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Noncommutative Riemann Surfaces by Embeddings in $\mathbb{R}^3$

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Abstract: We introduce *C-Algebras* of compact Riemann surfaces Σ as non-commutative analogues of the Poisson algebra of smooth functions on Σ. Representations of these algebras give rise to sequences of matrix-algebras for which matrix-commutators converge to Poisson-brackets as $N \to \infty$. For a particular class of surfaces, interpolating between spheres and tori, we completely characterize (even for the intermediate singular surface) all finite dimensional representations of the corresponding *C-algebras*.

Introduction

Attaching sequences of matrix algebras to a given manifold M to describe a non-commutative and approximate version of its ring of smooth functions has become a rather important tool in non-commutative field theory: more precisely, for each positive integer N let $Q_N : \mathcal{C}^\infty(M, \mathbb{C}) \to M_{N,N}(\mathbb{C})$ be a complex linear surjective map of the ring of smooth functions on M into the space of all complex $N \times N$-matrices such that products of functions are approximately mapped to products of matrices in the limit $N \to \infty$. In almost all cases, $\mathcal{C}^\infty(M, \mathbb{C})$ carries a Poisson bracket $\{\ ,\ \}$ (for instance if M is symplectic, such as every orientable Riemann surface), and one further demands that Poisson brackets are approximately mapped to matrix commutators in the limit $N \to \infty$ (see e.g. [BHSS91] for details).

For the 2-sphere $\mathbb{S}^2$ [GH82] one could use the fact that the space of all spherical harmonics of fixed l is in bijection with the space of all harmonic polynomials in $\mathbb{R}^3$ of degree l; substituting the three commuting variables by irreducible N-dimensional representations of the three-dimensional Lie algebra $\mathfrak{su}(2)$ allows to define a map from functions on $\mathbb{S}^2$ to $N \times N$ matrices, that sends Poisson brackets to matrix commutators up to corrections of order $1/N$ (see also [BHSS91], Example 3, p. 218). The result was dubbed “Fuzzy Sphere” in [Mad92]. The papers [KL92] prove that the (complexified) Poisson algebra of functions on *any* Riemann surface arises as a $N \to \infty$ limit of

$\mathfrak{gl}(N, \mathbb{C})$ – which had been conjectured in [BHSS91]. This result was extended to any quantizable compact Kähler manifold in [BMS94], the technical tool being geometric and Berezin-Toeplitz quantization. A thorough analysis of non-commutative Riemann surfaces of genus greater than or equal to 2 as a continuous field of simple C^*-algebras, strongly Morita equivalent to a reduced twisted group C^*-algebra of its fundamental group, has been given in [NN99]. Insight on how matrices can encode topological information (certain sequences having been identifiable as converging to a particular function, but $\mathfrak{gl}(N, \mathbb{C})$ lacking topological invariants) was gained in [Shi04].

Even though the above general results are constructive, there seem to be only two explicit formulas, for the two-sphere [GH82] and for the two-torus [FFZ89] (see also [Hop89/88]), which are quite different from each other; the former uses the natural embedding of the two-sphere into $\mathbb{R}^3$ whereas the latter relies on the fact that the two-torus is a quotient $\mathbb{R}^2/\mathbb{Z}^2$. The general results are based on the complex nature of any compact orientable Riemann surface.

In this paper, we should like to propose an approach which to the best of our knowledge does not seem to have been treated in the literature so far, despite its rather intuitive appeal: we are using the 'visualisable' embedding of a compact orientable Riemann surface Σ into $\mathbb{R}^3$ explicitly given by the set of all zeros of a real polynomial C. The function C, via

$$\{f, g\}_C := \nabla C \cdot \big(\nabla f \times \nabla g\big),$$

defines a Poisson bracket for all real-valued smooth functions f, g on $\mathbb{R}^3$. Since C is a Casimir function for the bracket $\{\ ,\ \}_C$, one gets a symplectic Poisson bracket on Σ by restriction. The idea now is to use the above Poisson bracket on $\mathbb{R}^3$ to first define an infinite-dimensional algebra as a quotient algebra of the free non-commutative algebra in three variables, involving a real parameter $\hbar$ and suitably ordered non-commutative analogues of $\{x, y\}_C$, $\{y, z\}_C$ and $\{z, x\}_C$. In a second step the resulting algebra is divided by an ideal generated by the constraint polynomial C thus giving a non-commutative version of the functions on Σ. In a third step matrix representations of any size N of this latter algebra are constructed where the parameter $\hbar$ takes specific values depending on N. It is noteworthy that the construction does not require the zero set of C to be a regular surface. Thus, even for a singular surface (e.g., in the transition from sphere to torus) the non-commutative analogue is still well defined.

The main result of this paper is an explicit construction of non-commutative (non-round) spheres and tori, including the transition region with a singular surface that emerges at the point of topology change. Encouraged by the explicit construction and by the fact that for the two-torus our results almost coincide with the older results of [BHSS91], we are quite optimistic that for the case of genus $g \geq 2$ this embedding approach may give more explicit constructions than the existence proof in [KL92] and [BMS94].

The paper is organized as follows: In Sect. 1 we describe Riemann surfaces of genus g embedded in $\mathbb{R}^3$ as inverse images of polynomial constraint-functions, $C(\vec{x})$. The above-mentioned Poisson bracket $\{\ ,\ \}_C$ on $\mathbb{R}^3$ is treated in Sect. 2, where the bracket restricts to a symplectic bracket on the embedded Riemann surface Σ.

In Sect. 3 step one and two of the above programme is explicitly proven for a polynomial constraint C describing the two-sphere, the two-torus, and a transition region: we give a system of relations (Eqs. (3.2), (3.3), (3.4)), and show that this system satisfies the hypothesis of the Diamond lemma, thus proving that the non-commutative algebra carries a multiplication which is a converging deformation of the point wise multiplication of polynomials in three commuting variables (see Proposition 3.1).

In the central Sect. 4 we completely classify all the finite-dimensional representations of the algebras constructed in the preceding section (two-sphere, two-torus, and transition) which are hermitian in the sense that the variables x, y, and z are sent to hermitian $N \times N$-matrices. The main technical tool is graph-theory describing the non-zero entries of the matrices. Next, in Sect. 5, we confirm that the eigenvalue sequences of these representations reflect topology in the sense suggested in [Shi04].

The final Sect. 6 compares the classification results of Sect. 4 with previously known matrix constructions for the sphere and the torus. In the case of the torus it is shown that our result agrees with what can be obtained by (variants of) Berezin-Toeplitz-Quantization, see e.g. [BHSS91].

1. Genus g Riemann Surfaces

The aim of this section is to present compact connected Riemann surfaces of any genus embedded in $\mathbb{R}^3$ by inverse images of polynomials. For this purpose we use the regular value theorem and Morse theory. Let C be a polynomial in 3 variables and define $\Sigma = C^{-1}(\{0\})$. What are the conditions on C, for Σ to be a genus g Riemann surface? If the restriction of C to Σ is a submersion, then Σ is an orientable submanifold of $\mathbb{R}^3$. Σ has to be compact and of the desired genus. For further details see [Hir76, Hof02].

The classification of 2-dimensional compact (connected) manifolds is well-known. In this case there is a one to one correspondence between topological and diffeomorphism classes. The result is that any compact orientable surfaces is homeomorphic (hence diffeomorphic) to a sphere or to a surface obtained by gluing tori together (connected sum). The number g of tori is called the genus and is related to the Euler-Poincaré characteristic by the formula $\chi = 2 - 2g$.

To compute $\chi(\Sigma)$ we apply Morse theory to a specific function. A point p of a (smooth) function f on Σ is a singular point if $Df_p = 0$ in which case $f(p)$ is a singular value. At any singular point p one can consider the second derivative $D^2 f_p$ of f and p is said to be non-degenerate if $\det(D^2 f_p) \neq 0$. Moreover, one can attach an index to each such point depending on the signature of $D^2 f$: 0 if positive, 1 if hyperbolic and 2 if negative. A Morse function is a function such that every singular point is non-degenerate and singular values all distinct. Then $\chi(\Sigma)$ is given by the formula:

$$\chi(\Sigma) = n(0) - n(1) + n(2),$$

where $n(i)$ is the number of singular points which have an index i.

The Cote_x function is defined as the restriction of the first coordinate on the surface. It is not necessarily a Morse function (one has to choose a "good" embedding for that), but the singular points are those for which the gradient grad C is parallel to the Ox axis. Moreover the Hessian matrix of Cote_x at such a point p is:

$$-\frac{1}{\frac{\partial C}{\partial x}(p)} \begin{pmatrix} \frac{\partial^2 C}{\partial y^2}(p) & \frac{\partial^2 C}{\partial y \partial z}(p) \\ \frac{\partial^2 C}{\partial y \partial z}(p) & \frac{\partial^2 C}{\partial z^2}(p) \end{pmatrix}.$$

Take

$$C(\vec{x}) = \frac{1}{2}\big(P(x) + y^2\big)^2 + \frac{1}{2}z^2 - \frac{1}{2}c,$$

where $c > 0$, $P(x) = a_{2k}x^{2k} + a_{2k-1}x^{2k-1} + \cdots + a_1 x + a_0$ with $a_{2k} > 0$ and $k > 0$. Obviously Σ is closed and bounded (even degree of P) hence compact. Σ is a submanifold of $\mathbb{R}^3$ if, and only if for each $p \in \Sigma$, $DC_p \neq 0$ which is equivalent to requiring that the polynomials $P - \sqrt{c}$ and $P + \sqrt{c}$ have only simple roots. The singular points of the Cote_x function on Σ are the points $(x, 0, 0)$ such that $P(x)^2 = c$ and the Hessian matrix is:

$$-\frac{1}{\frac{\partial C}{\partial x}(x,0,0)}\begin{pmatrix} 2P(x) & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence it is positive or negative if, and only if $P(x) = \sqrt{c}$ and hyperbolic if, and only if $P(x) = -\sqrt{c}$. Thanks to the fact that $P(x)$ never vanishes at a singular point, this also shows that Cote_x is a genuine Morse function. Finally,

$$n(0) + n(2) = \#\{P = \sqrt{c}\} \quad \text{and} \quad n(1) = \#\{P = -\sqrt{c}\}.$$

If the polynomial $P - \sqrt{c}$ has exactly 2 simple roots and the polynomial $P + \sqrt{c}$ has exactly $2g$ simple roots, then $\chi(\Sigma) = 2 - 2g$ and Σ is a surface of genus g.

Let $g > 0$. Set:

(i) $G(t) = (t-1)(t-2^2)\ldots(t-g^2)$ and $M = \max_{0 \le t \le g^2+1} G(t)$, $\alpha \in \left(0, \frac{2\sqrt{c}}{M}\right)$,

(ii) $Q(x) = \alpha G(x) - \sqrt{c}$ and $P(x) = Q(x^2)$.

One can directly see that $Q + \sqrt{c}$ has exactly g simple roots, hence $P + \sqrt{c}$ has exactly $2g$ simple roots. For $t \in [0; g^2 + 1]$, the function $Q(t) - \sqrt{c}$ has no zero. On the other hand, for $t \ge g^2 + 1$, $Q(t) - \sqrt{c}$ is strictly growing and has exactly one zero. Consequently the polynomial $P - \sqrt{c}$ has exactly 2 simple roots and the surface Σ defined above is a genus g compact Riemann surface. Note that non-compact, respectively non-polynomial, higher genus Riemann surfaces have been considered in [BKL05].

2. The Construction for General Riemann Surfaces

For arbitrary smooth $C : \mathbb{R}^3 \longrightarrow \mathbb{R}$,

$$\{f, g\}_{\mathbb{R}^3} := \nabla C \cdot (\nabla f \times \nabla g) \tag{2.1}$$

defines a Poisson bracket for functions on $\mathbb{R}^3$ (see e.g. Nowak [Now97] who studied the formal deformability of (2.1)).[1] Clearly, C is a Casimir function of the bracket, i.e. C commutes with every function. Let now, as in Sect. 1, $\Sigma_g \subset \mathbb{R}^3$ be described as $C^{-1}(0)$ with

$$C(\vec{x}) = \frac{1}{2}\left(P(x) + y^2\right)^2 + \frac{1}{2}z^2 - \frac{1}{2}c, \tag{2.2}$$

and $c > 0$. For this choice of C, the bracket $\{\cdot,\cdot\}_{\mathbb{R}^3}$ defines a Poisson bracket on Σ_g through restriction. The Poisson brackets between x,y and z read:

$$\begin{aligned} \{x, y\}_{\mathbb{R}^3} &= \partial_z C = z, \\ \{y, z\}_{\mathbb{R}^3} &= \partial_x C = P'(x)(P(x) + y^2), \\ \{z, x\}_{\mathbb{R}^3} &= \partial_y C = 2y(P(x) + y^2). \end{aligned} \tag{2.3}$$

[1] While we did not (yet find a way to) use his results, we are very grateful for his "New Year's Eve" explanations, as well as providing us with his Ph.D. Thesis.

We claim that fuzzy analogues of Σ_g can be obtained via matrix analogues of (2.3). Apart from possible "explicit $1/N$ corrections", direct ordering questions arise on the r.h.s. of (2.3), while on the l.h.s. one replaces Poisson brackets by commutators, i.e. $\{\cdot,\cdot\} \to \frac{1}{i\hbar}[\cdot,\cdot]$. We present the following Ansatz for the *C-algebra* of Σ_g, given as three relations in the free algebra generated by the letters X, Y, Z:

$$[X, Y] = i\hbar Z, \tag{2.4}$$

$$[Y, Z] = i\hbar \sum_{r=1}^{2g} a_r \sum_{i=0}^{r-1} X^i \left(P(X) + Y^2\right) X^{r-1-i} \equiv \hat{\phi}_X, \tag{2.5}$$

$$[Z, X] = i\hbar \left[2Y^3 + YP(X) + P(X)Y\right] \equiv \hat{\phi}_Y, \tag{2.6}$$

where $\hbar$ is a positive real number and $P(X) = \sum_{r=0}^{2g} a_r X^r$. The particular ordering in (2.5) and (2.6) is chosen such that the three equations are consistent, in the sense of the Diamond Lemma [Ber78].

Proposition 2.1. *Let $S = \{\sigma_X, \sigma_Y, \sigma_Z\}$ be a reduction system with*

$$\begin{aligned}
\sigma_X &= (W_X, f_X) = \left(ZY, YZ - \hat{\phi}_X\right),\\
\sigma_Y &= (W_Y, f_Y) = \left(ZX, XZ + \hat{\phi}_Y\right),\\
\sigma_Z &= (W_Z, f_Z) = \left(YX, XY - i\hbar Z\right).
\end{aligned}$$

This reduction system contains an ambiguity; i.e., there are two ways of reducing the word ZYX: Either we replace ZY by $YZ - \hat{\phi}_X$ or we replace YX by $XY - i\hbar Z$. The ambiguity is called resolvable *if these two reductions eventually reduce to the same expression, by using that we can replace any occurrence of W_X, W_Y, W_Z by f_X, f_Y, f_Z respectively.*

The statement of this proposition is that the ambiguity $(ZY)X = Z(YX)$ is resolvable if and only if $[X, \hat{\phi}_X] + [Y, \hat{\phi}_Y] = 0$, and that this relation is satisfied for the choice in (2.5) and (2.6).

Proof. The ambiguity is resolvable if we can show that $A := (YZ - \hat{\phi}_X)X - Z(XY - i\hbar Z) = 0$ using only the possibility to replace any occurrence of W_i with f_i, for $i = X, Y, Z$. We get

$$\begin{aligned}
A &= YZX - ZXY - \hat{\phi}_X X + i\hbar Z^2 = Y(XZ + \hat{\phi}_Y) - (XZ + \hat{\phi}_Y)Y - \hat{\phi}_X X + i\hbar Z^2\\
&= YXZ - XZY + [Y, \hat{\phi}_Y] - \hat{\phi}_X X + i\hbar Z^2\\
&= (XY - i\hbar Z)Z - X(YZ - \hat{\phi}_X) + [Y, \hat{\phi}_Y] - \hat{\phi}_X X + i\hbar Z^2 = [X, \hat{\phi}_X] + [Y, \hat{\phi}_Y].
\end{aligned}$$

It is then straightforward to check that $[Y, \hat{\phi}_Y] = -[X, \hat{\phi}_X]$ for the choice in (2.5) and (2.6). $\square$

Finding explicit representations of (2.4)–(2.6), let alone classifying them, is of course a very complicated task. We succeeded in doing so for a (continuously deformable) class of surfaces corresponding to spheres and tori.

3. The Torus and Sphere C-Algebras

Let us now take $P(x) = x^2 - \mu$, in which case $C^{-1}(0)$, with

$$C(x, y, z) = \frac{1}{2}\left(x^2 + y^2 - \mu\right)^2 + \frac{1}{2}z^2 - \frac{1}{2}c \qquad (c > 0), \tag{3.1}$$

describes a surface of revolution which is a torus for $\mu > \sqrt{c}$ and a sphere for $-\sqrt{c} < \mu < \sqrt{c}$.

As μ increases from $-\sqrt{c}$ to $\sqrt{c}$ the (almost) round sphere gets deformed by introducing two growing "sinks"; one at the north pole and one at the south pole. At the critical point $\mu = \sqrt{c}$ the two sinks meet and the surface develops a singularity. For larger μ the singularity vanishes and a hole appears, giving the topology of a torus.

The corresponding C-algebra is defined as the quotient of the free algebra $\mathbb{C}\langle X, Y, Z\rangle$ with the two-sided ideal generated by the relations

$$\left[X, Y\right] = i\hbar Z, \tag{3.2}$$

$$\left[Y, Z\right] = i\hbar\left[2X^3 + XY^2 + Y^2X - 2\mu X\right], \tag{3.3}$$

$$\left[Z, X\right] = i\hbar\left[2Y^3 + YX^2 + X^2Y - 2\mu Y\right]. \tag{3.4}$$

By introducing $W = X + iY$ and $V = X - iY$ one can rewrite (3.3) and (3.4) as

$$\left(W^2V + VW^2\right)(1 + \hbar^2) = 4\mu\hbar^2 W + 2(1 - \hbar^2)WVW, \tag{3.5}$$

$$\left(V^2W + WV^2\right)(1 + \hbar^2) = 4\mu\hbar^2 V + 2(1 - \hbar^2)VWV, \tag{3.6}$$

and we denote by $I(\mu, \hbar)$ the ideal generated by these relations. Through the "Diamond lemma" [Ber78] one can explicitly construct a basis of this algebra.

Proposition 3.1. *Let $C(\mu, \hbar) = \mathbb{C}\langle W, V\rangle / I(\mu, \hbar)$. Then a basis of $C(\mu, \hbar)$ is given by*

$$\{V^i(WV)^jW^k \;:\; i, j, k = 0, 1, 2, \ldots\}.$$

As a vector space, $C(\mu, \hbar)$ is therefore isomorphic to the space of commutative polynomials $\mathbb{C}[X, Y, Z]$.

Proof. In the notation of the Diamond Lemma, let $S = \{\sigma_1, \sigma_2\}$ be a reduction system with

$$\sigma_1 = (w_{\sigma_1}, f_{\sigma_1}) = \left(W^2V, \frac{4\mu\hbar^2}{1 + \hbar^2}W + \frac{2(1 - \hbar^2)}{1 + \hbar^2}WVW - VW^2\right),$$

$$\sigma_2 = (w_{\sigma_2}, f_{\sigma_2}) = \left(WV^2, \frac{4\mu\hbar^2}{1 + \hbar^2}V + \frac{2(1 - \hbar^2)}{1 + \hbar^2}VWV - V^2W\right),$$

and let $\leq$ be a partial ordering on $\langle W, V\rangle$ such that $p < q$ if either the total degree (in W and V) of p is less than the total degree of q or if p is a permutation of the letters in q and the *misordering index* of p is less than the misordering index of q. The misordering index of a word $a_1a_2\ldots a_k$ is defined to be the number of pairs $(a_k, a_{k'})$ with $k < k'$ such that $a_k = W$ and $a'_k = V$. This partial ordering is *compatible* with S in the sense that every word in f_{σ_i} is less than w_{σ_i}.

We will now argue that the partial ordering fulfills the descending chain condition, i.e. that every sequence of words such that $w_1 \geq w_2 \geq \cdots$ eventually becomes constant. Assume that w_1 has degree d and misordering index i. If $w_1 > w_k$, then d or i must decrease by at least 1. Since both the degree and the misordering index are non-negative integers, an infinite sequence of strictly decreasing words can not exist.

The reduction system S has one *overlap ambiguity*, namely, there are two ways to reduce the word W^2V^2; either you write it as $(W^2V)V$ and use σ_1, or you write it as $W(WV^2)$ and use σ_2. In an associative algebra, these must clearly be the same, and if they do reduce to the same expression, we call the ambiguity *resolvable*. It is now straightforward to check that the indicated ambiguity is in fact resolvable.

The above observations allow for the use of the Diamond lemma, which in particular states that a basis for $C(\mu, \hbar)$ is given by the set of irreducible words. In this particular case, it is clear that the words $V^i(WV)^kW^j$ are irreducible (since they do not contain W^2V or WV^2) and that there are no other irreducible words. □

By a straightforward calculation, using (3.5) and (3.6), one proves the following result.

Proposition 3.2. *Define* $D = WV$, $\tilde{D} = VW$ *and* $\hat{C} = (D + \tilde{D} - 2\mu)^2 + (D - \tilde{D})^2/\hbar^2$. *Then it holds that*

(i) $[D, \tilde{D}] = 0$,
(ii) $[W, \hat{C}] = [V, \hat{C}] = 0$.

In particular, this means that the direct non-commutative analogue of the constraint (3.1) is a Casimir of $C(\mu, \hbar)$.

Let us make a remark on the possibility of choosing a different ordering when constructing a non-commutative analogue of the Poisson algebra. Assume we choose to completely symmetrize the r.h.s of Eqs. (2.3). Then, the defining relations of the algebra become

$$\begin{aligned}
\big[X, Y\big] &= i\hbar Z,\\
\big[Y, Z\big] &= 2i\hbar\Big[X^3 + \frac{1}{3}\big(XY^2 + Y^2X + YXY\big) - \mu X\Big],\\
\big[Z, X\big] &= 2i\hbar\Big[Y^3 + \frac{1}{3}\big(YX^2 + X^2Y + XYX\big) - \mu Y\Big].
\end{aligned}$$

Again, defining $W = X + iY$ and $V = X - iY$, gives

$$\begin{aligned}
\Big(W^2V + VW^2\Big)(1 + 4\hbar^2/3) &= 4\mu\hbar^2 W + 2(1 - 2\hbar^2/3)WVW,\\
\Big(V^2W + WV^2\Big)(1 + 4\hbar^2/3) &= 4\mu\hbar^2 V + 2(1 - 2\hbar^2/3)VWV,
\end{aligned}$$

which, by rescaling $\hbar^2 = \frac{3\hbar'^2}{3-h'^2}$, can be brought to the form of Eqs. (3.5) and (3.6), with $\hbar'$ as the new parameter.

4. Representations of the Torus and Sphere Algebras

Let us now turn to the task of finding representations ϕ, of the algebra $C(\mu, \hbar)$, with $0 < \hbar < 1$, for which $\phi(X), \phi(Y), \phi(Z)$ are hermitian matrices, i.e. $\phi(W)^\dagger = \phi(V)$. First, we observe that any such representation is completely reducible; hence, in the following, we need only consider irreducible representations.

Proposition 4.1. *Any representation ϕ of $C(\mu, \hbar)$ such that $\phi(W)^\dagger = \phi(V)$ is completely reducible.*

Proof. Since the algebra of all complex $N \times N$ matrices equipped with the sup-norm is a C^*-algebra, it is clear that any *-subalgebra is completely reducible. For the convenience of the reader, we give the algebraic proof.

Let ϕ be a representation of $C(\mu, \hbar)$ fulfilling the conditions in the proposition. Moreover, let $\mathcal{A}$ be the subalgebra, of the full matrix-algebra, generated by $\phi(W)$ and $\phi(V)$. First we note that since $\phi(V) = \phi(W)^\dagger$, the algebra $\mathcal{A}$ is invariant under hermitian conjugation, thus given $M \in \mathcal{A}$ we know that $M^\dagger \in \mathcal{A}$.

We prove that $\mathrm{Rad}(\mathcal{A})$ (the radical of $\mathcal{A}$), i.e. the largest nilpotent ideal of $\mathcal{A}$, vanishes, which implies, by the Wedderburn-Artin theorem, see e.g. [ASS06], that ϕ is completely reducible. Let $M \in \mathrm{Rad}(\mathcal{A})$. Since $\mathrm{Rad}(\mathcal{A})$ is an ideal it follows that $M^\dagger M \in \mathrm{Rad}(\mathcal{A})$. For a finite-dimensional algebra, $\mathrm{Rad}(\mathcal{A})$ is nilpotent, which in particular implies that there exists a positive integer m such that $\left(M^\dagger M\right)^m = 0$. It follows that $M = 0$, hence $\mathrm{Rad}(\mathcal{A}) = 0$. □

In the following, we shall always assume that ϕ is an *hermitian* irreducible representation of $C(\mu, \hbar)$. For these representations, $\phi(D)$ and $\phi(\tilde{D})$ (as defined in Proposition 3.2) will be two commuting hermitian matrices and therefore one can always choose a basis such that they are both diagonal. We then conclude that the value of the Casimir $\hat{C}$ will always be a non-negative real number, which we will denote by $4c$. Finding hermitian representations of $C(\mu, \hbar)$ with $\phi(\hat{C}) = 4c\mathbb{1}$ thus amounts to solving the matrix equations

$$(WD + \tilde{D}W)\left(1 + \hbar^2\right) = 4\mu\hbar^2 W + \left(1 - \hbar^2\right)(W\tilde{D} + DW), \quad (4.1)$$

$$\left(D + \tilde{D} - 2\mu\mathbb{1}\right)^2 + \frac{1}{\hbar^2}\left(D - \tilde{D}\right)^2 = 4c\,\mathbb{1}, \quad (4.2)$$

with $D = WW^\dagger = \mathrm{diag}(d_1, d_2, \ldots, d_N)$ and $\tilde{D} = W^\dagger W = \mathrm{diag}(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_N)$ being diagonal matrices with non-negative eigenvalues. The "constraint" (4.2) constrains the pairs $\vec{x}_i = (d_i, \tilde{d}_i)$ to lie on the ellipse $(x + y - 2\mu)^2 + (x - y)^2/\hbar^2 = 4c$, e.g. as in Fig. 1.

Representations with $c = 0$, which we shall call *degenerate*, are particularly simple, and can be directly characterized.

Proposition 4.2. *Let ϕ be an hermitian representation of $C(\mu, \hbar)$ such that $\phi(\hat{C}) = 0$. Then $\mu \geq 0$ and there exists a unitary matrix U such that $\phi(W) = \sqrt{\mu}\,U$.*

Proof. When D and $\tilde{D}$ are non-negative diagonal matrices, $c = 0$ implies $D = \tilde{D} = \mu\mathbb{1}$ via (4.2), which necessarily gives $\mu \geq 0$. In this case, Eq. (4.1) is identically satisfied, and we are left with solving the equations $WW^\dagger = W^\dagger W = \mu\mathbb{1}$. Hence, there exists a unitary matrix U such that $W = \sqrt{\mu}\,U$. □

Assume in the following that $c > 0$. We note that any representation ϕ' of $C(\mu', \hbar)$, with $\phi'(\hat{C}) = 4c'\mathbb{1}$, can be obtained from a representation ϕ of $C(\mu, \hbar)$ with $\phi(\hat{C}) = 4c\mathbb{1}$, if $\mu/\sqrt{c} = \mu'/\sqrt{c'}$. Namely, one simply defines $\phi'(W) := \sqrt[4]{c'/c}\;\phi(W)$.

Proposition 4.3. *Let ϕ be an hermitian representation of $C(\mu, \hbar)$ with $\phi(\hat{C}) = 4c\mathbb{1}$. Then it holds that $-\sqrt{c} \leq \mu$.*

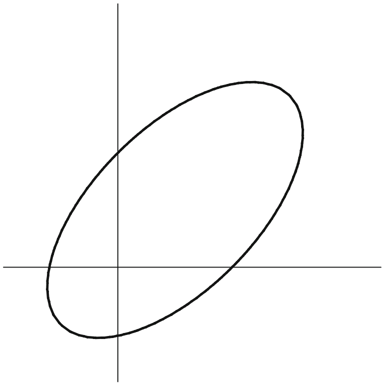

Fig. 1. The constraint ellipse

Proof. Assume that there exists a representation of $C(\mu, \hbar)$ with $-\sqrt{c} > \mu$. Then the diagonal components of Eq. (4.2) describes an ellipse in the $(d, \tilde{d})$-plane, for which all points $(d, \tilde{d})$ satisfy that either d or $\tilde{d}$ is strictly negative. This contradicts the fact that D and $\tilde{D}$ have non-negative eigenvalues. Hence, $-\sqrt{c} \leq \mu$. □

Writing out (4.1) in components gives

$$W_{ij}\Big(\big(\hbar^2+1\big)(\tilde{d}_i+d_j)+\big(\hbar^2-1\big)(d_i+\tilde{d}_j)-4\mu\hbar^2\Big)=0, \tag{4.3}$$

and we also note that $W\tilde{D} = DW$ yields $W_{ij}\big(d_i - \tilde{d}_j\big) = 0$. If $W_{ij} \neq 0$, the two equations give a relation between the pairs $\vec{x}_i = (d_i, \tilde{d}_i)$ and $\vec{x}_j = (d_j, \tilde{d}_j)$. Namely, $\vec{x}_j = s\,(\vec{x}_i)$ with

$$s\big(d, \tilde{d}\big) = \Big(4\mu \sin^2\theta + 2d\cos 2\theta - \tilde{d}, d\Big), \tag{4.4}$$

where $\hbar = \tan\theta$ for $0 < \theta < \pi/4$. The map s is better understood if we introduce coordinates $z(\vec{x}) = (d - \tilde{d})/\hbar$ and $\varphi(\vec{x}) = d + \tilde{d} - 2\mu$ in which case one finds that

$$\begin{pmatrix} z\big(s(\vec{x})\big) \\ \varphi\big(s(\vec{x})\big) \end{pmatrix} = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} z(\vec{x}) \\ \varphi(\vec{x}) \end{pmatrix}. \tag{4.5}$$

We conclude that s amounts to a "rotation" on the ellipse described by the constraint (4.2). Let us collect some basic facts about s in the next proposition.

Proposition 4.4. *Let $s : \mathbb{R}^2 \to \mathbb{R}^2$ be the map as defined above and let $q = e^{2i\theta}$. Then*

(i) *s is a bijection,*
(ii) *if $\vec{x}(\beta_0) = \sqrt{c}\left(\frac{\mu}{\sqrt{c}} + \frac{\cos\beta_0}{\cos\theta}, \frac{\mu}{\sqrt{c}} + \frac{\cos(\beta_0+2\theta)}{\cos\theta}\right)$ then $s^l\big(\vec{x}(\beta_0)\big) = \vec{x}\,(\beta_0 + 2l\theta)$,*
(iii) *$s(\vec{x}) = \vec{x}$ if and only if $\vec{x} = (\mu, \mu)$,*
(iv) *if $\vec{x} \neq (\mu, \mu)$, then $s^n(\vec{x}) = \vec{x}$ if and only if $q^n = 1$.*

From these considerations one realizes that it will be important to keep track of the pairs (i, j) for which $W_{ij} \neq 0$. This leads us to a graph representation of the matrix W.

4.1. Graph representation of matrices. In this section we will introduce the directed graph of the matrix W. See, e.g., [FH94] for the standard terminology concerning directed graphs.

Definition 4.5. *Let $G = (V, E)$ be a directed graph on N vertices with vertex set $V = \{1, 2, \ldots, N\}$ and edge set $E \subseteq V \times V$. We say that an $N \times N$ matrix W is* **associated** *to G (or G is associated to W) if it holds that $(ij) \in E \Leftrightarrow W_{ij} \neq 0$.*

Given an equation for W, we say that a graph G is a *solution* if G is associated to a matrix W, solving the equation. Needless to say, for a given solution G there might exist many different (matrix) solutions associated to G. A graph with several disconnected components is clearly associated to a matrix that is a direct sum of matrices; hence, it suffices to consider connected graphs. In the following, a solution will always refer to a solution of (4.1).

Given a connected solution G, we note that given the value of $\vec{x}_i = (d_i, \tilde{d}_i)$, for any i, we can compute $\vec{x}_k = (d_k, \tilde{d}_k)$, for all k, using (4.4). Namely, since G is connected, we can always find a sequence of numbers $i = i_1, i_2, \ldots, i_l = k$, such that $W_{i_j i_{j+1}} \neq 0$ or $W_{i_{j+1} i_j} \neq 0$, which will give us $\vec{x}_k = s^m(\vec{x}_i)$, where m is the difference between the number of edges (in the path) directed *from i* and the number of edges directed *towards i*.

Proposition 4.6. *Let $G = (V, E)$ be a connected non-degenerate solution. Then*

(i) *G has no self-loops (i.e. $(ii) \notin E$),*
(ii) *there is at most one edge between any pair of vertices.*

Proof. In both cases, assuming the opposite, it follows from (4.3) that there exists an i such that $d_i = \tilde{d}_i = \mu$. Since the graph is connected we will have $d_i = \tilde{d}_i = \mu$ for all i ((μ, μ) is indeed the fix-point of s), giving $c = 0$. Hence, a non-degenerate solution will satisfy the two conditions above. □

Any finite directed graph has a directed cycle, which we shall call *loop*, or a directed path from a transmitter (i.e. a vertex having no incoming edges) to a receiver (i.e. a vertex having no outgoing edges), which we shall call *string*. The existence of a loop or a string imposes restrictions on the corresponding representations. From Proposition 4.4 we immediately get:

Proposition 4.7. *Let G be a non-degenerate solution containing a loop on n vertices. Then $q^n = 1$.*

Lemma 4.8. *Let G be a solution. The vertex i is a transmitter if and only if $\tilde{d}_i = 0$. The vertex i is a receiver if and only if $d_i = 0$.*

Proof. Since $D = WW^\dagger$ and $\tilde{D} = W^\dagger W$, we have

$$d_i = \sum_k W_{ik} \overline{W}_{ik} = \sum_k |W_{ik}|^2,$$
$$\tilde{d}_i = \sum_k \overline{W}_{ki} W_{ki} = \sum_k |W_{ki}|^2,$$

and it follows that $d_i = 0$ if and only if $W_{ik} = 0$ for all k, i.e. i is a receiver. In the same way $\tilde{d}_i = 0$ if and only if $W_{ki} = 0$ for all k, i.e. i is a transmitter. □

Next we prove that if G is a solution, then G can not contain both a string and a loop.

Lemma 4.9. *Let G be a non-degenerate connected solution and assume that G has a transmitter or a receiver. Then G has no loop and therefore there exists a string.*

Proof. Let us prove the case when a transmitter exists. Let us denote the transmitter by $1 \in V$, and by Lemma 4.8 we have $\vec{x}_1 = (a, 0)$, for some $a > 0$. Assume that there exists a loop and let i be a vertex in the loop. Since G is connected there exists an integer i such that $\vec{x}_i = s^i(\vec{x}_1)$. Let l be the number of vertices in the loop. From Proposition 4.7 we know that $q^l = 1$, which means that there is at most l different values of $\vec{x}_k$ in the graph, and all values are assumed by vertices in the loop. In particular this means that there exists a vertex k in the loop, such that $\vec{x}_k = \vec{x}_1$. But this implies, by Lemma 4.8, that k is a transmitter, which contradicts the fact that k is part of a loop. Hence, if a transmitter exists, there exists no loop and therefore there must exist a string. □

The above result suggests to introduce the concept of *loop representations* and *string representations*, since all representations are associated to graphs that have either a loop or a string.

Let us now prove a theorem providing the general structure of the representations.

Theorem 4.10. *Let ϕ be an N-dimensional non-degenerate connected hermitian representation of $C(\mu, \hbar)$ with $\phi(\hat{C}) = 4c\mathbb{1}$. Then there exists a positive integer k dividing N, a unitary $N \times N$ matrix T, unitary $N/k \times N/k$ matrices $U_0, \dots, U_{k-1}$ and $\beta, \tilde{e}_0, \dots, \tilde{e}_{k-1} \in \mathbb{R}$ with $\tilde{e}_1, \dots, \tilde{e}_{k-1} > 0$, such that*

$$T\phi(W)T^\dagger = \begin{pmatrix} 0 & \sqrt{\tilde{e}_1}\,U_1 & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{\tilde{e}_2}\,U_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \sqrt{\tilde{e}_{k-1}}\,U_{k-1} \\ \sqrt{\tilde{e}_0}\,U_0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{4.6}$$

$$\tilde{e}_l = \sqrt{c}\left[\frac{\mu}{\sqrt{c}} + \frac{\cos(2l\theta + \beta)}{\cos\theta}\right]. \tag{4.7}$$

Proof. Let U be a unitary $N \times N$ matrix such that $UDU^\dagger$ and $U\tilde{D}U^\dagger$ are diagonal, set $\hat{W} = U\phi(W)U^\dagger$ and let G be the graph associated to $\hat{W}$. Define $\{\hat{x}_0, \dots, \hat{x}_{k-1}\}$ to be the set of pairwise different vectors out of the set $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_N\}$, such that $\hat{x}_{i+1} = s\left(\hat{x}_i\right)$ for $i = 0, \dots, k-2$ (which is always possible since G is connected), and write $\hat{x}_i = (e_i, \tilde{e}_i)$. We note that if G has a transmitter, it must necessarily correspond to the vector $\hat{x}_0$, in which case $\tilde{e}_0 = 0$. In particular this means that no vertex corresponding to $\hat{x}_i$, for $i > 0$, can be a transmitter and hence, by Lemma 4.8, $\tilde{e}_1, \dots, \tilde{e}_{k-1} > 0$. Now, define

$$V_i = \{j \in V : \vec{x}_j = \hat{x}_i\} \qquad i = 0, \dots, k-1,$$

and set $l_i = |V_i|$. Since $\hat{x}_{i+1} = s(\hat{x}_i)$, a necessary condition for $(ij) \in E$ is that $j = i+1$. This implies that there exists a permutation $\sigma \in S_N$ (permuting vertices to give the order $V_0, \dots, V_{k-1}$) such that

$$W' := \sigma\hat{W}\sigma^\dagger = \begin{pmatrix} 0 & W_1 & 0 & \cdots & 0 \\ 0 & 0 & W_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & W_{k-1} \\ W_0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

where W_i is a $l_{i-1} \times l_i$ matrix (counting indices modulo k). In this basis we get

$$\begin{aligned}
D &= \mathrm{diag}(\underbrace{e_0, \ldots, e_0}_{l_0}, \ldots, \underbrace{e_{k-1}, \ldots, e_{k-1}}_{l_{k-1}}) \\
&= W' W'^{\dagger} = \mathrm{diag}(W_1 W_1^{\dagger}, \ldots, W_{k-1} W_{k-1}^{\dagger}, W_0 W_0^{\dagger}), \\
\tilde{D} &= \mathrm{diag}(\underbrace{\tilde{e}_0, \ldots, \tilde{e}_0}_{l_0}, \ldots, \underbrace{\tilde{e}_{k-1}, \ldots, \tilde{e}_{k-1}}_{l_{k-1}}) \\
&= W'^{\dagger} W' = \mathrm{diag}(W_0^{\dagger} W_0, W_1^{\dagger} W_1, \ldots, W_{k-1}^{\dagger} W_{k-1}),
\end{aligned}$$

which gives $W_i W_i^{\dagger} = e_{i-1} \mathbb{1}_{l_{i-1}}$ and $W_i^{\dagger} W_i = \tilde{e}_i \mathbb{1}_{l_i}$. Since $\hat{x}_{i+1} = s(\hat{x}_i)$ we know that $\tilde{e}_{i+1} = e_i$, which implies that $W_i W_i^{\dagger} = \tilde{e}_i \mathbb{1}_{i-1}$ for $i = 1, \ldots, k-1$. Any matrix satisfying such conditions must be a square matrix, i.e. $l_i = l_{i-1}$ for $i = 1, \ldots, k-1$. Hence, W_i is a square matrix of dimension N/k, and there exists a unitary matrix U_i such that $W_i = \sqrt{\tilde{e}_i} U_i$. Moreover, we take T to be the unitary $N \times N$ matrix σU. Finally, since every point $\hat{x}_i = (e_i, \tilde{e}_i)$ lies on the ellipse, there exists a β_0 such that $\hat{x}_0$ corresponds to the point $\sqrt{c}\,(\cos(\beta_0 + \theta), \sin(\beta_0 + \theta))$ in the (z, φ)-plane, as in Proposition 4.4. By defining $\beta = \beta_0 + 2\theta$, we get, since $\hat{x}_{l+1} = s(\hat{x}_l)$, that $\tilde{e}_l = \sqrt{c}\left[\frac{\mu}{\sqrt{c}} + \frac{\cos(2l\theta+\beta)}{\cos\theta}\right]$. □

The above theorem proves the structure of any connected representation, but the question of irreducibility still remains. We will now prove that any representation is in fact equivalent to a direct sum of representations where the U_i's are 1×1-matrices.

Lemma 4.11. *Let W_1 and W_2 be matrices such that*

$$W_1 = \begin{pmatrix} 0 & w_1 U_1 & 0 & \cdots & 0 \\ 0 & 0 & w_2 U_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & w_{n-1} U_{n-1} \\ w_0 U_0 & 0 & \cdots & 0 & 0 \end{pmatrix}; \quad W_2 = \begin{pmatrix} 0 & w_1 \mathbb{1} & 0 & \cdots & 0 \\ 0 & 0 & w_2 \mathbb{1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & w_{n-1} \mathbb{1} \\ w_0 V & 0 & \cdots & 0 & 0 \end{pmatrix},$$

where $U_0, \ldots, U_{n-1}$ are unitary matrices, $w_0, \ldots, w_{n-1} \in \mathbb{C}$ and V a diagonal matrix such that

$$S V S^{\dagger} = U_1 U_2 \cdots U_{n-1} U_0$$

for some unitary matrix S. Then there exists a unitary matrix P such that

$$W_1 = P W_2 P^{\dagger} \quad \text{and} \quad W_1^{\dagger} = P W_2^{\dagger} P^{\dagger}.$$

Proof. Let us define P as $P = \mathrm{diag}(S, P_1, \ldots, \bar{P}_{n-1})$ with

$$P_l = (U_1 U_2 \ldots U_l)^{\dagger} S$$

for $l = 1, \ldots, n-1$. Then one easily checks that $W_1 = P W_2 P^{\dagger}$ and $W_1^{\dagger} = P W_2^{\dagger} P^{\dagger}$. □

Note that a graph associated to a matrix such as W_2, consists of n components, each being either a string ($\tilde{e}_0 = 0$) or a loop ($\tilde{e}_0 > 0$). Therefore, we have the following result.

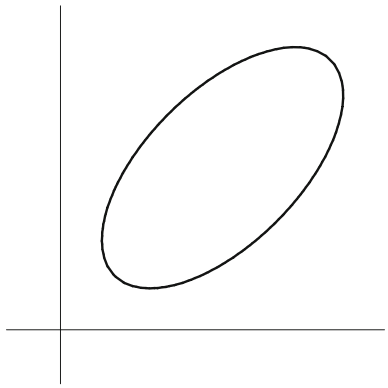

Fig. 2. The constraint ellipse of a Toral representation

Theorem 4.12. *Let ϕ be a non-degenerate hermitian representation of $C(\mu,\hbar)$. Then ϕ is unitarily equivalent to a representation whose associated graph is such that every connected component is either a string or a loop.*

The existence of strings or loops will depend on the ratio $\mu/\sqrt{c}$, and therefore we split all connected representations of $C(\mu,\hbar)$ into three subsets, in correspondence with the original surface described by the polynomial $C(x,y,z)$:

(a)	$-1 < \mu/\sqrt{c} \leq 1$	– Spherical representations,
(b)	$1 < \mu/\sqrt{c} \leq 1/\cos\theta$	– Critical toral representations,
(c)	$1/\cos\theta < \mu/\sqrt{c}$	– Toral representations.

4.2. Toral representations. For $\mu/\sqrt{c} > 1/\cos\theta$ the constraint ellipse lies entirely in the region where both d and $\tilde{d}$ are strictly positive, e.g. as in Fig. 2. In particular this implies, by Lemma 4.8, that a graph associated to a toral representation can not have any transmitters or receivers. Hence, it must have a loop, and by Proposition 4.7, there exists an integer k such that $q^k = 1$. We note that the restriction $0 < \theta < \pi/4$ necessarily gives $k \geq 5$.

Theorem 4.13. *Assume that $\mu/\sqrt{c} > 1/\cos\theta$ and let k be a positive integer such that $q^k = 1$. Furthermore, let $U_0,\ldots,U_{k-1}$ be unitary matrices of dimension N and let $\beta\in\mathbb{R}$. Then ϕ is an $N\cdot k$ dimensional hermitian toral representation of $C(\mu,\hbar)$, with $\phi(\hat{C}) = 4c\mathbb{1}$, if*

$$\phi(W) = \begin{pmatrix} 0 & \sqrt{\tilde{e}_1}\,U_1 & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{\tilde{e}_2}\,U_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \sqrt{\tilde{e}_{k-1}}\,U_{k-1} \\ \sqrt{\tilde{e}_0}\,U_0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{4.8}$$

and

$$\tilde{e}_l = \sqrt{c}\left[\frac{\mu}{\sqrt{c}} + \frac{\cos(2l\theta+\beta)}{\cos\theta}\right]. \tag{4.9}$$

Definition 4.14. *We define a* **single loop representation** ϕ_L *of* $C(\mu, \hbar)$ *to be a toral representation, as in Theorem 4.13, with* U_i *chosen to be* 1×1 *matrices and* k *to be the smallest positive integer such that* $q^k = 1$.

As a simple corollary to Theorem 4.12 we obtain

Corollary 4.15. *Let* ϕ *be a toral representation of* $C(\mu, \hbar)$. *Then* ϕ *is unitarily equivalent to a direct sum of single loop representations.*

Proposition 4.16. *A single loop representation of* $C(\mu, \hbar)$ *is irreducible.*

Proof. Given a single loop representation ϕ_L of dimension n, it holds that $q^n = 1$, and there exists no $n' < n$ such that $q^{n'} = 1$, by definition. Now, assume that ϕ_L is reducible. Then, by Proposition 4.1, ϕ_L is equivalent to a direct sum of at least two representations. In particular, this means that there exists a toral representation of $C(\mu, \hbar)$ of dimension $m < n$ which implies, by Proposition 4.7, that there exists an integer $n' < n$ such that $q^{n'} = 1$. But this is impossible by the above argument. Hence, ϕ_L is irreducible. □

For two loop representations of the same dimension, it is not only the value of the Casimir $\hat{C}$ that distinguishes them, but there is in fact a whole set of inequivalent representations - parametrized by a complex number.

Definition 4.17. *Let* ϕ_L *be a single loop representation in the notation of Theorem 4.13 with* $U_l = e^{i\alpha_l}$. *We define the* **index** $z(\phi_L)$ *as the complex number*

$$z(\phi_L) = \sqrt{\tilde{e}_0 \tilde{e}_1 \cdots \tilde{e}_{k-1}}\, e^{i\gamma}$$

with $\gamma = \alpha_0 + \alpha_1 + \cdots + \alpha_{k-1}$.

Lemma 4.18. *Let* k, n *be integers such that* $\gcd(k, n) = 1$ *and define*

$$A_l(\beta) = \cos\left(\beta + \frac{2\pi k l}{n}\right)$$

for $l = 0, 1, \ldots, n-1$. *Then there exists permutations* $\sigma_+, \sigma_- \in S_n$ *such that*

$$A_{\sigma_+(l)}(\beta) = A_l(\beta + 2\pi/n) \quad \text{and} \quad A_{\sigma_-(l)}(\beta) = A_l(2\pi/n - \beta)$$

for $l = 0, 1, \ldots, n-1$.

Proof. Let us prove the existence of σ_+; the proof that σ_- exists is analogous. We want to show that there exists a permutation σ_+ such that $A_{\sigma_+(l)}(\beta) = A_l(\beta + 2\pi/n)$. Let us make an Ansatz for the permutation; namely, we take it to be a shift with $\sigma_+(l) = l + \delta \pmod{n}$ for some $\delta \in \mathbb{Z}$. We then have to show that there exists a δ such that

$$\cos\left(\beta + \frac{2\pi k(l+\delta)}{n}\right) = \cos\left(\beta + \frac{2\pi(kl+1)}{n}\right).$$

This holds if for some $m \in \mathbb{Z}$,

$$\beta + \frac{2\pi k(l+\delta)}{n} = \beta + \frac{2\pi(kl+1)}{n} + 2\pi m \iff$$
$$k\delta - nm = 1.$$

Now, can we find δ such that this holds for some m? It is an elementary fact in number theory that such an equation has integer solutions for δ and m if $\gcd(k, n) = 1$. Hence, if we set $\sigma_+(l) = l + \delta \pmod{n}$, where δ is such a solution, then the argument above shows that $A_{\sigma_+(l)}(\beta) = A_l(\beta + 2\pi/n)$. □

Lemma 4.19. *Let $\theta = \pi k/n$ with $\gcd(k, n) = 1$, and set*

$$f(\beta) = \prod_{l=0}^{n-1} \left[\mu + \frac{\sqrt{c}\cos(2l\theta + \beta)}{\cos\theta} \right].$$

Then $f(\beta) = f(\beta + 2\pi/n)$, $f(\beta) = f(2\pi/n - \beta)$ and if $\beta, \beta' \in [0, \pi/n]$ then $\beta \neq \beta'$ implies that $f(\beta) \neq f(\beta')$.

Proof. It follows directly from Lemma 4.18 that $f(\beta) = f(\beta + 2\pi/n) = f(2\pi/n - \beta)$. Since f is periodic, with period $2\pi/n$, it can be expanded in a Fourier series as

$$f(\beta) = \sum_{l=-\infty}^{\infty} a_l e^{2\pi i l\beta/(2\pi/n)} = \sum_{l=-\infty}^{\infty} a_l e^{iln\beta}.$$

Comparing the Fourier series with the original expression for f, and introducing $q = e^{2i\theta}$, we get

$$f(\beta) = \left(\frac{\sqrt{c}}{\cos\theta}\right)^n \prod_{l=0}^{n-1} \left[\frac{\mu\cos\theta}{\sqrt{c}} + \frac{1}{2}\left(q^l e^{i\beta} + q^{-l} e^{-i\beta}\right) \right] = \sum_{l=-\infty}^{\infty} a_l e^{inl\beta}.$$

From this equality we deduce that there are only three non-zero coefficients in the Fourier series, namely a_{-1}, a_0, a_1. Comparing both sides, we obtain

$$\begin{aligned} a_{-1} &= \frac{1}{2^n} q^{-n(n-1)/2}, \\ a_1 &= \frac{1}{2^n} q^{n(n-1)/2}, \end{aligned}$$

which implies that

$$\begin{aligned} \left(\frac{\sqrt{c}}{\cos\theta}\right)^{-n} f(\beta) &= a_0 + \frac{1}{2^n} q^{-n(n-1)/2} e^{-in\beta} + \frac{1}{2^n} q^{n(n-1)/2} e^{in\beta} \\ &= a_0 + \left(-\frac{1}{2}\right)^{n-1} \cos n\beta. \end{aligned}$$

From this it is clear that $f(\beta) \neq f(\beta')$ when $\beta \neq \beta'$ and $\beta, \beta' \in [0, \pi/n]$. □

Proposition 4.20. *Let ϕ_L and ϕ'_L be single loop representations of dimension n, such that $\phi_L(\hat{C}) = \phi'_L(\hat{C})$. Then ϕ_L and ϕ'_L are equivalent if and only if $z(\phi_L) = z(\phi'_L)$.*

Proof. Then characteristic equation of $\phi_L(W)$ is $\lambda^n - z(\phi_L)$. Therefore, a necessary condition for ϕ_L and ϕ'_L to be equivalent is that $z(\phi_L) = z(\phi'_L)$. Now, to prove the opposite implication, assume that $z(\phi_L) = z(\phi'_L)$. Let us denote the β in Theorem 4.13 by β and β' for ϕ_L and ϕ'_L respectively. The fact that $z(\phi_L) = z(\phi'_L)$ gives directly $\gamma = \gamma'$, and in the notation of Lemma 4.19, we must have $f(\beta) = f(\beta')$. By the same Lemma, writing $\theta = \pi k/n$, this leaves us with three possibilities: Either $\beta' = \beta$, $\beta' = \beta + 2\pi m/n$ or $\beta' = 2\pi m/n - \beta$ for some $m \in \mathbb{Z}$. In all three cases, by Lemma 4.18, there exists a permutation σ such that for $W'' = \sigma\phi'_L(W)\sigma^\dagger$ it holds that $\tilde{e}''_l = \tilde{e}_l$. Then it is easy to construct a diagonal unitary matrix P such that $\phi_L(W) = P\sigma\phi'_L(W)\sigma^\dagger P^\dagger$. □

Hence, for a given dimension n and for a given value of the Casimir, such that toral representations exist, the set of inequivalent irreducible representations is parametrized by a complex number w such that $\pi/n \leq |w| \leq 2\pi/n$. We relate w to a single loop representation by setting $w = \beta e^{i\gamma}$.

4.3. Spherical representations. In contrast to the case of toral representations, we will show that, in a spherical representation, there can not exist any loops. The intuitive picture is that the part of the ellipse lying in the region where either d or $\tilde{d}$ is negative, is too large to skip by a rotation through the map s ; see, e.g. Fig. 1.

By Lemma 4.8, we know that the $\vec{x}$ corresponding to a transmitter or a receiver must lie on the d-axis or the $\tilde{d}$-axis respectively. For this reason, let us calculate the points where the ellipse crosses the axes.

Lemma 4.21. *Consider the ellipse $(x+y-2\mu)^2+(x-y)/\hbar^2 = 4c$. Then $x = 0$ implies $y = a_\pm$ and $y = 0$ implies $x = a_\pm$ with*

$$a_\pm = 2\sin\theta\Big[\mu\sin\theta \pm \sqrt{c-\mu^2\cos^2\theta}\Big] = 2\sin^2\theta\left[\mu \pm \sqrt{\mu^2 + \frac{c-\mu^2}{\sin^2\theta}}\right] \tag{4.10}$$

Lemma 4.22. *Let $\vec{x} = (0, a_+)$, with a_+ as in Lemma 4.21. Then $s(\vec{x}) = (a_-, 0)$.*

Lemma 4.23. *If ϕ is a spherical representation of $C(\mu, \hbar)$, that contains a string on n vertices, then*

$$0 < (n+1)\theta \leq \pi. \tag{4.11}$$

Proof. Let us denote the vectors corresponding to the vertices in the string by $\vec{x}_1, \ldots, \vec{x}_n$ and we define $0 < \beta, \theta_0 < 2\pi$ through $\vec{x}_1 = \vec{x}(\beta)$ and $\vec{x}_n = \vec{x}(\beta+\theta_0)$ in the notation of Proposition 4.4. Since $\vec{x}_n = s^{n-1}\big(\vec{x}(\beta)\big)$ we must have that $(n-1)2\theta = \theta_0 + 2\pi k$ for some integer $k \geq 0$. Let us prove that $k = 0$. For a spherical representation, $a_- \leq 0$, which implies, by Lemma 4.22, that $s\big(\vec{x}(\beta+\theta_0)\big) = (a_-, 0)$ can not correspond to a vertex of a connected representation. Hence, for any $\alpha \in (0, 2\theta)$, $s\big(\vec{x}(\beta+\theta_0-\alpha)\big)$ can not correspond to a vertex of a connected representation. This implies that $k = 0$, i.e. the string never crosses the $\tilde{d}$-axis. Therefore $0 < (n-1)2\theta = \theta_0 < 2\pi$. Again, by Lemma 4.22, both vectors $s(0, a_+)$ and $s^2(0, a_+)$ have non-positive components which implies that $0 < (n+1)2\theta \leq 2\pi$. In fact, equality is attained when $a_- = 0$. □

Proposition 4.24. *Let ϕ be a spherical representation of $C(\mu, \hbar)$. Then the associated graph has no loops.*

Proof. In the same way as in the proof of Lemma 4.23, we can argue that for $\alpha \in (0, 2\theta)$, $s\big(\vec{x}(\beta+\theta_0-\alpha)\big)$ has a negative component (or equals $(0, 0)$), which implies that it is impossible to have loops. □

Hence, we have excluded the possibility of loop representations and can conclude that all spherical representations are string representations. We therefore get the following corollary to Theorem 4.12.

Corollary 4.25. *Let ϕ be a spherical representation of $C(\mu, \hbar)$. Then ϕ is unitarily equivalent to a direct sum of string representations.*

Let us now investigate the conditions for the existence of strings.

Lemma 4.26. *Let $\vec{x}_1 = (a, 0)$ and $\vec{x}_n = (0, b)$. Then $s^{n-1}(\vec{x}_1) = \vec{x}_n$ if and only if*

(i) *$q^n = -1$, $\mu = 0$ and $a = b$,*
(ii) *$q^n = 1$ and $b = -a + 4\mu\sin^2\theta$,*
(iii) *$q^n \neq \pm 1$, and*

$$a = b = -\frac{2\mu \sin\theta \sin(n-1)\theta}{\cos n\theta}. \tag{4.12}$$

In particular, if $a = a_+$ and $q^n = 1$, then $b = a_-$.

Proposition 4.27. *Let ϕ be a spherical representation of $C(\mu, \hbar)$ containing a string on n vertices. Then*

$$\sqrt{c}\cos n\theta + \mu\cos\theta = 0. \tag{4.13}$$

Proof. Assume the existence of a string on n vertices. From Lemma 4.26 we can exclude the possibility that $q^n = 1$, since $a_- < 0$. Hence, either $q^n = -1$ and $\mu = 0$ or $q^n \neq \pm 1$. If $q^n = -1$ and $\mu = 0$ then (4.13) is clearly satisfied. Now, assume $q^n \neq \pm 1$ and $a = b = \frac{2\mu \sin\theta \sin(n-1)\theta}{\cos n\theta}$. Demanding that $(a, 0)$ and $(0, b)$ lie on the ellipse determines c as $c = \mu^2 \cos^2\theta / \cos^2 n\theta$. Let us set $\varepsilon = \operatorname{sgn}\mu$. Recalling that $0 < (n+1)\theta \leq \pi$, from Lemma 4.23, demanding $a > 0$ makes it necessary that $\operatorname{sgn}(\cos n\theta) = -\varepsilon$, which determines the sign of the root in the statement. □

As we have seen, the existence of a loop puts a restriction on $\hbar$ through the relation $q^n = 1$. For the case of strings, the restriction comes out as a restriction on the possible values of the Casimir.

In the next theorem we show that the necessary conditions for the existence of spherical representations are in fact sufficient.

Theorem 4.28. *Let n be a positive integer, c a positive real number such that $\sqrt{c}\cos n\theta + \mu\cos\theta = 0$ and $0 < (n+1)\theta \leq \pi$. Furthermore, let $U_1, \ldots, U_{n-1}$ be $N \times N$ unitary matrices. Then ϕ is a $N \cdot n$-dimensional spherical representation of $C(\mu, \hbar)$, with $\phi(\hat{C}) = 4c\mathbb{1}$, if*

$$\phi(W) = \begin{pmatrix} 0 & \sqrt{\tilde{e}_1}\, U_1 & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{\tilde{e}_2}\, U_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \sqrt{\tilde{e}_{n-1}}\, U_{n-1} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

and

$$\tilde{e}_l = \frac{2\sqrt{c}\sin l\theta \sin(n-l)\theta}{\cos\theta}.$$

Proof. It is easy to check that the matrix $\phi(W)$ satisfy (4.1), since $s(\tilde{e}_l, \tilde{e}_{l-1}) = (\tilde{e}_{l+1}, \tilde{e}_l)$. Moreover, it is clear that $\tilde{e}_l > 0$ since $0 < (n-1)\theta < \pi$. Let us show that it is indeed a spherical representation, i.e. $-1 < \mu/\sqrt{c} \leq 1$. Since $\sqrt{c}\cos n\theta + \mu\cos\theta = 0$, we get that

$$\frac{\mu}{\sqrt{c}} = -\frac{\cos n\theta}{\cos\theta}$$

and from $0 < (n+1)\theta \leq \pi$ we obtain $0 < n\theta \leq \pi - \theta$. From this it follows that $|\cos n\theta| \leq |\cos\theta|$ which implies that ϕ is a spherical representation. □

Remark. Let us note that the matrix elements of the diagonal matrix $Z = [W, W^\dagger]/2\hbar$ can be written as $z_l = \sqrt{c}\sin(n+1-2l)\theta$.

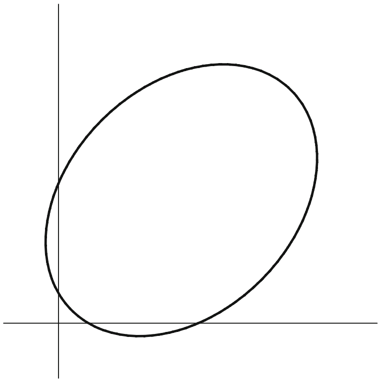

Fig. 3. The constraint ellipse of a critical toral representation

Definition 4.29. *We define a* **single string representation** ϕ_S *of* $C(\mu, \hbar)$ *to be a spherical representation, as in Theorem* 4.28*, with* U_i *chosen to be* 1×1 *matrices.*

Proposition 4.30. *Any single string representation of* $C(\mu, \hbar)$ *is irreducible.*

Proof. Assume that ϕ_S is reducible and has dimension n with $\phi_S(\hat{C}) = 4c\mathbb{1}$. Then, by Proposition 4.1, ϕ_S is equivalent to a direct sum of at least two representations of dimension $< n$. In particular, this implies that there exists a representation ϕ of dimension $m < n$ with $\phi(\hat{C}) = 4c\mathbb{1}$. But this is false, since there is at most one integer l such that $\vec{x}(\beta + 2l\theta) = \vec{x}(\beta + \theta_0)$, for $0 < (l+1)2\theta < 2\pi$ and $0 < \theta_0 < 2\pi$. □

We conclude that the single string representations are the only irreducible spherical representations. Moreover, two single string representations ϕ_S and ϕ'_S, of the same dimension, are equivalent if and only if $\phi_S(\hat{C}) = \phi'_S(\hat{C})$.

4.4. Critical toral representations. In the case of critical toral representations, the constraint ellipse intersects the positive d (resp. $\tilde{d}$) axis twice, as in Fig. 3. As we will show, there are both loop representations and string representations. String representations can be obtained from Theorem 4.28, by demanding that $1 < \mu/\sqrt{c} \leq 1/\cos\theta$ instead of $0 < (n+1)\theta \leq \pi$. Let us as well give an example of a loop representation.

Proposition 4.31. *Assume that* $\theta = \pi/N$, $N \geq 5$ *odd and* $1 < \mu/\sqrt{c} \leq 1/\cos\theta$*. If we define* ϕ *as in Theorem* 4.13 *with* $\beta = 0$*, then* ϕ *is a critical toral representation of* $C(\mu, \hbar)$.

Proof. One simply has to check that

$$\tilde{e}_l = \sqrt{c}\left[\frac{\mu}{\sqrt{c}} + \frac{\cos(2l\frac{\pi}{N})}{\cos\frac{\pi}{N}}\right] > 0$$

for $l = 0, \ldots, N-1$. If N is odd then $2\theta l \notin (\pi - \theta, \pi + \theta)$ and $2\theta l \notin (2\pi - \theta, 2\pi)$, which implies that $|\cos 2\theta l| < |\cos\theta|$. Since $\mu/\sqrt{c} > 1$ we conclude that $\tilde{e}_l > 0$ for $l = 0, \ldots, N-1$. □

In contrast to the previous cases, it is, for a given value of the Casimir, possible to have both string representations and loop representations. Namely, if we assume that $q^n = 1$ and let $\vec{x}_1$ correspond to the largest intersection with the d-axis, then $s^{n-1}(\vec{x}_1)$ will be the smallest intersection with the $\tilde{d}$-axis (cp. Lemma 4.26), and one can check that all pairs $\vec{x}_i$, for $i = 2, \ldots, n-1$ will be strictly positive.

4.5. Summary of representations. We have shown that every representation can be decomposed into a direct sum of irreducible representations of two types: string and loop representations. String representations correspond to matrices of the form

$$\phi(W) = \begin{pmatrix} 0 & W_{12} & 0 & \cdots & 0 \\ 0 & 0 & W_{23} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & W_{N-1,N} \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix},$$

and loop representations to matrices of the form

$$\phi(W) = \begin{pmatrix} 0 & W_{12} & 0 & \cdots & 0 \\ 0 & 0 & W_{23} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & W_{N-1,N} \\ W_{N,1} & 0 & \cdots & \cdots & 0 \end{pmatrix},$$

with $W_{12}, W_{23}, \ldots, W_{N-1,N}, W_{N,1} \neq 0$. Furthermore, existence of representations puts restrictions on the parameter $\hbar$ and the value $4c$ of the Casimir $\hat{C}$. A necessary condition for loop representations to exist is that there is a positive integer k such that $q^k = e^{2ik\theta} = 1$, where $\hbar = \tan\theta$. A necessary condition for string representations to exist is that $\sqrt{c}\cos n\theta + \mu\cos\theta = 0$, for some positive integer n.

The structure of representations respects the classical geometry as follows: In the region $-1 < \mu/\sqrt{c} \leq 1$ we have shown that there are only string representations and when $\mu/\sqrt{c} > 1/\cos\theta$ there are only loop representations. In the critical region, where $1 < \mu/\sqrt{c} \leq 1/\cos\theta$ (classically, one is close to the singular surface), there are in fact representations of both types.

5. Eigenvalue Distribution and Surface Topology

In this section we consider the eigenvalue distribution of the matrix X in the representations obtained in sect. 4, with the help of numerical computations. The eigenvalue distribution is of interest since in [Shi04] it was shown that the Morse theoretic information of topology manifests itself in certain branching phenomena of eigenvalue distribution of a single matrix. More precisely, critical points of the Morse function correspond to branching points of the eigenvalue distribution. (The meaning of the word "branching phenomena" will be illustrated below by using the eigenvalue distribution of the matrix X, plotted in Fig. 4, as an example.) This was achieved by using arguments analogous to those used in the WKB approximation in quantum mechanics, and is part of a more general correspondence between matrix elements and certain geometric quantities computed from the corresponding function on the surface. For a description of this analogy and also for more examples, we refer the reader to [Shi04].

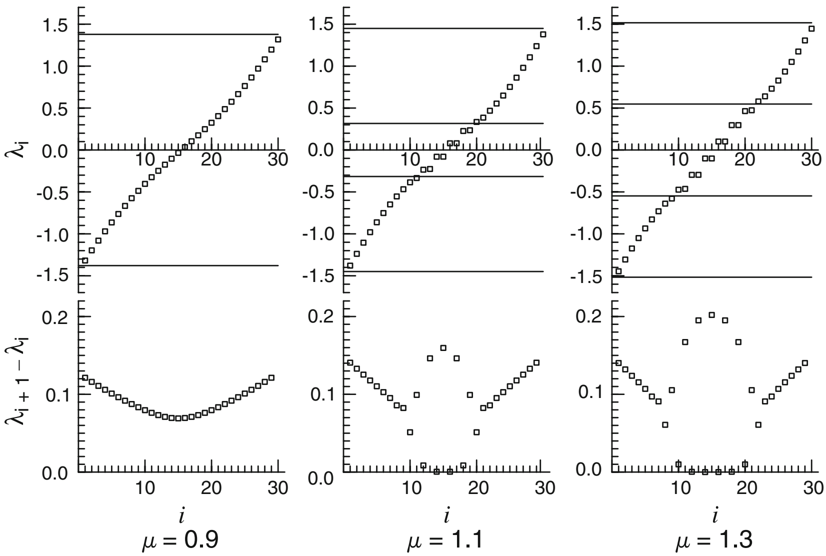

Fig. 4. Plot of λ_i and $\lambda_{i+1} - \lambda_i$ versus i, where $\lambda_1 < \lambda_2 < \ldots < \lambda_N$ are eigenvalues of X, for $\mu = 0.9$, 1.1, 1.3. The size of matrices is given by $N = 30$. Critical values of x are also shown by the horizontal lines

Eigenvalues of X (whose continuum counterpart, x, is a Morse function on the surface) in the representations obtained in Sect. 4, do exhibit this branching phenomena, as is consistent with the results in [Shi04]. In Fig. 4, eigenvalues of X, computed numerically, for the case $\mu = 0.9$, 1.1, 1.3 are shown. (We use the normalization convention in which $c = 1$, so that the transition between sphere and torus occurs at $\mu = 1$. The size of matrices is given by $N = 30$. For the toral representation, we have taken the additional "phase shift" parameter β to be zero. Using different β's does not change the plot qualitatively.) The horizontal lines correspond to the critical values of the function x on the surface.

The plots directly reflect the Morse theoretic information of topology, with x as the Morse function, for each case $\mu = 0.9$, 1.1, 1.3. For the case $\mu = 0.9$, there are two critical values which are connected by a single branch. Correspondingly, the eigenvalue plot shows that there is only one "sequence" of eigenvalues $\lambda_1 < \lambda_2 < \ldots < \lambda_N$ which increase smoothly. For the cases $\mu = 1.1$ and $\mu = 1.3$, there are four critical values of x, say $x_A < x_B < x_C < x_D$. For $x_A < x < x_B$ and $x_C < x < x_D$ the surface consists of a single branch, whereas for $x_B < x < x_C$, the surface consists of two branches. Correspondingly, in the plot of eigenvalues, one sees that eigenvalues $x_A < \lambda_i < x_B$ and $x_C < \lambda_i < x_D$ each consists of a single smoothly increasing eigenvalue sequence, whereas eigenvalues $x_C < \lambda_i < x_D$ are naturally divided into two sequences both of which increase smoothly. This branching phenomena of eigenvalues can be seen more manifestly if one plots the difference between eigenvalues, $\lambda_{i+1} - \lambda_i$, as is shown in the figure. From the figure it can also be seen that by decreasing the parameter μ from 1.3 to 1.1 the part of the surface which has two branches shrinks, as is consistent with the geometrical picture about the transition between torus and sphere.

6. Comparison with Other Quantization Methods

6.1. The torus. The purpose of this section is to compare matrix representations obtained in Sect. 4, in the torus case, with those one gets using Berezin-Toeplitz quantization. Full details and proofs can be found in [Hof07]. We shall use Theorem 5.1 from the paper [BHSS91] applied to $\mathbb{S}^1 \times \mathbb{S}^1$. Namely $n = 1$, $\tau = 1$ and we omit the Laplacian terms:

$$\frac{\pi}{m} \sum_{k=1}^{n} \tau_k \left(r_k^2 + \frac{r_{k+n}^2}{t_k^2} \right) \quad \text{and} \quad \prod_{s=1}^{n} \exp\left(-\frac{\pi \tau_s}{2m} \left(r_s^2 + \frac{r_{s+n}^2}{\tau_s^2} \right) \right).$$

We reformulate it for simplicity and to fix notations.

Theorem 6.1. *Let $r_1, r_2 \in \mathbb{Z}$ and $N \geq 5$ be an integer. Then the $N \times N$-matrix corresponding to the phase function $e^{2\pi i (r_1 \vartheta + r_2 \varphi)}$ is:*

$$\mathcal{M}\left(e^{2\pi i (r_1 \vartheta + r_2 \varphi)} \right) = \chi^{r_1 r_2} S^{-r_1} T^{r_2} \quad \text{and} \quad \chi := e^{-\frac{\pi i}{N}},$$

where the S and T are matrices such that:

$$S = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad T = \operatorname{diag}(1, q, \ldots, q^{N-1}) \quad \textit{where} \quad q := \chi^2 = e^{-\frac{2\pi i}{N}}.$$

Remark 6.2. The $\mathcal{M}$ map is not a morphism of algebras. However, $\mathcal{M}$ is continuous in the topology of uniform convergence.

To apply this theorem to the torus case, i.e. the regular values of the polynomial function $(x^2 + y^2 - \mu)^2 + z^2 - c$ (with $\mu/\sqrt{c} > 1$), one has to choose an embedding:

Proposition 6.3. *Assume that $\mu/\sqrt{c} > 1$. By using the parametrization:*

$$\begin{cases} x(\vartheta, \varphi) = \cos(2\pi\vartheta)\sqrt{\sqrt{c}\cos(2\pi\varphi) + \mu} \\ y(\vartheta, \varphi) = \sin(2\pi\vartheta)\sqrt{\sqrt{c}\cos(2\pi\varphi) + \mu} \\ z(\vartheta, \varphi) = \sqrt{c}\sin(2\pi\varphi) \end{cases}$$

one gets:

$$\mathcal{M}(x) = \frac{S}{2}\sqrt{\mathbb{1}\mu + \frac{\sqrt{c}}{2}\left(\chi^{-1}T + \chi T^{-1}\right)} + \frac{S^{-1}}{2}\sqrt{\mathbb{1}\mu + \frac{\sqrt{c}}{2}\left(\chi T + \chi^{-1}T^{-1}\right)}, \tag{6.1}$$

$$\mathcal{M}(y) = \frac{S}{2i}\sqrt{\mathbb{1}\mu + \frac{\sqrt{c}}{2}\left(\chi^{-1}T + \chi T^{-1}\right)} - \frac{S^{-1}}{2i}\sqrt{\mathbb{1}\mu + \frac{\sqrt{c}}{2}\left(\chi T + \chi^{-1}T^{-1}\right)}, \tag{6.2}$$

$$\mathcal{M}(z) = \frac{\sqrt{c}}{2i}\left(T - T^{-1}\right). \tag{6.3}$$

Proof. The key idea is an expansion in Fourier series of $\sqrt{\mu + \sqrt{c}\cos(2\pi\varphi)}$. We then replace phase functions by matrices T and S according to Theorem 6.1. Square roots of matrices are well defined since the matrices are positive definite. □

Lemma 6.4. *Let $D = \mathrm{diag}(d_1, \ldots, d_N)$ be a diagonal $N \times N$-matrix, then:*

$$S^{-1}DS = \mathrm{diag}(d_N, d_1, \ldots, d_{N-1}) \quad \textit{and} \quad SDS^{-1} = \mathrm{diag}(d_2, \ldots, d_N, d_1).$$

Let us denote:

$$D := \sqrt{\mathbb{1}\mu + \frac{\sqrt{c}}{2}\left(\chi T + \chi^{-1}T^{-1}\right)} \quad \textit{and} \quad \widetilde{D} := \sqrt{\mathbb{1}\mu + \frac{\sqrt{c}}{2}\left(\chi^{-1}T + \chi T^{-1}\right)}.$$

Then one can write (6.1) and (6.2) as:

$$\mathcal{M}(x) = \frac{1}{2}\left(S\widetilde{D} + S^{-1}D\right) \quad \textit{and} \quad \mathcal{M}(y) = -\frac{i}{2}\left(S\widetilde{D} - S^{-1}D\right).$$

It is easily seen that the matrices D and $\widetilde{D}$ are diagonal:

$$D = \mathrm{diag}\left(\sqrt{\mu + \sqrt{c}\cos\left(\frac{2\pi l}{N} + \frac{\pi}{N}\right)}\right)_{l=1,\ldots,N},$$
$$\widetilde{D} = \mathrm{diag}\left(\sqrt{\mu + \sqrt{c}\cos\left(\frac{2\pi l}{N} - \frac{\pi}{N}\right)}\right)_{l=1,\ldots,N}.$$

By Lemma 6.4,

$$S\widetilde{D} = S\widetilde{D}S^{-1}S = \mathrm{diag}\left(\sqrt{\mu + \sqrt{c}\cos\left(\frac{2\pi l}{N} + \frac{\pi}{N}\right)}\right)_{l=1,\ldots,N} \times S = DS.$$

As a consequence, $\mathcal{M}(x)$ and $\mathcal{M}(y)$ can be written as:

$$\mathcal{M}(x) = \frac{1}{2}\left(DS + S^{-1}D\right) \quad \text{and} \quad \mathcal{M}(y) = -\frac{i}{2}\left(DS - S^{-1}D\right).$$

Theorem 6.5. *The matrices $\mathcal{M}(x)$, $\mathcal{M}(y)$ and $\mathcal{M}(z)$ are:*

$$\mathcal{M}(x) = \frac{1}{2}\begin{pmatrix} 0 & x_1 & 0 & \cdots & 0 & x_N \\ x_1 & 0 & x_2 & \cdots & 0 & 0 \\ 0 & x_2 & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \ddots & 0 & x_{N-1} \\ x_N & 0 & \cdots & 0 & x_{N-1} & 0 \end{pmatrix},$$

$$\mathcal{M}(y) = -\frac{i}{2}\begin{pmatrix} 0 & y_1 & 0 & \cdots & 0 & -y_N \\ -y_1 & 0 & y_2 & \cdots & 0 & 0 \\ 0 & -y_2 & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \ddots & 0 & y_{N-1} \\ y_N & 0 & \cdots & 0 & -y_{N-1} & 0 \end{pmatrix},$$

$$\mathcal{M}(z) = \mathrm{diag}(z_1, z_2, \ldots, z_N),$$

where the x_l's, y_l's and z_l's (for $l = 1, \dots, N$) are:

$$x_l = y_l = \sqrt{\mu + \sqrt{c}\cos\left(\frac{2\pi l}{N} + \frac{\pi}{N}\right)} \quad \textit{and} \quad z_l = -\sqrt{c}\sin\left(\frac{2\pi l}{N}\right).$$

These matrices satisfy the following relations:

Theorem 6.6. *Let $\mu, \sqrt{c} \in \mathbb{R}$ and $N \geq 5$ such that $\mu/\sqrt{c} > 1$. If one assumes $\hbar = \tan(\theta)$ with $\theta := \pi/N$, then:*

$$\begin{aligned}
[\tilde{X}, \tilde{Y}] &= i\hbar(\cos(\theta)\tilde{Z}),\\
[\tilde{Y}, (\cos(\theta)\tilde{Z})] &= i\hbar\left(\tilde{X}(\tilde{X}^2 + \tilde{Y}^2 - \mu\mathbb{1}) + (\tilde{X}^2 + \tilde{Y}^2 - \mu\mathbb{1})\tilde{X}\right),\\
[(\cos(\theta)\tilde{Z}), \tilde{X}] &= i\hbar\left(\tilde{Y}(\tilde{X}^2 + \tilde{Y}^2 - \mu\mathbb{1}) + (\tilde{X}^2 + \tilde{Y}^2 - \mu\mathbb{1})\tilde{Y}\right),\\
(\tilde{X}^2 + \tilde{Y}^2 - \mu\mathbb{1})^2 + (\cos(\theta)\tilde{Z})^2 &= (\sqrt{c}\cos(\theta))^2\mathbb{1},
\end{aligned}$$

where $\tilde{X} := \mathcal{M}(x)$, $\tilde{Y} := \mathcal{M}(y)$ and $\tilde{Z} := \mathcal{M}(z)$ are the matrices obtained in Theorem 6.5.

Proof. This is a direct computation on matrices. □

This proves that the matrices $X = \tilde{X}$, $Y = \tilde{Y}$ and $Z = \cos(\theta)\tilde{Z}$ satisfy exactly the relations (3.2), (3.3) and (3.4). The Casimir identity is satisfied with c replaced by $c\cos^2\theta$. Note that $\cos\theta$ converges to 1 as N goes to infinity.

6.2. The sphere. Let us start by constructing a parametrization for the deformed sphere described by (3.1) with $-\sqrt{c} < \mu < \sqrt{c}$. Recall the well-known parametrization of the sphere $\xi_1 = \sin\vartheta\cos\varphi$, $\xi_2 = \sin\vartheta\sin\varphi$, $\xi_3 = \cos\vartheta$. We would like to keep the axial symmetry and therefore we make the following Ansatz outside the poles for a map $(x, y, z) : S^2 \to \Sigma$,

$$\begin{aligned}
x &= f(\vartheta)\cos\varphi,\\
y &= f(\vartheta)\sin\varphi.
\end{aligned}$$

Keeping the relation $\{x, y\}_{S^2} = z$, using the (round sphere) Poisson bracket,

$$\{f, g\}_{S^2} = \frac{\lambda}{\sin\vartheta}\left(\partial_\vartheta f\,\partial_\varphi g - \partial_\varphi f\,\partial_\vartheta g\right)$$

(where $\lambda > 0$ is an arbitrary parameter scaling the round sphere volume as $4\pi/\lambda$), yields

$$z = \frac{\lambda}{\sin\vartheta} f'(\vartheta) f(\vartheta).$$

Demanding that these functions satisfy the constraint $(x^2 + y^2 - \mu)^2 + z^2 - c = 0$ gives the following differential equation for $f(\vartheta)$:

$$\left(f^2(\vartheta) - \mu\right)^2 + \frac{\lambda^2}{\sin^2\vartheta}\left(f'(\vartheta) f(\vartheta)\right)^2 - c = 0,$$

which is solved by

$$f(\vartheta) = \sqrt{\mu + \sqrt{c}\cos\left(\frac{2}{\lambda}\cos\vartheta + B\right)},$$

for arbitrary B. As ϑ goes to 0 or π we need that x and y go to zero; there are two ways of achieving this (giving conditions on λ and B) but only the following leads to an embedding:

Proposition 6.7. *The map $\Lambda : S^2 \to \mathbb{R}^3$ defined by*

$$\vartheta \in (0,\pi) : \begin{cases} x = \cos\varphi\sqrt{\mu + \sqrt{c}\cos\left(\frac{2}{\lambda}\cos\vartheta\right)} \\ y = \sin\varphi\sqrt{\mu + \sqrt{c}\cos\left(\frac{2}{\lambda}\cos\vartheta\right)}, \\ z = \sqrt{c}\sin\left(\frac{2}{\lambda}\cos\vartheta\right) \end{cases}$$

$$\vartheta = 0 : \quad x = y = 0,\ z = \sqrt{c}\sin(2/\lambda); \qquad \vartheta = \pi : \quad x = y = 0,\ z = -\sqrt{c}\sin(2/\lambda),$$

with $\frac{\mu}{\sqrt{c}} = -\cos\left(\frac{2}{\lambda}\right)$, $-1 < \mu/\sqrt{c} < 1$, and $0 < 2/\lambda < \pi$, is an embedding of the (round) sphere into $\mathbb{R}^3$ whose image coincides with Σ. Moreover, it holds that

$$\{x,y\}_{S^2} = z,\ \{y,z\}_{S^2} = 2x(x^2+y^2-\mu) \quad \textit{and} \quad \{z,x\}_{S^2} = 2y(x^2+y^2-\mu).$$

The embedding is therefore a Poisson map and hence volume preserving (where Σ is equipped with the volume defined by the inverse of the restriction of the C-bracket (2.1)).

Proof. Outside the poles all the assertions are computed in a straight forward manner. Around the poles we can express x,y and z by the local round sphere charts ξ_1 and ξ_2 to see that the map is a smooth embedding. □

Let us introduce the hermitian $n \times n$ matrices S_1, S_2, S_3, whose nonzero matrix elements are

$$\begin{aligned} (S_1)_{k,k+1} &= \frac{1}{2}\sqrt{k(n-k)} = (S_1)_{k+1,k}, \qquad k = 1,\ldots,n-1, \\ (S_2)_{k,k+1} &= -\frac{i}{2}\sqrt{k(n-k)} = -(S_2)_{k+1,k}, \qquad k = 1,\ldots,n-1, \\ (S_3)_{k,k} &= \frac{1}{2}(n+1-2k), \qquad k = 1,\ldots,n, \end{aligned}$$

satisfying $[S_a, S_b] = i\epsilon_{abc}S_c$ and $S_1^2 + S_2^2 + S_3^2 = \frac{n^2-1}{4}\mathbb{1}$. We then define rescaled matrices $X_a = A(n)S_a$, for some function $A(n)$.

In analogy with the case of the torus, we would like to compare the Berezin-Toeplitz quantization of the embedding functions with the results obtained in Sect. 4.3. Quantizing the function $\cos\vartheta$ will (up to scaling) give the diagonal matrix S_3. However, a function of $\cos\vartheta$ is in general *not* mapped to the same function of S_3 and one can numerically check that the quantization of $z(\vartheta)$ (in Proposition 6.7) is not equal to the matrix Z obtained in Sect. 4.3. However, they agree up to corrections of order $1/n$.

In [GH82] the following prescription for replacing functions on S^2 by matrices was introduced: Smooth functions on S^2 are expanded in terms of the spherical harmonics $Y_{lm}(\vartheta, \varphi)$, resp. $\mathcal{Y}_{lm} = r^l Y_{lm}$, written as

$$\mathcal{Y}_{lm}(x_1, x_2, x_3) = \sum_{a_1,\dots,a_l=1}^{3} c^{(m)}_{a_1\cdots a_l} x_{a_1} \cdots x_{a_l},$$

($x_1 = r\sin\vartheta\cos\varphi$, $x_2 = r\sin\vartheta\sin\varphi$, $x_3 = r\cos\vartheta$) with $c^{(m)}_{a_1\cdots a_l}$ chosen to be totally symmetric with respect to the lower indices. A function is then mapped to a $n \times n$ via

$$T^{(n)}\big(Y_{lm}\big) = B(n,l) \sum_{a_1,\dots,a_l=1}^{3} c^{(m)}_{a_1\cdots a_l} X_{a_1} \cdots X_{a_l},$$

where $B(n,l) = \sqrt{4\pi}\sqrt{\frac{(n^2-1)^l (n-1-l)!}{(n+l)!}}$ and X_1, X_2, X_3 are defined as above, with the choice $A(n) = 2/\sqrt{n^2-1}$. Disregarding multiplication by an overall n-dependent function, the map $T^{(n)}$ will act on the basic functions in the following way:

$$\begin{aligned} T^{(n)}(x_1) &= T^{(n)}\big(\sin\vartheta\cos\varphi\big) \sim S_1,\\ T^{(n)}(x_2) &= T^{(n)}\big(\sin\vartheta\sin\varphi\big) \sim S_2,\\ T^{(n)}(x_3) &= T^{(n)}\big(\cos\vartheta\big) \sim S_3. \end{aligned}$$

We will now show that, for some scaling of X_1, X_2, X_3, the following hermitian matrices:

$$\begin{aligned} \hat{X} &= \frac{1}{2}\sqrt{\mu + \sqrt{c}\cos\left(\frac{2}{\lambda}X_3\right)}\left(\sqrt{\mathbb{1} - X_3^2}\right)^{-1} X_1\\ &\quad + \frac{1}{2}X_1\sqrt{\mu + \sqrt{c}\cos\left(\frac{2}{\lambda}X_3\right)}\left(\sqrt{\mathbb{1} - X_3^2}\right)^{-1},\\ \hat{Y} &= \frac{1}{2}\sqrt{\mu + \sqrt{c}\cos\left(\frac{2}{\lambda}X_3\right)}\left(\sqrt{\mathbb{1} - X_3^2}\right)^{-1} X_2\\ &\quad + \frac{1}{2}X_2\sqrt{\mu + \sqrt{c}\cos\left(\frac{2}{\lambda}X_3\right)}\left(\sqrt{\mathbb{1} - X_3^2}\right)^{-1},\\ \hat{Z} &= \sqrt{c}\sin\left(\frac{2}{\lambda}X_3\right), \end{aligned}$$

being noncommutative analogues of the embedding functions in Proposition 6.7, agree with the results obtained in Sect. 4.3 up to corrections of order $1/n$; moreover, the matrix $\hat{Z}$ will have an exact agreement. (Actually, all orderings we tried for $x + iy$ gave matrices with nonzero elements only on the first off-diagonal; furthermore, they also agreed with our results up to order $1/n$, as we have seen from numerical computations.) For a spherical representation, it holds that

$$Z_{ll} = \frac{1}{2\hbar}[W, W^\dagger]_{ll} = \sqrt{c}\sin\big((n+1-2l)\theta\big),$$

with $\mu\cos\theta + \sqrt{c}\cos n\theta = 0$. Furthermore, the matrix elements of $\hat{Z}$ are given by

$$\hat{Z}_{ll} = \sqrt{c}\sin\left((n+1-2l)\frac{A(n)}{\lambda}\right).$$

As one can easily check, the relation $\mu\cos\theta + \sqrt{c}\cos n\theta = 0$ defines a unique smooth function $\theta(n)$ such that $0 < \theta(n) < \pi/(n+1)$. Defining $A(n) = \lambda\theta(n)$ gives directly that $Z_{ll} = \hat{Z}_{ll}$, and one can show that the matrices $\hat{X}, \hat{Y}$ will agree with the matrices X, Y up to corrections of order $1/n$. The main ingredient is the following lemma.

Lemma 6.8.

$$\frac{\mu}{\sqrt{c}} + \cos\left(\frac{2}{\lambda}\big(X_3\big)_{l,l}\right) = \frac{2\sin(n-l)\theta\sin l\theta}{\cos\theta} + O\left(\frac{1}{n}\right).$$

Proof. Setting $\theta = A(n)/\lambda$, we can rewrite

$$\cos\left(\frac{2}{\lambda}\big(X_3\big)_{l,l}\right) = \cos(n-l)\theta\cos l\theta + \sin(n-l)\theta\sin l\theta$$
$$+\big(\cos\theta - 1\big)\cos(n-2l)\theta - \sin\theta\sin(n-2l)\theta.$$

Since $-\cos(2/\lambda) = \mu/\sqrt{c} = -\cos n\theta/\cos\theta$, it follows that $\theta(n) = 2/(\lambda n) + O(1/n^2)$ and we conclude that

$$\cos\left(\frac{2}{\lambda}\big(X_3\big)_{l,l}\right) = \cos(n-l)\theta\cos l\theta + \sin(n-l)\theta\sin l\theta + O\left(\frac{1}{n}\right).$$

Since $\mu = -\sqrt{c}\cos\big(2/\lambda\big)$, we have $-\cos n\theta = -\cos(2/\lambda + O(1/n)) = \mu/\sqrt{c} + O(1/n)$, which implies that

$$\frac{\mu}{\sqrt{c}} + \cos\left(\frac{2}{\lambda}\big(X_3\big)_{l,l}\right) = 2\sin(n-l)\theta\sin l\theta + O\left(\frac{1}{n}\right),$$

from which the statement of the lemma follows. □

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