



# The boundary of the irreducible components for invariant subspace varieties

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**Abstract** Given partitions  $\alpha, \beta, \gamma$ , the short exact sequences

$$0 \longrightarrow N_\alpha \longrightarrow N_\beta \longrightarrow N_\gamma \longrightarrow 0$$

of nilpotent linear operators of Jordan types  $\alpha, \beta, \gamma$ , respectively, define a constructible subset  $\mathbb{V}_{\alpha, \gamma}^\beta$  of an affine variety. Geometrically, the varieties  $\mathbb{V}_{\alpha, \gamma}^\beta$  are of particular interest as they occur naturally and since they typically consist of several irreducible components. In fact, each Littlewood–Richardson tableau  $\Gamma$  of shape  $(\alpha, \beta, \gamma)$  contributes one irreducible component  $\overline{\mathbb{V}}_\Gamma$ . We consider the partial order  $\Gamma \leq_{\text{boundary}} \tilde{\Gamma}$  on LR-tableaux which is the transitive closure of the relation given by  $\mathbb{V}_{\tilde{\Gamma}} \cap \overline{\mathbb{V}}_\Gamma \neq \emptyset$ . In this paper we compare the boundary relation with partial orders given by algebraic, combinatorial and geometric conditions. It is known that in the case where the parts of  $\alpha$  are at most two, all those partial orders are equivalent. We prove that those partial orders are also equivalent in the case where  $\beta \setminus \gamma$  is a horizontal and vertical strip. Moreover, we discuss how the orders differ in general.

**Keywords** Nilpotent operator · Invariant subspace · Partial order · Degeneration · Littlewood–Richardson tableau

**Mathematics Subject Classification** 14L30 (primary) · 16G20 · 47A15

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### 1 Introduction

Often in geometry, naturally occurring conditions define subsets of varieties which are either very big in size or tiny. For example, among all linear operators acting on a given finite dimensional vector space, the invertible ones form an open and dense subset. And so do, among all nilpotent operators, those which have only one Jordan block. A notable exception to this rule occurs in the variety of short exact sequences of nilpotent linear operators; it can be written, by means of Littlewood–Richardson tableaux, as a union of components of equal dimension. They are the topic of this paper.

Throughout we assume that  $k$  is an algebraically closed field. A nilpotent  $k$ -linear operator is a finite dimensional module over the localized polynomial ring  $k[T]_{(T)}$  in one indeterminate, hence it has the form  $N_\alpha = \bigoplus_{i=1}^s k[T]/(T^{\alpha_i})$  for a uniquely determined partition  $\alpha = (\alpha_1, \dots, \alpha_s)$  which represents the sizes of its Jordan blocks, (see Notation 2.12).

The Theorem of Green and Klein [8] states that for given partitions  $\alpha, \beta, \gamma$ , there exists a short exact sequence

$$0 \longrightarrow N_\alpha \longrightarrow N_\beta \longrightarrow N_\gamma \longrightarrow 0$$

of nilpotent linear operators if and only if there is a Littlewood–Richardson (LR-) tableau of shape  $(\alpha, \beta, \gamma)$ . The collection of all such short exact sequences forms a variety  $\mathbb{V}_{\alpha,\gamma}^\beta(k)$  which can be written as a disjoint union, using LR-tableaux, as follows. Consider the affine variety  $\text{Hom}_k(N_\alpha, N_\beta)$  of  $k$ -linear maps endowed with the Zariski topology, and assume that all subsets carry the induced topology. Define

$$\mathbb{V}_{\alpha,\gamma}^\beta(k) = \{f : N_\alpha \rightarrow N_\beta \mid f \text{ monomorphism of } k[T]\text{-modules with cokernel isomorphic to } N_\gamma\}.$$

The irreducible components of  $\mathbb{V}_{\alpha,\gamma}^\beta(k)$  are counted by the Littlewood–Richardson coefficient. Namely, to each monomorphism in  $\mathbb{V}_{\alpha,\gamma}^\beta$  one can associate an LR-tableau  $\Gamma$  of shape  $(\alpha, \beta, \gamma)$ , as we will see in Sect. 2. The subset  $\mathbb{V}_\Gamma$  of  $\text{Hom}_k(N_\alpha, N_\beta)$  of all such monomorphisms is constructible and irreducible as a space. All the  $\mathbb{V}_\Gamma$  have the same dimension. We denote by  $\overline{\mathbb{V}}_\Gamma$  the closure of  $\mathbb{V}_\Gamma$  in  $\mathbb{V}_{\alpha,\gamma}^\beta$ ; the sets  $\overline{\mathbb{V}}_\Gamma$  define the irreducible components of  $\mathbb{V}_{\alpha,\gamma}^\beta$ , they are indexed by the set  $\mathcal{T}_{\alpha,\gamma}^\beta$  of all LR-tableaux of shape  $(\alpha, \beta, \gamma)$  (see [16, Theorem 4.3] and [18]).

Our aim in this paper is to shed light on the geometry in the variety

$$\mathbb{V}_{\alpha,\gamma}^\beta = \dot{\bigcup}_{\Gamma \in \mathcal{T}_{\alpha,\gamma}^\beta} \mathbb{V}_\Gamma;$$

by studying the boundary relation given as follows.

$$\Gamma \preceq_{\text{boundary}} \tilde{\Gamma} \Leftrightarrow \mathbb{V}_{\tilde{\Gamma}} \cap \overline{\mathbb{V}}_\Gamma \neq \emptyset \text{ where } \Gamma, \tilde{\Gamma} \in \mathcal{T}_{\alpha,\gamma}^\beta. \tag{1.1}$$

We illustrate the boundary relation in the simplest example. Consider the tableaux

$$\Gamma : \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 2 & \\ \hline 1 & & \\ \hline \end{array}, \quad \tilde{\Gamma} : \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 1 & \\ \hline 2 & & \\ \hline \end{array}.$$

It turns out that the variety  $\mathbb{V}_\Gamma = \mathcal{O}_M$  consists of a single orbit ( $M = P_1^3 \oplus P_2^2 \oplus P_0^1$ , see Sect. 2), while  $\mathbb{V}_{\tilde{\Gamma}} = \mathcal{O}_{M'} \cup \mathcal{O}_{M''}$  is the disjoint union of two orbits ( $M' = P_2^3 \oplus P_0^2 \oplus P_1^1, M'' =$

$P(0, 2) \oplus P_1^2$ ). One can deduce the following, see [11]. The orbit  $\mathcal{O}_{M'}$  is dense. Moreover,  $\mathbb{V}_{\tilde{f}}$  is closed while  $\overline{\mathbb{V}}_{\Gamma} = \mathcal{O}_M \cup \mathcal{O}_{M''}$ . Thus,  $\mathbb{V}_{\tilde{f}} \cap \overline{\mathbb{V}}_{\Gamma} \neq \emptyset$  and hence  $\Gamma \preceq_{\text{boundary}} \tilde{\Gamma}$ .

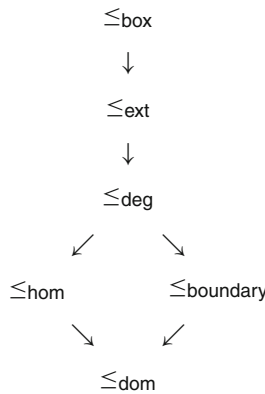
Obviously,  $\preceq_{\text{boundary}}$  is reflexive and we will see that it is anti-symmetric. We denote by  $\preceq_{\text{boundary}}$  the transitive closure of  $\preceq_{\text{boundary}}$ . In general, for a reflexive and anti-symmetric relation  $\preceq_x$  we denote its transitive closure by  $\preceq_x$ .

On the set

$$\mathcal{P} = \mathcal{P}_{\alpha,\gamma}^\beta = \left\{ \mathbb{V}_{\Gamma} ; \Gamma \in \mathcal{T}_{\alpha,\gamma}^\beta \right\}$$

of irreducible components of the representations space  $\mathbb{V}_{\alpha,\gamma}^\beta$ , there are several relations of algebraic, geometric and combinatorial nature: the hom- and the ext-order, the degeneration order and the boundary condition, the box relation and the dominance order. By taking the reflexive and transitive closure of each relation, if necessary, we obtain six partial orders on the set  $\mathcal{P}$ . It is the aim of the paper to compare those partial orders.

Given two partial orders  $(\mathcal{P}, \preceq_x)$ ,  $(\mathcal{P}, \preceq_y)$  on the same set, we say  $(\mathcal{P}, \preceq_x)$  is *finer* than  $(\mathcal{P}, \preceq_y)$  if  $P \preceq_y Q$  implies  $P \preceq_x Q$  for all  $P, Q \in \mathcal{P}$ . With respect to the fineness relation, we obtain the following diagram (whenever the box-relation is defined):



Examples 3.6 and 3.7 show that, in general, the ext and the deg relation, and the boundary and the dom relation are not equivalent, respectively. Even in the case where  $\beta \setminus \gamma$  is a horizontal strip, the box and the dom relation may be different (Example 2.8).

However, if  $\beta \setminus \gamma$  is a horizontal and vertical strip, then all the above relations are equal.

Since the set  $\mathcal{P}$  of irreducible components of  $\mathbb{V}_{\alpha,\gamma}^\beta$  is in bijection with the set  $\mathcal{T}_{\alpha,\gamma}^\beta$ , we will work with posets  $(\mathcal{T}_{\alpha,\gamma}^\beta, \preceq_x)$  instead of  $(\mathcal{P}, \preceq_x)$ .

We are ready to present the main results of the paper.

### 1.1 Two algebraic tests

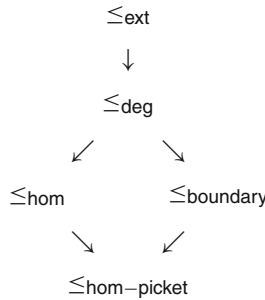
The algebraic group  $G = \text{Aut}_{k[T]}(N_\alpha) \times \text{Aut}_{k[T]}(N_\beta)$  acts on  $\mathbb{V}_{\alpha,\gamma}^\beta$  via  $(a, b) \cdot f = bfa^{-1}$ . The orbits of this group action are in one-to-one correspondence with the isomorphism classes of embeddings  $f : N_\alpha \rightarrow N_\beta$ .

We consider the following reflexive relation for LR-tableaux. We say  $\Gamma \prec_{\text{deg}} \tilde{\Gamma}$  if there are embeddings  $f \in \mathbb{V}_{\Gamma}$ ,  $\tilde{f} \in \mathbb{V}_{\tilde{\Gamma}}$  such that  $f \preceq_{\text{deg}} \tilde{f}$ , that is,  $\mathcal{O}_{\tilde{f}} \subset \mathcal{O}_f$ , where  $\mathcal{O}_f$  is the orbit of  $f$  under the action of  $G$  on  $\mathbb{V}_{\alpha,\gamma}^\beta$ .

The degeneration relation is under control algebraically as the ext-relation  $\prec_{\text{ext}}$  implies the deg-relation  $\prec_{\text{deg}}$ , which in turn implies the hom-relation  $\prec_{\text{hom}}$  (see Sect. 4). We show

that the boundary relation implies the restriction  $\leq_{\text{hom-picket}}$  of the hom order to certain objects called *pickets*.

In the diagram below, the relations introduced so far on the set  $\mathcal{T}_{\alpha,\gamma}^\beta$  are ordered vertically by containment.



We show that the restriction of the hom-order to pickets is an anti-symmetric relation. As a consequence, all the relations in the diagram are partial orders on  $\mathcal{T}_{\alpha,\gamma}^\beta$ . We have algebraic tests both for the validity and for the failure of the boundary relation:

**Theorem 1.2** *Suppose  $\alpha, \beta, \gamma$  are partitions. The following implications hold for LR-tableaux  $\Gamma, \tilde{\Gamma}$  of shape  $(\alpha, \beta, \gamma)$ :*

$$\Gamma \leq_{\text{ext}} \tilde{\Gamma} \implies \Gamma \leq_{\text{boundary}} \tilde{\Gamma} \implies \Gamma \leq_{\text{hom-picket}} \tilde{\Gamma}.$$

We present proofs in Sect. 4.

### 1.2 Two combinatorial criteria

On the set  $\mathcal{T}_{\alpha,\gamma}^\beta$ , there are two partial orders of combinatorial nature. The *dominance relation*  $\leq_{\text{dom}}$  is given by the natural partial orders of the partitions defining the tableaux. The second relation is the *box-order*  $\leq_{\text{box}}$ , it is given by repeatedly exchanging two entries in the tableau in such a way that the smaller entry moves up and such that the lattice permutation condition is preserved. We introduce the two orders formally in Sect. 2.1.

The following result presents a necessary and a sufficient criterion of combinatorial nature for two LR-tableaux to be in boundary relation:

**Theorem 1.3** *Given partitions  $\alpha, \beta, \gamma$ , the following implications hold for LR-tableaux  $\Gamma, \tilde{\Gamma}$  of shape  $(\alpha, \beta, \gamma)$ .*

- (a) *If  $\Gamma \leq_{\text{boundary}} \tilde{\Gamma}$  then  $\Gamma \leq_{\text{dom}} \tilde{\Gamma}$ .*
- (b) *Suppose  $\beta \setminus \gamma$  is a horizontal strip. If  $\Gamma \leq_{\text{box}} \tilde{\Gamma}$  then  $\Gamma \leq_{\text{boundary}} \tilde{\Gamma}$ .*

We show in Sect. 3.1 that the dominance relation is in fact equivalent to the restriction of the hom-order to pickets. The second part follows from a result in [13].

**Proposition 1.4** *Suppose  $\Gamma, \tilde{\Gamma}$  are LR-tableaux which have the same shape and which are horizontal strips. If  $\Gamma \leq_{\text{box}} \tilde{\Gamma}$  then  $\Gamma \leq_{\text{ext}} \tilde{\Gamma}$ .*

### 1.3 Horizontal and vertical strips

Of particular interest is the case where the partitions are such that  $\beta \setminus \gamma$  is both a horizontal and a vertical strip. In this situation, the combinatorial relations  $\leq_{\text{box}}$  and  $\leq_{\text{dom}}$  are in fact equivalent. In [14] we give two proofs for this statement; below in Sect. 2.1 we sketch the algorithmic approach in one of them. We deduce the following result.

**Theorem 1.5** *Suppose  $\alpha, \beta, \gamma$  are partitions such that  $\beta \setminus \gamma$  is a horizontal and vertical strip. The following relations are partial orders which are equivalent to each other:*

$$\leq_{\text{box}}, \leq_{\text{ext}}, \leq_{\text{deg}}, \leq_{\text{hom}}, \leq_{\text{boundary}}, \leq_{\text{dom}} .$$

For comparison we note that there is a related result about the six partial orders in the case where all parts of  $\alpha$  are at most two. In this situation the orbits and the boundary relation are given combinatorially in terms of arc diagrams and of resolution of crossings, respectively [11, 12].

**Theorem 1.6** *Suppose  $\alpha, \beta, \gamma$  are partitions such that all parts of  $\alpha$  are at most two. The relations  $\leq_{\text{dom}}, \leq_{\text{hom}}, \leq_{\text{boundary}}, \leq_{\text{deg}}, \leq_{\text{ext}}, \leq_{\text{box}}$  are all partial orders which are equivalent to each other.*

## 1.4 Related results

The Theorem of Gerstenhaber and Hesselink shows that the natural partial order of partitions is equivalent to the degeneration order of nilpotent linear operators, see [5, 6, 15]. We investigate a similar problem: connections of the dominance order of LR-tableaux with the boundary order defined below. Also extensions of nilpotent linear operators are of interest as they are connected with the classical Hall algebras and Hall polynomials, see [17]. Well understood are generic extensions and their relationships with the specializations to  $q = 0$  of the Ringel-Hall algebras, see [3, 4, 10, 19, 20].

## 1.5 Organization of this paper

In Sect. 2, we describe how partitions and tableaux describe short exact sequences of linear operators, or equivalently of *embeddings* or *invariant subspaces* of linear operators. Moreover, we introduce pickets as special types of embeddings.

In Sect. 3, we show that the boundary relation in Formula (1.1) implies the dominance order (Part (a) of Theorem 1.3). As a consequence we obtain that the boundary relation is anti-symmetric. We present an example showing that  $\leq_{\text{boundary}}$  may not be transitive. Example 3.6 shows that the dominance order does not imply the boundary relation in general, not even for vertical strips. But note that the two relations are equivalent when we are dealing with horizontal and vertical strips (Theorem 1.5).

In Sect. 4, we adapt the ext- deg- and hom-relations for modules to tableaux. As for modules, the ext-order implies the degeneration order, which implies the hom-order. Moreover, the hom-relation implies the dominance order. This completes the proof of Theorem 1.2. Using results given in [13] and in [14], we show part (b) of Theorem 1.3 and complete the proof of Theorem 1.5.

## 2 Littlewood–Richardson tableaux

Given three partitions,  $\alpha, \beta, \gamma$ , we consider the set  $\mathcal{T}_{\alpha, \gamma}^{\beta}$  of all Littlewood–Richardson tableaux of shape  $(\alpha, \beta, \gamma)$ . We define the dominance order on the set  $\mathcal{T}_{\alpha, \gamma}^{\beta}$ . Moreover, we introduce the LR-tableau of a short exact sequence, and determine the tableaux for certain types of short exact sequences, in particular pickets. For the case where the skew diagram  $\beta \setminus \gamma$  is a horizontal strip, we also introduce the box-order.

### 2.1 Combinatorial orders on the set of LR-tableaux

**Notation 2.1** Recall that a *partition*  $\alpha = (\alpha_1, \dots, \alpha_s)$  is a finite non-increasing sequence of natural numbers; we picture  $\alpha$  by its Young diagram which consists of  $s$  columns of length given by the parts of  $\alpha$ . The *transpose*  $\alpha'$  of  $\alpha$  is given by the formula

$$\alpha'_j = \#\{i : \alpha_i \geq j\},$$

it is pictured by the transpose of the Young diagram for  $\alpha$ . Two partitions  $\alpha, \tilde{\alpha}$  are in the *natural partial order*, in symbols  $\alpha \leq_{\text{nat}} \tilde{\alpha}$ , if the inequality

$$\alpha'_1 + \dots + \alpha'_j \leq \tilde{\alpha}'_1 + \dots + \tilde{\alpha}'_j$$

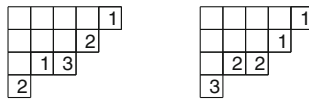
holds for each  $j$ .

Given three partitions  $\alpha, \beta, \gamma$ , an *LR-tableau* of shape  $(\alpha, \beta, \gamma)$  is a Young diagram of shape  $\beta$  in which the region  $\beta \setminus \gamma$  contains  $\alpha'_1$  entries  $\boxed{1}$ ,  $\dots$ ,  $\alpha'_t$  entries  $\boxed{t}$ , where  $t = \alpha_1$  is the largest entry, such that

- in each row, the entries are weakly increasing,
- in each column, the entries are strictly increasing,
- for each  $\ell > 1$  and for each column  $c$ : on the right hand side of  $c$ , the number of entries  $\ell - 1$  is at least the number of entries  $\ell$ .

The skew diagram  $\beta \setminus \gamma$  is said to be a *horizontal strip* if  $\beta_i \leq \gamma_i + 1$  holds for all  $i$ , and a *vertical strip* if  $\beta' \setminus \gamma'$  is a horizontal strip.

*Example 2.2* Let  $\alpha = (3, 2)$ ,  $\beta = (4, 3, 3, 2, 1)$ ,  $\gamma = (3, 2, 2, 1)$ . Then the transpose of  $\alpha$  is  $\alpha' = (2, 2, 1)$ , so we have to fill the skew diagram  $\beta \setminus \gamma$  with two  $\boxed{1}$ 's, two  $\boxed{2}$ 's, and one  $\boxed{3}$ . Due to the conditions on an LR-tableau, this can be done in exactly two ways.



In this example,  $\beta \setminus \gamma$  is a horizontal but not a vertical strip.

**Notation 2.3** One can represent an LR-tableau  $\Gamma$  by a sequence of partitions

$$\Gamma = [\gamma^{(0)}, \dots, \gamma^{(t)}]$$

where  $\gamma^{(i)}$  denotes the region in the Young diagram  $\beta$  which contains the entries  $\boxed{i}, \boxed{1}, \dots, \boxed{i}$ . If  $\Gamma$  has shape  $(\alpha, \beta, \gamma)$ , then  $\gamma = \gamma^{(0)}$ ,  $\beta = \gamma^{(t)}$ , and  $\alpha'_i = |\gamma^{(i)} \setminus \gamma^{(i-1)}|$  for  $i = 1, \dots, t$ .

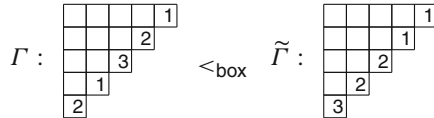
In the example above, the first tableau is given by the sequence of partitions  $\Gamma = [(3, 2, 2, 1), (3, 3, 2, 1, 1), (4, 3, 2, 2, 1), (4, 3, 3, 2, 1)]$ .

We introduce two partial orders on the set  $\mathcal{T}_{\alpha, \gamma}^{\beta}$  of all LR-tableaux of shape  $(\alpha, \beta, \gamma)$ .

**Definition 2.4** Two LR-tableaux  $\Gamma = [\gamma^{(0)}, \dots, \gamma^{(t)}]$ ,  $\tilde{\Gamma} = [\tilde{\gamma}^{(0)}, \dots, \tilde{\gamma}^{(t)}]$  of the same shape are in the *dominance order*, in symbols  $\Gamma \leq_{\text{dom}} \tilde{\Gamma}$ , if for each  $i$ , the corresponding partitions  $\gamma^{(i)}, \tilde{\gamma}^{(i)}$  are in the natural partial order, i.e.  $\gamma^{(i)} \leq_{\text{nat}} \tilde{\gamma}^{(i)}$ .

**Definition 2.5** Suppose  $\Gamma, \tilde{\Gamma}$  are LR-tableaux of the same shape which we assume to be a horizontal strip. We say  $\tilde{\Gamma}$  is obtained from  $\Gamma$  by a box move if after two entries in  $\Gamma$  have been exchanged in such a way that the smaller entry is in the higher position in  $\tilde{\Gamma}$ , we obtain  $\tilde{\Gamma}$  by re-sorting the list of columns if necessary. We denote by  $\leq_{\text{box}}$  the partial order generated by box moves.

Here is an example:



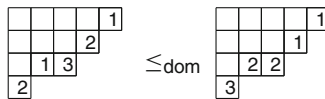
*Remark 2.6* In [13] the box-order is defined in a more general case: in the case when LR-tableaux are unions of so called columns. For simplicity, we present definitions and results for horizontal strips.

**Lemma 2.7** For LR-tableaux of the same shape, the  $\leq_{\text{box}}$ -order implies the  $\leq_{\text{dom}}$ -order.

*Proof* Suppose the LR-tableau  $\tilde{\Gamma} = [\tilde{\gamma}^{(0)}, \dots, \tilde{\gamma}^{(t)}]$  is obtained from  $\Gamma = [\gamma^{(0)}, \dots, \gamma^{(t)}]$  by a box move based on entries  $i$  and  $j$  with, say,  $i < j$ . The process of reordering the entries in each row will not affect entries less than  $i$  or larger than  $j$ , so the partitions  $\gamma^{(0)}, \dots, \gamma^{(i-1)}$ , and  $\gamma^{(j)}, \dots, \gamma^{(t)}$  remain unchanged. The partitions  $\gamma^{(\ell)}, \tilde{\gamma}^{(\ell)}$  for  $i \leq \ell < j$  are different and satisfy  $\gamma^{(\ell)} <_{\text{nat}} \tilde{\gamma}^{(\ell)}$  (since the defining partial sums can only increase). This shows that  $\Gamma <_{\text{dom}} \tilde{\Gamma}$ .  $\square$

The converse does not always hold, not even for horizontal strips:

*Example 2.8* Let  $\beta = (4, 3, 3, 2, 1), \gamma = (3, 2, 2, 1)$  and  $\alpha = (3, 2)$ . We have seen that there are two LR-tableaux of type  $(\alpha, \beta, \gamma)$ . They are incomparable in  $\leq_{\text{box}}$ -relation, but



However for horizontal and vertical strips, the two partial orders are equivalent [14]:

**Theorem 2.9** Suppose  $\alpha, \beta, \gamma$  are partitions such that  $\beta \setminus \gamma$  is a horizontal and vertical strip. Then the two partial orders  $\leq_{\text{dom}}, \leq_{\text{box}}$  are equivalent on  $\mathcal{T}_{\alpha, \gamma}^{\beta}$ .  $\square$

In [14] we present two proofs of the fact that  $\leq_{\text{dom}}$  implies  $\leq_{\text{box}}$  (for horizontal and vertical strips). Both are algorithmic. Below we present one of these algorithms without any proof of its correctness. The reader is referred to [14] for details and proofs.

**Algorithm 2.10** For an LR-tableau  $\Gamma$  we denote by  $\omega(\Gamma)$  the list of entries when read from left to right. Clearly,  $\Gamma$  is determined uniquely by its shape and by the list of its entries.

**Input:** Two LR-tableaux  $\Gamma, \tilde{\Gamma}$  of shape  $(\alpha, \beta, \gamma)$  such that  $\beta \setminus \gamma$  is a horizontal and vertical strip and such that  $\Gamma <_{\text{dom}} \tilde{\Gamma}$ .

**Output:** An LR-tableau  $\hat{\Gamma}$  of the shape  $(\alpha, \beta, \gamma)$  such that  $\Gamma \leq_{\text{dom}} \hat{\Gamma}$  and  $\hat{\Gamma} <_{\text{box}} \tilde{\Gamma}$ .

**Step 1.** Find the smallest  $k$  such that  $\omega(\Gamma)_k \neq \omega(\tilde{\Gamma})_k$  and put  $x = \omega(\Gamma)_k$ .

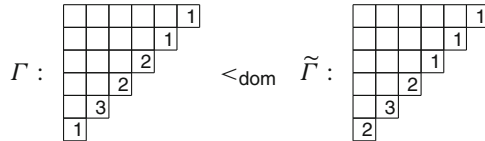
**Step 2.** Choose the minimal  $m \geq k + 1$  such that  $x = \omega(\tilde{\Gamma})_m$ .

**Step 3.** Let  $y = \min\{\omega(\tilde{\Gamma})_i > x : k \leq i < m\}$ .

**Step 4.** Choose  $k \leq l < m$  such that  $y = \omega(\tilde{\Gamma})_l$ .

**Step 5.** Define  $\hat{\Gamma}$  such that  $\omega(\hat{\Gamma})_i = \omega(\tilde{\Gamma})_i$ , for  $i \neq l, m$ , and  $\omega(\hat{\Gamma})_l = x, \omega(\hat{\Gamma})_m = y$ .

*Example 2.11* Let  $\beta = (6, 5, 4, 3, 2, 1), \gamma = (5, 4, 3, 2, 1)$  and  $\alpha = (3, 2, 1)$ . Consider two LR-tableaux  $\Gamma$  and  $\tilde{\Gamma}$  of the shape  $(\alpha, \beta, \gamma)$  such that  $\omega(\Gamma) = (1, 3, 2, 2, 1, 1)$  and  $\omega(\tilde{\Gamma}) = (2, 3, 2, 1, 1, 1)$ . It is straightforward to check that  $\beta \setminus \gamma$  is a horizontal and vertical strip and  $\Gamma <_{\text{dom}} \tilde{\Gamma}$ .



We apply the algorithm. Note that  $k = 1, x = 1$  and  $m = 4$ . Now we can choose  $y = \omega(\tilde{\Gamma})_1 = 2$  or  $y = \omega(\tilde{\Gamma})_3 = 2$ . If we choose  $y = \omega(\tilde{\Gamma})_1$ , then  $\hat{\Gamma} = \Gamma$ . In the second case, i.e. if  $y = \omega(\tilde{\Gamma})_3$ , we get  $\omega(\hat{\Gamma}) = (2, 3, 1, 2, 1, 1)$ . It is easy to see that  $\Gamma <_{\text{dom}} \hat{\Gamma}$  and we can continue.

### 2.2 The LR-tableau of a short exact sequence

**Notation 2.12** By a nilpotent operator we understand a pair  $(V, T)$  where  $V$  is a finite dimensional  $k$ -vector space and  $T : V \rightarrow V$  a  $k$ -linear nilpotent operator. Each such pair is determined uniquely, up to isomorphism, by the partition  $\alpha = (\alpha_1, \dots, \alpha_s)$  which records the sizes of the Jordan blocks. We consider  $(V, T)$  as the module over the polynomial ring

$$N_\alpha := \bigoplus_{i=1}^s k[T]/(T^{\alpha_i}).$$

Conversely, given a  $k[T]$ -module  $M$  on which the variable  $T$  acts nilpotently, the transpose of the partition  $\beta$  such that  $M \cong N_\beta$  is given by  $\beta'_\ell = \dim \frac{T^{\ell-1}M}{T^\ell M}$ .

Given three partitions  $\alpha, \beta, \gamma$ , there is a short exact sequence  $E : 0 \rightarrow N_\alpha \rightarrow N_\beta \rightarrow N_\gamma \rightarrow 0$  if and only if there is an LR-tableau of shape  $(\alpha, \beta, \gamma)$  [8]. The tableau  $\Gamma$  corresponding to the sequence  $E$  is obtained as follows. Let  $B$  be the  $k[T]$ -module  $N_\beta$  and  $A$  the submodule given by the image of the monomorphism  $N_\alpha \rightarrow N_\beta$ . The partitions defining  $\Gamma = [\gamma^{(0)}, \dots, \gamma^{(t)}]$ , where  $t = \alpha_1$ , are obtained as the isomorphism types of the nilpotent operators [17, II, (1.4)]:

$$N_{\gamma^{(i)}} = B/T^i A.$$

**Definition 2.13** Given two partitions  $\gamma, \tilde{\gamma}$ , the union  $\gamma \cup \tilde{\gamma}$  has as Young diagram the sorted union of the columns in the Young diagrams for  $\gamma$  and  $\tilde{\gamma}$ , in symbols,  $(\gamma \cup \tilde{\gamma})'_i = \gamma'_i + \tilde{\gamma}'_i$ .

For two tableaux  $\Gamma = [\gamma^{(0)}, \dots, \gamma^{(s)}], \tilde{\Gamma} = [\tilde{\gamma}^{(0)}, \dots, \tilde{\gamma}^{(t)}]$ , the union of the tableaux is given rowwise:

$$\Gamma \cup \tilde{\Gamma} = [\gamma^{(0)} \cup \tilde{\gamma}^{(0)}, \dots, \gamma^{(m)} \cup \tilde{\gamma}^{(m)}]$$

where  $m = \max\{s, t\}$  and  $\gamma^{(i)} = \gamma^{(s)}$  for  $i > s$  and  $\tilde{\gamma}^{(i)} = \tilde{\gamma}^{(t)}$  for  $i > t$ .

**Lemma 2.14** Suppose the exact sequences  $E, \tilde{E}$  have LR-tableaux  $\Gamma, \tilde{\Gamma}$ , respectively. Then the LR-tableau of the direct sum  $E \oplus \tilde{E}$  is  $\Gamma \cup \tilde{\Gamma}$ .

*Proof* Suppose  $E, \tilde{E}$  are given by the embeddings  $A \subset B, \tilde{A} \subset \tilde{B}$ . The  $j$ -th partition in the LR-tableau for  $E \oplus \tilde{E}$  is the Jordan type for  $B/T^j A \oplus \tilde{B}/T^j \tilde{A}$ , which is  $\gamma^{(j)} \cup \tilde{\gamma}^{(j)}$ .  $\square$



Thus, the LR-tableau of a direct sum is obtained by merging the rows of the LR-tableaux of the summands, starting at the top, and by sorting the entries in each row.

We present a formula for the number  $\mu_{\ell,r}$  of boxes  $\boxed{\ell}$  in the  $r$ -th row in the LR-tableau  $\Gamma = [\gamma^{(0)}, \dots, \gamma^{(\ell)}]$  of an embedding  $(A \subset B)$ . We refer to [23, Theorem 1] for a module-theoretic and homological interpretation of this number.

Denote by  $\gamma_{\leq r} = (\gamma'_1, \dots, \gamma'_r)'$  the partition which consists of the first  $r$  rows of  $\gamma$ . Thus, if a  $k[T]$ -module  $C$  has type  $\gamma$ , then  $C/T^r C$  has type  $\gamma_{\leq r}$ . In particular, the first  $r$  rows of the partitions  $\gamma^{(\ell)}$  are given as follows.

$$\gamma_{\leq r}^{(\ell)} = \text{type } \frac{B}{T^\ell A + T^r B} \tag{2.15}$$

As an immediate consequence, the number of boxes  $\boxed{\ell}$  in the first  $r$  rows of  $\Gamma$  is given by

$$|\gamma_{\leq r}^{(\ell)} \setminus \gamma_{\leq r}^{(\ell-1)}| = \dim \frac{T^{\ell-1} A + T^r B}{T^\ell A + T^r B},$$

and the formula for  $\mu_{\ell,r}$  is as follows.

$$\begin{aligned} \mu_{\ell,r}(A \subset B) &= |\gamma_{\leq r}^{(\ell)} \setminus \gamma_{\leq r}^{(\ell-1)}| - |\gamma_{\leq r-1}^{(\ell)} \setminus \gamma_{\leq r-1}^{(\ell-1)}| \\ &= \dim \frac{T^{\ell-1} A + T^r B}{T^\ell A + T^r B} - \dim \frac{T^{\ell-1} A + T^{r-1} B}{T^\ell A + T^{r-1} B} \end{aligned} \tag{2.16}$$

In the remainder of this section we study two types of examples.

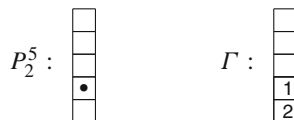
### 2.3 Example 1: pickets

**Definition 2.17** A short exact sequence  $E : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is a *picket* if  $B$  is indecomposable as a  $k[T]$ -module (so the partition  $\beta$  has only one part). A picket  $E$  is *empty* if  $A = 0$ .

*Remark 2.18* Recall that the invariant subspaces of a linear operator with only one Jordan block are determined uniquely by their dimension. As a consequence, a picket  $E$  as above is determined uniquely, up to isomorphism, by the dimensions  $n = \dim B$  and  $m = \dim A$ . We write

$$P_m^n := (0 \rightarrow (T^{n-m}) \subset k[T]/(T^n) \rightarrow k[T]/(T^{n-m}) \rightarrow 0).$$

We picture pickets as follows. In the diagram, the column represents the Jordan block of  $B$  and the dot in the  $(n - m + 1)$ -st box the submodule generator  $T^{n-m}$  in  $B$ .



To determine the LR-tableau  $\Gamma = [\gamma^{(0)}, \dots, \gamma^{(\ell)}]$  of a picket, note that  $t = m$ ,  $\gamma^{(0)} = \text{type } B/A = (n - m)$ ,  $\gamma^{(1)} = \text{type } B/TA = (n - m + 1)$ ,  $\dots$ ,  $\gamma^{(m)} = \text{type } B = (n)$ .

### 2.4 Example 2: poles

**Definition 2.19** A short exact sequence  $E : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is a *pole* if  $A$  is indecomposable as a  $k[T]$ -module and  $E$  is indecomposable as a short exact sequence.

Poles have been classified, up to isomorphy, by Kaplansky [7, Theorem 24].

**Theorem 2.20** A pole with submodule generator  $a$  is determined uniquely, up to isomorphy, by the radical layers of the elements  $T^i a$ . □

For a nonempty, strictly increasing sequence  $S = (x_0, \dots, x_{L-1})$  of nonnegative integers we construct the pole  $P(S)$  for which the submodule generator  $a$  satisfies that each  $T^i a$  occurs in the  $x_i$ -st power of the radical.

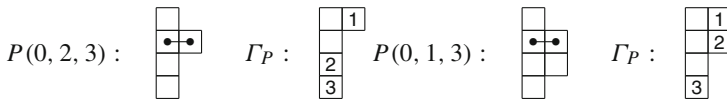
Partition the sequence into intervals of subsequent numbers,

$$S = (y_1, y_1 + 1, \dots, y_1 + \ell_1 - 1, y_2, \dots, y_2 + \ell_2 - 1, \dots, y_u, \dots, y_u + \ell_u - 1),$$

so  $y_{i+1} > y_i + \ell_i$  for  $1 \leq i < u$ . Let  $\beta$  be the partition  $\beta = (y_u + \ell_u, y_{u-1} + \ell_{u-1}, \dots, y_1 + \ell_1)$ , and put  $B = N_\beta$  and  $a = (T^{y_u - \ell_{u-1} - \dots - \ell_1}, \dots, T^{y_2 - \ell_1}, T^{y_1}) \in B$ . Then  $A = \langle a \rangle$  is an indecomposable  $k[T]$ -module and  $P(S) : 0 \rightarrow A \subset B \rightarrow B/A \rightarrow 0$  is an indecomposable short exact sequence such that for each  $i, 0 \leq i < L$ , the element  $T^i a$  is in the  $x_i$ -th radical of  $B$ .

The LR-tableau for  $P(S) = [\gamma^{(0)}, \dots, \gamma^{(L)}]$  is easily computed as  $\gamma^{(i)} \setminus \gamma^{(i-1)}$  consists of a single box  $\boxed{i}$  in row  $x_{i-1} + 1$ .

For examples, note that each picket  $P_\ell^m$  with  $\ell > 0$  is a pole, more precisely,  $P_\ell^m = P(m - \ell, m - \ell + 1, \dots, m - 1)$ . We picture here the poles  $P(0, 2, 3)$  and  $P(0, 1, 3)$  and their LR-tableaux as they will occur in an example below. For the first pole,  $\beta = (4, 1)$ ,  $a = (T, 1)$ ; for the second  $\beta = (4, 2)$ ,  $a = (T, 1)$ .



### 3 The boundary relation and its properties

In this section we present properties of the boundary relation defined in Formula 1.1.

#### 3.1 The boundary relation is anti-symmetric

We show that the boundary relation for LR-tableaux is anti-symmetric by verifying that it implies the dominance order. This is Part (a) in Theorem 1.3.

**Lemma 3.1** Suppose  $A, B$  are vector spaces and  $\mathcal{M} \subseteq \text{Hom}_k(A, B)$  is a set of monomorphisms. For subspaces  $U \subseteq A, V \subseteq B$  and a natural number  $n$ , the set

$$\{f \in \mathcal{M} : \dim(f(U) \cap V) \geq n\}$$

is closed in  $\mathcal{M}$ .

*Proof* Let  $\mathcal{M} \subseteq \text{Hom}_k(A, B)$  be a set of monomorphisms,  $U \subseteq A$  and  $V \subseteq B$  be subspaces, and let  $n \in \mathbb{N}$ . Recall that for a natural number  $m$ , the condition  $\text{rank}(f) > m$  defines an open subset in  $\text{Hom}_k(A, B)$  since it is given by the non-vanishing of a minor in the matrix representing  $f$ . By restricting that matrix to a basis for  $U$  and a basis for the complement of  $V$ , we see that the condition  $\dim \frac{f(U)+V}{V} > m$  also defines an open subset in  $\text{Hom}_k(A, B)$ . Let now  $m = \dim U - n$ . From the isomorphism  $\frac{f(U)+V}{V} \cong \frac{f(U)}{f(U) \cap V}$  we obtain that the subset defined by  $\dim \frac{f(U)}{f(U) \cap V} > m$  is open, in particular it is open when restricted to  $\mathcal{M}$ . Since on  $\mathcal{M}$ , all spaces  $f(U)$  have the same dimension ( $f$  is a monomorphism), the condition is equivalent to

$$\dim f(U) \cap V < \dim f(U) - m = n.$$

The complementary condition  $\dim f(U) \cap V \geq n$  defines a closed subset of  $\mathcal{M}$ . □

**Proposition 3.2** *For all natural numbers  $i, \ell, n$ , the subset*

$$\bigcup \{ \mathbb{V}_\Gamma : \Gamma \text{ satisfies } (\gamma^{(i)})'_1 + \dots + (\gamma^{(i)})'_\ell \geq n \}$$

in  $\mathbb{V}_{\alpha, \gamma}^\beta(k)$  is closed.

*Proof* Suppose  $f : A \rightarrow B$  is an embedding in  $\mathbb{V}_\Gamma$ . Recall that the partitions in  $\Gamma$  are given by  $B/f(T^i A) = N_{\gamma^{(i)}}$ .

Also recall that  $\dim \text{Hom}_{k[T]}(N_{(\ell)}, N_{(m)}) = \min\{\ell, m\} = \dim \frac{N_{(\ell)}}{T^m N_{(\ell)}}$ . Thus:

$$\begin{aligned} (\gamma^{(i)})'_1 + \dots + (\gamma^{(i)})'_\ell &= \sum_j \min\{\gamma_j^{(i)}, \ell\} \\ &= \dim \text{Hom}_{k[T]}(B/f(T^i A), N_{(\ell)}) \\ &= \dim \frac{B/f(T^i A)}{T^\ell(B/f(T^i A))} \\ &= \dim \frac{B/f(T^i A)}{(T^\ell B + f(T^i A))/f(T^i A)} \end{aligned}$$

Using the isomorphism  $\frac{T^\ell B + f(T^i A)}{f(T^i A)} \cong \frac{T^\ell B}{T^\ell B \cap f(T^i A)}$  we obtain

$$(\gamma^{(i)})'_1 + \dots + (\gamma^{(i)})'_\ell = \dim B - \dim f(T^i A) - \dim T^\ell B + \dim T^\ell B \cap f(T^i A).$$

Since  $\dim B - \dim f(T^i A) - \dim T^\ell B = c$  is constant on  $\mathbb{V}_{\alpha, \gamma}^\beta$ , Lemma 3.1 implies that the set

$$\bigcup \{ \mathbb{V}_\Gamma : (\gamma^{(i)})'_1 + \dots + (\gamma^{(i)})'_\ell \geq n \} = \left\{ f \in \mathbb{V}_{\alpha, \gamma}^\beta : \dim T^\ell B \cap f(T^i A) \geq n - c \right\}$$

is a closed subset of  $\mathbb{V}_{\alpha, \gamma}^\beta$ . □

We can now show that the boundary relation implies the dominance order.

*Proof [of Part (a) of Theorem 1.3]* We assume that  $\Gamma \not\prec_{\text{dom}} \tilde{\Gamma}$  and show that  $\mathbb{V}_{\tilde{\Gamma}} \cap \overline{\mathbb{V}}_\Gamma = \emptyset$ . By assumption, there exist  $i, \ell$  such that

$$n = (\gamma^{(i)})'_1 + \dots + (\gamma^{(i)})'_\ell > (\tilde{\gamma}^{(i)})'_1 + \dots + (\tilde{\gamma}^{(i)})'_\ell$$

holds. By the proposition,  $\mathbb{U} = \bigcup \{ \mathbb{V}_{\widehat{f}} : (\widehat{y}^{(i)})'_1 + \dots + (\widehat{y}^{(i)})'_\ell \geq n \}$  is a closed subset of  $\mathbb{V}_{\alpha, \gamma}^\beta$  such that

$$\mathbb{V}_\Gamma \subseteq \mathbb{U} \quad \text{and} \quad \mathbb{U} \cap \mathbb{V}_{\widehat{f}} = \emptyset.$$

Thus,  $\mathbb{V}_{\widehat{f}} \cap \overline{\mathbb{V}}_\Gamma = \emptyset$ . □

As a consequence we obtain:

**Corollary 3.3** *The boundary relation is reflexive and antisymmetric.* □

We conclude this section with a result for later use.

**Lemma 3.4** *Suppose  $f, g : N_\alpha \rightarrow N_\beta$  are objects in  $\mathbb{V}_{\alpha, \gamma}^\beta$ . Let  $W$  be a subspace of  $N_\beta$  which is invariant under all automorphisms of  $N_\beta$  as a  $k[T]$ -module. If  $\mathcal{O}_f \subset \mathcal{O}_g$  then*

$$\dim \text{Im } f \cap W \geq \dim \text{Im } g \cap W.$$

Examples of possible invariant submodules of  $N_\beta$  are the powers of the radical  $T^r N_\beta$ , powers of the socle  $T^{-s} 0$ , and their intersections  $T^r N_\beta \cap T^{-s} 0$ .

*Proof* Let  $h_\lambda : N_\alpha \rightarrow N_\beta$  be a one-parameter family of objects in  $\mathbb{V}_{\alpha, \gamma}^\beta$  such that  $h_\lambda \cong g$  for  $\lambda \neq 0$  and  $h_0 \cong f$ . Put  $n = \dim \text{Im } g \cap W$ .

Any isomorphism  $h_\lambda \cong g$  ( $\lambda \neq 0$ ) induces an isomorphism  $\text{Im } h_\lambda \cap W \cong \text{Im } g \cap W$  since  $W$  is invariant under automorphisms of  $N_\beta$ . By Lemma 3.1, the set

$$\left\{ h \in \mathbb{V}_{\alpha, \gamma}^\beta : \dim \text{Im } h \cap W \geq n \right\}$$

is closed in  $\mathbb{V}_{\alpha, \gamma}^\beta$ , so with  $h_\lambda, \lambda \neq 0$ , also  $h_0$  is in the set. This shows  $\dim \text{Im } f \cap W = \dim \text{Im } h_0 \cap W \geq n$ . □

### 3.2 The boundary relation and the dominance relation

We have seen in Sect. 3.1 that the boundary relation implies the dominance relation. Here we give an example that in general, the boundary relation is strictly stronger than the dominance relation.

In this and in the following section, we determine all isomorphism types of objects which realize a given tableau that has at most 4 rows. Such objects occur in the category  $\mathcal{S}(4)$  studied in [22, (6.4)] of all pairs consisting of a nilpotent linear operator with nilpotency index at most 4 and an invariant subspace.

**Lemma 3.5** *Each object in the category  $\mathcal{S}(4)$  is a direct sum of indecomposables. There are 20 indecomposable objects, up to isomorphism: Four empty pickets  $P_0^1, \dots, P_0^4$ , fifteen poles  $P(S)$ , where  $S$  is a non-empty subset of  $\{0, 1, 2, 3\}$ , and a remaining object  $X$  which has the property that the invariant subspace has two Jordan blocks:*

$$X : \begin{array}{|c|} \hline \square \\ \hline \bullet \quad \bullet \\ \hline \bullet \\ \hline \square \\ \hline \end{array} \quad \Gamma_X : \begin{array}{|c|} \hline \square & 1 \\ \hline \square & 2 \\ \hline 1 & \\ \hline 3 & \\ \hline \end{array}$$

□

Recall that the LR-tableau of a direct sum is obtained by merging the rows of the LR-tableaux of the summands, see Lemma 2.14.

*Example 3.6* For  $\alpha = (3, 1)$ ,  $\beta = (4, 3, 1)$ ,  $\gamma = (3, 1)$ , there are two LR-tableaux of shape  $(\alpha, \beta, \gamma)$ :

$$\Gamma_1 : \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 1 & \\ \hline & 2 & \\ \hline 3 & & \\ \hline \end{array} \quad \Gamma_2 : \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 2 & \\ \hline & 3 & \\ \hline 1 & & \\ \hline \end{array}$$

We determine the possible isomorphism types of embeddings which have LR-tableaux  $\Gamma_1$  and  $\Gamma_2$ , respectively. For each tableau, there is only one realization, up to isomorphism.

$$M_1 = P_3^4 \oplus P_0^3 \oplus P_1^1 : \begin{array}{|c|} \hline \\ \hline \bullet \\ \hline \\ \hline \end{array} \begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \\ \hline \\ \hline \end{array} \quad M_2 = P_1^4 \oplus P_3^3 \oplus P_0^1 : \begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \\ \hline \\ \hline \end{array} \begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \end{array}$$

There are no other realizations: Any such embedding occurs in the category  $\mathcal{S}(4)$ , so Lemma 3.5 can be used. Considering the LR-tableau for  $X$ , this module cannot occur as a summand (since, for example, the LR-tableau for  $X$  has a  $\boxed{1}$  in the third row, but neither  $\Gamma_1$  nor  $\Gamma_2$  does). Hence any realization is a direct sum of poles and empty pickets. Note that the pole  $P(0, 2, 3)$  cannot occur in a decomposition for  $\Gamma_1$  since this would require that  $P_1^2$  is a summand, which is not possible since there is no column of length 2 in  $\Gamma_1$ . Since each pole  $P(S)$  is determined by the sequence  $S$ , up to isomorphism, and since  $S$  determines the entries in the LR-tableau, there are no other choices.

As a consequence, the varieties  $\mathbb{V}_{\Gamma_1}$  and  $\mathbb{V}_{\Gamma_2}$  have the same dimension, and each consists of only one orbit. Hence

$$\mathbb{V}_{\Gamma_1} \cap \overline{\mathbb{V}}_{\Gamma_2} = \emptyset = \mathbb{V}_{\Gamma_2} \cap \overline{\mathbb{V}}_{\Gamma_1}.$$

Thus,  $\Gamma_1$  and  $\Gamma_2$  are not in boundary relation, but clearly  $\Gamma_1 >_{\text{dom}} \Gamma_2$ .

### 3.3 The boundary relation may not be transitive

In general, the boundary relation given by

$$\mathbb{V}_{\tilde{\Gamma}} \cap \overline{\mathbb{V}}_{\Gamma} \neq \emptyset$$

is not transitive. In this section, we provide an example.

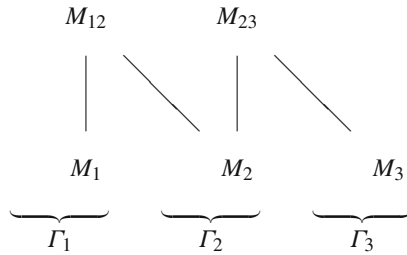
*Example 3.7* Let  $\alpha = (3, 1)$ ,  $\beta = (4, 3, 2, 1)$ ,  $\gamma = (3, 2, 1)$ . There are three LR-tableaux:

$$\Gamma_1 : \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & 1 & \\ \hline & 2 & & \\ \hline 3 & & & \\ \hline \end{array} \quad \Gamma_2 : \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & 2 & \\ \hline & 1 & & \\ \hline 3 & & & \\ \hline \end{array} \quad \Gamma_3 : \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & 2 & \\ \hline & 3 & & \\ \hline 1 & & & \\ \hline \end{array}$$

Distributed over those three tableaux are five pairwise nonisomorphic embeddings which can be determined using Lemma 3.5.

$$\begin{aligned} M_1 &= P_3^4 \oplus P_0^3 \oplus P_0^2 \oplus P_1^1, \\ M_{12} &= P(0, 2, 3) \oplus P_0^3 \oplus P_1^2, \\ M_2 &= X \oplus P_0^3 \oplus P_0^1, \\ M_{23} &= P(0, 1, 3) \oplus P_1^3 \oplus P_0^1, \\ M_3 &= P_1^4 \oplus P_3^3 \oplus P_0^2 \oplus P_0^1 \end{aligned}$$

The notation is such that  $M_i$  or  $M_{i,x}$  has LR-tableau  $\Gamma_i$ .  
 We show that the containment relation of orbit closures is as follows.



The short exact sequence

$$0 \longrightarrow P_1^2 \longrightarrow M_2 \longrightarrow P(0, 2, 3) \oplus P_0^3 \longrightarrow 0$$

shows that  $\mathcal{O}(M_{12}) \subset \overline{\mathcal{O}}(M_2)$  (since the ext-order implies the degeneration order, see Sect. 4).  
 Hence  $\mathbb{V}_{\Gamma_1} \cap \overline{\mathbb{V}}_{\Gamma_2} \neq \emptyset$  and  $\Gamma_1 >_{\text{boundary}} \Gamma_2$ .

Similarly, the short exact sequence

$$0 \longrightarrow P_1^3 \longrightarrow M_3 \longrightarrow P(0, 1, 3) \oplus P_0^1 \longrightarrow 0$$

shows that  $\mathcal{O}(M_{23}) \subset \overline{\mathcal{O}}(M_3)$ , hence  $\mathbb{V}_{\Gamma_2} \cap \overline{\mathbb{V}}_{\Gamma_3} \neq \emptyset$  and  $\Gamma_2 >_{\text{boundary}} \Gamma_3$ .

However,  $\mathbb{V}_{\Gamma_1} \cap \overline{\mathbb{V}}_{\Gamma_3} = \emptyset$ . The only possible orbit in the intersection is  $\mathcal{O}(M_{12})$ , since there are only two orbits in  $\mathbb{V}_{\Gamma_1}$ , and since the other orbit  $\mathcal{O}(M_1)$  has the same dimension as  $\mathbb{V}_{\Gamma_3} = \mathcal{O}(M_3)$ .

Note that the module  $M_{12} = (U \subset V)$  has the property that  $\dim U \cap T^2V \cap \text{soc } V = 1$ , while for the module  $M_3$ , the corresponding dimension is 2. It follows from Lemma 3.4 with  $W = T^2V \cap \text{soc } V$  that  $\mathcal{O}(M_{12}) \not\subseteq \overline{\mathcal{O}}(M_3)$ .

This finishes the example which illustrates that in general, the condition for LR-tableaux that  $\mathbb{V}_{\tilde{\Gamma}} \cap \overline{\mathbb{V}}_{\Gamma} \neq \emptyset$  may not define a partial order. □

### 4 The algebraic orders for LR-tableaux

For modules of a fixed dimension over a finite dimensional algebra the three partial orders

$$\leq_{\text{ext}}, \leq_{\text{deg}}, \leq_{\text{hom}}$$

have been studied extensively, see for example [1, 2, 9, 21, 24]. In particular, the partial orders are available for invariant subspaces in  $\mathbb{V}_{\alpha, \gamma}^\beta$ , see [11, Section 3.2]. For the convenience of the reader we recall these definitions. Let  $f, g \in \mathbb{V}_{\alpha, \gamma}^\beta$ .

- The relation  $f \leq_{\text{ext}} g$  holds if there exist embeddings  $h_i, u_i, v_i$  of linear operators and short exact sequences  $0 \rightarrow u_i \rightarrow h_i \rightarrow v_i \rightarrow 0$  of embeddings such that  $f \cong h_1, u_i \oplus v_i \cong h_{i+1}$  for  $1 \leq i \leq s$ , and  $g \cong h_{s+1}$ , for some natural number  $s$ .
- The relation  $f \leq_{\text{deg}} g$  holds if  $\mathcal{O}_g \subseteq \overline{\mathcal{O}}_f$  in  $\mathbb{V}_{\alpha, \gamma}^\beta(k)$ .
- The relation  $f \leq_{\text{hom}} g$  holds if

$$[f, h] \leq [g, h]$$

for any embedding  $h$ , where  $[f, h]$  denotes the dimension of the linear space  $\text{Hom}(f, h)$  of all homomorphisms of embeddings.

They induce three reflexive and anti-symmetric relations on the set  $\mathcal{T}_{\alpha,\gamma}^\beta$ .

**Definition 4.1** Suppose  $\Gamma, \tilde{\Gamma}$  are two LR-tableaux of shape  $(\alpha, \beta, \gamma)$ . We write  $\Gamma \leq_{\text{ext}} \tilde{\Gamma}$  ( $\Gamma \leq_{\text{deg}} \tilde{\Gamma}; \Gamma \leq_{\text{hom}} \tilde{\Gamma}$ ) if there is a sequence

$$\Gamma = \Gamma^{(0)}, \Gamma^{(1)}, \dots, \Gamma^{(s)} = \tilde{\Gamma}$$

such that for each  $1 \leq i \leq s$  there are  $f \in \mathbb{V}_{\Gamma^{(i-1)}}, g \in \mathbb{V}_{\Gamma^{(i)}}$  with  $f \leq_{\text{ext}} g$  ( $f \leq_{\text{deg}} g; f \leq_{\text{hom}} g$ ).

It follows from the corresponding properties for modules that:

- $\Gamma \leq_{\text{ext}} \tilde{\Gamma}$  implies  $\Gamma \leq_{\text{deg}} \tilde{\Gamma}$  and
- $\Gamma \leq_{\text{deg}} \tilde{\Gamma}$  implies  $\Gamma \leq_{\text{hom}} \tilde{\Gamma}$ .

Also, it is clear from the definitions that

- $\Gamma \leq_{\text{deg}} \tilde{\Gamma}$  implies  $\Gamma \leq_{\text{boundary}} \tilde{\Gamma}$ .

We observe that if there are only finitely many isomorphism classes of embeddings in  $\mathbb{V}_{\alpha,\gamma}^\beta$ , then the converse is true:

- $\Gamma \leq_{\text{boundary}} \tilde{\Gamma}$  implies  $\Gamma \leq_{\text{deg}} \tilde{\Gamma}$ .

Indeed, assume  $\Gamma <_{\text{boundary}} \tilde{\Gamma}$ . By definition,  $\mathbb{V}_{\tilde{\Gamma}} \cap \overline{\mathbb{V}}_\Gamma \neq \emptyset$ . Note that  $\mathbb{V}_\Gamma = \bigcup \mathcal{O}_f$  and  $\overline{\mathbb{V}}_\Gamma = \bigcup \overline{\mathcal{O}}_f$ , where the (finite!) union runs over all isomorphism classes of embeddings in  $\mathbb{V}_\Gamma$ . It follows that there exist  $g \in \mathbb{V}_{\tilde{\Gamma}}$  and  $f \in \mathbb{V}_\Gamma$  such that  $\mathcal{O}_g \subseteq \overline{\mathcal{O}}_f$ .

We have seen in Sect. 3.1 that the boundary relation implies the dominance order  $\leq_{\text{dom}}$ . In the following section we show that also the hom-relation implies the dominance order. As a consequence, each of the three relations  $\leq_{\text{ext}}, \leq_{\text{deg}}, \leq_{\text{hom}}$  is anti-symmetric, hence a partial ordering.

### 4.1 The Hom-relation implies the dominance order

We start with an abstract result.

Denote by  $\mathcal{N}$  the category  $\text{mod } k[T]_{(T)}$  of all nilpotent linear operators, and by  $\mathcal{S} = \mathcal{S}(k[T]_{(T)})$  the category of all invariant subspaces. For each  $i \in \mathbb{N}$ , there is a pair of functors

$$R_i : \mathcal{S} \rightarrow \mathcal{N}, (A \subset B) \mapsto \frac{B}{T^i A}$$

$$L_i : \mathcal{N} \rightarrow \mathcal{S}, X \mapsto (\text{soc}^i X \subset X).$$

**Lemma 4.2** For each  $i \in \mathbb{N}$ , the functors  $R_i, L_i$  form an adjoint pair.

*Proof* Given an operator  $X \in \mathcal{N}$  and an invariant subspace  $(A \subset B) \in \mathcal{S}$ , we need to show that there is a natural isomorphism

$$\text{Hom}_{\mathcal{S}}((A \subset B), L_i(X)) \cong \text{Hom}_{\mathcal{N}}(R_i(A \subset B), X).$$

A morphism in  $\mathcal{S}$  is given by a commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{f|_A} & \text{soc}^i X \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & X \end{array}$$

It gives rise to the commutative diagram:

$$\begin{array}{ccc} \text{rad}^i A & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & X \end{array}$$

Hence we obtain a morphism in  $\mathcal{N}$ :

$$\bar{f} : \frac{B}{\text{rad}^i A} \longrightarrow X.$$

Conversely, the morphism in  $\mathcal{N}$  gives rise to a commutative diagram and hence to a morphism in  $\mathcal{S}$ . Clearly, the two constructions are inverse to each other.  $\square$

We recognize that the objects of the form  $P_i^\ell = L_i(N_{(\ell)})$  are pickets.

**Proposition 4.3** *Suppose the objects  $(A \subset B)$  and  $(\tilde{A} \subset \tilde{B})$  have LR-tableaux  $\Gamma$  and  $\tilde{\Gamma}$ , respectively. The following assertions are equivalent:*

1.  $\Gamma \leq_{\text{dom}} \tilde{\Gamma}$
2. For each picket  $P_i^\ell$  the inequality holds:

$$\dim \text{Hom}_{\mathcal{S}}((A \subset B), P_i^\ell) \leq \dim \text{Hom}_{\mathcal{S}}((\tilde{A} \subset \tilde{B}), P_i^\ell)$$

*Proof* By the definition given in Sect. 2.1, the condition  $\Gamma \leq_{\text{dom}} \tilde{\Gamma}$  is equivalent to

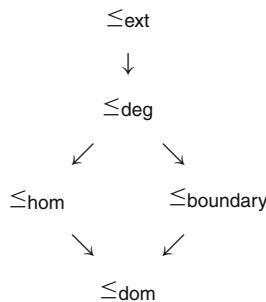
$$(\gamma^{(i)})'_1 + \dots + (\gamma^{(i)})'_\ell \leq (\tilde{\gamma}^{(i)})'_1 + \dots + (\tilde{\gamma}^{(i)})'_\ell \quad \text{for each } i \text{ and } \ell.$$

Let  $i$  and  $\ell$  be natural numbers. We obtain from Lemma 4.2 and from the equality in the proof of Proposition 3.2 that

$$(\gamma^{(i)})'_1 + \dots + (\gamma^{(i)})'_\ell = \dim \text{Hom}_{\mathcal{N}}(B/T^i A, N_{(\ell)}) = \dim \text{Hom}_{\mathcal{S}}((A \subset B), P_i^\ell)$$

The claim follows from this and from the corresponding equality for  $(\tilde{A} \subset \tilde{B})$ .  $\square$

It follows that the restriction  $\leq_{\text{hom-picket}}$  of the hom order to pickets and the dominance relation are equivalent. Hence, the hom-relation implies the dominance order. Without imposing any conditions on the triple  $(\alpha, \beta, \gamma)$ , we have established the following implications:





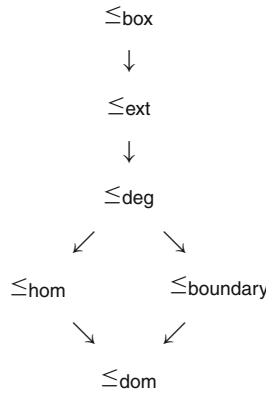
### 4.2 The box-order implies the ext-order (for horizontal strips)

Assume that  $\Gamma, \tilde{\Gamma}$  are LR-tableaux of the shape  $(\alpha, \beta, \gamma)$  such that  $\beta \setminus \gamma$  is a horizontal strip. In [13] we prove the following. Suppose that the Littlewood–Richardson tableau  $\tilde{\Gamma}$  is obtained from  $\Gamma$  by an increasing box move. By [13, Proposition 4.1] there exist embeddings  $M, \tilde{M}$  that realize tableaux  $\Gamma, \tilde{\Gamma}$ , respectively, and such that  $M = R \oplus R' \oplus N, \tilde{M} = \tilde{R} \oplus \tilde{R}' \oplus N$  for certain suitable embeddings  $R, R', \tilde{R}, \tilde{R}', N$ . In [13, Section 4.2] it is shown that there exists a short exact sequence

$$0 \longrightarrow \tilde{R} \longrightarrow Q \longrightarrow \tilde{R}' \longrightarrow 0$$

for some embedding  $Q$  with the same LR-tableau as  $R \oplus R'$ . Therefore  $Q \leq_{\text{ext}} \tilde{R} \oplus \tilde{R}'$  and  $\Gamma \leq_{\text{ext}} \tilde{\Gamma}$ .

With the assumption that  $\beta \setminus \gamma$  is a horizontal strip we have the following implications:

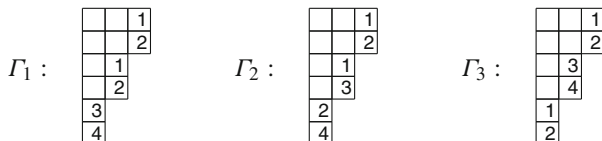


- Remark 4.4* 1. In [13] the implication  $\leq_{\text{box}} \implies \leq_{\text{ext}}$  is proved in a more general case.  
 2. Example 2.8 shows that these orders are not equivalent in general (even for horizontal strips).  
 3. Results of [14] prove the equivalence of all these orders in the case  $\beta \setminus \gamma$  is a horizontal and vertical strip (compare Theorem 1.5).

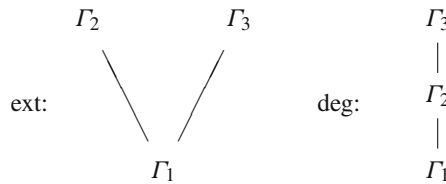
### 4.3 The ext- and deg-relations are not equivalent

It is well-known that for modules, the ext-relation  $\leq_{\text{ext}}$  implies the deg-relation  $\leq_{\text{deg}}$ . In general for modules, the converse is not the case. Here we give an example for embeddings of linear operators.

*Example 4.5* For  $\alpha = (4, 2), \beta = (6, 4, 2), \gamma = (4, 2)$ , there are three LR-tableaux:

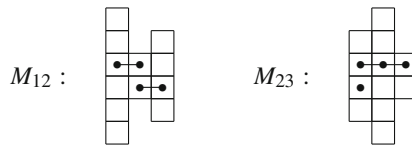


We show that the partial orders given by  $\leq_{\text{ext}}$  and  $\leq_{\text{deg}}$  are as follows:

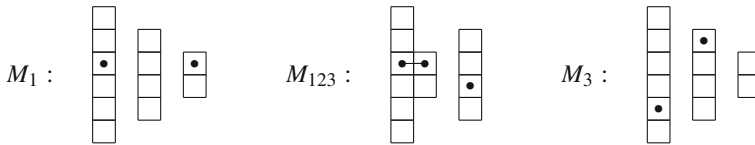


(In each case,  $\Gamma_1$  is the largest element in the poset.)

First we describe the embeddings which realize the tableaux. From [22] we know that there is a one-parameter family of indecomposable embeddings  $M_2(\lambda)$  occurring on the mouths of the homogeneous tubes with tubular index 0; they all have type  $\Gamma_2$ . The first modules in each tube are pictured in the figure in [22, p. 27, before (2.4)]. Note that the abbreviation  $c'$  in [22] is to be taken as  $c' = c - 1$  instead of  $c' = 1 - c$ . There are two additional indecomposables, they occur in the tube of circumference 2 at index 0; the modules are dual to each other and have type  $\Gamma_1$  and  $\Gamma_2$ , respectively. We sketch the modules, using the conventions as in [22].



In addition, there are three decomposable configurations; note that  $M_1$  is the dual of  $M_3$  while  $M_{123}$  is self dual.



The modules  $M_1 = P_4^6 \oplus P_0^4 \oplus P_2^2$  and  $M_{123} = P(0, 1, 4, 5) \oplus P_2^4$  have type  $\Gamma_1$ , and  $M_3 = P_2^6 \oplus P_4^4 \oplus P_0^2$  has type  $\Gamma_3$ .

We claim that there are no further isomorphism types of objects in  $\mathbb{V}_{\alpha,\gamma}^\beta$ .

For finite fields, the Hall polynomial  $g_{\alpha,\gamma}^\beta$  counts the number of submodules of  $N_\beta$  which are isomorphic to  $N_\alpha$  and have factor  $N_\gamma$ . For each of the isomorphism types of embeddings (that is,  $M_1, M_{12}, M_{123}, M_2(\lambda)$  ( $\lambda \neq 0, 1$ ),  $M_{23}, M_3$ ), we can count the corresponding numbers of submodules of  $N_\beta$ . It is straightforward to verify that the sum, taken over the isomorphism types, is exactly  $g_{\alpha,\gamma}^\beta$ .

For algebraically closed fields, the embeddings  $M_1, M_{123}, M_3$  are sums of exceptional objects in the covering category  $\mathcal{S}(\tilde{6})$  studied in [22], the others are indecomposable non-exceptional objects. The  $M_2(\lambda)$  occur in the homogeneous tubes,  $M_{12}$  and  $M_{23}$  in the tube of circumference 2 in the tubular family of index 0; the remaining tubes of index 0 are pictured in [22, (2.3)], they contain no non-exceptional objects in  $\mathbb{V}_{\alpha,\gamma}^\beta$ . All non-exceptional objects in tubes of index different from 0 have higher dimension. (Namely, an indecomposable module of index different from zero occurs as the image under the covering functor of a regular module over the tubular algebra  $\mathcal{O}_0$  corresponding to a tubular index  $\gamma = (p : q) \in \mathbb{Q}^+$  [22, (1.4),(1.1)]. Since the modules in an extended tube in  $\mathcal{T}_\gamma$  which have distance from the mouth less than the rank of the tube are all exceptional, one deduces that each non-exceptional module in  $\mathcal{T}_\gamma$  has dimension pair at least  $(p + q) \cdot (12, 6)$ .)

It follows that each remaining object in  $\mathbb{V}_{\alpha,\gamma}^\beta$  is a direct sum of exceptional modules. Each exceptional object  $X$  is determined uniquely by its dimension vector in  $\mathcal{S}(\tilde{6})$  and can be realized over any field. The dimension of the homomorphism spaces  $\text{Hom}(P, X)$  where  $P$  is a picket, and hence the LR-tableau for  $X$  ([23]) do not depend on the base field. Hence  $M_1, M_{123}$  and  $M_3$  are the only objects in  $\mathbb{V}_{\alpha,\gamma}^\beta$  which have an exceptional direct summand.

We determine the ext-order and the deg-order on  $\mathcal{T}_{\alpha,\gamma}^\beta$ .

Consider the short exact sequences

$$0 \longrightarrow P_2^4 \longrightarrow M_{23} \longrightarrow P(0, 1, 4, 5) \longrightarrow 0$$

and

$$0 \longrightarrow P_2^4 \longrightarrow M_3 \longrightarrow P(0, 1, 4, 5) \longrightarrow 0.$$

In each, the sum of the end terms is  $M_{123}$ . It follows that  $\Gamma_1 \geq_{\text{ext}} \Gamma_2$  and  $\Gamma_1 \geq_{\text{ext}} \Gamma_3$ , respectively. Note that  $\Gamma_2 \not\geq_{\text{ext}} \Gamma_3$  since there is no decomposable module of type  $\Gamma_2$ .

Since the ext-relation implies the deg-relation, it remains to show that  $\Gamma_2 \geq_{\text{deg}} \Gamma_3$ . As mentioned, the modules  $M_1$  and  $M_3$  are dual to each other, so their orbits have the same dimension. As  $\mathcal{O}_{M_3} = \mathbb{V}_{\Gamma_3}$ , and since all varieties given by LR-tableaux are irreducible of the same dimension, it follows that  $\mathcal{O}_{M_1}$  is dense in  $\mathbb{V}_{\Gamma_1}$ . In particular,  $\mathcal{O}_{M_1}$  contains  $\mathcal{O}_{M_{12}}$  in its closure. Applying duality again, we obtain that  $\mathcal{O}_{M_3}$  contains  $\mathcal{O}_{M_{23}}$  in its closure. Thus,  $\mathcal{O}_{M_{23}}$  is in the closure of  $\mathbb{V}_{\Gamma_3}$ .  $\square$

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