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Erratum

Erratum to: Hausdorff and Packing Spectra, Large Deviations, and Free Energy for Branching Random Walks in $\mathbb{R}^d$

Najmeddine Attia[1], **Julien Barral**[2]

[1] Département de Mathématiques, Faculté des Sciences de Monastir, Route de l'Environnement, 5000 Monastir, Tunisie. E-mail: najmeddine.attia@fsm.rnu.tn
[2] LAGA (UMR 7539), Département de Mathématiques, Institut Galilée, Université Paris 13, Paris Sorbonne Cité, 99 Avenue Jean-Baptiste Clément, 93430 Villetaneuse, France. E-mail: barral@math.univ-paris13.fr

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Abstract: The paper contains a gap in the justification of the uniform non degeneracy of the measures μ_ϱ, $\varrho \in \mathcal{J}$, asserted in Proposition 2.8.1; we used a 0-1 law argument which cannot hold in general due to a conditioning problem; we provide an alternative argument. In the terminology of branching processes, this argument yields new sufficient conditions for an additive martingale on a Galton-Watson tree in varying environment to be almost surely positive conditionally on non extinction of the tree.

Uniform non degeneracy of the measures μ_ϱ, $\varrho \in J$. We use the notations of Sects. 2.1 and 2.4. The property to be established is equivalent to $\mathbb{P}(\{\partial\widetilde{T} \neq \emptyset\} \cap \{Y(\cdot) > 0\}) = \mathbb{P}(\{\partial\widetilde{T} \neq \emptyset\})$.

For $n \geq 1$ we denote by f_n the generating function associated with the random integer $N_{A_{jn}}$.

If $k \geq 0$, we denote by p_k the probability of extinction of an inhomogeneous Galton-Watson tree in which the offspring distribution from generation $n-1$ ($n \geq 1$) is given by f_{k+n}. It follows that

$$p_k = f_k \circ \cdots \circ f_\ell(p_{\ell+1}) \quad \forall\, \ell \geq k \geq 0.$$

Moreover, by construction, $p_0 = \mathbb{P}(\partial\widetilde{T} = \emptyset)$. Also, $(f_k)_{k\geq 0}$ converges uniformly to f, the generating function of N, as $k \to \infty$. Another fact is that for any $\varrho \in \mathcal{J}$, for any $k \geq 0$ and $u \in \mathbb{N}_+^k$, the law of $Y(\varrho, u)$ is independent of u, and if we set $q_k(\varrho) = \mathbb{P}(Y(\varrho, u) = 0)$, we have $q_k(\varrho) \geq p_k$ and

$$q_k(\varrho) = f_k \circ \cdots \circ f_\ell(q_{\ell+1}(\varrho)) \quad \forall\, \ell \geq k \geq 0.$$

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Lemma. *For all* $\varrho \in \mathcal{J}$, *we have* $q_0(\varrho) = p_0 = \lim_{k\to\infty} f_0 \circ \cdots \circ f_k(0)$ *(more generally,* $q_k(\varrho) = p_k = \lim_{\ell\to\infty} f_k \circ \cdots \circ f_\ell(0)$*, for all* $k \geq 0$*).*

Consequently, since $\{\partial\widetilde{T} = \emptyset\} \subset \{Y(\varrho) = 0\}$*, we get* $\mathbb{P}(\{\partial\widetilde{T} = \emptyset\}\Delta\{Y(\varrho) = 0\}) = 0$ *(recall that* $Y(\varrho, \emptyset)$ *is simply denoted by* $Y(\varrho)$*).*

Suppose the lemma is proven. Recall that for each $n \geq 1$ we set $\mathcal{J}_n = \{\varrho_{|n} : \varrho \in \mathcal{J}\}$, and for $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathcal{J}_n$, $E_\gamma = \{\omega \in \{\partial\widetilde{\mathsf{T}} \neq \emptyset\} : \exists\, \varrho \in \mathcal{J},\ Y(\varrho) = 0,\ \varrho_{|n} = \gamma\}$. By definition of $Y(\cdot)$, conditionally on $\{\partial\widetilde{\mathsf{T}} \neq \emptyset\}$, the fact that $Y(\cdot)$ vanishes does not depend on the n first generations of the construction, for all $n \geq 1$. Then, due to the product structure $\mathcal{J}$, the previous events are all equal. Fix $\varrho \in \mathcal{J}$. We have that the events $E_{\varrho_{|n}}$, $n \geq 1$, are equal, and by almost sure continuity of $Y(\cdot)$ their intersection is equal to the event $\{\partial\widetilde{\mathsf{T}} \neq \emptyset\} \cap \{Y(\varrho) = 0\}$ up to a set of probability 0. However, due to the lemma, this event has a probability equal to 0. It follows that $\mathbb{P}(E_\gamma) = 0$ for all γ, hence the desired conclusion $\mathbb{P}(\{\partial\widetilde{T} \neq \emptyset\} \cap \{Y(\cdot) > 0\}) = \mathbb{P}(\{\partial\widetilde{T} \neq \emptyset\})$.

Proof of the lemma. For $k \geq 0$, set $g_k = f_0 \circ \cdots \circ f_k$. The functions g_k are convex and non decreasing, and uniformly bounded by 0 and 1 since the same holds true for the f_k. We need the following claims:

Claim 1: $(g_k)_{k\geq 0}$ *converges uniformly to a constant function* g *over the compact subintervals of* $[0, 1)$.

Claim 2: If $\varrho \in \mathcal{J}$, $q_k(\varrho)$ *does not converge exponentially fast to 1, i.e. there is no* $\gamma \in (0, 1)$ *such that* $1 - q_k(\varrho) \leq \gamma^k$ *for k large enough.*

Assuming these claims, we first notice that if x_k stands for the unique fixed point of f_k in $[0, 1)$, we have $x_k \geq p_k$. Moreover, by Proposition 2.1, we have $\lim_{k\to\infty} x_k = 0$. Consequently, the sequence $(p_k)_{k\geq 1}$ converges towards 0, hence by Claim 1 we have $p_0 = \lim_{k\to\infty} f_0 \circ \cdots \circ f_k(0) = g(0)$.

Next we fix $\epsilon \in (0, 1/2)$ and $k_0 \geq 1$ such that for all $k \geq k_0$, $f_k'(x) \geq 1 + \epsilon$ over $[1 - \epsilon, 1]$ (this is possible since f_k converges uniformly to f and $\mathbb{E}(N) = f'(1) > 1$ by our assumptions). Using Claim 2, let $(k_j)_{j\geq 1}$ be an increasing sequence of integers such that $1 - q_{k_j+1}(\varrho) \geq (1+\epsilon)^{-(k_j+1)/4}$. We claim that for j large enough, there exists $\lfloor k_j/2\rfloor \leq k_j' \leq k_j$ such that $z_{k_j'} := f_{k_j'+1} \circ \cdots \circ f_{k_j}(q_{k_j+1}(\varrho)) \leq 1-\epsilon$. Otherwise, since

$$1 - f_{\lfloor k_j/2\rfloor} \circ \cdots \circ f_{k_j}(q_{k_j+1}(\varrho)) = (1 - q_{k_j+1}(\varrho)) \prod_{k=\lfloor k_j/2\rfloor}^{k_j} f_k'(f_{k+1} \circ \cdots f_{k_j+1}(c))$$

for some $c \in [q_{k_j+1}(\varrho), 1]$, so that $f_{k+1}\circ \cdots f_{k_j+1}(c) \geq f_{k+1}\circ \cdots f_{k_j+1}(q_{k_j+1}(\varrho)) > 1-\epsilon$ for all $\lfloor k_j/2\rfloor \leq k \leq k_j$, we should have

$$1 - f_{\lfloor k_j/2\rfloor} \circ \cdots \circ f_{k_j}(q_{k_j+1}(\varrho)) \geq (1+\epsilon)^{-(k_j+1)/4}(1+\epsilon)^{(k_j+1-\lfloor k_j/2\rfloor}, $$

hence $1 - f_{\lfloor k_j/2\rfloor} \circ \cdots \circ f_{k_j}(q_{k_j+1}(\varrho))$ would tend to ∞, which is a contradiction.

Finally, due to Claim 1, we get the convergence of $g_{k_j}(q_{k_j+1}(\varrho)) = g_{k_j'}(z_{k_j'})$ to $g(0)$, hence the desired value of $q_0(\varrho)$.

Proof of Claim 1. At first, it follows from Ascoli-Arzela's theorem that there exists an increasing sequence of integers $(k_j)_{j\geq 1}$ such that $(g_{k_j})_{j\geq 1}$ converges uniformly to a limit g over all the compact subintervals of $[0, 1)$.

Set $\widetilde{x}_k = \max\{x_n : n \geq k\}$ and fix $\epsilon \in (0, 1/2)$. By our assumptions, f is convex, $f(0) = 0$, $f(1) = 1$ and $f'(1) > 1$, hence $f(x) < x$ over $(0, 1)$. Also f_k converges to f uniformly as $k \to \infty$, so we can fix $\gamma \in (0, 1)$ and $k_0 \in \mathbb{N}$ such that $f_k(x) \leq \gamma x$ for $k \geq k_0$ and $x \in [\epsilon, 1-\epsilon]$. Then, for $x \in [0, 1-\epsilon]$, for any $j \geq 1$ such that $k_j \geq k_0$ and any $k \geq k_j$ we have (setting $\widetilde{x}_k = \max\{x_n : n \geq k\}$)

$$g_{k_j}(0) \leq f_1 \circ \cdots \circ f_k(x) \leq g_{k_j}(\max(\widetilde{x}_{k_j+1}, \epsilon, \gamma^{k-k_j} x)).$$

Since x_n converges to 0 as $n \to \infty$ and we can take the difference $k - k_j$ arbitrarily large as $k \to \infty$, it follows that for any $\eta > 0$, for k large enough one has, uniformly in $x \in [0, 1-\epsilon]$, $g(0) - \eta \leq f_1 \circ \cdots \circ f_k(x) \leq g(\epsilon) + \eta$. Due to the continuity of g at 0, this yields the pointwise convergence to $g(0)$ of $f_1 \circ \cdots \circ f_k$ over $[0, 1)$, which must be equal to $\lim_{k\to\infty} f_1 \circ \cdots \circ f_k(0)$. Finally, since the $g_k = f_1 \circ \cdots \circ f_k$ are convex, we get the uniform convergence over the compact subintervals of $[0, 1)$.

Proof of Claim 2. This is a consequence of the property

$$\mathbb{E}\Big(\sup_{\varrho' \in \mathcal{J}} Y(\varrho', u) \exp(\sqrt{\log(Y(\varrho', u) + 3)})\Big) = O(\exp(\epsilon_{|u|}\sqrt{|u|})),$$

where $\lim_{k\to\infty} \epsilon_k = 0$ and O depends on $\mathcal{J}$ only (see (2.22)). Suppose that there exists $\gamma \in (0, 1)$ such that $q_{|u|}(\varrho) = \mathbb{P}(Y(\varrho, u) = 0) \geq 1 - \gamma^{|u|}$ for $|u|$ large enough. In this case, as $|u| \to \infty$ we have $\mathbb{E}(Y(\varrho, u)) = 1 \sim \mathbb{E}(\mathbf{1}_{\{Y(\varrho,u)\geq\gamma^{-|u|+\sqrt{|u|}}\}} Y(\varrho, u))$. Consequently,

$$\begin{aligned}
&\mathbb{E}\Big(Y(\varrho, u) \exp(\sqrt{\log(Y(\varrho, u) + 3)})\Big) \\
&\quad \geq \exp\Big(\sqrt{\log \gamma^{-|u|+\sqrt{|u|}} + 3}\Big)\mathbb{E}(\mathbf{1}_{\{Y(\varrho,u)\geq\gamma^{-|u|+\sqrt{|u|}}\}} Y(\varrho, u)) \geq \exp(\sqrt{|u|}\sqrt{\log(1/\gamma)}/2)
\end{aligned}$$

for all large enough $|u|$, which contradicts (2.22). □

Remark 1. We notice that the literature contains studies of the coincidence of the non extinction of a Galton-Watson process in varying environment and the positivity of the limit of the naturally associated martingale measuring the growth of the process (see e.g., J.C. D'Souza, J.D. Biggins, The Supercritical Galton-Watson process in varying environments, Stoc. Proc. Appl. (1992) 42, 39–47). However, our approach, which concerns a more general type of inhomogeneous Mandelbrot martingales, is different.

Remark 2. We also mention that our estimates for the dimensions of the measures μ_ϱ, as well as Hausdorff and packing dimensions and measures of sets assume implicitly that we first work in $(\mathbb{N}_+^{\mathbb{N}_+}, d)$ after extending the measures μ_ϱ to $(\mathbb{N}_+^{\mathbb{N}_+}, d)$; then all the results automatically transfer to $(\partial T, d)$. It results that in the Appendix we must replace the σ-compactness assumption by separability property. Also, we should have added that the connection between the lower and upper packing dimensions of a measure and its upper local dimension holds as soon as the Besicovich covering property holds (for the Hausdorff dimension, the basic "5-r covering theorem", which always holds, is enough), and on boundaries of countable trees the Besicovich covering property obviously holds.

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