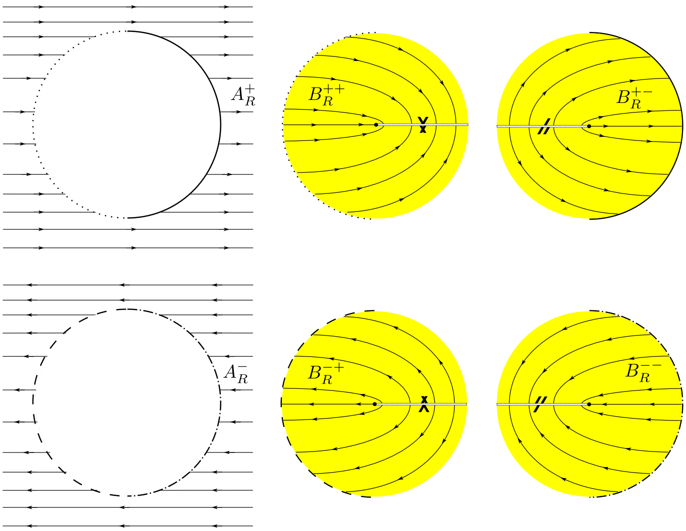

Fig. 19. Phase space of the horizontal flow in a neighborhood of an Eaton lens

Denote by B_R the interior of $\overline{B}_R$. Through every point of B_R pass exactly four trajectories of $(\mathfrak{g}_t)_{t\in\mathbb{R}}$, while through every point of $A_R := \mathbb{R}^2 \setminus B_R$ pass exactly two, in direction 0 and π, see Fig. 18. It follows, that $\mathscr{P}_\theta$ consists of four copies of B_R ($B_R^{\pm\pm}$) and two copies of A_R ($A_R^{\pm}$).

Let us take a closer look at the dynamics $(\mathfrak{g}_t)_{t\in\mathbb{R}}$ on the four copies of $B_R^{\pm\pm}$. Since they are related by a reflective symmetry or the reversal of time, we can restrict our considerations to one of them, say B_R^{++} in Fig. 19. Consider the transversal curve for the flow $(\mathfrak{g}_t)_{t\in\mathbb{R}}$ represented by the dotted semicircle C_R parameterized by $(-\sqrt{R^2-s^2}, s)$ for $s \in (-R, R)$. In view of (A.1) and (A.2) the trajectory of the point $(-\sqrt{R^2-s^2}, s) \in C_R$ travels on an ellipse that is in polar coordinates given by

$$-\cos\theta = \frac{Rr - s^2}{r\sqrt{R^2-s^2}} \quad \text{and} \quad \frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{r\sqrt{r(2R-r)-s^2}}{s} \tag{A.3}$$

before it escapes B_R^{++}. Let us write $(r(t,s), \theta(t,s))$ for the polar coordinates of $\mathfrak{g}_{t+\sqrt{R^2-s^2}}(-\sqrt{R^2-s^2}, s)$, then $(r(-\sqrt{R^2-s^2}, s), \theta(-\sqrt{R^2-s^2}, s))$ are the polar coordinates of the point $(-\sqrt{R^2-s^2}, s)$. Since the velocity vectors of the geodesic flow have unit length with respect to the Riemannian metric g, we obtain

$$\Big(\frac{\partial r}{\partial t}\Big)^2 + r^2\Big(\frac{\partial \theta}{\partial t}\Big)^2 = \frac{1}{n^2(r)} = \frac{r}{2R-r}.$$

Because of (A.3), we have

$$\frac{\partial \theta}{\partial t} = \frac{\partial r}{\partial t}\frac{s}{r\sqrt{r(2R-r)-s^2}} \tag{A.4}$$

and hence

$$\frac{r}{2R-r} = \Big(\frac{\partial r}{\partial t}\Big)^2\Big(1 + r^2\frac{s^2}{r^2(r(2R-r)-s^2)}\Big) = \Big(\frac{\partial r}{\partial t}\Big)^2\frac{r(2R-r)}{r(2R-r)-s^2}.$$

Therefore,

$$\frac{\partial r}{\partial t} = -\frac{\sqrt{r(2R-r)-s^2}}{2R-r}. \tag{A.5}$$

Hence

$$\begin{aligned} t+\sqrt{R^2-s^2} &= \int_R^{r(t,s)} \frac{u-2R}{\sqrt{u(2R-u)-s^2}}\,\mathrm{d}u \\ &= \Big[-\sqrt{u(2R-u)-s^2} + R\arcsin\frac{R-u}{\sqrt{R^2-s^2}}\Big]_R^{r(t,s)} \\ &= -\sqrt{r(2R-r)-s^2} + R\arcsin\frac{R-r}{\sqrt{R^2-s^2}} + \sqrt{R^2-s^2} \end{aligned}$$

and

$$t = -\sqrt{r(2R-r)-s^2} + R\arcsin\frac{R-r}{\sqrt{R^2-s^2}}.$$

Let t_s be the exit time of $(-\sqrt{R^2-s^2}, s)$ from B_R^{++}. Since $r(t_s,s)$ minimizes the distance to the origin, we have $s^2 = R^2-(R-r(t_s,s))^2 = r(t_s,s)(2R-r(t_s,s))$. It follows that

$$t_s = R\arcsin 1 = \frac{1}{2}\pi R.$$

Introduce new coordinates on B_R^{++} given by (t,s). Then the set $E_R = B_R \cup ([0,\pi R/2)\times(-R,R))$ is the domain of these coordinates and they coincide with the cartesian coordinates on C_R. Moreover, by definition, the geodesic flow $(\mathfrak{g}_t)_{t\in\mathbb{R}}$ in the new coordinates is the unit horizontal translation in positive direction.

One can define the same type of coordinates on the other copies B_R^{+-}, B_R^{-+} and B_R^{--}. Let us consider a measure μ on $\mathscr{P}_0$ that coincides with the Lebesgue measure on $A_R^{\pm}$ and the Lebesgue measure in the new coordinates on each $B_R^{\pm\pm}$. This is a $(\mathfrak{g}_t)_{t\in\mathbb{R}}$-invariant measure and we will calculate its density in the next paragraph.

In view of (A.5) and (A.4) we have

$$\frac{\partial r}{\partial t} = -\frac{\sqrt{r(2R-r)-s^2}}{2R-r} \quad\text{and}\quad \frac{\partial \theta}{\partial t} = -\frac{s}{r(2R-r)}. \tag{A.6}$$

As

$$t = -\sqrt{r(2R-r)-s^2} + R \arcsin \frac{R-r}{\sqrt{R^2-s^2}},$$

differentiating it in direction s we obtain

$$0 = -\frac{\frac{\partial r}{\partial s}(R-r)-s}{\sqrt{r(2R-r)-s^2}} + R\frac{-\frac{\partial r}{\partial s}+\frac{(R-r)s}{R^2-s^2}}{\sqrt{r(2R-r)-s^2}}.$$

Hence

$$\frac{\partial r}{\partial s} = s\frac{R(2R-r)-s^2}{(R^2-s^2)(2R-r)}. \tag{A.7}$$

Differentiating the first equality of (A.3) in direction s we obtain

$$\frac{s\sqrt{r(2R-r)-s^2}}{r\sqrt{R^2-s^2}}\frac{\partial \theta}{\partial s} = \sin\theta \cdot \frac{\partial \theta}{\partial s} = \frac{\partial r}{\partial s}\frac{s^2}{r^2\sqrt{R^2-s^2}} - \frac{s(R(2R-r)-s^2)}{r\sqrt{R^2-s^2}(R^2-s^2)}.$$

In view of (A.7), it follows that

$$\begin{aligned}\frac{\partial \theta}{\partial s} &= \frac{1}{\sqrt{r(2R-r)-s^2}}\Big(\frac{s^2(R(2R-r)-s^2)}{r(R^2-s^2)(2R-r)} - \frac{(R(2R-r)-s^2)}{R^2-s^2}\Big)\\ &= -\frac{R(2R-r)-s^2}{(R^2-s^2)}\frac{\sqrt{r(2R-r)-s^2}}{r(2R-r)}.\end{aligned} \tag{A.8}$$

Putting (A.6), (A.7) and (A.8) together, we have

$$\begin{aligned}\Big|\frac{\partial r}{\partial s}\cdot\frac{\partial \theta}{\partial t} - \frac{\partial r}{\partial t}\cdot\frac{\partial \theta}{\partial s}\Big| &= \Big|\frac{s^2(R(2R-r)-s^2)}{r(R^2-s^2)(2R-r)^2} + \frac{(r(2R-r)-s^2)(R(2R-r)-s^2)}{r(R^2-s^2)(2R-r)^2}\Big|\\ &= \frac{(R(2R-r)-s^2)}{(R^2-s^2)(2R-r)} = \frac{1}{2R-r}\Big(1+\frac{R(R-r)}{R^2-s^2}\Big).\end{aligned}$$

By (A.3),

$$\sqrt{R^2-s^2} = \frac{\sqrt{r^2\cos^2\theta+4R(R-r)}-r\cos\theta}{2} = \frac{2R(R-r)}{\sqrt{r^2\cos^2\theta+4R(R-r)}+r\cos\theta}.$$

Hence,

$$\Big|\frac{\partial r}{\partial s}\cdot\frac{\partial \theta}{\partial t} - \frac{\partial r}{\partial t}\cdot\frac{\partial \theta}{\partial s}\Big| = \frac{1}{2R-r}\Big(1+\frac{\big(\sqrt{r^2\cos^2\theta+4R(R-r)}+r\cos\theta\big)^2}{4R(R-r)}\Big)$$

Therefore, the density of the invariant measure μ restricted to B_R^{++} in the cartesian coordinates is

$$\xi_R(x,y) = \frac{2R-r}{r}\frac{4R(R-r)}{(\sqrt{x^2+4R(R-r)}+x)^2+4R(R-r)}.$$

On the other copies $B_R^{\pm\pm}$ the measure μ is given by $\xi_R(\pm x, y)\,\mathrm{d}\,x\,\mathrm{d}\,y$.

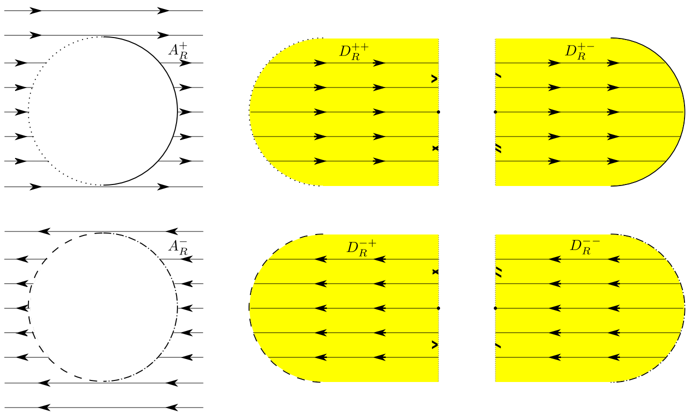

Fig. 20. Linearized Eaton lens flow in phase space

For every $\theta \in \mathbb{R}/\pi\mathbb{Z}$ the flows $(\mathfrak{g}_t^\theta)_{t\in\mathbb{R}}$ phase space $\mathscr{P}_\theta$ is given by the rotation of $\mathscr{P}_0$ by θ and the invariant measure μ_θ is the rotation of μ by the same angle.

Generally instead of one Eaton lens on the plane we deal with a pattern $\mathcal{L}$ of infinitely many pairwise disjoint Eaton lenses on $\mathbb{R}^2$. We are interested in the dynamics of the light rays provided by the geodesic flow $(\mathfrak{g}_t^{\mathcal{L}})_{t\in\mathbb{R}}$ on $\mathbb{R}^2$ without the centers of lenses; the Riemann metric is given by $g_{(x,y)} = n(x, y)\cdot(dx\otimes dx + dy\otimes dy)$. The local behavior of the flow around any lens was described in detail previously. For every $\theta \in \mathbb{R}/\pi\mathbb{Z}$ there exists an invariant set $\mathscr{P}_{\mathcal{L},\theta}$ in the unit tangent bundle, such that all trajectories on $\mathscr{P}_{\mathcal{L},\theta}$ are tangent to $\pm e^{i\theta}$ outside the lenses. The restriction of $(\mathfrak{g}_t^{\mathcal{L}})_{t\in\mathbb{R}}$ to $\mathscr{P}_{\mathcal{L},\theta}$ is denoted by $(\mathfrak{g}_t^{\mathcal{L},\theta})_{t\in\mathbb{R}}$. Moreover, $(\mathfrak{g}_t^{\mathcal{L},\theta})_{t\in\mathbb{R}}$ possesses a natural invariant measure $\mu_{\mathcal{L},\theta}$ equivalent to the Lebesgue measure on $\mathscr{P}_{\mathcal{L},\theta}$. The density of $\mu_{\mathcal{L},\theta}$ is equal to one outside lenses and inside every lens of radius R centered at (c_1, c_2) is determined by $\xi_R(\pm(x-c_1), y-c_2)$ depending on its copy in the phase space. Moreover, the density is continuous on $\mathscr{P}_{\mathcal{L},\theta}$ and piecewise C^∞.

A.1. From the geodesic flow to translation surfaces and measured foliations. For simplicity we return to a single lens and the horizontal flow $(\mathfrak{g}_t)_{t\in\mathbb{R}}$ on $\mathscr{P}_0$. Representing $\mathscr{P}_0$ in (t, s) coordinates, we can treat it as the union on $A_R^\pm$ and $D_R^{\pm\pm}$, see Fig. 20. Moreover, the new coordinates give rise to a translation structure on the surface $\mathscr{P}_0$. Since the horizontal sides of $D_R^{\pm\pm}$ do not belong to $\mathscr{P}_0$, the surface is not closed. However, we can complete the surface by adding the horizontal sides as in Fig. 21. Let us denote the completed surface by $\overline{\mathscr{P}}_0$. It has two singular points with the cone angle 6π which are connected by two horizontal saddle connections labeled by A and B in Fig. 21. Moreover, the flow $(\mathfrak{g}_t)_{t\in\mathbb{R}}$ is measure-theoretically isomorphic to the horizontal translation flow on the translation surface $\overline{\mathscr{P}}_0$.

Let us consider an involution $\sigma : \overline{\mathscr{P}}_0 \to \overline{\mathscr{P}}_0$ given by the translation between upper and lower parts of $\overline{\mathscr{P}}_0$ in Fig. 20. Then the quotient surface $\mathscr{Q}_0 = \overline{\mathscr{P}}_0/ < \sigma >$ is a half-translation surface. It has two singular points (marked by circles) having cone angle 3π connected by a horizontal saddle connection labeled by A (and then continued as A') and two poles (marked by squares), see Fig. 22.

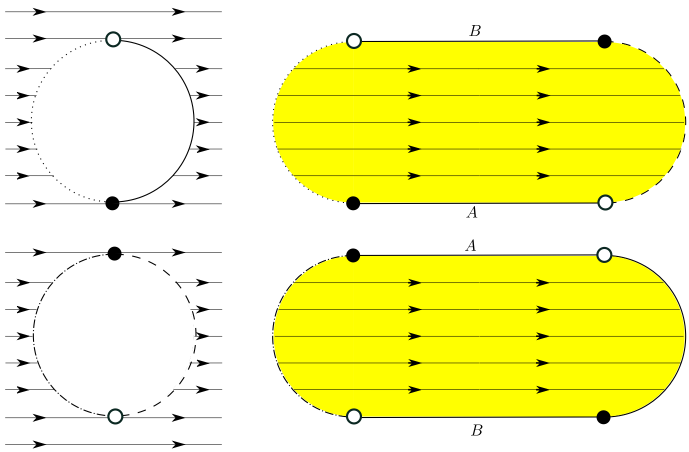

Fig. 21. The completed linearized phase space is a translation surface

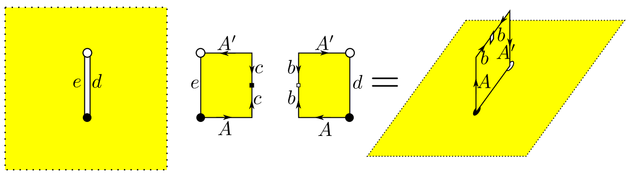

Fig. 22. The quadratic surface $\mathscr{Q}_0$

If we consider an infinite pattern $\mathcal{L}$ of Eaton lenses on $\mathbb{R}^2$, then for every $\theta \in \mathbb{R}/\pi\mathbb{Z}$ we can similarly represent the space $\mathscr{P}_{\mathcal{L},\theta}$ as a translation surface which after a completion is a closed translation surface $\overline{\mathscr{P}}_{\mathcal{L},\theta}$. The translation flow $(\varphi_t^{\mathcal{L},\theta})_{t\in\mathbb{R}}$ on $\overline{\mathscr{P}}_{\mathcal{L},\theta}$ in direction θ is measure-theoretically isomorphic to the flow $(\mathfrak{g}_t^{\mathcal{L},\theta})_{t\in\mathbb{R}}$. Moreover, the surface $\overline{\mathscr{P}}_{\mathcal{L},\theta}$ has an natural involution σ which maps a unit vector to the vector at the same foot-point but oppositely directed. The quotient surface $\mathscr{Q}_{\mathcal{L},\theta} = \overline{\mathscr{P}}_{\mathcal{L},\theta}/ < \sigma >$ is a half-translation surface that is the euclidian plane with a system of pockets each attached at the place of the corresponding lens. Each pocket is a rotated (by θ) version of the pocket in Fig. 22. Its length is equal to the diameter of the corresponding lens and is perpendicular to θ. Most relevant for us, the ergodicity of measured foliation $\mathcal{F}_\theta^{\mathcal{L}}$ in direction θ on $\mathscr{Q}_{\mathcal{L},\theta}$ is equivalent to the ergodicity of $(\varphi_t^{\mathcal{L},\theta})_{t\in}$, and hence to the ergodicity of the flow $(\mathfrak{g}_t^{\mathcal{L},\theta})_{t\in\mathbb{R}}$.

The measured foliation $\mathcal{F}_\theta^{\mathcal{L}}$ is Whitehead equivalent to the foliation $\mathcal{FL}_\theta^{\mathcal{L}}$ where each attached briefcase is replaced by the slit-fold stemming from the "flat lens" representation of the same Eaton lens in direction θ, as in Fig. 23.

In summary, instead of studying the ergodic properties of the geodesic flow $(\mathfrak{g}_t^{\mathcal{L},\theta})_{t\in\mathbb{R}}$ on the plane with a system of Eaton lenses it suffices to pass to the measured foliation $\mathcal{FL}_\theta^{\mathcal{L}}$ where each Eaton lens is replaced by the corresponding flat lens of the same center and diameter as the lens attached perpendicular to θ.

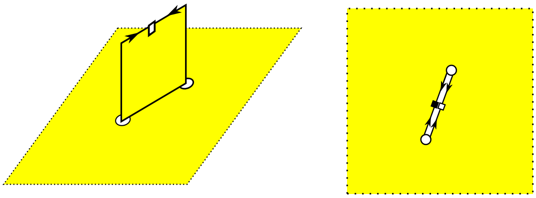

Fig. 23. The half-translation equivalent to an Eaton lens

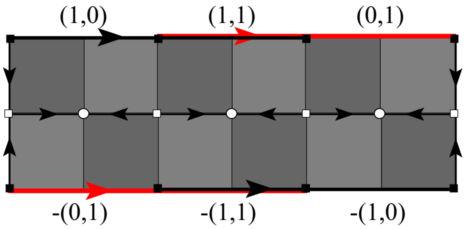

Fig. 24. Homology generators and deck changes

Appendix B. Folds and Skeletons

In this section we describe examples of ergodic curves obtained from other torus differentials. Starting with some of the quadratic differentials in our table one obtains quadratic differentials on the plane that are not pre-Eaton differentials. This section shows ways how to convert those into pre-Eaton differentials. In particular, we need to consider quadratic differentials on the plane that cut out holes and those need to be removed. The holes are described by pillow-folds, that will be converted to an appropriate union of slit-folds.

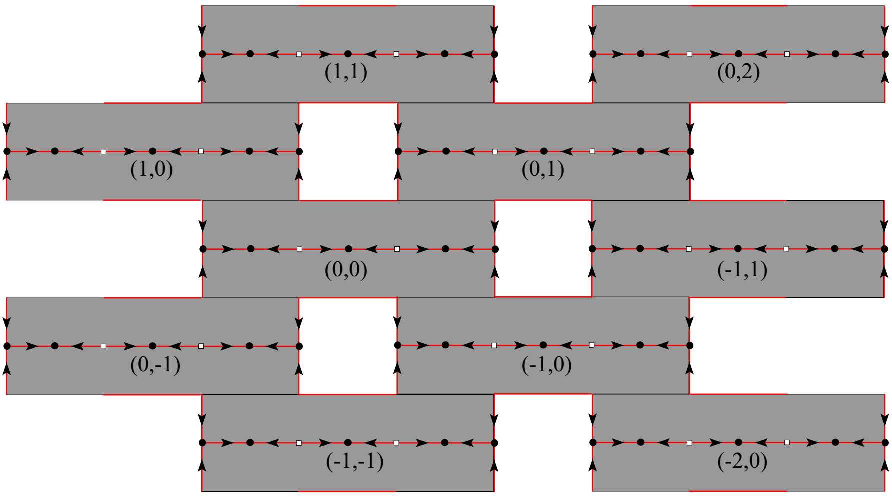

Fig. 25. The universal homology cover $\widetilde{X}_6(3, 1)$

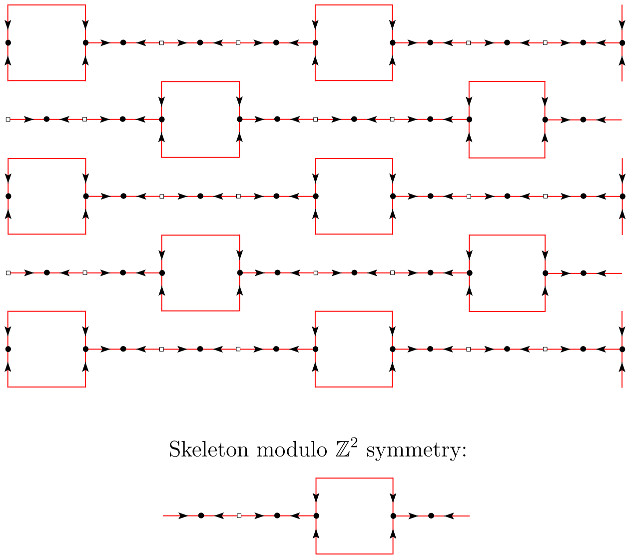

Fig. 26. Skeleton representation of $\mathbb{C}_6(3, 1)$

B.1. Skeleton representation for $\mathbb{Z}^2$-covers of $X_d(a, b)$. First, we convert the standard polygonal representation of a pillowcase cover $X_d(a, b)$ into a pre-Eaton differential. Recall from Sect. 3 that for $X_3(2, 1)$, $X_4(2, 1)$ and $X_6(3, 1)$ this can be done by a central cut followed by turning one half underneath the other. After the half-turn the absolute homology generators are arranged as shown in Fig. 24 for $X_6(3, 1)$. The arrangement for the absolute homology of $X_3(2, 1)$ after the half-turn is similar: The two homology generators overlap in the middle third of the rectangle representing the surface. Now we can represent the universal cover $\widetilde{X}_6(3, 1) \to X_6(3, 1)$ determined by the pair of homology generators. Let us label the deck shifts as in Fig. 24, and the decks by $\mathbb{Z}^2$. Then, starting at deck $(0, 0)$ we reach deck $(1, 0)$, once crossing the left third of the rectangles upper edge and we reach deck $(0, 1)$ when crossing the right third. We enter deck $(1, 1)$ when crossing the middle third of the upper edge and so forth. The labeled tiles of Fig. 25 show the cover. The rectangular holes lead to jumps of the directional dynamics in the plane. In particular the skeleton describing the quadratic differential contains boundaries of the spared rectangles besides the slit-folds, see Fig. 26. Let us now forget the covering and just consider the skeleton in the plane. While previously only the dynamics outside the spared rectangles was defined and investigated we now extend the quadratic differential to the rectangles inside as follows: The folded parts of any rectangle are genuine slit-folds and the edge pair identified by a translations are now translation identified from the inside as well. That way we obtain a quadratic differential on the whole plane that we denote by $\mathbb{C}_6(3, 1)$. The notation, a combination of the standard complex plane notation together with the weight notation of the pillow case cover, will be used for other surfaces below. The "inside" of each rectangle is a pillowcase carrying invariant foliations. The natural extension promotes an easy geometric definition of the foliation: Given a direction $\theta \in \mathbb{R}/\pi\mathbb{Z}$ consider the non oriented lines parallel to $\pm e^{i\theta}$ in $\mathbb{C}$. Then put a skeleton in the plane and identify the intersection points of the leaves with the skeleton according to the respective rules, i.e. translation or central rotation.

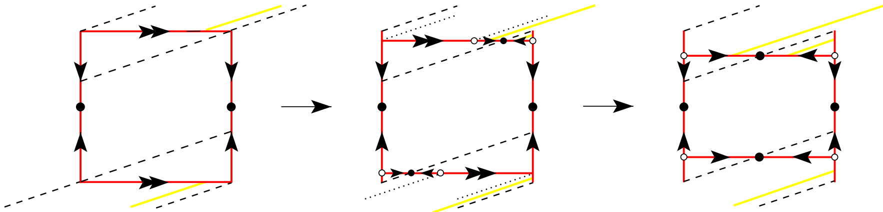

Fig. 27. A railed deformation applied to both horizontal segments of a pillow-fold. The dotted lines indicate some leaves of the direction foliation. The endpoints of the line segments, marked by a black dot, move along a leaf

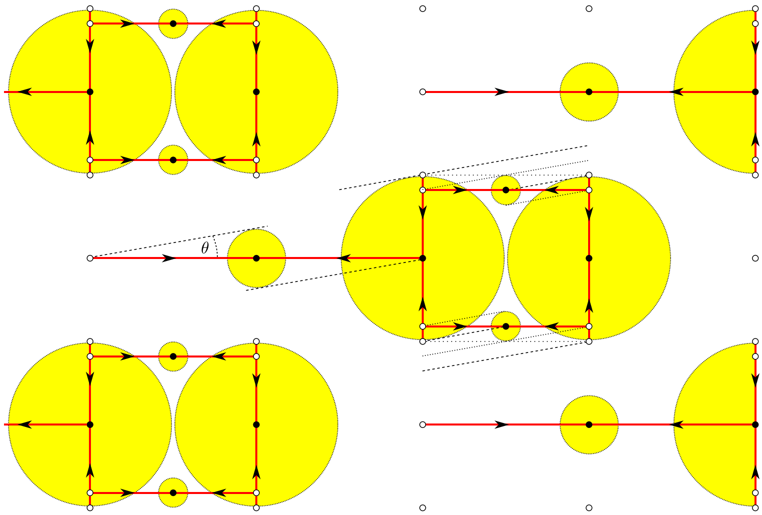

Fig. 28. Eaton lens configuration for $\mathbb{C}_3(2, 1)$

The extension of the quadratic differential, i.e. $\widetilde{X}_6(3, 1)$, to the whole complex plane, i.e. $\mathbb{C}_6(3, 1)$, is a step towards the conversion of the skeleton into an admissible Eaton lens configuration. In fact, this first step allows us to deform the "outside" quadratic differential into a pre-Eaton differential, as shown in Fig. 27. In view of both, Eaton lens dynamics and the respective quadratic differential foliation in the plane, the conversion removes the jump of leaves over the rectangular gaps and replaces it by an equivalent jumpfree dynamics. This deformation is shown in Fig. 27. It is a railed deformation moving a pair of translation identified lines through two slit-folds, ultimately changing the lines into a pair of slit-folds. Because singular points are broken up, this is not a railed deformation in the strict sense of our definition, but it does not change the measurable dynamics.

Pillow-folds and chip-folds. All we did so far for rectangular folds can be done for folds built from a parallelogram.

Let $[a, b]$ and $[c, d]$ are two non-parallel line segments in $\mathbb{C}$, so that $c \in [a, (b+a)/2]$. By $[a, b] \boxtimes [c, d]$ we denote the union of segments $[a, b]$, $[a+(d-c), b+(d-c)]$, $[c, d]$ and $[b-(c-a), b-(c-a)+(d-c)]$. Replacing all line segments in $[a, b] \boxtimes [c, d]$ by slit-folds, we get a what we call a *chip-fold* denoted by $\rangle a, b\langle \boxtimes \rangle c, d\langle$. A *pillow fold*

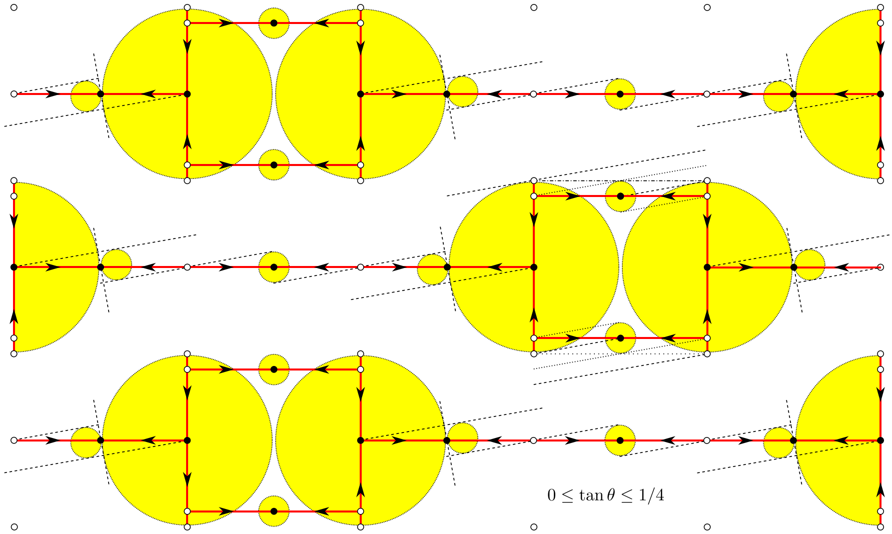

Fig. 29. Eaton lens configuration for $\mathbb{C}_6(3, 1)$

on the other hand is obtained by identifying two segments parallel to, say $[c, d]$, with a translation parallel to $[a, b]$ and the two other segments with slit-folds. Let us denote this fold by $\rangle a, b\langle \boxtimes | c, d|$. Chip-folds are parts of the skeletons in Figs. 28 and 29. Chip-folds and their generalization are necessary to replace the jumps, created by the translation identification in pillow-folds.

If $[a, b]$ and $[c, d]$ are line segments, so that $c \in [a, (b+a)/2]$, then for $n \geq 2$ define $\rangle a, b\langle \boxtimes\, n{\cdot}\rangle c, d\langle$ to be

$$\left(\bigcup_{k=0}^{n-1} \big(\rangle a, b\langle \cup \rangle c, d\langle \cup \rangle b + a - c, b + a + d - 2c\langle\big) + k(d - c)\right) \cup \big(\rangle a, b\langle + n(d - c)\big),$$

this is the fold configuration with $n + 1$ slit-folds parallel to $[a, b]$. We call this object *n-chip-fold*. In particular, a 1-chip-fold is a chip-fold. Analogously $\rangle a, b\langle \boxtimes\, n \cdot |c, d|$ denotes the *n-pillow-fold* obtained by replacing all slit-folds of an n-chip-fold that are parallel to $[c, d]$ by line segments. These line segments are identified by a translation in the direction of the vector $\overrightarrow{ab}$.

Proposition B.1. *Take a plane equipped with a single pillow-fold and consider a fixed direction foliation on the outside of a pillow-fold. Then there is either an n-chip-fold, or a pair of parallel slit-folds which has an outer measured foliation Whitehead equivalent (up to a finite number of leaves) to the given measured foliation.*

Proof. Since the problem is invariant under affine transformations we can consider a pillow-fold $\rangle 0, ib\langle \boxtimes |0, a|$ in the complex plane where $a, b \in \mathbb{R}_+$, so the segment $[0, a]$ is horizontal, and $[0, ib]$ is vertical. For fixed $\theta \in \mathbb{R}/\pi\mathbb{Z}$ consider the outer foliation for $\rangle 0, ib\langle \boxtimes |0, a|$. If $\theta = \pi/2$, we translate both horizontal sides to the center of the rectangle. That is a Whitehead move and so the outer foliations are equivalent. The resulting skeleton consists of two vertical slit-folds.

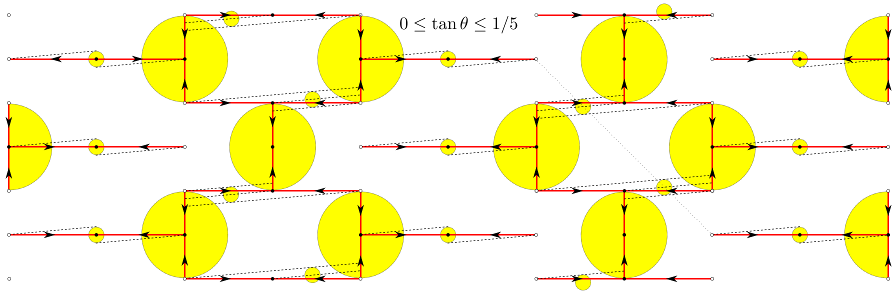

Fig. 30. Eaton lens configuration for $\mathbb{C}_6(3, 2)$

Suppose $|\tan\theta| \leq \frac{b}{a}$. That is, the (absolute) slope of the foliation is bounded by the slope of the diagonal $[0, a + ib]$ in the rectangle $[0, ib] \times [0, a]$. In this case translate the two horizontal edges parallel to the foliation into the rectangle and through the vertical slit-folds. This is a railed deformation, so every point on the edges remains on the same line (leaf) (including the slit-fold identification) of slope $\tan\theta$, as shown in Fig. 27. Note, that the two horizontal edges form a loop at any time. Two slit-folds appear, unless $|\tan\theta| = \frac{b}{a}$. In that case, both slit-folds fall together and we regard it as a single slit-fold located at the center of the rectangle. By construction both outer measured foliations differ by a Whitehead move that breaks up the singular point at the vertex of the pillow-fold, so they are equivalent.

For larger angles we need to use an intermediate step. In fact, if θ is not covered by the previous case(s), then there is a minimal $n \geq 2$ such that $|\tan\theta| \leq n\frac{b}{a}$. Then change the given pillow-fold $\rangle 0, ib\langle \boxtimes |0, a|$ into an n-pillow-fold $\rangle 0, ib\langle \boxtimes n \cdot |0, a/n|$ by putting $n - 1$ successive a/n translates of the left vertical slit-fold into the rectangle. Then for each of the n (translation equivalent) pillow-folds the previous conversion into a union of slit-folds applies. This process changes those finitely many leaves hitting the endpoints of the $n - 1$ new slit-folds that we need to put into the pillow case. Again we find a measurably equivalent outer foliation. Note, that the inner foliation is changed by this procedure, but this is irrelevant for our claim. □

Let us call a skeleton in the plane *standard skeleton*, if it is a countable union of pillow-folds and slit-folds, so that no pillow-fold contains other folds. For those we can use Proposition B.1 inductively to obtain:

Corollary B.2. *For any quadratic differential defined by a standard skeleton and any direction $\theta \in \mathbb{R}/\pi\mathbb{Z}$ the outer measured foliation tangential to θ in the plane is up to countably many leaves Whitehead equivalent to the direction foliation of a pre-Eaton-differential in the plane.*

The skeletons we consider are special, they have exactly one unbounded component. With the boundary identifications given by the skeleton the unbounded component is homeomorphic to a plane.

B.2. Other ergodic Eaton curves. Using the $X_3(2, 1)$, $X_6(3, 1)$ and $X_6(3, 2)$ torus differentials, we present more examples of admissible ergodic Eaton lens curves. Skeletons of the torus differentials allow us to write down differentials on the plane and represent them geometrically by arrow diagrams as in Figs. 28, 29 and 30. In those particular cases all folds will be horizontal and vertical in cartesian coordinates. Because the skeleton depends on the angle, see Proposition B.1, we only present the ergodic curve for small angles. We do not give a formal proof of admissibility for those Eaton lens distributions, it would go along the same lines as done in Proposition 2.4 for the Wollmilchsau differential. The figures give some clues how to work out the details, such as dividing tangent lines between some lenses.

References

1. Aaronson, J.: An Introduction to Infinite Ergodic Theory. Mathematical Surveys and Monographs, vol. 50. AMS, Providence, RI (1997)
2. Artigiani, M.: Exceptional ergodic directions in Eaton lenses. Isr. J. Math. **220**, 29–56 (2017)
3. Chaika, J., Eskin, A.: On Birkhoff and Osceledets genericity for all flat surfaces in almost all directions. J. Mod. Dyn. **9**, 1–23 (2015)
4. Conze, J.-P., Frączek, K.: Cocycles over interval exchange transformations and multivalued Hamiltonian flows. Adv. Math. **226**, 4373–4428 (2011)
5. Eskin, A., Kontsevich, M., Zorich, A.: Lyapunov spectrum of square-tiled cyclic covers. J. Mod. Dyn. **5**, 319–353 (2011)
6. Eskin, A., Mirzakhani, M.: Invariant and stationary measures for the $SL(2, \mathbb{R})$ action on moduli space. Publ. IHES **127**, 95–324 (2018)
7. Eskin, A., Mirzakhani, M., Mohammadi, A.: Isolation, equidistribution and orbit closures for the $SL(2, \mathbb{R})$ action on moduli space. Ann. Math. **182**, 673–721 (2015)
8. Filip, S.: Semisimplicity and rigidity of the Kontsevich–Zorich cocycle. Invent. Math. **205**, 617–670 (2016)
9. Forni, G.: Deviation of ergodic averages for area-preserving flows on surfaces of higher genus. Ann. Math. (2) **155**, 1–103 (2002)
10. Forni, G.: A geometric criterion for the nonuniform hyperbolicity of the Kontsevich–Zorich cocycle. With an appendix by Carlos Matheus. J. Mod. Dyn. **5**, 355–395 (2011)
11. Forni, G., Matheus, C.: Introduction to Teichmüller theory and its applications to dynamics of interval exchange transformations, flows on surfaces and billiards. J. Mod. Dyn. **8**, 271–436 (2014)
12. Forni, G., Matheus, C., Zorich, A.: Square-tiled cyclic cover. J. Mod. Dyn. **5**, 285–318 (2011)
13. Forni, G., Matheus, C., Zorich, A.: Lyapunov spectrum of invariant subbundles of the Hodge bundle. Ergod. Theory Dyn. Syst. **34**, 353–408 (2014)
14. Frączek, K., Hubert, P.: Recurrence and non-ergodicity in generalized wind-tree models. Math. Nachr. https://doi.org/10.1002/mana.201600480. arXiv:1506.05884
15. Frączek, K., Ulcigrai, C.: Non-ergodic $\mathbb{Z}$-periodic billiards and infinite translation surfaces. Invent. Math. **197**, 241–298 (2014)
16. Frączek, K., Shi, R., Ulcigrai, C.: Genericity on curves and applications: pseudo-integrable billiards, Eaton lenses and gap distributions. J. Mod. Dyn. **12**, 55–122 (2018)
17. Frączek, K., Schmoll, M.: Directional localization of light rays in a periodic array of retro-reflector lenses. Nonlinearity **27**, 1689–1707 (2014)
18. Glasmachers, E., Knieper, G.: Characterization of geodesic flows on $\mathbb{T}^2$ with and without positive topological entropy. GAFA **20**, 1259–1277 (2010)
19. Glasmachers, E., Knieper, G.: Minimal geodesic foliation on $\mathbb{T}^2$ in case of vanishing topological entropy. J. Topol. Anal. **3**, 511–520 (2011)
20. Grivaux, J., Hubert, P.: Loci in strata of meromorphic differentials with fully degenerate Lyapunov spectrum. J. Mod. Dyn. **8**, 61–73 (2014)
21. Hannaya, J.H., Haeusserab, T.M.: Retroreflection by refraction. J. Mod. Opt. **40**, 1437–1442 (1993)
22. Hooper, P.: The invariant measures of some infinite interval exchange maps. Geom. Topol. **19**, 1895–2038 (2015)
23. Hubert, P., Weiss, B.: Ergodicity for infinite periodic translation surfaces. Compos. Math. **149**, 1364–1380 (2013)
24. Kerckhoff, S., Masur, H., Smillie, J.: Ergodicity of billiard flows and quadratic differentials. Ann. Math. (2) **124**, 293–311 (1986)
25. Masur, H.: Hausdorff dimension of the set of nonergodic foliations of a quadratic differential. Duke Math. J. **66**, 387–442 (1992)
26. Masur, H.: Ergodic Theory of Translation Surfaces. Handbook of Dynamical Systems, vol. 1B, pp. 527–547. Elsevier, Amsterdam (2006)

27. Papadopoulos, A., Théret, G.: On Teichmüller's Metric and Thurston's Asymmetric Metric on Teichmüller Space. Handbook of Teichmüller Theory, vol. I, pp. 111–204. IRMA Lectures in Mathematics and Theoretical Physics, 11. European Mathematical Society, Zürich (2007)
28. Schmidt, K.: Cocycle of Ergodic Transformation Groups. Lecture Notes in Mathematics, vol. 1. Mac Milan Co. of India (1977)
29. Viana, M.: Dynamics of Interval Exchange Transformations and Teichmüller Flows, Lecture Notes Available from http://w3.impa.br/~viana/out/ietf.pdf
30. Yoccoz, J.-C.: Interval Exchange Maps and Translation Surfaces. Homogeneous Flows, Moduli Spaces and Arithmetic. Clay Mathematics Proceedings, vol. 10, pp. 1–69. American Mathematical Society, Providence, RI (2010)
31. Zorich, A.: Flat Surfaces, Frontiers in Number Theory, Physics, and Geometry I, pp. 437–583. Springer, Berlin (2006)

Communicated by J. Marklof