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Erratum

Erratum to: Diffusion at the Random Matrix Hard Edge

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As stated, Theorem 2 of [2] is valid only for $a \geq 0$. While this restriction does not affect the main result of [2], we still would like to communicate the correct statement of Theorem 2 and its proof.

The starting point remains Theorem 1 of [2]. There it was proved that the scaled limiting points of the (β, a)-Laguerre ensemble[1] corresponds to the eigenvalues of $\mathfrak{G}_{\beta,a}$, where $-\mathfrak{G}_{\beta,a}$ generates the diffusion process with random speed and scale measures

$$m(dx) = e^{-(a+1)x - \frac{2}{\sqrt{\beta}} b(x)} dx, \quad s(dx) = e^{ax + \frac{2}{\sqrt{\beta}} b(x)} dx.$$

Here $x \mapsto b(x)$ is a standard Brownian motion. More precisely, the reciprocal of the (β, a)-Laguerre points converge to the ordered eigenvalues of the almost surely trace class integral operator

$$(\mathfrak{G}_{\beta,a})^{-1}\psi(x) = \int_0^\infty \int_0^{x\wedge y} s(dz)\psi(y)m(dy), \quad \text{for } \psi \in L^2[\mathbb{R}_+, m].$$

The corresponding boundary conditions for an eigenfunction f may be read off from the above. That one must have $\psi(0) = 0$ is clear, but it was a misunderstanding of the boundary "at infinity" which led to the error in question. For $\mathfrak{G}_{\beta,a}$, $+\infty$ is entrance, not exit if $a \geq 0$, while it is entrance *and* exit when $a \in (-1, 0]$ (see [1] for definitions). In fact, the theorem in question remains correct for $a \geq 0$ (though for reasons slightly different than stated). For $a < 0$, the process generated by $\mathfrak{G}_{\beta,a}$ can reach $+\infty$, and this has consequences not addressed in [2].

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[1] This is the measure with density proportional to $\prod_{j<k} |\lambda_j - \lambda_k|^\beta \times \prod_{k=0}^{n-1} \lambda_k^{\frac{\beta}{2}(a+1)-1} e^{-\frac{\beta}{2}\lambda_k}$ on $(\mathbb{R}_+)^n$. Here $\beta > 0$ and $a > -1$.

Consider now the "Riccati diffusion" for $\mathfrak{G}_{\beta,a}$, given by the Itô equation:

$$dp(x) = \frac{2}{\sqrt{\beta}} p(x)db(x) + \left(\left(a + \frac{2}{\beta}\right) p(x) - p^2(x) - \lambda e^{-x}\right) dx, \tag{1}$$

(x is the time parameter). The process p may be begun at $+\infty$, which it leaves instantaneously, and there is a positive probability of explosion to $-\infty$. Note also, if $p(x) = 0$ at some $x < \infty$, then $p(x') < 0$ for all $x < x'$.

With $\Lambda_0(\beta, a) < \Lambda_1(\beta, a) < \cdots$ the eigenvalues of $\mathfrak{G}_{\beta,a}$, the corrected Theorem 2 ([2]) reads as follows.

Theorem. *Let $P_{\infty,x}$ denote the law induced by $p(\cdot; \beta, a, \lambda)$ started at $+\infty$ at time x, and restarted at $+\infty$ and time $\mathfrak{m}$ upon any $\mathfrak{m} < \infty$, $p(\mathfrak{m}) = -\infty$. Then,*

$$P(\Lambda_0(\beta, a) > \lambda) = P_{\infty,0}(p \textit{ never hits } 0),$$
$$P(\lambda_k(\beta, a) < \lambda) = P_{\infty,0}(p \textit{ hits } 0 \textit{ at least } k + 1 \textit{ times}).$$

The proof still comes down to Sturm oscillation. The difference between $a \geq 0$ and $-1 < a < 0$ is connected to the next simple fact pointed out to us by O. Zeitouni.

Claim. *With m_c the passage time to position c, $P(\mathfrak{m}_{-\infty} < \infty | \mathfrak{m}_0 < \infty) = 1$ whenever $a \geq 0$.*

In words, when $a \geq 0$ the process $x \mapsto p(x)$ will hit $-\infty$ with probability one once it hits 0. Hence we also have the following.

Corollary. *Let $a \geq 0$, and set $\nu_x(dc) = P_{\infty,x}(\mathfrak{m}_{-\infty} \in dc)$. Then, $P(\Lambda_0(\beta, a) > \lambda) = \nu_0(\{\infty\})$ and $P(\lambda_k(\beta, a) < \lambda) = \int_{\mathbb{R}^{k+1}} \nu_0(dx_1)\nu_{x_1}(dx_2) \cdots \nu_{x_k}(dx_{k+1})$.*

This corollary is just how Theorem 2 ([2]) was stated, erroneously, for all $a > -1$. Effectively, when $a \geq 0$ we can think of there being a Dirichlet condition at $+\infty$. Said better, for $a \geq 0$ the half-line eigenvalue problem is the $L \uparrow \infty$ limit of the problem on $[0, L]$ with either Dirichlet or Neumann conditions at L (and Dirichlet at 0). For $-1 < a < 0$ however the boundary point at $+\infty$ must be viewed as Neumann – it can only be approximated by a sequence of problems on $[0, L]$ with a Neumann condition at L. In terms of $x \mapsto p(x)$, this results in counting passages to 0 (Neumann) rather than to $-\infty$ (Dirichlet).

Finally note that Theorem 3 of [2] (which is a direct corollary of Theorem 2) is unaffected, this being a statement about $a \uparrow +\infty$.

Proof of the theorem. Bring in the approximating operator $\mathfrak{G}^L_{\beta,a}$ $(0 < L < \infty)$ defined via

$$(\mathfrak{G}^L_{\beta,a})^{-1}\psi(x) = \int_0^\infty s_L(x, y)\psi(y)m(dy), \quad s_L(x, y) = \left[\int_0^{x \wedge y} s(dz)\right] 1_{x,y \in [0,L]}, \tag{2}$$

acting on $\psi \in L^2([0, L], m)$. One may readily check that any solution to $\psi(x) = \lambda(\mathfrak{G}^L_{\beta,a})^{-1}\psi(x)$ must satisfy $\psi(0) = 0$ and $\psi'(L) = 0$. That is, this defines the eigenvalue problem for $\mathfrak{G}_{\beta,a}$, cut down to $[0, L]$ with Dirichlet/Neumann conditions at 0 and L.

In the manner of Lemma 12 of [2] one may check that $(\mathfrak{G}^L_{\beta,a})^{-1} \to (\mathfrak{G}_{\beta,a})^{-1}$ in trace norm, which provides convergence of the ordered eigenvalues. As a bit of amplification, in Lemma 12 of [2] we had applied the argument to the approximating $\widetilde{\mathfrak{G}}^L_{\beta,a}$ prescribing

Dirichlet eigenvalues at both 0 and L. In that case the integral kernel $s_L(x, y)$ (with respect to m) from (2) is changed to

$$\widetilde{s_L}(x, y) = \left[\int_0^{x\wedge y} s(dz)\right] \times \left[\frac{\int_{x\vee y}^L s(dz)}{\int_0^L s(dz)}\right] 1_{x,y\in[0,L]}.$$

The crux of the problem is that $(\widetilde{\mathfrak{G}}^L_{\beta,a})^{-1} \to (\mathfrak{G}_{\beta,a})^{-1}$ only for $a \geq 0$.

Differentiating both sides of $\psi(x) = \lambda \int_0^\infty s_L(x, y)\psi(y)m(dy)$ leads to the same system for $x \mapsto (\psi(x), \psi'(x))$ found in [2] (see Eq. (3.1) in that reference). Further, $x \mapsto p(x) = \psi'(x)/\psi(x)$, sensible away from the zeros of ψ, again solves the stochastic differential equation (1).

With this setup, λ is an eigenvalue of $\mathfrak{G}^L_{\beta,a}$ only if $(\psi(x, \lambda), \psi'(x, \lambda))$ satisfying the differential system with initial condition $(0, 1)$ at $x = 0$ comes to $(\cdot, 0)$ at $x = L$. In terms of p, λ is an eigenvalue if $p(L, \lambda) = 0$ (after possible passages to $-\infty$ and subsequent "re-starts"). As λ increases, one sees that the zeros of p on $[0, L]$ move from right to left, additional zeros appearing at L. At this point, the proof proceeds exactly as in [2], with the understanding that the passages of p to 0, not to $-\infty$, comprises the eigenvalue counting function. □

References

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