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Large Deviations for Gibbs Measures with Singular Hamiltonians and Emergence of Kähler–Einstein Metrics

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Abstract: In the present paper and the companion paper (Berman, Kähler–Einstein metrics, canonical random point processes and birational geometry. arXiv:1307.3634, 2015) a probabilistic (statistical-mechanical) approach to the construction of canonical metrics on complex algebraic varieties X is introduced by sampling "temperature deformed" determinantal point processes. The main new ingredient is a large deviation principle for Gibbs measures with singular Hamiltonians, which is proved in the present paper. As an application we show that the unique Kähler–Einstein metric with negative Ricci curvature on a canonically polarized algebraic manifold X emerges in the many particle limit of the canonical point processes on X. In the companion paper (Berman in 2015) the extension to algebraic varieties X with positive Kodaira dimension is given and a conjectural picture relating negative temperature states to the existence problem for Kähler–Einstein metrics with positive Ricci curvature is developed.

Contents

1. Introduction

In the present paper and the companion paper [9] a probabilistic approach to the construction of canonical metrics on a complex algebraic varieties X is introduced by sampling random point processes defined in terms of algebro-geometric data, canonically attached to X. The processes are "positive temperature deformations" of determinantal (fermionic) point processes and the main new ingredient is a large deviation principle for Gibbs measures with singular Hamiltonians, which is proved in the present paper. As an application we show that the unique Kähler–Einstein metric with negative Ricci curvature on a canonically polarized algebraic manifold X emerges in the many particle limit of the canonical point processes on X. More generally, in the presence of a stress-energy tensor on X it is shown that the unique Kähler metric solving Einstein's equation on X with negative cosmological constant (in Euclidean signature) emerges in the many particle limit.

The generalization to the construction of canonical metrics and measures on a general algebraic variety X of positive Kodaira dimension are given in the companion paper [9], by exploiting the global pluripotential theory and variational calculus in [7,12,14,20]. This leads to a new probabilistic link between algebraic geometry on one hand (in particular the Minimal Model Program) and Kähler–Einstein geometry on the other. A conjectural picture is also developed describing the relation between the existence of negative temperature states and the existence problem for Kähler–Einstein metrics with positive Ricci curvature. In particular, relations to algebro-geometric stability properties, as in the Yau-Tian-Donaldson conjecture are described in [9]. See also [8,40] for connections to optimal transport in the real setting (corresponding to the case when X is a toric and abelian variety, respectively) and [6] for connections to physics.

1.1. A large deviation principle for Gibbs measures. Let X be a compact Riemannian manifold and denote by dV the corresponding volume form. Given a sequence of symmetric lower semi-continuous functions $H^{(N)}$ on the N-fold products X^N the corresponding Gibbs measures at inverse temperature $\beta \in]0,\infty[$ is defined as the following sequence of symmetric probability measures on X^N :

$$\mu_\beta^{(N)} := e^{-\beta H^{(N)}} dV^{\otimes N}/Z_{N,\beta},$$

where the normalizing constant

$$Z_{N,\beta} := \int_{X^N} e^{-\beta H^{(N)}} dV^{\otimes N}$$

is called the *(N-particle) partition function.* The ensemble $(X^N, \mu_\beta^{(N)})$ defines a random point process with N particles on X which, from the point of view of statistical mechanics, models N identical particles on X interacting by the *Hamiltonian* (interaction energy) $H^{(N)}$ in thermal equilibrium at *inverse temperature* β. The corresponding *empirical measure* is the random measure

$$\delta_N : \ X^N \to \mathcal{M}_1(X), \ \ (x_1, \dots, x_N) \mapsto \delta_N(x_1, \dots, x_N) := \frac{1}{N}\sum_{i=1}^N \delta_{x_i} \qquad (1.1)$$

taking values in the space $\mathcal{M}_1(X)$ of all normalized positive measures on X, i.e., the space of all probability measures on X.

A classical problem is to establish conditions for the existence of a macroscopic limit of the empirical measures δ_N in the many particle limit $N \to \infty$. More precisely, the problem is to show that the random measures δ_N admit a deterministic limit $\mu_\beta \in \mathcal{M}_1(X)$ in the sense that the law

$$\Gamma_N := (\delta_N)_* \mu_\beta^{(N)} \tag{1.2}$$

of δ_N, defining a probability measure on $\mathcal{M}_1(X)$, converges, as $N \to \infty$, weakly to a Dirac mass concentrated on some μ_β in $\mathcal{M}_1(X)$. Equivalently, the marginals $(\mu_\beta^{(N)})_j$ of $\mu_\beta^{(N)}$ on X^j satisfy

$$(\mu_\beta^{(N)})_j := \int_{X^{N-j}} \mu_\beta^{(N)} \to \mu_\beta^{\otimes j},$$

weakly as probability measures on X^j as $N \to \infty$, which in the terminology of Kac and Snitzmann [50] means that the sequence $\mu_\beta^{(N)}$ is *chaotic*. A stronger exponential notion of convergence of δ_N, with an explicit speed and rate functional, is offered by the theory of large deviations, by demanding that the laws Γ_N satisfy a *Large Deviation Principle (LDP)* with *speed* r_N and a *rate functional* F, symbolically expressed as

$$\Gamma_N(\mu) \sim e^{-r_N F(\mu)}, \ \ N \to \infty$$

and assuming that F admits a unique minimizer μ_β in $\mathcal{M}_1(X)$. Loosely speaking this means that the probability of finding a cloud of N points $x_1, \dots, x_N$ on X such that the corresponding measure $\frac{1}{N}\sum_i \delta_{x_i}$ approximates a volume form μ is exponentially small unless μ is the minimizer μ_β of F_β.

Our main general result establishes such a LDP for a class of singular Hamiltonians:

Theorem 1.1. *Let $H^{(N)}$ be a sequence of functions (Hamiltonians) on X^N as above.* Assume that

- there exists a sequence $\beta_N \to \infty$ of positive numbers β_N such that for any continuous function u on X

$$\mathcal{F}_{\beta_N}(u) := -\frac{1}{N\beta_N} \log \int_{X^N} e^{-\beta_N \left(H^{(N)}(x_1,\dots,x_N)+u(x_1)+\dots+u(x_N)\right)} dV^{\otimes N}$$

  converges, as $N \to \infty$, to a *Gateaux differentiable functional $\mathcal{F}(u)$ on $C^0(X)$*
- *$H^{(N)}$ is uniformly quasi-superharmonic, i.e. $\Delta_{x_1} H^{(N)}(x_1, x_2, \dots x_N) \leq C$* on X^N

Then, for any fixed $\beta > 0$, the measures $(\delta_N)_*(e^{-\beta H^{(N)}} dV^{\otimes N})$ on $\mathcal{M}_1(X)$ satisfy, as $N \to \infty$, a large deviation principle (LDP) with speed βN *and good* rate functional

$$F_\beta(\mu) = E(\mu) + \frac{1}{\beta} D_{dV}(\mu), \tag{1.3}$$

where the functional $E(\mu)$ is the Legendre-Fenchel transform of $-\mathcal{F}(-\cdot)$ and $D_{dV}(\mu)$ is the entropy of μ relative to dV. In particular, the empirical measures δ_N of the corresponding random point processes on X converge in law to the deterministic measure given by the unique minimizer μ_β of F_β. Moreover, if the equation

$$d\mathcal{F}_{|u} = \frac{e^{\beta u} dV}{\int_X e^{\beta u} dV} \tag{1.4}$$

on $C^0(X)$ admits a solution u_β, then the corresponding differential $\mu_\beta := d\mathcal{F}_{|u_\beta}$ is the minimizer of F_β.

It follows from the previous theorem that the LDP indeed also holds for the corresponding Gibbs measures with the rate functional $F_\beta - C_\beta$, where C_β is the following constant:

$$C_\beta := \inf_{\mathcal{M}_1(X)} F_\beta = -\lim_{N\to\infty} \frac{1}{N\beta_N} \log Z_{N,\beta_N}, \tag{1.5}$$

It should be stressed that even the convergence of the first marginals of $\mu_\beta^{(N)}$, implied by the previous theorem, appears to be a new result.

As explained in Sect. 4.1 the asymptotics in the first assumption of the theorem may be replaced by the weaker assumption that there exists a functional $E(\mu)$ on $\mathcal{M}_1(X)$ such that

$$H^{(N)}(x_1,\ldots,x_N)/N \to E(\mu)$$

in the sense of Gamma convergence. Moreover, Theorem 1.1 can be viewed as a generalization of the Gärtner–Ellis theorem in the setting of Gibbs measures (see Sect. 4.2). Let us also point out that the restriction that X be compact can be removed if suitable growth-assumptions of $H^{(N)}$ "at infinity" are made. But since our main application concerns the case of compact complex manifolds, we have, for simplicity, taken X to be compact.

It may be illuminating to point out that in thermodynamical terms the content of Theorem 1.1 can be heuristically expressed as follows. Imagine that we know the macroscopic ground state (i.e., the state of zero energy E) of a system of a large number N of particles in thermal equilibrium at zero temperature (i.e. at $\beta = \infty$). If we can rule out any *first order phase transitions* at zero-temperature (which essentially means that the macroscopic equilibrium states is unique), then increasing the temperature leads to a new macroscopic equilibrium state, minimizing the corresponding *free energy* functional $E - S/\beta$, where S is the physical entropy (i.e., $S = -D$ with our sign conventions). In fact, in the complex geometric setting to which we next turn the zero-temperature limit $\beta \to \infty$ is reminiscent of a (second order) gas-liquid phase transition [10].

1.2. Application to Kähler–Einstein geometry. Let now X be an n-dimensional complex algebraic projective variety of positive Kodaira dimension. This means that the plurigenera N_k of X are increasing:

$$N_k := \dim_{\mathbb{C}} H^0(X, kK_X) \to \infty,$$

where $H^0(X, kK_X)$ denotes, as usual, the complex vector space of all pluricanonical (holomorphic) n-forms of X at level k, i.e. $H^0(X, kK_X)$ is the space of all global holomorphic sections of the k tensor power of the canonical line bundle

$$K_X := \Lambda^n(T^*X)$$

of X (using additive notation of tensor powers). In terms of local holomorphic coordinates $z_1,\ldots,z_n$ on X this simply means that the elements $s^{(k)}$ of $H^0(X, kK_X)$ may be represented by local holomorphic functions $s^{(k)}$ on X, such that $|s^{(k)}|^{2/k}$ transforms as a density on X and thus defines a measure on X. To any such algebraic variety X we can associate the following canonical sequence of probability measures $\mu^{(N_k)}$ on X^{N_k} :

$$\mu^{(N_k)} := \frac{1}{Z_{N_k}} \left|(\det S^{(k)})(z_1,\ldots,z_{N_k})\right|^{2/k}, \tag{1.6}$$

where $\det S^{(k)}$ is a generator of the top exterior power $\Lambda^{N_k}(H^0(X^{N_k}, kK_{X^{N_k}})$, i.e., totally antisymmetric (and thus defined up to a multiplicative complex number) and Z_{N_k} is the normalizing constant. The probability measure $\mu^{(N_k)}$ thus defined is symmetric, i.e., invariant under the natural action of the permutation group S_{N_k}, independent of the choice of generator $\det S^{(k)}$ and hence defines a canonical random point process on X with N_k points.

As shown in the companion paper [9], it follows from Theorem 1.1, combined with the asymptotics in [12] that the corresponding empirical measures δ_{N_k} converge in law, as $k \to \infty$, towards a deterministic measure μ_{can} on X, which is thus canonically attached to X. In fact, using the pluripotential theory and variational calculus in [7,14] the limiting measure μ_{can} is shown to coincide with the canonical measure of Song-Tian [51] and Tsuji [52] previously defined in terms of Kähler–Einstein geometry or equivalently as solutions to certain complex Monge–Ampère equations. In the present paper we will show how to apply Theorem 1.1 in the special case when K_X is positive (i.e. ample) to deduce the following

Theorem 1.2. *Let X be a compact complex manifold with positive canonical line bundle K_X. Then the empirical measures δ_{N_k} of the corresponding canonical random point processes on X converge in law, as $N_k \to \infty$, towards the normalized volume form dV_{KE} of the unique Kähler–Einstein metric ω_{KE} on X. More precisely, the law of δ_{N_k} satisfies a large deviation principle with speed N_k whose rate functional may be identified with Mabuchi's K-energy functional on the space of Kähler metrics in $c_1(K_X)$.*

By the celebrated Aubin–Yau theorem [3,53] the canonical line bundle K_X of a compact complex manifold X is positive precisely when X admits a Kähler–Einstein metric ω_{KE} with negative Ricci curvature, i.e., a Kähler metric with constant negative Ricci curvature:

$$\operatorname{Ric}\omega_{KE} = -\omega_{KE} \tag{1.7}$$

However, there are very few examples where the Kähler–Einstein metric can be obtained explicitly. The previous theorem provides a canonical sequence of quasi-explicit Kähler forms ω_k approximating ω_{KE} :

Corollary 1.3. *Let X be a complex compact manifold such that K_X is positive. Then the sequence*

$$\omega_k := dd^c \log \int_{X^{N_k-1}} \left|(\det S^{(k)})(\cdot, x_1, \ldots, x_{N_k-1})\right|^{2/k} \tag{1.8}$$

(consisting of Kähler forms, for k sufficiently large) converges, as $k \to \infty$, to the Kähler–Einstein metric ω_{KE} in the weak topology of currents on X.

Theorem 1.2 fits into a more general setting of "temperature deformed" determinantal point processes attached to a polarized manifold (X, L), i.e., a compact complex manifolds X endowed with a positive line bundle L (Theorem 5.7). More precisely, in the general setting the point processes are attached to the data $(\|\cdot\|, dV, \beta_k)$ consisting of a Hermitian metric $\|\cdot\|$ on a L, a volume form dV on X and a sequence of positive numbers $\beta_k \to \beta \in]0, \infty]$. Then the corresponding probability measures on X^{N_k} are defined by

$$\mu^{(N_k,\beta)} := \frac{\left\|(\det S^{(k)})(x_1, x_2, \ldots x_{N_k})\right\|^{2\beta_k/k} dV^{\otimes N_k}}{Z_{N_k,\beta}}, \tag{1.9}$$

where $\det S^{(k)}$ is a generator of the top exterior power $\Lambda^{N_k} H^0(X, kL)$. Concretely, the corresponding LDP is equivalent to the following asymptotics for the $L^{2\beta_k/k}$-norm of the generator $\det S^{(k)}$ of the determinant line of $H^0(X, kL)$ which is orthonormal with respect to the L^2-product determined by $(\|\cdot\|, dV)$:

$$\frac{1}{N_k} \log \left\| \det S^{(k)} \right\|_{L^{2\beta_k/k}(X^{N_k}, \mu_0^{\otimes N_k})} \to - \inf_{\mu \in \mathcal{M}_1(X)} F_\beta(\mu)$$

(by Lemma 4.7). In this general setting the limiting deterministic measure μ_β minimizing F_β is the volume form of the unique Kähler metric ω_β in the first Chern class of L solving the twisted Kähler–Einstein equation

$$\text{Ric}\,\omega = -\beta\omega + \eta, \tag{1.10}$$

where the twisting form η is explicitly determined by $(\|\cdot\|, dV, \beta)$. The point is that when $L = K_X$ any given volume form dV naturally defines a metric $\|\cdot\|_{dV}$ on L and the probability measures on X^{N_k} attached to $(\|\cdot\|_{dV}, dV, 1)$ are precisely the canonical ones defined by formula 1.6. Moreover, in this special case η vanishes and the Eq. 1.10 thus reduces to the usual Kähler–Einstein equation 1.7. The more general twisted version of the equation has previously appeared in various situations in Kähler geometry [37, 51,52]. From the physics point of view the twisting form η corresponds to the (trace-reversed) stress-energy tensor in Einstein's equations on X (with Euclidean signature).

The Hamiltonians

$$H^{(N_k)}(x_1, \ldots, x_{N_k}) := -k^{-1} \log \left\| (\det S^{(k)})(x_1, x_2, \ldots x_{N_k}) \right\|^2 \tag{1.11}$$

corresponding to the probability measures 1.9 are strongly non-linear unless X is a Riemann surface, i.e. unless $n = 1$. In fact, in the simplest latter case, i.e., when X is the Riemann sphere, $H^{(N_k)}(x_1, \ldots, x_{N_k})$ is a sum of identical pair interactions $W(x_i, x_j)$, where W is the Green function of the corresponding Laplace operator and then the corresponding functional $E(\mu)$ is the Dirichlet energy (Remark 5.11). In general, the connection to the Kähler–Einstein geometry of (X, L) will be shown to arise from the fact that the Eq. 1.4 is intimately related to the complex Monge–nAmpère equation

$$(\omega_0 + i\partial\bar{\partial}u)^n = e^{\beta u} dV, \tag{1.12}$$

where ω_0 is the normalized curvature two form of the given metric $\|\cdot\|$ on L. More precisely, the two equations coincide for smooth functions u such that $\omega_0 + i\partial\bar{\partial}u$ is a Kähler form (i.e., smooth and positive). In this complex geometric setting the strong non-linearity of the Hamiltonians $H^{(N)}$ when $n \geq 2$ is reflected in the non-linearity of the complex Monge–Ampère operator appearing in the left hand side of Eq. 1.12 (coinciding with the Laplacian when $n = 1$). Furthermore, the singularity of $H^{(N)}$ (which is present for any dimension n) is a reflection of the fact that solutions to the (generalized) Calabi–Yau equation

$$(\omega_0 + i\partial\bar{\partial}u)^n = \mu \tag{1.13}$$

are, in general, singular when μ is a probability measure on X (as is clear already for the Laplace equation appearing when $n = 1$).

Finally, let us point out that the extension to general complex algebraic manifolds X with positive Kodaira dimension, established in the companion paper [9], relies on an extension of Theorem 5.7 to line bundles L which are big (but not necessarily positive); see Sect. 5.4.

1.3. Comparison with previous results. First a comment on relations to the physics literature: in the case $n = 1$ (i.e., in two real dimensions) the quasi-linear Laplace type Eq. 1.12 arises as the macroscopic equilibrium equation in a range of statistical mechanical models of mean field type: it is called the Joyce–Montgomery equation in Onsager's vortex model for 2D turbulence, the Poisson–Boltzmann equation in the Debye–Hückel theory of plasmas and electrolytes and the Lane–Emden equation in stellar physics (see [35]). But the Monge–Ampère equation ($n > 1$) does not seem to have appeared in any statistical mechanical model before. On the other hand, in the case when $\beta_k := k$ the density of the corresponding probability measure has a natural quantum mechanical interpretation: it is the squared amplitude of the Slater determinant representing a maximally filled many particle state of N free fermions on X, subject to an exterior magnetic field (the corresponding single particle wave functions are elements of $H^0(X, kL)$ and represent the corresponding lowest Landau levels). The case when $\beta_k = \nu k$, for a given positive integer ν, also appears in the fractional Quantum Hall Effect, where the corresponding probability density is the squared amplitude of the Laughlin state (see the review [43] and references therein).

1.3.1. Large deviations. The LDP in Theorem 1.1 in the case when $H^{(N)}$ is uniformly equicontinuous is essentially well-known in the setting of mean field models [8,34] (it then also applies to the case of negative β, by replacing $H^{(N)}$ with -$H^{(N)}$). But the key feature of Theorem 1.1 is that it applies to a large class of *singular* Hamiltonians and in particular $H^{(N)}$ is allowed to be strongly repulsive in the sense that it blows up, as two points merge (and hence the Gibbs measure may be ill-defined when β is negative). It seems that the only previous class where a convergence result as in Theorem 4.6 has been established for singular Hamiltonians is in the "linear" case when $H^{(N)}$ is a sum of pair interactions with a mean field scaling:

$$H^{(N)}(x_1, \ldots, x_N) = \frac{1}{(N-1)} \sum_{1 \leq i < j \leq N} W(x_i, x_j), \tag{1.14}$$

where the pair interaction W is allowed to be singular along the diagonal, as long it is lower semi-continuous and in L^1_{loc} (this is indeed a mean field interaction in the sense that each particle x_i is exposed to the average of the pair interactions $W(x_i, x_j)$ for the $N - 1$ remaining particles). Then the asymptotics of the partitions functions 1.5 can be obtained using the method of Messer–Spohn [45], which is based on the Gibbs variational principle and which crucially relies on the existence of the *mean energy* $\bar{E}(\mu)$ corresponding to $H^{(N)}$ (see [26,41] for the case of a logarithmic singularity which is motivated by Onsager's vortex model for 2D turbulence [35,46]). A similar argument applies in the case of "finite order", i.e., when $H^{(N)}$ is a sum of j-point interactions for a uniformly bounded j (then $E(\mu)$ depends polynomially on μ). However, the main point of the previous theorem is to avoid the latter assumption, which is not satisfied in the application to Kähler–Einstein geometry (apart from the classical lowest dimensional setting of Riemann surfaces). In particular, the present proof bypasses the problem of the existence of the limiting mean energies. Instead, the main idea of the proof is to exploit the Riemannian orbifold geometry of the space of configurations of N points on X, viewed as the singular quotients X^N/S_N, where S_N is the symmetric group acting on X^N by permuting the factors. The key result is a submean inequality for positive quasi-subharmonic functions on X^N/S_N with a distortion coefficient that is sub-exponential in the dimension (Theorem 2.1), which is closely related to an inequality of Li-Schoen [44].

There is also another approach to large deviation principles for mean field Hamiltonians of the form 1.14 originating in the literature on random matrices and Coulomb gases [17,18,28,48], which as explained in [48], is closely related to the notion of Gamma convergence (see also [55,56] for applications to univariat random polynomials). This approach seems to be limited to the case when $\beta_N \gg \log N$ and in particular $\beta = \infty$ so that the entropy contributions can be neglected.[1] See also [33] for a general LDP for Hamiltonians of the form 1.14 using weak convergence methods.

Let us also point out that the role of $(\det S^{(k)})(x_1, x_2, \ldots x_{N_k})$ appearing in formula 1.9 is played by the classical Vandermonde determinant in the random matrix literature (see Example 5.6). In fact, there is a non-compact analogue of Theorem 5.7 in Euclidean $\mathbb{C}^n$ which specializes to the setting of random matrix theory and the 2D log gas when $n = 1$ and $\beta = \infty$ and to the 2D vortex model (for $n = 1$ and $\beta < \infty$) and which can be proved by supplementing the proof of Theorem 5.7 with a tightness estimate, as in the non-compact setting considered for $\beta = \infty$ in [5] (see also [19] for the case $\beta = \infty$). Details will appear elsewhere.

1.3.2. Kähler geometry. A statistical mechanics approach has previously been applied to conformal geometry [42], as opposed to the present complex-geometric setting. The role of the "determinantal" Hamiltonian 1.11 is in the conformal setting played by a mean field Hamiltonian of the form 1.14 with a logarithmic pair interaction and the role of the fully non-linear complex Monge–Ampère operator is played by a linear conformally invariant operator, which is zero-order perturbation of a power of the Laplacian (the Paneitz operator). Accordingly, previous results in [26,41] concerning such Hamiltonians can be applied in the conformal setting (compare the discussion above), while the present setting seems to require new methods.

The present probabilistic should be viewed in the light of the pervasive philosophy in Kähler geometry, going back to Yau [54], of approximating metrics on a complex algebraic manifold with algebraically defined Bergman metrics, which may be identified with elements of the symmetric space $GL(N, \mathbb{C})/U(N)$. For example, the quasi-explicit Kähler metrics ω_k in formula 1.8, approximating the Kähler–Einstein metric ω_{KE} on a canonically polarized manifold X, are analogs of Donaldson's balanced metrics in $GL(N, \mathbb{C})/U(N)$ [32]. One advantage of the present approach is that, as shown in the companion paper [9], the approximation also applies when X is of general type, where the role of ω_{KE} is played by the canonical Kähler–Einstein current on X (which is singular along a subvariety of X) [14,20]. In another direction it would be interesting to see if the present approach can be implemented to construct numerical simulations of Kähler–Einstein metrics, using Monte Carlo type methods, complementing the different numerical approaches in [31,32] (see [4] for relations between Monte Carlo simulations and similar polynomial determinantal point processes).

Even if the connection between canonical random point processes on a complex algebraic manifold X does not seem to have been studied before, there are some connections to previous work on random polynomials/holomorphic sections in a given back-ground geometry [49]; in particular in the one-dimensional setting where an LDP was obtained in [55,56]. Another probabilistic approach to the space of Kähler metrics has been introduced in a series of papers by Ferrari, Klevtsov and Zelditch [36], motivated by Quantum Field Theory. The approach aims at approximating random Kähler metrics with random Bergman metrics. Accordingly, the role of the N-particle space X^N/S_N is in [36] played

[1] The Hamiltonians in the random matrix and Coulomb gas literature are usually scaled in a different way so that our zero-temperature ($\beta = \infty$) corresponds to a fixed inverse temperature.

by the symmetric space $GL(N.\mathbb{C})/U(N)$. In conclusion, it would be very interesting to understand the precise connections between [36] and the present setting, as well as the connection to Donaldson's balanced metrics [32].

Organization. In Sect. 2, we prove the submean inequality in large dimensions, which plays a key role in the subsequent Sect. 3, where the general LDP in Theorem 1.1 is proved. In Sect. 4, we make a digression on relations to previous methods and notions used in the literature on large deviations. The applications to Kähler–Einstein geometry are given in Sect. 5. For the convenience of readers lacking background in Kähler geometry we start the section by giving a reasonably self-contained account of the Kähler geometry setup (including some rudiments of pluripotential theory). The article is concluded with an outlook in Sect. 6 on some open problems and an appendix where the dimension dependence on the constant in the Cheng–Yau gradient estimate is obtained by tracing through the usual proof.

2. Submean Inequalities in Large Dimension

2.1. Setup. Let (X, g) be a n-dimensional Riemannian manifold and assume that

$$\text{Ric } g \geq -\kappa^2(n-1)g$$

for some positive constant κ (sometimes referred to as the normalized lower bound on the Ricci curvature). Let G a finite group acting by isometries on X and denote by $M := X/G$ the corresponding quotient equipped with the distance function induced by the metric g, i.e.

$$d_M(x, y) := \inf_{\gamma \in G} d_X(x, \gamma y),$$

where d_X is the Riemannian distance function on (X, g). Even though the quotient M is not a manifold in general (since G will in general have fixed points) it still comes with a smooth structure in the following sense. Denote by p the natural projection map from X to M. Using the projection p we can identify a function f on M with G-invariant function $p^* f$ on X and accordingly we say that f is smooth if $p^* f$ is. Similarly, there is a natural notion of Laplacian Δ on the quotient M : the Laplacian Δu of a locally integrable function u on M is the signed Radon measure defined by

$$\int_M (\Delta u) f := \frac{1}{|G|} \int_X p^* u \Delta(p^* f)$$

for any smooth function f on M. More generally, by localization, this setup naturally extends to the setting of Riemannian orbifolds (see [21]), but the present setting of global quotients will be adequate for our purposes.

2.2. Statement of the submean inequality.

Theorem 2.1. *Let (X, g) be a Riemannian manifold of dimension n such that Ric $g \geq -\kappa^2(n-1)g$ and G a finite group acting by isometries on X. Denote by $M := X/G$ the corresponding quotient equipped with the distance function induced by the metric g and let v be a non-negative function on M such that $\Delta_g v \geq -\lambda^2 v$ for* *some non-negative constant* λ. *Then, for any* $\delta \in]0,1[$ *and* $\epsilon \in]0,1]$ *there exist constants* A *and* C *such that*

$$\sup_{B_{\epsilon\delta}(x_0)} v^2 \leq A e^{2\lambda\delta} e^{Cn(\delta+\epsilon)} \frac{\int_{B_\delta(x_0)} v^2 dV}{\int_{B_{\epsilon\delta}(x_0)} dV},$$

where C *only depends on an upper bound on* κ *and* A *only depends on* δ *and* ϵ *(assuming that the balls above are contained in a compact subset of* M*).*

Note that by the G-invariance we may as well replace the functional v and the balls on M with their pull-back to X.

2.3. Proof of the submean inequality in Theorem 2.1. We will follow closely the elegant proof of Li-Schoen [44] of a similar submean inequality. But there are two new features here that we have to deal with:

- We have to make explicit the dependence on the dimension n of all constants and make sure that the final contribution is sub-exponential in n
- We have to adapt the results to the singular setting of a Riemannian quotient

Before turning to the proof we point out that it is well-known that submean inequalities with a multiplicative constant $C(n)$ do hold in the more general singular setting of Alexandrov spaces (with a strict lower bound $-\kappa$ on the sectional curvature). But it seems that the current proofs (see for example [38]), which combine local Poincaré and Sobolev inequalities with the Moser iteration technique, do not give the subexponential dependence on $C(n)$ that we need.

We recall that the two main ingredients in the proof of the result of Li-Schoen referred to above is the gradient estimate of Cheng–Yau [29] and a Poincaré-Dirichlet inequality on balls. Let us start with the gradient estimate that we will need:

Proposition 2.2. *Let* u *be a harmonic function on the ball* $B_a(x_0)$ *of radius* a *centered at* $x_0 \in M$ *and assume that* $a \leq 1$ *Set* $\rho_{x_0}(x) := d(x, x_0)$ *(the distance between* x *and* x_0*). Then*

$$\sup_{B_a(x_0)} \left(|\nabla \log u| \left(a - \rho_{x_0}\right)\right) \leq Cn,$$

where the constant C *only depends on an upper bound on* κ.

Proof. In the smooth case this is the celebrated Cheng–Yau gradient estimate [29]. The result is usually stated without an explicit estimate of the multiplicative constant C_n in terms of n, but tracing through the proof in [29] gives $C_n \leq Cn$ (see the appendix in the present paper and also [2] for a probabilistic proof providing an explicit constant). We claim that the same estimate holds in the present setting using a lifting argument. To see this recall that the usual proof of the gradient estimate proceeds as follows (see the appendix). Set $\phi(x) := |\nabla \log u| (= |\nabla u| / u)$ and $F(x) := \phi(x)(\rho_{x_0} - a)^2$. Then F attains its maximum in a point x_1 in the interior of $B_a(x_0)$ (otherwise $|\nabla u|$ vanishes identically and then we are trivially done). Hence, $F(x) \leq F(x_1)$ on some neighborhood U of x_1. Now, in case F (or equivalently ρ_{x_0}) is smooth on U we get $\Delta F \leq 0$ and $\nabla F = 0$ at x_1. Calculating ΔF and using Bochner formula and Laplacian comparison then gives

$$\phi(x_1)(a - \rho_{x_0}(x_1)) \leq Cn \tag{2.1}$$

which is the desired estimate. In the case when ρ_{x_0} is not smooth on U, i.e. x_1 is contained in the cut locus of x_0 one first replaces ρ_{x_0} with a smooth approximation $\rho_{x_0}^{(\epsilon)}$ of ρ_{x_0} (which is a local barrier for ρ_{x_0}) and then lets $\epsilon \to 0$ to get the same conclusion as before. In the singular case $M = X/G$ we proceed as follows. First we identify F with a G-invariant function on the inverse image of $B_R(x_0)$ in X (and x_0 and x_1 with a choice of lifts in the corresponding G-orbits) and set $\tilde{F} := \phi(x)(a - \tilde{\rho}_{x_0})^2$, where $\tilde{\rho}_{x_0}(x) := d_X(x_0, x)$. By definition $\tilde{\rho}_{x_0} \geq \rho_{x_0}$ on X and, after possibly changing the lift of the point x_1 we may assume that $\tilde{\rho}_{x_0} = \rho_{x_0}$ at $x = x_1$ and hence $\tilde{\rho}_{x_0} < a$ (after perhaps shrinking U). In particular, $\tilde{F} \leq F$ on U and $\tilde{F} = F$ at x_1 and hence $\tilde{F}$ also has a local maximum at x_1. But then the previous argument in the smooth case gives that 2.1 holds with ρ_{x_0} replaced by $\tilde{\rho}_{x_0}$. But since the two functions agree at x_1 this concludes the proof in the general case. □

Corollary 2.3. *Let h be a positive harmonic function on $B_\delta(x_0)$. Then there exists a constant C only depending on an upper bound on κ such that*

$$\sup_{B_{\epsilon\delta}(x_0)} h^2 \leq e^{C\epsilon n} \frac{\int_{B_{\epsilon\delta}(x_0)} h^2 dV}{\int_{B_{\epsilon\delta}(x_0)} dV}$$

for $0 < \epsilon < 1$.

Proof. Set $v := \log h$ and fix $x \in B_{\epsilon\delta}(x_0)$. Integrating along a minimizing geodesic connecting x_0 and x and using the gradient estimate in the previous proposition gives

$$|v(x) - v(x_0)| \leq Cn \int_0^{\epsilon\delta} \frac{1}{\delta - t} dt = Cn\,(\log(\delta - 0) - \log(\delta - \epsilon\delta)) = -\,Cn\log(1-\epsilon).$$

In particular, for any two points $x, y \in B_{\epsilon\delta}(x_0)$ we get $|v(x) - v(y)| \leq |v(x) - v(x_0)| + |v(y) - v(x_0)| \leq -2Cn\log(1-\epsilon)$, i.e. $h(x) \leq (1-\epsilon)^{-2Cn} h(y)$. In particular, $\sup_{B_{\epsilon\delta}(x_0)} h^2 \leq (1-\epsilon)^{-4Cn} \inf_{B_{\epsilon\delta}(x_0)} h^2$, which implies the proposition after renaming the constant C. □

The second key ingredient in the proof of Theorem 2.1 is the following Poincaré-Dirichlet inequality:

Proposition. *Let f be a smooth function on $B_\delta(x_0)$ vanishing on the boundary. Then*

$$\int_{B_{\delta(x_0)}} |f|^2 dV_g \leq 4e^{Cn\delta} \int_{B_{\delta(x_0)}} |\nabla f|^2 dV_g$$

where the constant C only depends on an upper bound on κ.

Proof. We follow the proof in [44] with one crucial modification (compare the remark below). To fix ideas we first consider the case of a Riemannian manifold. Fix a point p in the boundary of the ball $B_1(x_0)$ and denote by $r_1(x)$ the distance between $x \in M$ and p. From the standard comparison estimate for the Laplacian we get

$$\Delta r_1 \leq (n-1)(\frac{1}{r_1} + \kappa) \tag{2.2}$$

(in the weak sense and point-wise away from the cut locus of p). In particular, for any positive number a we deduce the following inequality on $B_\delta(x_0)$ (using that $g(\nabla r_1, \nabla r_1) = 1$) a.e.)

$$\Delta_g(e^{-ar_1}) = ae^{-ar_1}(a - \Delta r_1) \geq ae^{-a(1+\delta)} \left(a - (n-1)(\frac{1}{(1-\delta)} + \kappa) \right)$$

Hence, setting $a := n(\frac{1}{(1-\delta)} + \kappa)$ gives

$$\Delta_g(e^{-ar_1}) \geq ae^{-a(1+\delta)}(\frac{1}{(1-\delta)} + \kappa) > 0$$

Multiplying by $|f|$ and integrating once by parts (and using that $\|\nabla r_1\| \leq 1$) we deduce that

$$a\int_{B_{\delta(x_0)}} |\nabla f| e^{-ar_1} dV \geq a(\frac{1}{(1-\delta)} + \kappa) \int_{B_{\delta(x_0)}} |f| e^{-a(1+\delta)} dV.$$

Estimating $e^{-ar_1} \leq e^{-a(1-\delta)}$ in the left hand side above and rearranging gives

$$\int_{B_{\delta(x_0)}} |\nabla f| dV e^{2a\delta}(\frac{1}{(1-\delta)} + \kappa)^{-1} \geq \int_{B_{\delta(x_0)}} |f| dV,$$

(using that $g(\nabla r_1, \nabla r_1) \leq 1$ in the sense of upper gradients). This shows that the L^1-version of the Poincaré inequality in question holds with the constant $(\frac{1}{(1-\delta)} + \kappa)^{-1} e^{\delta 2n(\frac{1}{(1-\delta)} + \kappa)}$, which for δ sufficiently small is bounded from above by $e^{n(4+2\kappa)\delta}$. The general Riemannian L^2-Poincare inequality now follows from replacing $|f|$ with $|f|^2$ and using Hölder's inequality. Finally, in the case of the a Riemannian quotient M we can proceed exactly as above using that the Laplacian comparison estimate in formula 2.2 is still valid. Indeed, the pull-back p^*r_1 of r_1 to X is an infimum of functions for which the corresponding estimate holds (by the usual Laplacian comparison estimate and the assumption that G acts by isometries). But then the estimate also holds for the function p^*r_1, by basic properties of Laplacians. More generally, the required Laplacian comparison estimate was shown in [21] for general Riemannian orbifolds. □

Remark 2.4. The only difference from the argument used in [44] is that we have taken the point p to be of distance 1 from x_0 rather than distance 2δ, as used in [44]. For δ small this change has the effect of improving the exponential factor from $e^{n(1+\delta\kappa)}$ to $e^{n(\delta+\delta\kappa)}$, which is crucial as we need a constant in the Poincare inequality which has subexponential growth in n as $\delta \to 0$.

2.3.1. End of proof of Theorem 2.1. Let us first consider the case when $\lambda = 0$. Denote by h the harmonic function on B_δ coinciding with v on ∂B_δ. By Cor 2.3 and the subharmonicity of v

$$\sup_{B_{\epsilon\delta}(x_0)} v^2 \leq e^{Cn\epsilon} \frac{\int_{B_{\epsilon\delta}(x_0)} |h|^2 dV_g}{\int_{B_{\epsilon\delta}(x_0)} dV_g}.$$

Next, by the triangle inequality

$$\int_{B_{\epsilon\delta}(x_0)} |h|^2 dV_g/2 \leq \int_{B_{\epsilon\delta}(x_0)} |h - v|^2 dV + \int_{B_{\epsilon\delta}(x_0)} |v|^2 dV$$

Since $h-v$ vanishes on the boundary of $B_\delta(x_0)$ applying the Poincare inequality in Prop 2.3 then gives $\int_{B_{\epsilon\delta}(x_0)} |h-v|^2 dV \leq$

$$\leq \int_{B_\delta(x_0)} |h-v|^2 dV \leq Ae^{Bn\delta} \int_{B_\delta(x_0)} |\nabla h - \nabla v|^2 dV \leq 2Ae^{Bn\delta} \int_{B_\delta(x_0)} |\nabla h|^2 + |\nabla v|^2 dV$$

But h is the solution to a Dirichlet problem and as such minimizes the Dirichlet norm $\int_{B_\delta(x_0)} |\nabla h|^2$ over all subharmonic functions with the same boundary values as h. Accordingly,

$$\int_{B_{\epsilon\delta}(x_0)} |h-v|^2 dV \leq 4Ae^{Bn\delta} \int_{B_\delta(x_0)} |\nabla v|^2 dV$$

Finally, using that v is subharmonic we get

$$\int_{B_\delta(x_0)} |\nabla v|^2 dV \leq C_\delta \int_{B_{2\delta}(x_0)} |v|^2 dV$$

(as is seen by multiplying with a suitable smooth function χ supported on $B_{2\delta}$ such that $\chi = 1$ on B_δ). All in all this concludes the proof of Theorem 2.1 in the case $\lambda = 0$.

Finally, to handle the general case (i.e. $\lambda \neq 0$) we set $N := M \times]-1, 1[$ equipped with the standard product metric and apply the previous case to the function $ve^{\lambda t}$ to get

$$\sup_{B_{\epsilon\delta}(x_0,0)\subset N} v^2 e^{2\lambda t} \leq A_\delta e^{Bn(\delta+\epsilon)} \frac{\int_{B_{2\delta}(x_0,0)\subset N} v^2 e^{2\lambda t} dV}{\int_{B_{\epsilon\delta}(x_0,0)\subset N} dV},$$

But restricting the sup in the left hand side to $B_{\epsilon\delta}(x_0) \times \{0\}$ and using that $B_{\epsilon\delta/2}(x_0, 0) \times [-\epsilon\delta/2, \epsilon\delta/2] \subset B_{\epsilon\delta}(x_0, 0)$ and $B_{2\delta}(x_0, 0) \subset B_{2\delta}(x_0, 0) \times [2\delta, 2\delta]$ gives

$$\sup_{B_{\epsilon\delta}(x_0)\subset M} v^2 \leq A_{\delta,\epsilon} e^{2\lambda\delta} e^{Bn(\delta+\epsilon)} \frac{\int_{B_{2\delta}(x_0)\subset M} v^2 dV_g}{\int_{B_{\epsilon\delta/2}(x_0,0)\subset M} dV_g},$$

which concludes the proof of the general case (after a suitable rescaling).

3. Proof of the Large Deviation Principle for Gibbs Measures

Given a compact topological space X we will denote by $C^0(X)$ the space of all continuous functions u on X, equipped with the sup-norm and by $\mathcal{M}(X)$ the space of all signed (Borel) measures on X. The subset of $\mathcal{M}(X)$ consisting of all probability measures will be denoted by $\mathcal{M}_1(X)$. We endow $\mathcal{M}(X)$ with the weak topology, i.e. μ_j is said to converge to μ weakly in $\mathcal{M}(X)$ if

$$\langle \mu_j, u_j \rangle \to \langle \mu, u \rangle$$

for any continuous function u on X, i.e. for any $u \in C^0(X)$, where $\langle u, \mu \rangle$ denotes the standard integration pairing between $C^0(X)$ and $\mathcal{M}(X)$ (equivalently, the weak topology is precisely the weak*-topology when $\mathcal{M}(X)$ is identified with the topological dual of $C^0(X)$). A functional $\mathcal{F}$ on $C^0(X)$ will be said to be *Gateaux differentiable* if it is differentiable along affine lines and for any u in $C^0(X)$ there exists an element $d\mathcal{F}_{|u}$ in $\mathcal{M}(X)$, called the differential of $\mathcal{F}$ at u, such that for any v in $C^0(X)$

$$\frac{d\mathcal{F}(u+tv)}{dt}_{|t=0} = \langle d\mathcal{F}_{|u}, v \rangle$$

3.1. Setup: the Gibbs measure $\mu_\beta^{(N)}$ associated to the Hamiltonian $H^{(N)}$. A *random point process* with N particles is by definition a probability measure $\mu^{(N)}$ on the N-particle space X^N which is symmetric, i.e. invariant under permutations of the factors of X^N. The *empirical measure* of a given random point process is the following random measure

$$\delta_N : X^N \to \mathcal{M}_1(X), \mapsto (x_1, \dots, x_N) \mapsto \delta_N(x_1, \dots, x_N) := \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i} \quad (3.1)$$

on the ensemble $(X^N, \mu^{(N)})$. By definition the *law* of δ_N is the push-forward of $\mu^{(N)}$ to $\mathcal{M}_1(X)$ under the map δ_N, which thus defines a probability measure on $\mathcal{M}_1(X)$.

Now fix a back-ground measure μ_0 on X and let $H^{(N)}$ be a given *N-particle Hamiltonian,* i.e. a symmetric function on X^N, which we will assume is lower semi-continuous (and in particular bounded from below, since X is assumed compact). Also fixing a positive number β the corresponding *Gibbs measure* (at inverse temperature β) is the symmetric probability measure on X^N defined as

$$\mu_\beta^N := e^{-\beta H^{(N)}} \mu_0^{\otimes N}/Z_N,$$

where the normalizing constant

$$Z_{N,\beta} := \int_{X^N} e^{-\beta H^{(N)}} \mu_0^{\otimes N}$$

is called the *(N-particle) partition function.* In our setting we will take μ_0 to be the volume form dV of a fixed Riemannian metric. Given a continuous function u on X we will also write

$$Z_{N,\beta}[u] := \int_{X^N} e^{-\beta(H^{(N)}+u)} \mu_0^{\otimes N},$$

where u has been identified with the following function on the product X^N :

$$u(x_1, .., x_N) := \sum_{i=1}^{N} u(x_i)$$

3.2. Preliminaries on large deviation principles and Legendre transforms. Let us start by recalling the general definition of a Large Deviation Principle (LDP) for a sequence of measures.

Definition 3.1. Let $\mathcal{P}$ be a Polish space, i.e. a complete separable metric space.

(i) A function $I : \mathcal{P} \to]-\infty, \infty]$ is a *rate function* if it is lower semi-continuous. It is a *good rate function* if it is also proper.

(ii) A sequence Γ_k of measures on $\mathcal{P}$ satisfies a *large deviation principle* with *speed* r_k and *rate function* I if

$$\limsup_{k\to\infty} \frac{1}{r_k} \log \Gamma_k(\mathcal{F}) \leq -\inf_{\mu\in\mathcal{F}} I$$

for any closed subset $\mathcal{F}$ of $\mathcal{P}$ and

$$\liminf_{k\to\infty} \frac{1}{r_k} \log \Gamma_k(\mathcal{G}) \geq -\inf_{\mu\in G} I(\mu)$$

for any open subset $\mathcal{G}$ of $\mathcal{P}$.

Remark 3.2. The LDP is said to be *weak* if the upper bound is only assumed to hold when $\mathcal{F}$ is compact. Anyway, we will be mainly interested in the case when $\mathcal{P}$ is compact and hence the notion of a weak LDP and an LDP then coincide (and moreover any rate functional is automatically good).

We will be mainly interested in the case when Γ_k is a probability measure (which implies that $I \geq 0$ with infimum equal to 0). Then it will be convenient to use the following alternative formulation of a LDP (see Theorems 4.1.11 and 4.1.18 in [30]):

Proposition 3.3. *$\mathcal{P}$ be a metric space and denote by $B_\epsilon(\mu)$ the ball of radius ϵ centered at $\mu \in \mathcal{P}$. Then a sequence Γ_N of probability measures on $\mathcal{P}$ satisfies a weak LDP with speed r_N and a rate functional I iff*

$$\lim_{\epsilon\to 0} \liminf_{N\to\infty} \frac{1}{r_N} \log \Gamma_N(B_\epsilon(\mu)) = -I(\mu) = \lim_{\epsilon\to 0} \limsup_{N\to\infty} \frac{1}{r_N} \log \Gamma_N(B_\epsilon(\mu)) \tag{3.2}$$

We note the following simple lemma which allows one to extend the previous proposition to the non-normalized measures $(\delta_N)_* e^{-\beta H^{(N)}} \mu_0^{\otimes N}$:

Lemma 3.4. *Assume that the following bound for the partition functions holds: $\left|\log Z_{N,\beta}\right| \leq CN$. Then the measures*

$$\Gamma_N := (\delta_N)_* e^{-\beta H^{(N)}} \mu_0^{\otimes N} \tag{3.3}$$

satisfy the asymptotics 3.2 for any $\mu \in \mathcal{M}_1(X)$ with rate functional $\tilde{I}(\mu)$ and $r_N = N$ iff the probability measures $(\delta_N)_* \mu_\beta^{(N)}$ on $\mathcal{M}_1(X)$ satisfy an LDP at speed N with rate functional $I := \tilde{I} - C_\beta$, where $C_\beta := \inf_{\mu\in\mathcal{M}(X)} \tilde{I}(\mu)$.

Proof. Set $\tilde{\Gamma}_N := (\delta_N)_* e^{-\beta H^{(N)}} \mu_0^{\otimes N}$ and $C_{N,\beta} := -\frac{1}{N} \log Z_{N,\beta}$. By assumption $C_{N,\beta}$ is uniformly bounded and we denote by C_β a given limit point of the sequence obtained by replacing N with a subsequence N_j. Since $\frac{1}{N} \log \Gamma_N(B_\epsilon(\nu)) = \frac{1}{N} \log \tilde{\Gamma}_N(B_\epsilon(\nu)) + C_{N,\beta}$ we obtain that after replacing N with the subsequence N_j the probability measures Γ_N satisfy (by Prop 3.3) an LDP with rate functional $\tilde{I} - C_\beta$. As a consequence $0 = \inf(\tilde{I} - C_\beta)$, showing that C_β is independent of the subsequence. Hence, the whole sequence converges towards C_β, which proves one direction in the Lemma. The converse is proved in a similar way. □

We will also use the following classical result of Sanov, which is the standard example of a LDP for point processes [30] (the result follow, for example, from the Gärtner–Ellis theorem; see Sect. 4.2).

Proposition 3.5. *Let X be a topological space and μ_0 a finite measure on X. Then the law Γ_N of the empirical measures of the corresponding Gibbs measure $\mu_0^{\otimes N}$ (i.e. $H^{(N)} = 0$) satisfies an LDP with speed N and rate functional the relative entropy D_{μ_0}.*

We recall that the *relative entropy* D_{μ_0} (also called the *Kullback–Leibler divergence* or the *information divergence* in probability and information theory) is the functional on $\mathcal{M}_1(X)$ defined by

$$D_{\mu_0}(\mu) := \int_X \log \frac{\mu}{\mu_0} \mu, \tag{3.4}$$

when μ has a density $\frac{\mu}{\mu_0}$ with respect to μ_0 and otherwise $D_{\mu_0}(\mu) := \infty$. When μ_0 is a probability measure, $D_{\mu_0}(\mu) \geq 0$ and $D_{\mu_0}(\mu) = 0$ iff $\mu = \mu_0$ (by Jensen's inequality).

3.2.1. Legendre–Fenchel transforms. Let f be a function on a topological vector space V. Then its The Legendre-Fenchel transform is defined as following convex lower semi-continuous function f^* on the topological dual V^*

$$f^*(w) := \sup_{v \in V} \langle v, w \rangle - f(v)$$

in terms of the canonical pairing between V and V^*. In the present setting we will take $V = C^0(X)$ and $V^* = \mathcal{M}(X)$, the space of all signed Borel measures on a compact topological space X. We will use the following variant of the Brøndsted-Rockafellar property A^*[25]:

Lemma 3.6. *Let f be function on $C^0(X)$ which is Gateaux differentiable. Then, for any $\mu \in \mathcal{M}(X)$ such that $f^*(\mu) < \infty$ there exists a sequence of $u_j \in C^0(X)$ such that*

$$\mu_j := df_{|u_j} \to \mu, \quad f^*(\mu_j) \to f^*(\mu) \tag{3.5}$$

Proof. First recall that a convex function g on a topological vector space E is said to be *subdifferentiable* at $x \in E$ if $g(x) < \infty$ and g admits a *subgradient* x^* at x, i.e. an element x^*in the topological dual E^* such that for any $y \in E$

$$g(y) \geq g(x) + \left\langle (y - x), x^* \right\rangle$$

The set of all such subgradients is denoted by $(\partial g)(x)$. Now assume that $g = f^*$ for a convex function f on a Banach space V. Then g is a lower semi-continuous function convex function on the topological vector space $E := V^*$ equipped with its weak topology. According to [25, Thm 2] any element $\mu \in V^*$ such that $f^*(\mu) < \infty$ has the property that there exists a sequence $\mu_j \to \mu$ in V^* such that $f^*(\mu_j) \to f^*(\mu)$ and f^* is subdifferentiable at μ_j with a subgradient in V. The latter property equivalently means that there exists $u_j \in V$ such that $\mu_j \in (\partial f)(u_j)$ (as follows from the definition of the Legendre-Fenchel transform). The proof is now concluded by setting $V := C^0(X)$ and observing that if f is Gateaux differentiable at $u \in V$, then $(\partial f)(u) = \{df_{|u}\}$ (as is seen by restricting f to any affine line). □

Remark 3.7. By convexity, if $\mu = df_{|u}$ for some $u \in V := C^0(X)$, then $f^*(\mu) = \left\langle u, df_{|u} \right\rangle - f(u)$, which is essentially the classical definition of the *Legendre transform* of f at μ. Accordingly, the previous lemma may be reformulated as the statement that the Legendre-Fenchel transform is the greatest lower semi-continuous extension to all of V^* of the Legendre transform, originally defined on $(df)(V) \subset V^*$.

3.3. The proof of Theorem 5.7. We start with the following simple

Lemma 3.8. *Assume that $H^{(N)}$ satisfies the quasi-superharmonicity assumption in the second point of Theorem 1.1*. Then, for any sequence of positive numbers $\beta_N \to \infty$

$$-\mathcal{F}_{\beta_N}(u) := \frac{1}{N\beta_N}\log\int_{X^N} e^{-\beta_N(H^{(N)}+u)}dV^{\otimes N} = -\inf_{X^N}\frac{H^{(N)}+u}{N} + o(1)$$

Proof. The inequality $\leq$ is trivial and to prove the reversed inequality we fix a sequence of $x^{(N)} \in X^N$ realizing the infimum appearing the right hand side above. Then replacing the integral of X^N with an integral over the L^∞-ball $B_\epsilon := \{(x_1, \ldots, x_N) : d_g(x, x_i^{(N)}) \leq \epsilon\}^N$, for a fixed number ϵ and a fixed metric g with distance function d_g, and using the classical submean inequality in each variable with a fixed multiplicative constant C gives

$$\int_{X^N} e^{-\beta_N(H^{(N)}+u)}dV^{\otimes N} \geq C^N e^{-\beta_N(H^{(N)}+u)(x^{(N)})}\int_{B_\epsilon} dV^{\otimes N} e^{-N\beta_N\delta_\epsilon}$$

δ_ϵ is the modulus of continuity of u, tending to 0 as $\epsilon \to 0$. Finally, since $\int_{B_\epsilon} dV^{\otimes N} \geq (C'\epsilon)^N$ letting $N \to \infty$ concludes the proof. □

To handle the case when $\beta_N = \beta + o(1)$ for a finite β we will need to use the subexponential dependence on the dimensions of the multiplicative constant appearing in Theorem 2.1. To this end we first recall that, since X is assumed compact, the weak topology on $\mathcal{M}_1(X)$ is metrized by the Wasserstein 2-metric d induced by a given Riemannian metric g on X, where

$$d(\mu, \nu)^2 := \inf_{\Gamma\in\Gamma(\mu,\nu)}\int d_g(x, y)^2 d\Gamma,$$

where $\Gamma(\mu, \nu)$ is the space of all *couplings* between μ and ν, i.e. all probability measures Γ on $X \times X$ such that the push forward of Γ to the first and second factor is equal to μ and ν, respectively.

Proposition 3.9. *For any given $\epsilon > 0$ there exists a positive constant $C > 0$ such that the following submean inequality holds on X^N, for any N :*

$$e^{-\beta H^{(N)})}(x^{(N)}) \leq Ce^{C\epsilon N}\frac{\int_{B_\epsilon(x^{(N)})} e^{-\beta H^{(N)}}dV^{\otimes N}}{\int_{B_{\epsilon^2}(x^{(N)})} dV^{\otimes N}}, \tag{3.6}$$

where $B_r(x^{(N)})$ denotes the inverse image in X^N, under the map δ_N, of the Wasserstein ball of radius r centered at $\delta_N(x^{(N)})$

Proof. First observe that the pull-back of d on $\mathcal{M}_1(X)$ to the quotient space $X^{(N)} := X^N/S_N$ under the map δ_N defined by the empirical measure (formula 3.1) coincides with $1/N^{1/2}$ times the quotient distance function on $X^{(N)}$, induced by the product Riemannian metric on X^N :

$$\delta_N^* d = \frac{1}{N^{1/2}} d_{X^{(N)}} := d^{(N)} \tag{3.7}$$

Indeed, this is well-known and follows from the Birkhoff-Von Neumann theorem which gives that for any symmetric function $c(x, y)$ on $X \times X$ we have that if $\mu = \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}$ and $\nu = \frac{1}{N}\sum_{i=1}^{N}\delta_{y_i}$ for given $(x_1, \dots, x_N), (y_1, \dots, y_N) \in X^N$, then

$$\inf_{\Gamma(\mu,\nu)}\int c(x,y)d\Gamma = \inf_{\Gamma_N(\mu,\nu)}\int c(x,y)d\Gamma$$

where $\Gamma_N(\mu, \nu) \subset \Gamma(\mu, \nu)$ consists of couplings of the form $\Gamma_\sigma := \frac{1}{N}\sum \delta_{x_i} \otimes \delta_{y_{\sigma(i)}}$, for $\sigma \in S_N$, where S_N is the symmetric group on N letters.

Now consider the metric space $(X^{(N)}, d^{(N)})$ which is the quotient space defined with respect to the finite group S_N acting isometrically on the Riemannian manifold (X^N, g_N), where g_N denotes $1/N$ times the product Riemannian metric. By assumption $H^{(N)}$ is S_N-invariant and $\Delta_{g_N} H^{(N)} \leq C$ on X^N (using the obvious scaling of the Laplacian). Moreover, since X is compact there exists a non-negative number k such that Ric $g \geq -kg$ on X and hence rescaling gives Ric $g_N \geq -kNg_N$ on (X^N, g_N). But the dimension of X^N is equal to nN and hence the assumptions in Theorem 2.1 are satisfied for $u := e^{-\beta H^{(N)}}$ and (X, g) replaced by (X^N, g_N) with a constant κ independent of N. Applying the latter theorem with $\delta = \epsilon$ and using the pull-back property in formula 3.7 then shows that the submean property 3.6 indeed holds. □

We will also rely on the following simple but very useful lemma (which was used in the similar context of Fekete points in [13]).

Lemma 3.10. *Fix $u_* \in C^0(X)$ and assume that $x_*^{(N)} \in X^N$ is a minimizer of the function $(H^{(N)} + u_*)/N$ on X^N. If the corresponding large N-limit $\mathcal{F}(u)$ exists for all $u \in C^0(X)$ and $\mathcal{F}$* is Gateaux differentiable at *u_*, then $\delta_N(x_*^{(N)})$ converges weakly towards $\mu_* := d\mathcal{F}_{|u_*}$.*

Proof. Fix $v \in C^0(X)$ and a real number t. Let $f_N(t) := \frac{1}{N}(H^{(N)} + u + tv)(x_*^{(N)})$ and $f(t) := \mathcal{F}(u + tv)$. By assumption $\lim_{N\to\infty} f_N(0) = f(0)$ and $\liminf_{N\to} f_N(t) \geq f(t)$. Note that f is a concave function in t (since it is defined as an inf of affine functions) and $f_N(t)$ is affine in t. But then it follows from the differentiability of f at $t = 0$ that $\lim_{N\to\infty} df_N(t)/dt_{|t=0} = df(t)/dt_{|t=0}$, i.e. that

$$\lim_{N\to\infty}\left\langle \delta_N(x_*^{(N)}), v\right\rangle = \left\langle d\mathcal{F}_{|u}, v\right\rangle,$$

which thus concludes the proof of the lemma (see [13, Lemma 3.1]). □

The upper bound in the LDP. By Lemma 3.4 it will be enough to establish the LDP for the non-normalized measures Γ_N in formula 3.3. To prove the upper bound of the integrals appearing in the equivalent formulation of the LDP in Prop 3.3 we fix a function $u \in C^0(X)$ and rewrite

$$e^{-\beta H^{(N)}} = e^{-\beta(H^{(N)}+u)}e^{\beta u},$$

Then, trivially, for any fixed $\epsilon > 0$,

$$\int_{B_\epsilon(\mu)} e^{-\beta H^{(N)}} dV^{\otimes N} \leq \sup_{B_\epsilon(\mu)}\left(e^{-\beta(H^{(N)}+u)}\right)\int_{B_\epsilon(\mu)} \mu_u^{\otimes N}, \ \ \mu_u := e^{\beta u}dV \tag{3.8}$$

Replacing the sup over $B_\epsilon(\mu)$ in the first factor above with the sup over all of X^N and applying Sanov's theorem relative to the tilted volume form μ_u to the second factor gives

$$\lim_{\epsilon\to 0}\limsup_{k\to\infty}\frac{1}{\beta N}\log\int_{B_\epsilon(\mu)} e^{-\beta H^{(N)}}dV^{\otimes N}$$
$$\leq -\liminf_{N\to\infty} N^{-1}\inf_{X^N}(H^{(N)}+u)+\int u\mu-\frac{1}{\beta}D_{dV}(\mu),$$

using that $D_{e^{\beta u}dV}(\mu)=-\beta\int u\mu+D_{dV}(\mu)$. According to Lemma 3.8 and the definition of the functional $\mathcal{F}(u)$ this means that

$$\lim_{\epsilon\to 0}\limsup_{k\to\infty}\frac{1}{\beta N}\log\int_{B_\epsilon(\mu)} e^{-\beta H^{(N)}}dV^{\otimes N}\leq -\mathcal{F}(u)+\int u\mu-\frac{1}{\beta}D_{dV}(\mu),$$

Finally, taking the infimum over all $u\in C^0(X)$ shows that the lim sup in the previous formula is bounded from above by $-F(\mu)$,

$$F(\mu):=f^*(\mu)+\frac{1}{\beta}D_{dV}(\mu),\quad f(u):=-\mathcal{F}(-u)$$

Remark 3.11. In the argument above dV can be replaced by any finite measure μ_0 on X.

The lower bound in the LDP. As usual, the proof of the lower bound in the LDP is the hardest. We first assume that

$$\mu=d\mathcal{F}_{|u}$$

for some $u\in C^0(X)$. Denote by $x^{(N)}\in X^N$ a sequence of minimizers of $H^{(N)}+u$. By Lemma 3.10 we have that

$$\delta_N(x^{(N)})\to\mu$$

By the submean inequality 3.6

$$\frac{1}{N}\log\Gamma_N(B_{2\epsilon}(\mu))\geq\frac{1}{N}\log\int_{B_{\epsilon^2}(x^{(N)})}dV^{\otimes N}-\beta H^{(N)}(x^{(N)})/N-\epsilon-\frac{C_\epsilon}{N}$$

Since $\delta(x^{(N})\to\mu$ it follows that $\frac{1}{N}\log\Gamma_N(B_{2\epsilon}(\mu))\geq$

$$\geq\frac{1}{N}\log\int_{B_{\epsilon^2}(x^{(N)})}dV^{\otimes N}-\beta(H^{(N)}+u)(x^{(N)})/N+\beta\langle u,\mu\rangle-\delta(\epsilon)-\epsilon-\frac{C_\epsilon}{N},$$

where $\delta(\epsilon)\to 0$ as $\epsilon\to 0$. Moreover, fixing $\epsilon>0$ we may for N sufficiently large assume that $B_{\epsilon^2/2}(\mu)\subset B_{\epsilon^2}(\delta_N(x^{(N)}))$ and hence letting $N\to\infty$ and using Sanov's theorem (i.e. Prop 3.5) for ϵ fixed and the convergence of $(H^{(N)}+u)(x^{(N)})/N$ gives

$$\liminf_{N\to\infty}\frac{1}{N}\log\Gamma_N(B_{2\epsilon}(\mu))\geq\beta\langle u,\mu\rangle-\delta(\epsilon)-\inf_{B_{\epsilon^2/2}(\mu)}D_{dV}-\beta\mathcal{F}(u)-\epsilon$$

(after perhaps replacing the original family $\delta(\epsilon)$ with a smaller one). Since μ is a candidate for the inf in the right hand side above the inf in question may be estimated from above by $D_{dV}(\mu)$ and hence letting $\epsilon \to 0$ concludes the proof under the assumption that $\mu := d\mathcal{F}_{|u}$ for some $u \in C^0(X)$. To prove the general case we invoke Lemma 3.6 to write μ as a weak limit of $\mu_j := d\mathcal{F}_{|u_j}$ for $u_j \in C^0(X)$. We may then replace u in the previous argument with u_j for a fixed j and replace μ with μ_j in the previous argument to get, for $j \geq j_\delta$, $\liminf_{N\to\infty} \frac{1}{N} \log \Gamma_N(B_{3\epsilon}(\mu)) \geq$

$$\geq \liminf_{N\to\infty} \frac{1}{N} \log \Gamma_N(B_{2\epsilon}(\mu_j)) \geq \beta \left\langle u_j, \mu_j \right\rangle - \delta_j(\epsilon) - \inf_{B_{\epsilon^2/2}(\mu_j)} D_{dV} + \beta\mathcal{F}(u_j) - \epsilon$$

But for j sufficiently large μ_j is in the ball $B_{\epsilon^2/2}(\nu)$ and hence the inf above is bounded from above by $D_{dV}(\mu)$ giving

$$\liminf_{N\to\infty} \frac{1}{N} \log \Gamma_N(B_{3\epsilon}(\mu)) \geq \beta \left\langle u_j, \mu_j \right\rangle - \delta_j(\epsilon) - D_{dV}(\mu) + \beta\mathcal{F}(u_j) - \epsilon$$

Letting first $\epsilon \to 0$ and then $j \to \infty$ gives

$$\liminf_{N\to\infty} \frac{1}{N} \log \Gamma_N(B_{3\epsilon}(\mu)) \geq -\beta(\lim_{j\to\infty} (E(\mu_j) + \frac{1}{\beta} D_{dV}(\mu))$$

Finally, by Lemma 3.6 we may assume that $E(\mu_j) \to E(\mu)$ and that concludes the proof.

The equation for the minimizer μ_β. Finally, the Eq. 1.4 follows immediately from the following general convex analytical result:

Lemma 3.12. *Let X be a compact topological space and f and g* be Gateaux differentiable convex functionals on $C^0(X)$ such that the differentials dg and df takes values in $\mathcal{M}_1(X)$. Then

- The following identity holds:

$$\inf_{\mathcal{M}_1(X)} \left(f^* + g^*\right) = \sup_{u\in C^0(X)} (-f(-u) - g(u)) \tag{3.9}$$

- if the sup in the right hand side above is attained at some u_0 in $C^0(X)$ (i.e. if $-f(-u) - g(u)$ admits a critical point u_0), then, setting $\mathcal{F}(u) := -f(-u)$, the measure $\mu_0 := d\mathcal{F}_{|u_0}$ minimizes the functional $f^* + g^*$ on $\mathcal{M}_1(X)$.

Proof. First observe that f and g are Lipschitz continuous on the Banach space $C^0(X)$. Indeed, setting $u_t := u_0(1-t) + tu_1$, for $t \in [0, 1]$, gives

$$|f(u_1) - f(u_0)| = \left| \int_0^1 dt \int_X df_{u_t}(u_1 - u_0) \right| \leq \sup_X |u_1 - u_0|$$

and similarly for g. The first point in the lemma is then obtained as a special case of the Fenchel-Rockafeller duality theorem which only requires that f and g be convex on a Banach space V and that f and g be finite at some point u where f is moreover assumed continuous [24, Thm 1.12]. To prove the second point we let u_0 be a critical point of $\mathcal{F}(u) - g(u)$ on $C^0(X)$, i.e.

$$d\mathcal{F}_{|u_0} = dg_{|u_0}, \tag{3.10}$$

which, by convexity, means that u_0 realizes the sup in the right hand side of formula 3.9. We rewrite,

$$f^*(\mu) := \sup_{u \in C^0(X)} \langle u, \mu \rangle - f(u) = \sup_{u \in C^0(X)} \mathcal{F}(u) - \langle u, \mu \rangle \tag{3.11}$$

(by replacing u with $-u$ in the sup). Hence, if $\mu := d\mathcal{F}_{|u}$ then, by concavity, $f^*(\mu) := F(u) - \langle u, \mu \rangle$. Similarly, if $\mu = dg_{|v}$ then, by convexity, $g^*(\mu) := \langle u, \mu \rangle - g(u)$. All in all this means that if u_0 satisfies the critical point Eq. 3.10, then we can take $u = v = u_0$ to get

$$f^*(\mu_0) + g^*(\mu_0) = \mathcal{F}(u_0) + 0 - g(u_0),$$

which concludes the proof, using the first point. □

4. Relations to Γ-Convergence, the Gärtner–Ellis Theorem and Mean Energy

Before turning to the applications of Theorem 1.1 in the complex geometric setting we explore some relations to previous results and methods in the literature.

4.1. Relations to Gamma-convergence. We recall that a sequence of functions E_N on a topological space $\mathcal{P}$ *is said to* Γ*-converge* to a function E on $\mathcal{P}$ if

$$\begin{array}{ccc} \mu_N \to \mu \text{ in } \mathcal{P} & \Longrightarrow & \liminf_{N\to\infty} E_N(\mu_N) \geq E(\mu) \\ \forall \mu & \exists \mu_N \to \mu \text{ in } \mathcal{P}: & \lim_{N\to\infty} E_N(\mu_N) = E(\mu) \end{array} \tag{4.1}$$

(such a sequence μ_N is called a *recovery sequence*); see [23]. It then follows that E is lower semi-continuous on $\mathcal{P}$. In the present setting we take, as before, $\mathcal{P} = \mathcal{M}(X)$ and define E_N by setting $E_N = \infty$ on the complement of the image of the mapδ_N and

$$E_N(\delta_N(x_1, \ldots, x_N)) := H^{(N)}(x_1, \ldots, x_N)/N \tag{4.2}$$

We can now formulate the following variant of Theorem 1.1:

Theorem 4.1. *Let $H^{(N)}$ be a sequence of lower semi-continuous symmetric functions on X^N, where X is a compact Riemannian manifold.* Assume that

- The functions E_N on $\mathcal{M}_1(X)$ determined by $H^{(N)}$ converge to a function E, in the sense of Γ-convergence on $\mathcal{M}_1(X)$.
- $H^{(N)}$ *is uniformly quasi-superharmonic, i.e.* $\Delta_{x_1} H^{(N)}(x_1, x_2, \ldots x_N) \leq C$ on X^N

Then, for any sequence of positive numbers $\beta_N \to \beta \in]0, \infty]$ the measures $\Gamma_N := (\delta_N)_* e^{-\beta_N H^{(N)}}$ on $\mathcal{M}_1(X)$ satisfy, as $N \to \infty$, a LDP with speed $\beta_N N$ *and good* rate functional

$$F_\beta(\mu) = E(\mu) + \frac{1}{\beta} D_{dV}(\mu) \tag{4.3}$$

Proof. Using the characterization of a LDP in Proposition 3.3, the upper bound in the LDP follows almost immediately from the liminf property of the Gamma-convergence together with Sanov's theorem. To prove the lower bound fix $\mu \in \mathcal{M}_1(X)$ and take a recovery sequence μ_N corresponding to a sequence $x^{(N)} \in X^N$. Then, using the same notation for the balls as in the proof of Theorem 1.1 we have, for $\epsilon > 0$ fixed and N large,

$$\int_{B_{2\epsilon}(\mu)} e^{-\beta H^{(N)}} dV^{\otimes N} \geq \int_{B_\epsilon(x^{(N)})} e^{-\beta H^{(N)}} dV^{\otimes N} \geq e^{NC\epsilon} e^{-N\beta E_N(\mu_N)} \int_{B_{\epsilon^2}(x^{(N)})} dV^{\otimes N},$$

using the submean inequality in Theorem 2.1 in the last inequality. Letting first $N \to \infty$ and then $\epsilon \to 0$ then concludes the proof, using Sanov's theorem again. □

It should be stressed that, in general, the functional $E(\mu)$ in the previous theorem will not be convex and hence the subset $\mathcal{C}_\beta \subset \mathcal{M}_1(X)$ consisting of the minima of F_β will, in general, consist of more than one element. By general principles the LDP then implies that any limit point $\Gamma_\infty \in \mathcal{M}_1(\mathcal{M}_1(X))$ of the laws Γ_N is concentrated on $\mathcal{C}_\beta$ (in the terminology of statistical mechanics Γ_∞ is thus a *mixed state* defined as a superposition of the pure states δ_μ where $\mu \in \mathcal{C}_\beta$).

Remark 4.2. The proof of the previous theorem in the case $\beta = \infty$ is much simpler as it is does not require the sub-exponential dependence on the dimension in the sub-mean inequality in Theorem 2.1. Indeed, the rough exponential bound used in the proof of Lemma 3.8 is enough. Moreover, all that is used in the proof for $\beta < \infty$ is that $\Delta_{x_1}(e^{-\beta_N H^{(N)}}) \geq -\lambda_\beta e^{-\beta_N H^{(N)}}$ for a constant λ_β independent on N (but the assumption that $\Delta_{x_1} H^{(N)} \leq C$ is a convenient way of ensuring that the previous inequality holds for any β).

Example 4.3. In the case when $X = \mathbb{R}^n$ equipped with the Euclidean distance it is known that the mean field Hamiltonian with pair interaction of the form $W(x, y) = w(|x - y|)$ (formula 1.14) Γ-convergences towards $E(\mu) := \int_{X^2} W\mu \otimes \mu$, if w is lower semi-continuous and increasing close to 0 (see [48, Prop 2.8, Remark 2.19] and [15,17,28] for similar results). The proof exploits the explicit nature of $E(\mu)$ and a similar argument applies on a compact manifold when W is continuous away from the diagonal with a singularity of the local form $w(|x - y|)$ close to the diagonal (compare [55,56]).

In contrast to the previous example, for the "determinantal" Hamiltonian 5.16 appearing in the complex geometric setting there is no explicit candidate for a limit $E(\mu)$. Instead the Gamma convergence is a consequence of the following dual criterion.

4.1.1. A criterion for Gamma convergence using duality. Next we separate out the convex analysis used in the proof of Theorem1.1 to get the following criterion for Γ-convergence:

Proposition 4.4. *Let E_N a sequence of functions on $\mathcal{M}_1(X)$ and assume that*

$$\lim_{N\to\infty} E_N^*(u) = f(u)$$

where f is a Gateaux differentiable convex function on $C^0(X)$. Then E_N converges to $E := f^*$ in the sense of Γ-convergence on the space $\mathcal{M}_1(X)$, equipped with the weak topology.

Proof. First suppose that $\mu_N \to \mu$ weakly in $\mathcal{M}_1(X)$. Fix u in $C^0(X)$. Then $-E_N(\mu_N) = \langle u, \mu_N \rangle - E_N(\mu_N) - \langle u, \mu \rangle + o(1)$ and hence taking the sup over all $\mu \in \mathcal{M}_1(X)$ gives

$$-E_N(\mu_N) \leq f_N(u) - \langle u, \mu \rangle + o(1) = f(u) - \langle u, \mu \rangle + o(1).$$

Finally, letting first $N \to \infty$ and then taking the sup over all $u \in C^0(X)$ concludes the proof of the lower bound for $E_N(\mu_N)$.

To prove the existence of a recovery sequence we first assume that $\mu = df_{|u_\mu}$ for some $u_\mu \in C^0(X)$. Then,

$$f^*(\mu) = \langle u_\mu, \mu \rangle - f(u_\mu) = \langle u_\mu, \mu \rangle - f_N(u_\mu) + o(1),$$

Now, by the weak compactness of $\mathcal{M}_1(X)$ the sup defining f_N is attained at some $\mu_N \in \mathcal{M}_1(X)$ and hence

$$f^*(\mu) + o(1) = \langle u_\mu, \mu \rangle - \left(\langle u_\mu, \mu_N \rangle - E_N(\mu_N) \right)$$

Next, by a minor generalization of Lemma 3.10 $\mu_N \to \mu (:= df_{|u_\mu})$ and hence $f^*(\mu) = 0 + E_N(\mu_N) + o(1)$, as desired. Finally, the proof of the existence of recovery sequence for any μ such that $E(\mu) < \infty$ is concluded by a simple diagonal argument based on Lemma 3.6 applied to $E := f^*$. $\square$

Now, if E_N is of the form 4.2, then

$$f_N(u) := \sup_{X^N} \frac{1}{N} u(x_1) + \ldots + \frac{1}{N} u(x_N) - \frac{1}{N} H^{(N)}(x_1, \ldots, x_N) \tag{4.4}$$

Thanks to the previous proposition the first assumption in Theorem 1.1 thus implies (also using Lemma 3.8) that $E_N \to E$ in the sense of Γ-convergence on $\mathcal{M}(X)$. Accordingly we recover Theorem 1.1 from Theorem 4.1.

Remark 4.5. In general, if E_N gamma converges to a function E on $\mathcal{M}_1(X)$, then it follows (almost directly) that $E_N^* \to E^*$ point-wise on $C^0(X)$. Hence, the point of the previous proposition is that it gives a converse statement under the assumption that E^* is Gateaux differentiable. By basic convex duality it thus follows from the previous proposition that E_N converges to a strictly convex functional E on $\mathcal{M}_1(X)$ iff $E_N^* \to E^*$ point-wise on $C^0(X)$, with E^* Gateaux differentiable.

4.2. Relations to the Gärtner–Ellis theorem. First observe that

$$\int_{X^N} e^{-\beta_N (H^{(N)} + u)} dV^{\otimes N} = \widehat{\Gamma_N}(-r_N u),$$

where Γ_N is the measure

$$\Gamma_N := (\delta_N)_* (e^{-\beta H^{(N)}} dV^{\otimes N})$$

on $\mathcal{M}_1(X)$ and $\widehat{\Gamma_N}$ denotes its Laplace transform on $C^0(X)$. In this context the Gärtner–Ellis theorem may be formulated as follows (see [30, Cor 4.6.14, p. 148] and references therein):

Theorem 4.6 (Gärtner–Ellis). *Let $H^{(N)}$ be a sequence of Hamiltonians on X^N and β_N a sequence of positive numbers such that $\beta_N \to \beta \in]0,\infty]$. Assume that, for any $u \in C^0(X)$, as $N \to \infty$,*

$$\mathcal{F}_{\beta_N}(u) := -\frac{1}{N\beta_N}\log\int_{X^N} e^{-\beta_N(H^{(N)}+u)}dV^{\otimes N} \to \mathcal{F}_\beta(u), \tag{4.5}$$

where $\mathcal{F}$ is a Gateaux differentiable function. *Then the measures Γ_N on $\mathcal{M}_1(X)$ satisfy, as $N \to \infty$, a LDP with* speed $\beta_N N$ *and good* rate functional $f^*(\mu)$, where $f(u) := -\mathcal{F}(-u)$.

Compared with the Gärtner–Ellis theorem the main point of Theorem 1.1 is thus that the assumption that the convergence of the partition functions in formula 4.5 holds for $\beta = \infty$ is enough to ensure that one gets an LDP for any $\beta \in]0,\infty[$ (under the quasi-subharmonicity assumption). As a consequence, the convergence of the partition functions then also hold for any $\beta \in]0,\infty[$ with the limiting functional $-\mathcal{F}_\beta(\cdot)$ defined as the Legendre-Fenchel transform of the rate functional F_β appearing in Theorem 1.1. In fact, the latter convergence is equivalent to the LDP in question, as made precise by the following

Lemma 4.7. *Let $H^{(N)}$ be a sequence of Hamiltonians on X^N and β_N a sequence of positive numbers such that $\beta_N \to \beta \in]0,\infty[$. Assume that, for any given volume form dV, the corresponding partition functions $Z_{N.\beta_N}$ satisfy*

$$\lim_{N\to\infty} -\frac{1}{N\beta_N}\log Z_{N.\beta_N} := \inf_\mu F_\beta, \quad F_\beta := E + D_{dV}/\beta,$$

with $E(\mu)$ convex. Then the measures $(\delta_N)_*(e^{-\beta H^{(N)}}dV^{\otimes N})$ on $\mathcal{M}_1(X)$ satisfy, as $N \to \infty$, an LDP with speed $\beta_N N$ *and good* rate functional F_β. Moreover, if the asymptotics above also holds for $\beta = \infty$ with $E(\mu)$ strictly convex, then the LDP holds for $\beta = \infty$, as well.

Proof. Fixing a volume form dV and applying the asymptotics in the lemma to the volume forms $e^{-\beta u}dV$ for any $u \in C^0(X)$ reveals that the asymptotics 4.5 hold with f_β given by the Legendre-Fenchel transform of $E + D_{dV}/\beta$. Now, if E is convex, then $E + D_{dV}/\beta$ is strictly convex (since D_{dV} is) and hence it follows from basic convex duality that f_β is Gateaux differentiable. In fact, the differential $\mu_u := df_{\beta|u}$ is the unique minimizer attaining the sup defining $f_\beta(u)$, viewed as the Legendre-Fenchel transform of $E + D_{dV}/\beta$. Equivalently, μ_u is the unique minimizer of the strictly convex functional $\mu \mapsto E(\mu) + \langle u,\mu\rangle + D_{dV}(\mu)/\beta$. □

Remark 4.8. Let β_N be sequence tending to ∞. By convex duality the Gärtner–Ellis theorem may in the present setting, be formulated as follows (also using Varadhan's lemma [30] in the converse): let E_N be a sequence of functions on $\mathcal{M}_1(X)$. Then $e^{-\beta_N N E_N}(\delta_N)_*(dV^{\otimes N}) \sim e^{-\beta_N N E(\mu)}$ in the sense of a LDP, with $E(\mu)$ strictly convex iff $\beta_N N$ times the log of the Laplace transform of $e^{-\beta_N N E_N}(\delta_N)_*(dV^{\otimes N})$ converges to the Gateaux differentiable function E^*on $C^0(X)$.

4.3. Relations to the existence of the mean energy. Given a sequence of Hamiltonians $H^{(N)}$ on X^N we set

$$\bar{E}_N(\mu) := \frac{1}{N}\int_{X^N} H^{(N)}\mu^{\otimes N},$$

If the limit as $N \to \infty$ exists then we will call it the *mean energy* of μ, denoted by $\bar{E}(\mu)$.

Example 4.9. If $H^{(N)}$ is the mean field Hamiltonian associated to the pair interaction potential W (formula emph1.14) then, trivially, $E(\mu) = \bar{E}_N(\mu)$ for any μ such that $W \in L^1(\mu)$.

It follows immediately from the definition that if the limit of $E_N^*(:= f_N)$ appearing in formula 4.4 exists then

$$\bar{E}(\mu) \geq f^*(\mu).$$

(but, in general this is a strict inequality, for example if $\bar{E}(\mu)$ is not convex). In particular, under the assumptions in Theorem 1.1 we have $\bar{E}(\mu) \geq E(\mu)$, where $E(\mu)$ appears as the rate functional in Theorem 1.1 for $\beta = \infty$ (using Lemma 3.8). Motivated by the complex geometric applications discussed in Sect. 6 this leads one to consider the following

Problem 4.10. Show that the assumptions on $H^{(N)}$ in Theorem 1.1 imply that the corresponding mean energy $\bar{E}(\mu)$ exists when μ is a volume for (perhaps under additional appropriate assumptions on $H^{(N)}$).

As illustrated by the following lemma this problem turns out to be related to the asymptotics of the Gibbs measures with β negative:

Lemma 4.11. *Assume that there exists some negative β_0 such that for any $\beta \geq \beta_0$ the corresponding Gibbs measures are well-defined, i.e. $Z_{N.\beta_N} < \infty$ for N sufficiently large. Moreover, assume that there exists a functional $E(\mu)$ such that*

$$-\lim_{N\to\infty}\frac{1}{N}\log Z_{N.\beta_N} = \inf_{\mu\in\mathcal{M}_1(X)} \beta E(\mu) + D_{dV}(\mu) > -\infty, \tag{4.6}$$

for any volume form dV. Then the mean energy $\bar{E}(\mu)$ exists for any volume form μ and $\bar{E}(\mu) = E(\mu)$.

Proof. First observe that, by Jensen's inequality, the number $f_N(\beta) := -\frac{1}{N}\log Z_{N.\beta}$ appearing in the right hand side above for N is concave in β (and, by assumption, finite). Moreover, $\partial f_N(\beta)/\partial\beta = \bar{E}_N(dV)$ at $\beta = 0$. Further more, the finite function $f(\beta)$ defined by the right hand side in formula 4.6 is also concave, as it is an infimum of a family of linear functions and for $\beta = 0$ the infimum is attained precisely at $\mu = dV$. Hence, by basic convex analysis, the derivative of f at $\beta = 0$ exists and is given by $E(dV)$. Finally, the proof is concluded by using that if f_N is a sequence of convex functions converging point-wise to convex function f such that f_N and f are differentiable at 0 then the corresponding derivatives at 0 also converge. □

It should, however, be stressed that, if $H^{(N)}$ is too singular then the partition function $Z_{N.\beta}$ is equal to ∞, for any $\beta < 0$ (even if $H^{(N)}$ is quasi-superharmonic as in the assumptions of Theorem 1.1). Indeed, for the mean field Hamiltonian corresponding to a pair interaction W this happens as soon as W has a repulsive power-law singularity, i.e. $W(x, y) \sim |x - y|^\alpha$ with $\alpha < 0$ close to the diagonal. On the other hand, in the case of a *logarithmic* singularity $Z_{N.\beta_N}$ is indeed finite for $\beta_0 < 0$ and sufficiently close to 0 (see [22] for the corresponding LDP in the setting of the 2D vortex model).

Using the Gibbs variational principle some converses to Lemma 4.11 can be established [11], where the existence of the mean energy is assumed (and some additional assumptions), by extending the approach of Messer–Spohn [45]. However is should be stressed that the main point of our proof of Theorem 1.1 is that it does note rely on the existence of the mean energy $\bar{E}(\mu)$, which, as pointed above, is an open problem in the present setting.

5. Applications to Kähler–Einstein Geometry

In this section we will apply Theorem 1.1 to complex manifolds X equipped with a line bundle L, assuming that L is positive. The extension to big line bundles (and varieties of positive dimension) is given in the companion paper [9], using the full power of the pluripotential theory developed in [7,14,20] (see the discussion in Sect. 5.4).

5.1. Kähler geometry setup. Let X be an n-dimensional compact complex manifold and denote by J the corresponding complex structure viewed as an endomorphism of the real tangent bundle satisfying $J^2 = -I$.

5.1.1. Kähler forms/metrics. On a complex manifold (X, J) anti-symmetric two forms ω and symmetric two tensors g on $TX \otimes TX$, which are J-invariant, may be identified by setting

$$g := \omega(\cdot, J\cdot)$$

Such a real two form ω is said to be Kähler if $d\omega = 0$ and the corresponding symmetric tensor g is positive definite (i.e. defines a Riemannian metric).[2] Conversely, a Riemannian metric g is said to be *Kähler* if it arises in this way (in Riemannian terms this means that parallel transport with respect to g preserves J). By the local $\partial\bar{\partial}$-lemma a J-invariant two form ω is closed, i.e. $d\omega = 0$ if and only if ω may be locally expressed as $\omega = \frac{i}{2\pi}\partial\bar{\partial}\phi$, in terms of a local smooth function ϕ (called a local potential for ω). In real notation this means that

$$\omega = dd^c\phi,\ \ d^c := -\frac{1}{4\pi}J^*d$$

(and hence ω is Kähler iff ϕ is strictly plurisubharmonic). The normalization above ensures that $dd^c \log|z|^2$ is a probability measure on $\mathbb{C}$. We will denote by $[\omega] \in H^2(X, \mathbb{R})$ the de Rham cohomology represented by ω. If ω_0 is a fixed Kähler form then, according to the global $\partial\bar{\partial}$-lemma, any other Kähler metric in $[\omega_0]$ may be globally expressed as

$$\omega_\varphi := \omega_0 + dd^c\varphi, \quad \varphi \in C^\infty(X),$$

[2] In the complex analysis literature a J-invariant two form ω is usually said to be of type $(1, 1)$ since $\omega = \sum_{i,j}\omega_{ij}dz_i \wedge d\bar{z}_j$ in local holomorphic coordinates.

where φ is determined by ω_0 up to an additive constant. We set

$$\mathcal{H}(X,\omega) := \left\{\varphi \in C^\infty(X) : \omega_\varphi > 0\right\}$$

The association $\varphi \mapsto \omega_\varphi$ thus allows one to identify $\mathcal{H}(X,\omega)/\mathbb{R}$ with the space of all Kähler forms in $[\omega_0]$.

5.1.2. Metrics on line bundles and curvature. Let L be a holomorphic line bundle on X and $\|\cdot\|$ a Hermitian metric on L. The normalized curvature two form of $\|\cdot\|$ may be (locally) written as

$$\omega := -dd^c \log \|s\|^2,$$

in terms of a given local trivialization holomorphic section s of L. The corresponding cohomology class $[\omega]$ is independent of the metric $\|\cdot\|$ on L and coincides with the first Chern class $c_1(L)$ in $H^2(X,\mathbb{R}) \cap H^2(X,\mathbb{Z})$ (conversely, any such cohomology class is the first Chern class of line bundle L). A line bundle L is said to be *positive* if it admits a metric with positive curvature, i.e. such that the curvature form ω is Kähler. Fixing a reference metric $\|\cdot\|$on L with curvature form ω_0 any other metric on L may be expressed as $\|\cdot\| e^{-u/2}$, for $u \in C^\infty(X)$ and its curvature is positive iff $u \in \mathcal{H}(X,\omega)$. When L is the canonical line bundle K_X, i.e. the top exterior power of the holomorphic cotangent bundle of X:

$$K_X := \det(T^*X)$$

any volume form dV on X induces a smooth metric $\|\cdot\|_{dV}$ on K_X, by locally setting $\|dz\|^2_{dV} := c_n dz \wedge d\bar{z}/dV$, where $dz := dz_1 \wedge \cdots \wedge dz_n$ in terms of local holomorphic coordinates and $c_n dz \wedge d\bar{z}$ is a short hand for the local Euclidean volume form $\frac{i}{2}dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge \frac{i}{2}dz_n \wedge d\bar{z}_n$. When dV is the volume form of a given Kähler metric ω on X, i.e. $dV = \omega^n/n!$, then its curvature form may be identified with minus the Ricci curvature of ω, i.e.

$$\mathrm{Ric}\omega = -\, dd^c \log \frac{dV}{c_n dz \wedge d\bar{z}}. \tag{5.1}$$

By a slight abuse of notation we will also write Ric (dV) for the right hand side in formula 5.1.

5.1.3. Twisted Kähler–Einstein metrics. A Kähler metric ω_β is said to be a *twisted Kähler–Einstein metric* if it satisfies the twisted Kähler–Einstein equation

$$\mathrm{Ric}\omega = -\beta\omega + \eta, \tag{5.2}$$

where the form η is called the *twisting form.* Since ω_β is Kähler the form η is necessarily closed and J-invariant. The corresponding equation at the level of cohomology classes is

$$[\eta] = -c_1(K_X) + \beta[\omega]$$

Fixing, once and for all, a volume form dV on X gives the following one-to-one corresponds between twisting forms η and Kähler forms ω_0 solving the cohomological equation above:

$$\eta := \beta\omega_0 + \mathrm{Ric}dV. \tag{5.3}$$

The following lemma then follows directly from the expression 5.1 for the Ricci curvature of a Kähler metric:

Lemma 5.1. *Let X be a compact complex manifold endowed with a J-invariant and closed form η. Then a Kähler form ω_β solves the corresponding twisted Kähler–Einstein equation 5.2 iff $\omega_\beta := \omega_0 + dd^c\varphi_\beta$ for a unique $\varphi_\beta \in \mathcal{H}(X,\omega)$ solving the PDE*

$$\omega_\varphi^n = e^{\beta\varphi} dV \tag{5.4}$$

The celebrated Aubin–Yau theorem may now be formulated as follows:

Theorem 5.2 (Aubin–Yau). [3,53] *Given a compact complex manifold X, endowed with a Kähler form ω_0 and a volume form dV, the PDE 5.4 admits, for any positive number β, a unique solution $\varphi_\beta \in \mathcal{H}(X,\omega)$. Equivalently, given a J-invariant and closed form η such that the class $[\eta] + c_1(K_X)$ is positive (i.e. contains a Kähler form) there exists a unique Kähler metric ω_β solving the twisted Kähler–Einstein equation 5.2.*

Example 5.3. A complex manifold X admits a Kähler–Einstein metric with negative Ricci curvature iff K_X is positive (and the metric is unique). Indeed, if K_X is positive then, by the very definition of positivity, we can take $\omega_0 := -\text{Ric}dV$ for some volume form on X, ensuring that $\eta = 0$ above, with $\beta = 1$ (and the converse is trivial).

Remark 5.4. When $n \geq 2$ the equation is 5.2 precisely the trace-reversed formulation of Einstein's equations on X (with Euclidean signature): $-\beta$ is the cosmological constant and η is the trace-reversed stress-energy tensor. Here we are only concerned with the solutions which are Kähler metrics.

5.1.4. The projection operator P_{ω_0} to the space $PSH(X,\omega_0)$. Next, we recall the definition of the operator P introduced in [12] (which turns out to be related to the limit as $\beta \to \infty$ of the equations 5.4). Given $u \in C^0(X)$ we set

$$(Pu)(x) := \sup_{\varphi \in \mathcal{H}(X,\omega_0)} \{\varphi(x) : \ \varphi \leq u\} \tag{5.5}$$

which defines an operator

$$P : C^0(X) \to PSH(X,\omega_0)$$

from $C^0(X)$ to the space $PSH(X,\omega_0)$ of all ω_0-psh functions on X, i.e. all upper semi-continuous functions φ in $L^1(X)$ such that $\omega_\varphi \geq 0$ in the sense of currents. In fact, the operator P defines a projection operator from $C^0(X)$ onto $PSH(X,\omega_0) \cap C^0(X)$. More generally, if $u \in C^\infty(X)$, the current $dd^c(Pu)$ has coefficients in L^∞_{loc}, i.e.

$$\omega_{Pu} \in L^\infty_{\text{loc}} \tag{5.6}$$

In fact, as shown in [10], taking dV in Eq. 5.4 to be of the form $dV = e^{-\beta u} dV$ one has

$$\lim_{\beta\to\infty} \varphi_\beta = Pu$$

uniformly on X and with a uniform upper bound on the corresponding Kähler forms ω_{φ_β}.

5.1.5. The Monge–Ampère operator and the functionals $\mathcal{E}$ and $\mathcal{F}$. The second order operator

$$\varphi \mapsto MA(\varphi) := \omega_\varphi^n, \tag{5.7}$$

appearing in the Eq. 5.4, is the complex *Monge–Ampère operator* (with respect to the reference form ω_0).[3] By Stokes theorem

$$\int_X \omega_\varphi^n = \int_X \omega_0^n := V$$

which is hence a positive number independent of $\varphi \in C^\infty(X)$. Up to a trivial scaling we may and will assume that $V = 1$. When $n = 1$ the operator MA may be identified with the Laplacian, but when $n \geq 2$ it is fully non-linear. The one-form on $C^\infty(X)$ defined by MA is closed and hence admits a primitive $\mathcal{E}$, i.e. a functional on $C^\infty(X)$ whose differential is given by

$$d\mathcal{E}_{|\varphi} = MA(\varphi). \tag{5.8}$$

The functional $\mathcal{E}$ is only determined up to an additive constant which may be fixed by the normalization condition $\mathcal{E}(0) = 0$.

Using pluripotential theory [14,20] the operator MA can be extended from $\mathcal{H}_{\omega_0}$ to all of $PSH(X, \omega_0)$ giving a positive measures satisfying

$$\int_X MA(\varphi) \leq V (:= 1)$$

Similarly the functional $\mathcal{E}$ also extends from $\mathcal{H}(X, \omega_0)$ to an increasing lower-semi continuous functional on $PSH(X, \omega_0)$. We then set

$$\mathcal{F}(u) := (\mathcal{E} \circ P)(u), \tag{5.9}$$

which by [12] defines a Gateaux differentiable functional on $C^0(X)$. More precisely,

$$(d\mathcal{F})_{|u} = MA(Pu). \tag{5.10}$$

This setup leads to a direct variational approach for solving complex Monge–Ampère equations, including the Aubin–Yau equation 5.4, in the more general setting of big cohomology classes and singular volume forms dV [14] (compare Sect. 5.4). However, in the present setting where L is positive the pluripotential theory can be dispensed with by observing that $MA(\varphi)$ is a well-defined probability measure as long as ω_φ is in L^∞_{loc} (using that $MA(\varphi)$ is point-wise defined almost everywhere on X). Then $\mathcal{F}(u)$ may be defined by first taking u to be in $C^\infty$and using the regularity result 5.6 for Pu. One then defines $\mathcal{F}$ on $C^0(X)$ as the unique continuous extension of $\mathcal{F}$ from $C^\infty(X)$, using that $\mathcal{F}(u)$ is Lipschitz continuous on $C^\infty(X)$ with respect to the C^0-norm (as follows form general principles; see the beginning of the proof of Lemma 3.12).

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[3] The terminology stems from the fact that when $\omega_0 = 0$ (which can always be locally arranged by shifting φ) the density of $MA(\varphi)$ is proportional to the determinant of the complex Hessian $\partial\bar{\partial}\varphi$.

5.2. The "temperature deformed" determinantal point processes on X. Let (X, L) be a polarized manifold, i.e. an n-dimensional complex compact manifold X endowed with a positive holomorphic line bundle L. We will denote by $H^0(X, kL)$ the space of all global holomorphic sections with values in the k th tensor power of L (using additive notation for tensor powers). By the Hilbert-Samuel theorem

$$N_k := \dim H^0(X, kL) = Vk^n + o(k^n),$$

where $V = \int_X c_1(L)^n > 0$.

To the data $(\|\cdot\|, dV, \beta_k)$ consisting of a Hermitian metric $\|\cdot\|$ on L, a volume form dV on X and a sequence of positive number β_k we can associate the following sequence of symmetric probability measures on X^{N_k} :

$$\mu^{(N_k,\beta)} := \frac{\left\|(\det S^{(k)})(x_1, x_2, \ldots x_{N_k})\right\|^{2\beta_k/k} dV^{\otimes N_k}}{Z_{N_k,\beta}} \tag{5.11}$$

where $\det S^{(k)}$ is a generator of the top exterior power $\Lambda^{N_k} H^0(X, kL)$, viewed as a one-dimensional subspace of $H^0(X^{N_k}, (kL)^{\boxtimes N_k})$ under the usual isomorphism between $H^0(X^{N_k}, (kL)^{\boxtimes N_k})$ and the N_k-fold tensor product of $H^0(X, kL)$. The number $Z_{N_k,\beta}$ is the normalizing constant

$$Z_{N_k,\beta} := \int_{X^{N_k}} \left\|\det S^{(k)}\right\|^{2\beta/k} dV^{\otimes N_k} \tag{5.12}$$

By homogeneity the probability measure $\mu^{(N_k,\beta)}$ is independent of the choice of generator $\det S^{(k)}$ and thus only depends on the data $(\|\cdot\|, dV, \beta_k)$. We will refer to to the corresponding random point processes on X, as the *temperature deformed determinantal point processes* on X attached to $(\|\cdot\|, dV, \beta_k)$ (the special case $\beta_k = k$ defines a bona fide determinantal point process, as recalled below).

Remark 5.5. Since the transformation $(\|\cdot\|, dV, \beta_k) \mapsto (\|\cdot\| e^{-u/2}, e^{u\beta_k} dV, \beta_k)$, for $u \in C^0(X)$, leaves the probability measure 5.11 invariant, the processes above only depend on the data $(\|\cdot\|, dV, \beta_k)$ through the corresponding two form η, defined by formula 5.3. Moreover, any twisting form η such that the cohomology class $([\eta] + c_1(K_X))/\beta_k$ defines a positive class in $H^2(X, \mathbb{R}) \cap H^2(X, \mathbb{Z})$ arises from a suitable choice of data $(\|\cdot\|, dV, \beta_k)$ (compare Sect. 5.1.3).

It will be convenient to take $\det S^{(k)}$ to be the generator determined by a basis $s_1, \ldots, s_{N_k}$ in $H^0(X, kL)$ which is orthonormal with respect to the L^2-product determined by $(\|\cdot\|, dV)$ for any fixed volume form dV on X :

$$\langle s, s\rangle_{L^2} := \int_X \|s\|^2 dV$$

We then take $(\det S^{(k)})(x_1, x_2, \ldots, x_{N_k}) :=$

$$= \det(s_i(x_j)) := \sum_{\sigma \in S_{N_k}} (-1)^{\text{sign}(\sigma)} s_1(x_{\sigma(1)}) \cdots s_{N_k}(x_{\sigma(N_k)}) \tag{5.13}$$

Example 5.6. The model case of a polarized manifold is $(X, L) = (\mathbb{P}^m, \mathcal{O}(1))$, where $\mathbb{P}^m (:= \mathbb{C}^{m+1} - \{0\})/\mathbb{C}^*$ is m-dimensional complex projective space and $\mathcal{O}(1)$ is the hyperplane line bundle over $\mathbb{P}^m$ (the model positively curved metric on $\mathcal{O}(1)$ is the Fubini-Study metric induced from the Euclidean metric on $\mathbb{C}^{m+1}$). More generally, taking X to be a non-singular algebraic variety of $\mathbb{P}^m$ and L as the restriction to X of $\mathcal{O}(1)$ gives a polarized where the elements in $H^0(X, kL)$ are, for k sufficiently large, the restrictions to X of homogeneous polynomials of degree k on $\mathbb{P}^m$ (in fact, by the Kodaira embedding theorem any polarized manifold (X, L) may, after replacing L with a sufficiently high tensor power, be concretely realized as $(X, \mathcal{O}(1)_{|X}))$. In the case of $X = \mathbb{P}^1$ (=the Riemann sphere) with $\|\cdot\|$ denoting the Fubini-Study metric on $\mathcal{O}(1)$ whose curvature form ω_0 is the invariant measure on $\mathbb{P}^1$ one can take the base $\{s_i\}$ to consist of monomials and factorize

$$\left\|\det S^{(k)}\right\|(x_1, x_2, \ldots, x_N) = Z_N \prod_{1\leq i<j\leq N} |x_i - x_j|,$$

where $N = k + 1$ and X has been identified with the unit-sphere in Euclidean $\mathbb{R}^3$ and where $Z_N = N^N\binom{N-1}{0}\ldots\binom{N-1}{N-1}/N!$. In the physics literature the corresponding ensemble appears as a *Coulomb gas* of N unit-charge particles (i.e a one component plasma) confined to the sphere in a neutralizing uniform background ω (see for example [27]). More generally, on any Riemann surface of genus g the bosonization formula [1] gives

$$\left\|\det S^{(k)}\right\|(x_1, \ldots x_N) = Z_N \exp\left(-\sum_{i\neq j} G(x_i, x_j) + r(x_1, \ldots, x_N)\right) \tag{5.14}$$

where G is the Green function of the Laplacian induced by the metric ω_0 and where the second term r appearing above vanishes for genus $g = 0$, while for $g > 0$ it may be expressed in terms of the Riemann theta function on the Jacobian torus of the Riemann surface X (giving a contribution which is lower order than the first term; see [56] and references therein). However, when $n > 1$ it should be stressed that there is no tractable formula for $\left\|\det S^{(k)}\right\|(x_1, \ldots x_N)$, even to the leading order.

When $\beta_k = k$ the probability measure $\mu^{(N_k,\beta_k)}$ in formula 5.11defines a *determinantal point process* i.e. its density can be written as

$$\left\|\det_{i,j\leq N}(K^{(k)}(x_i, x_j))\right\|/N_k!,$$

where $K^{(k)}(x, y)$ denotes the kernel of the orthogonal projection onto the space $H^0(X, kL)$ viewed as a subspace of the space $C^\infty(X, kL)$ of all smooth sections equipped with the L^2-norm determined by $(\|\cdot\|, dV)$ [5,39].

The following result generalizes the LDP in [5] for determinantal point processes (or more generally for the case $\beta = \infty$) to the general case where $\beta_k \to \beta \in]0, \infty]$:

Theorem 5.7. *Let (X, L) be a polarized manifold and assume given the data $(\|\cdot\|, dV, \beta_k)$ consisting of a Hermitian metric $\|\cdot\|$ on L, a volume form dV on X and a sequence of positive number $\beta_k \to \beta \in]0, \infty]$. Then the law of the empirical* *measures δ_{N_k} of the corresponding deformed determinantal point processes with N_k particles satisfies a LDP with speed $\beta_k N_k$ and rate functional*

$$F_\beta(\mu) = E_{\omega_0}(\mu) + \frac{1}{\beta} D_{dV}(\mu) - C_\beta,$$

where $E_{\omega_0}(\mu)$ is the pluricomplex energy of μ with respect to the curvature form ω_0 of $\|\cdot\|$ and

$$C_\beta = \inf_{\mathcal{M}_1(X)} F_\beta = - \lim_{N\to\infty} \frac{1}{N_k \beta_k} \log Z_{N.\beta_k},$$

In particular, δ_{N_k} converges in law to the deterministic measure given by the unique minimizer μ_β of F_β. Moreover, when $\beta < \infty$ the measure μ_β is the normalized volume form ω_β of the twisted Kähler–Einstein metric corresponding to the twisting form $\eta := \beta\omega_0 + Ric dV$.

In fact, the Kähler form ω_β may be recovered directly from the limiting volume form μ_β by differentiation twice (as follow from the very definition of the twisted Kähler–Einstein equation 5.2):

$$\omega_\beta := \frac{i}{2\pi} \frac{1}{\beta} \partial\bar{\partial} \log \frac{\mu_\beta}{dV} + \omega_0,$$

Using basic compactness properties of the space $PSH(X, \omega_0)$ one then arrives at the following corollary (see [9] for the proof):

Corollary 5.8. *Given data as in the previous theorem with $\beta \in]0, \infty[$, the following sequence of Kähler forms on X*

$$\omega^{(k)} := dd^c \frac{1}{\beta} \log \frac{\int_{X^{N_k-1}} \left\| \det S^{(k)}(\cdot, x_2, \ldots x_{N_k}) \right\|^{2\beta/k} dV^{\otimes(N_k-1)}}{dV} + \omega_0,$$

converges to the unique solution ω_β of the twisted Kähler–Einstein metric corresponding to the twisting form $\eta := \beta\omega_0 + Ric dV$.

Remark 5.9. The previous corollary yields a quasi-explicit way of approximating the solution ω_β to the twisted KE equation in question (or equivalently the solution φ_β of the corresponding complex Monge–Ampère equation 5.4), by performing integrals over the spaces X^{N_k-1} of increasing dimension. The procedure becomes explicit as soon as one has constructed bases in the spaces $H^0(X, kL)$, for k sufficiently large.

5.2.1. The canonical random point processes on X. We start by recalling the basic fact that, by the very definition of the canonical line bundle K_X, any holomorphic section s_k of the k th tensor power of K_X (i.e. $s_k \in H^0(X, kK_X)$ induces a measure on X, symbolically denoted by $(s_k \wedge \bar{s}_k)^{1/k}$. Concretely, given an open set $U \subset X$ with holomorphic coordinates $(z_1, \ldots, z_n)$ and writing $s_{k|U} = f_k dz^{\otimes k}$ for a holomorphic function f_k on U, where $dz := dz_1 \wedge \cdots \wedge dz_n$ trivializes K_X over U,

$$(s_k \wedge \bar{s}_k)^{1/k}_{|U} = |f_k|^{2/k} i^{n^2} dz \wedge d\bar{z},$$

which is independent of U and thus defines a global measure on X (using any holomorphic atlas on X). We also recall that any volume form dV on X induces a metric

$\|\cdot\|_{dV}$ on the canonical line bundle K_X with the property that, if $s_k \in H^0(X, kK_X)$ then $(s_k \wedge \bar{s}_k)^{1/k}_{|U}$ may be expressed as

$$(s_k \wedge \bar{s}_k)^{1/k}_{|U} = \|s_k\|^{2/k}_{dV} \, dV, \tag{5.15}$$

as follows immediately from the definitions.

Now, fixing a volume form dV on X we can apply the relation 5.15 to X^N equipped with the induced volume form $dV^{\otimes N}$ and the corresponding metric on L and deduce that the canonical probability measure $\mu^{(N_k)}$ on X^{N_k}defined by formula 1.6 coincides with the probability measured in formula 5.11 corresponding to the data $(\|\cdot\|_{dV}, dV, 1)$ Hence, Theorem 1.2 is indeed a special case of Theorem 5.7 (also using that $\eta = 0$ for this particular data).

5.3. Proof of Theorem 5.7. To apply Theorem 1.1 in the present setting first note that the Hamiltonian is given by

$$E^{(N_k)}(x_1, x_2, \ldots x_{N_k}) := -\frac{1}{k} \log \left\| (\det S^{(k)})(x_1, x_2, \ldots x_{N_k}) \right\|^2, \tag{5.16}$$

where $\det S^{(k)}$ is defined by formula 5.13. The validity of the first assumption in Theorem 1.1 is then a consequence of the following result from [12], where $\beta_{N_k} = k$:

Theorem 5.10. [12]. *Let $L \to X$ be a positive line bundle equipped with a smooth Hermitian metric $\|\cdot\|$ on L with curvature form ω_0 and dV a volume form on X. Then*

$$\lim_{k\to\infty} -\frac{1}{kN_k}\left(\log \int_{X^{N_k}} \left\|\det S^{(k)}\right\|^2 (x_1, \ldots, x_N) e^{-ku(x_1)-\cdots-ku(x_N)}\right) = \mathcal{F}(u),$$

where $\mathcal{F}$ is the Gateaux differentiable functional defined by formula 5.9

To verify the second assumption in Theorem 1.1, concerning quasi-superharmonicity, we first observe that we may as well assume that dV is the volume form dV_g of the metric g defined by the Kähler form ω_0. Indeed, $dV = e^{-u\beta} dV_g$ for some smooth function u and hence changing dV corresponds to changing the metric $\|\cdot\|$ to $\|\cdot\| e^{-u/2}$. Next, we recall that, in general, $\log \|s\|^2$ is $k\omega-$psh for any holomorphic section s of $kL \to X$ (where ω is the curvature form of $\|\cdot\|$). Hence, we get,

$$\Delta_g \log \|s\|^{2/k} \geq -\lambda$$

for some positive constant λ. Applying the latter inequality to $\left\|\det(s^{(k)}(\cdot, x_2, \ldots, x_N)\right\|$ for $x_2, \ldots, x_N$ thus shows that Theorem 1.1 can be applied to get the LDP in Theorem 5.7.

Next, we will show that the unique minimizer μ_β of the rate functional F_β appearing in Theorem 1.1 coincides with the normalized volume form ω_β of the corresponding twisted Kähler–Einstein metric, by applying the general Lemma 3.12. It should however be stressed that while the infimum in the left hand side of formula 3.9 is always attained at some $\mu_0 \in \mathcal{M}_1(X)$ (by weak compactness and lower-semi continuity) this is not so for the right hand side, in general. But in the present setting the sup is attained, when L is assumed to be positive, thanks to the Aubin–Yau theorem. Indeed, first setting

$$g(u) = \beta^{-1} \log \int e^{\beta u} dV,$$

for a given $\beta \in]0, \infty[$ gives $g^*(\mu) = \beta^{-1} D_{dV}(\mu)$ if $\mu \in \mathcal{M}_1(X)$ and $g^*(\mu) = \infty$ otherwise, as is well-known [30] (and follows from Jensen's inequality applied to the log). Moreover, by the dominated convergence theorem

$$dg_{|u} = \frac{e^{\beta u} dV}{\int_X e^{\beta u} dV} \in \mathcal{M}_1(X)$$

Letting $\mathcal{F}$ be the functional on $C^0(X)$ defined by formula 5.9 the critical point Eq. 3.10 thus becomes

$$MA(Pu) = \frac{e^{\beta u} dV}{\int_X e^{\beta u} dV},$$

when u is smooth, say. Up to replacing u by $u + C$ we may as well assume that the denominator above is equal to 1. In particular, when $u \in \mathcal{H}(X, \omega)$ the equation above is precisely the Aubin–Yau equation 5.4, which, by the Aubin–Yau theorem admits a (unique) solution $u_\beta \in \mathcal{H}(X, \omega)$. Hence, by the previous lemma $\mu_\beta := MA(u_\beta)$ is the unique minimizer of the rate functional F_β appearing in Theorem 1.1, in the present setting. Finally, as explained in Sect. 5.1.3 μ_β is the volume form of the Kähler form ω_β solving the twisted Kähler–Einstein equation 5.2.

Remark 5.11. To see the relation to the pluricomplex energy introduced in [14] we write, as in formula 3.11,

$$f^*(\mu) = \sup_{u \in C^0(X)} \mathcal{E}(Pu) - \langle u, \mu \rangle ,$$

when $\mu \in \mathcal{M}_1(X)$, which coincides with the *pluricomplex energy* of μ, with respect to ω_0 in [14] (using the notation in [7]). More concretely, a direct calculation reveals that when μ is a volume form

$$E(\mu) = \frac{1}{V} \sum_{j=0}^{n-1} \frac{1}{j+2} \int_X d\varphi_\mu \wedge d^c \varphi_\mu \wedge \frac{(dd^c \varphi_\mu + \omega_0)^j}{j!} \wedge \frac{\omega_0^{n-1-j}}{(n-1-j)!}, \qquad (5.17)$$

where $\varphi_\mu \in \mathcal{H}(X, \omega_0)$ is the solution to the Calabi–Yau equation 1.13, which in Aubin's notation [3] means that $E(\mu) = c_n(I - J)(\varphi_\mu)$ (using [20] the formula above holds for any μ such that $E(\mu) < \infty$, by letting $\wedge$ denote the non-pluripolar products [20]). Thus $E(\mu)$ is a generalization of the classical Dirichlet energy on a Riemann surface. The relation $F_\beta(\omega^n) = \kappa(\omega)$, where κ denotes the twisted version of *Mabuchi's K-energy* then follows from the Chen-Tian formula for the K-energy (see [7] and [9] for a direct proof using convex analysis). Moreover, the restriction to $\mathcal{H}(X, \omega_0)$ of the dual functional $f(-u) + g(u)$ appearing in Lemma 3.12 coincides with the *Ding functional* in Kähler geometry [7]. An alternative proof of the fact that ω_β^n minimizes F_β on $\mathcal{M}_1(X)$ can then be given by using that ω_β is a critical point of κ and hence, by convexity, minimizes κ on $\mathcal{H}(X, \omega_0)$. Accordingly, the Calabi–Yau isomorphism $\omega \mapsto \omega^n$ shows that ω_β^n minimizes the restriction of F_β to the subspace of all volume forms in $\mathcal{M}_1(X)$. However, showing that the infimum of F_β over all of $\mathcal{M}_1(X)$ coincides with the infimum over the subspace of volume forms requires the following non-trivial fact: any μ such that $E(\mu) < \infty$ can be written as a weak limit of volume forms μ_j such that $E(\mu_j) \to E(\mu)$ and $D_{dV}(\mu_j) \to D_{dV}(\mu)$ (see [16] where more general results are obtained).

5.4. The generalization to big line bundles and varieties of positive Kodaira dimension. Let us briefly give some indications about the extension of Theorem 5.7 to line bundles L which are merely assumed big, established in the companion paper [9]. In analytic terms L is big iff $c_1(L)$ contains a positive current on X which is strictly positive in the sense that it is bounded from below by a Kähler form. However, in general, there is a proper open subset $\Omega \subset X$ such that *all* positive currents in $c_1(L)$ are equal to $-\infty$ on the complement $X - \Omega$ (which can be taken to be a complex subvariety of X). Fixing a reference smooth Hermitian metric $\|\cdot\|$ on L with curvature form ω_0 in $c_1(L)$ the space of positive currents in $c_1(L)$ gets identified, as before, with the space $PSH(X, \omega_0)$ of all ω_0-psh functions, modulo constants (however, in general all elements in $PSH(X, \omega_0)$ will be singular along the subvariety $X - \Omega$). Moreover, the non-pluripolar Monge–Ampère operator can be defined on $PSH(X, \omega_0)$, by restricting to Ω [20]. Then the functional $\mathcal{F}$ can be defined essentially as before and Theorem5.7 still holds (again using [12] to verify the first assumption in Theorem 1.1) Invoking, the general Theorem 1.1 thus establishes an LDP with a rate functional F_β, admitting a unique minimizer μ_β as before. However, one new difficulty is to show that μ_β can be written as $MA(\varphi_\beta)$ for the solution to the Eq. 5.4 with minimal singularities, whose existence is provided by the general results in [14,20]. The problem is that Lemma 3.12 cannot be applied as it is not clear that φ_β is of the form Pu for some u in $C^0(X)$ (even if u can be taken to be in $L^\infty(X)$). But using the variational calculus in [7,14] shows that μ_β is of the desired form.

In particular, when K_X is big, i.e. X is a variety of general type, the corresponding positive current ω_β is the canonical Kähler–Einstein current in X [14,20]. In the general case of a variety of positive Kodaira dimension $\kappa \leq n$ (where $\kappa = n$ iff K_X is big) one can use the Ithaka fibration $X \to Y$ to represent K_X as the pull-back of a big line bundle L on the κ-dimensional manifold Y. Using the Fujino–Mori canonical bundle formula this reduces the proof of the convergence on X to the application of a generalization of Theorem 5.7 concerning big line bundles on Y endowed with a singular volume form dV. As shown in [9] this realizes the corresponding canonical limiting current ω_β as the pull-back to X of a (singular) Kähler form on Y solving a twisted Kähler–Einstein equation of the form 5.2, where η is a current on Y determined by the geometry of X (the canonical current ω_β first appeared in a different geometric context in [51,52]).

6. Outlook

6.1. $\beta = 0$. Let (X, L) be a polarized manifold and fix a Kähler metric ω_0 in $c_1(L)$. By Corollary 5.8 (and well-known stability properties of the complex Monge–Ampère operator) one can recover the unique (normalized) smooth solution to the Calabi–Yau equation

$$(\omega_0 + i\partial\bar{\partial}\varphi)^n = dV, \tag{6.1}$$

[53] as the double limit $\varphi := \lim_{\beta\to\infty} \lim_{k\to\infty} \varphi_\beta^{(k)}$, where

$$\varphi_\beta^{(k)} := \frac{1}{\beta} \log \frac{\int_{X^{N_k-1}} \left\| \det S^{(k)}(\cdot, x_2, \ldots x_{N_k}) \right\|^{2\beta/k} dV^{\otimes(N_k-1)}}{dV} - \log Z_N$$

Formally interchanging the two limits thus suggests the following

Conjecture 6.1. *Let (X, L) be a polarized manifold and ω_0 a Kähler metric in $c_1(L)$. Then the unique smooth solution φ to the Calabi–Yau equation 6.1, normalized so that $\int_X \varphi dV = 0$, may be represented as the following limit in $L^1(X)$:*

$$\varphi := \lim_{k\to\infty} \varphi^{(k)}, \quad \varphi^{(k)} := \frac{1}{k}\frac{\int_{X^{N_k-1}} \log \left\| \det S^{(k)}(\cdot, x_2, \dots x_{N_k}) \right\|^2 dV^{\otimes(N_k-1)}}{dV} - C_k,$$

where the constant C_k ensures that $\int_X \varphi^{(k)} dV = 0$.

The conjectural formula above can be seen as a generalization to the non-linear complex Monge–Ampère operator of the classical Green's formula for the solution of the Poisson equation for the Laplacian on a Riemann surface. Indeed, when X is a Riemann surface the limit φ above is precisely given by the Green formula in question (as follows from the bosonization formula 5.14). It turns out that the validity of the conjecture above would follow from the existence of the corresponding mean energy $\bar{E}(\mu)$, for any volume form μ (see Problem 4.10). This is shown precisely as in the setting of the *real* Monge–Ampère operator considered in [8,40] where the analog of the previous conjecture was established using permanents as a replacements of the determinants appearing in the present setting. In particular, when X is a Calabi–Yau manifold, i.e., K_X is trivial, the conjecture would imply a quasi-explicit formula for the unique Ricci flat Kähler metric $\omega \in c_1(L)$, i.e., solving the Kähler–Einstein equation with vanishing cosmological constant, $\Lambda = 0$.

6.2. $\beta < 0$. By Lemma 4.11 the existence of the mean energy (and thus the resolution of the conjecture above) would follow if one could establish the asymptotics in formula 4.6 of the corresponding partition functions $Z_{N_k,\beta/k}$ (assumed finite) for all $\beta > \beta_0$, for some negative number β_0. It can be shown that Z_{N,β_N} is indeed finite for some negative β_0, sufficiently close to zero. In fact, both sides of formula 4.6 are finite when $\beta > \beta_0$ (where the critical negative β_0 depends on (X, L)). This motivates the following

Conjecture 6.2. *Let (X, L) be a polarized manifold and assume given the data $(\|\cdot\|, dV)$ consisting of a Hermitian metric $\|\cdot\|$ on L, a volume form dV on X. For a given negative number β_0 the following is equivalent:*

- *For any $\beta > \beta_0$ the partition functions $Z_{N_k,\beta}$ are finite for k sufficiently large*
- *For any $\beta > \beta_0$ the functional βF_β admits a minimizer on $\mathcal{M}_1(X)$*
- *For any $\beta > \beta_0$ the measures $(\delta_N)_* \left(e^{-\beta H^{(N_k)}} dV^{\otimes N_k}\right)$ on $\mathcal{M}_1(X)$ satisfy a LDP with speed$N$ and rate functional*

$$\beta F_\beta(\mu) = \beta E_{\omega_0}(\mu) + D_{dV}(\mu),$$

where $E_{\omega_0}(\mu)$ is the pluricomplex energy of μ with respect to the curvature form ω_0 of $\|\cdot\|$.

In particular, if the conjectural LDP above holds then the functional βF_β is lower semi-continuous and the large N-limit of the laws of δ_{N_k} for the corresponding random point processes is concentrated on the (non-empty) set of minimizers of βF_β. By [7] any such minimizer is the volume form of a Kähler metric ω_β solving the twisted Kähler–Einstein equation 5.2 corresponding to the data (ω_0, dV, β) and βF_β may be identified with the corresponding twisted K-energy functional. Moreover, if the LDP holds then it follows that $Z_{N_k,\beta} \leq C_\beta^N$, when $\beta > \beta_0$. The conjecture should be contrasted with the fact that, in general, βF_β is unbounded from below if β is sufficiently negative and even when βF_β is bounded from below there exist, in general, twisted Kähler–Einstein metrics whose volume forms do not minimize βF_β.

In the case when L is the dual $-K_X$ of the canonical line bundle, i.e., X is a Fano manifold (which equivalently means that η can be taken to be zero) the equivalence between the first two points in the conjecture above can be seen as a probabilistic analog of the Yau-Tian-Donaldson conjecture saying that a Fano manifold X admits a Kähler–Einstein metric with positive Ricci curvature ((i.e. $\Lambda > 0$) iff X is K-stable in the algebro-geometric sense; see the companion paper [9] for more detailed explanations of these relations.

Interestingly, the notion of negative temperature has already appeared in Onsager's work on the 2D vortex model [46]. Using the bosonization formula 5.14 on a Riemann surface and large N-results for vortex models (as in [22,26,41]) it can be shown that the conjecture above holds when X is a Riemann surface. Moreover, then the critical β_0 is equal to 2, when the volume (degree) of L is normalized to be one. In our normalizations this corresponds to the critical negative temperature 8π in the vortex model [26,41] (a detailed proof of this will appear elsewhere).

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7. Appendix: The Constant in the Cheng–Yau Gradient Estimate

Set $\phi := |\nabla u/u|$ and $F := \phi(a^2 - \rho^2)$, where $\rho \leq a \leq 1$. We will follow the exposition in [47]. First, Bochner's identity gives after some calculations that, for any x,

$$\frac{\Delta\phi}{\phi} \geq \frac{\phi^2}{(n-1)} - (n-1)k^2 - (2 - \frac{2}{(n-1)})\frac{\nabla\phi}{\phi} \cdot \frac{\nabla u}{u} \tag{7.1}$$

(see the bottom of [47, page 141]). Let now x_1 be a point in the interior of $B_a(x_0)$ where F attains it maximum and assume that $\rho(:= d(x, x_0))$ is smooth close to x_1. Next $\nabla F = 0$ at x_1 gives

$$\frac{\nabla\phi}{\phi} = \frac{\nabla\rho^2}{a^2 - \rho^2} = \frac{2\rho\nabla\rho}{a^2 - \rho^2} \tag{7.2}$$

(in the following all (in-)equalities are evaluated at $x = x_1$) and $\Delta F \leq 0$ at x_1 gives

$$\frac{\Delta\phi}{\phi} - \frac{\Delta\rho^2}{a^2 - \rho^2} - \frac{2|\nabla\rho^2|^2}{(a^2 - \rho^2)^2} \leq 0$$

Now, by the Laplacian comparison

$$\Delta\rho^2 \leq 2 + 2(n-1)(1+k\rho)$$

Substituting this into the previous inequality we get (using $|\nabla\rho| \leq 1$)

$$\frac{\Delta\phi}{\phi} - \frac{2+2(n-1)(1+k\rho)}{a^2-\rho^2} - \frac{8\rho^2}{(a^2-\rho^2)^2} \leq 0 \tag{7.3}$$

By 7.2

$$-\frac{\nabla\phi}{\phi}\cdot\frac{\nabla u}{u} \geq -\frac{2\rho\phi}{a^2-\rho^2}$$

Hence, Eq. 7.1 combined with Eqs. 7.3 and the previous inequality gives

$$0 \geq \frac{\phi^2}{(n-1)} - (n-1)k^2 - \frac{4(n-2)}{(n-1)}\frac{2\rho\phi}{a^2-\rho^2} - \frac{(2+2(n-1))(1+k\rho)}{a^2-\rho^2} - \frac{8\rho^2}{(a^2-\rho^2)^2},$$

Equivalently, multiplying by $(a^2-\rho^2)^2$ gives

$$0 \geq \frac{F^2}{(n-1)} - (n-1)k^2(a^2-\rho^2)^2 - \frac{4(n-2)}{(n-1)}2\rho F - (2+2(n-1))(1+k\rho)(a^2-\rho^2) - 8\rho^2,$$

Since we are only interested in the large n behaviour (and $0 \leq \rho \leq a$) we deduce from the previous inequality that

$$0 \geq \frac{F^2}{(n-1)} - 8\rho F - nk^2(a^2)^2 - 2n(1+k)a^2 - 8\rho^2$$

giving, after multiplication by n,

$$0 \geq F^2 - 8anF - n^2k^2(a^2)^2 - 2n^2(1+k)a^2 - 8a^2n,$$

which we write as

$$(4an)^2 + n^2k^2(a^2)^2 + 2n^2(1+k)a^2 + 8a^2n \geq (F-4an)^2,$$

giving

$$a^2n^2\left(26 + k^2a^2 + 2k\right) \geq (F-4an)^2$$

Hence

$$an\left(\left(26+k^2a^2+2k\right)^{1/2}+4\right) \geq F := \phi(a-\rho)(a+\rho) \geq \phi(a-\phi)a,$$

so that

$$n\left(\left(26+k^2a^2+2k\right)^{1/2}+4\right) \geq \phi(a-\phi),$$

This shows that, if $a \leq 1$, there exists a constant C, only depending on an upper bound on k, such that

$$Cn \geq \phi(a-\phi),$$

as desired.

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