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Erratum

Erratum to: Robust Exponential Decay of Correlations for Singular-Flows

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We correct the proof of the main statement in Sect. 4.2.3 of the original article.

It has been pointed out to the authors by Butterley and Melbourne that the arguments in the original article are not enough to prove the following:

Theorem A. *Given any compact* 3*-manifold* M*, for each* $s \geq 2$ *there exists* $N = N(s) \geq 2$ *and an open subset* $\mathcal{U}$ *of* $\mathfrak{X}^N(M)$ *such that each* $X \in \mathcal{U}$ *exhibits a geometric Lorenz flow, which is* C^s*-smoothly semi-conjugated to a good hyperbolic skew-product semi-flow.*

In this erratum, we show how to complete the proof of Theorem A. Consequently, the main result in the original article remains valid. The need for possible higher smoothness of the vector field to be in a certain conjugation class is a direct consequence of a classical result of Hartman [3, Theorem 12.1, p. 257] stated also in Theorem 3.2 of the original article. Let $s \geq 2$ and let $\mathcal{U}$ be a C^N-open set of vector fields that exhibit a Lorenz attractor and let $r : \Delta \to \mathbb{R}^+$ denote the roof function obtained by conjugation to a suspension flow with a global cross-section Δ.

The good hyperbolic skew-product semiflow condition consists of exponential tail estimates for the roof function, smoothness of the disintegration of the SRB measure and a non-uniform integrability (UNI) condition; see Sect. 2 of the original article. The argument in Sect. 4.4 of the original article to prove smoothness of the disintegration of the SRB measure along stable manifolds lacks some details that were recently completed by Butterley and Melbourne in [2]. Since the last estimate in the argument in Sect. 4.2.3 of the original article is not enough to guarantee the uniform non-integrability condition claimed, the goal of this erratum is to provide an alternative proof to the uniform non-integrability condition. The argument in Sect. 4.2.3 of the original article must

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be replaced by the following result asserting that the non-integrability condition is a C^{1+}-open and C^k-dense condition, for all $N \le k < \infty$.

Proposition 1. *For any $N \le k < \infty$ and $\varepsilon > 0$ there exists a $C^{1+\varepsilon}$-open and C^k-dense set of vector fields $\mathcal{V} \subset \mathcal{U}$, so that the roof function r of every $X \in \mathcal{V}$ satisfies (UNI): it is not possible to write $r = v + u \circ F - u$ where $v : \Delta \to \mathbb{R}$ is constant on each Δ^ℓ and $u : \Delta \to \mathbb{R}$ is a C^1-function.*

The remainder of this corrigendum is devoted to the proof of this proposition. For any fixed $N \le k < \infty$ and $\varepsilon > 0$, if there exists a C^1 function $u : \Delta \to \mathbb{R}$ and a measurable function $v : \Delta \to \mathbb{R}$ constant on each element ω of $\mathcal{Q}$ satisfying $r = u \circ F - u + v$, for geometric Lorenz flow constructed as in the original article with $X \in \mathfrak{X}^k(M)$, then we show that, up to a C^k arbitrarily small perturbation of the original vector field X, this is impossible for all $C^{1+\varepsilon}$ nearby vector fields.

For this we choose two distinct periodic points x_1, x_2 for $F : \Delta \to \Delta$ of the same period n whose orbits are (i) distinct, and (ii) each orbit visits each of the elements of the Markov partition the same number of times as the other, but (iii) necessarily in some different order to each other. The existence of such a pair of periodic orbits is a consequence of F being a full branch Markov map: if ω_1, ω_2 are two elements of the Markov partition, we can choose the period $p = 4$ and x_i, $i = 1, 2$ such that

$$x_1, F(x_1) \in \omega_1, F^2(x_1), F^3(x_1) \in \omega_2 \quad \text{and} \quad x_2, F^2(x_2) \in \omega_1, F(x_2), F^3(x_2) \in \omega_2.$$

Furthermore, x_1, x_2 can be chosen in the interior of ω_1. The cohomological equation implies $S_p r(x_1) = S_p v(x_1) = S_p v(x_2) = S_p r(x_2)$ since x_1 and x_2 visit the same Markov partition elements an equal number of times and v is constant on each partition element. Hence it is enough to modify the roof function $\rho : S \to \mathbb{R}^+$ in a small neighbourhood of x_1 that does not intersect the orbit of x_2 to ensure that the induced roof function r satisfies $S_p r(x_1) > S_p r(x_2)$ and so is not cohomologous to a piecewise constant roof function. Note that $S_p r(x_i)$ is the period of x_i as a periodic orbit of the vector field X. This modification can be done by changing the size of the vector field X in a small neighborhood around x_1.

Indeed, let U_0, U_1 be small open neighborhoods of x_2 that do not intersect the orbit of x_1 with $U_0 \subset \overline{U_0} \subsetneq U_1$ and assume that on U_1 we have C^k-linearized the flow via the Tubular Neighborhood Theorem, that is, up to a C^k coordinate change, X in U_1 is constant and given by $e_1 = (1, 0, 0)$. Take a C^∞ function $\varphi : M \to [0, 1]$ that vanishes in the complement of U_1 and is constant equal to one in U_0 and consider $Z = \varphi \cdot e_1 \in \mathfrak{X}^\infty(M)$. Now consider $Y_\delta = X + \delta Z \in \mathfrak{X}^k(M)$, which is δ close to X in the C^k-topology: Y_δ equals X in $M \backslash U_1$ and equals (with a slight abuse of notation) $X + \delta e_1$ for points in U_0. For Y_δ both x_1 and x_2 are still periodic points, but the period of x_2 decreases while the period of x_1 is kept the same. By some abuse of notation we shall denote by X the perturbed vector field.

We need to show that this conclusion holds for all vector fields Y that are $C^{1+\varepsilon}$ close to X. The orbits of these points involve only finitely many iterates (the inducing time R) of the Lorenz transformations f: both orbits are contained in $\{R \le L\}$ for some fixed $L \ge 1$. Moreover, these points belong to two distinct hyperbolic periodic orbits (of saddle type) of the geometric Lorenz attractor and are away from a neighborhood of the singularity at the origin. Hence, these orbits admit a smooth continuation to all $C^{1+\varepsilon}$ nearby vector fields Y that admit a similar construction of smooth cross-section S_Y and induced transformation F_Y, following the inductive procedure detailed in [1].

In particular, since the periodic points x_1, x_2 belong to the interior of the partition elements, we can control finitely many iterates of the Lorenz transformation f_Y and obtain that the corresponding partition of $\{R_Y \leq L\}$ associated to the induced transformation $\tilde{F}$, up to the inducing time L, is close enough to the partition of $\{R \leq L\}$ so that the continuation $\tilde{x}_i$ of the orbits of x_i, $i = 1, 2$ have the same combinatorics as before, visiting the same elements of the Markov partition the same number of times as the other but in a different order, and with the same inducing times. In this argument it is crucial that all the ingredients in the inductive construction be preserved for Y. In particular, the size of hyperbolic balls obtained from hyperbolic times of the one-dimensional Lorenz transformation f_Y, which depend on the rates of expansion of f_Y but also on the Hölder exponent of Df_Y (see [1]). In its turn, the Hölder exponent of Df_Y depends on the smoothness of the strong-stable foliation of the Lorenz attractor. Hence, the procedure described here demands that the vector fields Y and X are at least $C^{1+\varepsilon}$-close.

The rest of the inducing map for f_Y is obtained following an inductive construction and we do not use in this argument the elements of the partition whose induction time is higher than L.

We can then use the same expression as before obtaining $S_p\tilde{r}(\tilde{x}_1) > S_p\tilde{r}(\tilde{x}_2)$ for Y sufficiently $C^{1+\varepsilon}$ close to X, showing that the induced roof function $\tilde{r}$ for the vector field Y cannot be cohomologous to a piecewise constant roof function.

This completes the proof of the proposition by proving that this function satisfies the uniform non-integrability condition needed to obtain exponential decay of correlations for the flow of Y on the geometric Lorenz attractor.

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