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CORRECTION

Correction to: Geometric Reid's recipe for dimer models

Raf Bocklandt[1] · Alastair Craw[2] · Alexander Quintero Vélez[3]

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The main results of [1], especially Theorems 1.1, 1.4 and Corollary 1.2, are correct as written. However, the final sentence in the statement of Proposition 1.3 is false when the quiver Q contains a loop at a vertex $i \in Q_0$. When this is the case, there exist points $y \in \mathcal{M}_\vartheta$ for which the corresponding A-module V_y contains a submodule of dimension vector S_i that is not isomorphic to S_i; note that any such V_y is not nilpotent. This situation is very rare,[1] but it does occur.

Example 1 For the action of type $\frac{1}{2}(1, 1, 0)$, let y be a generic point in the (noncompact) exceptional divisor in G-Hilb$(\mathbb{C}^3)$, so $y \notin \tau^{-1}(x_0)$. The nonzero maps in the

[1] If Q has a loop at vertex $i \in Q_0$, then the locus $\tau^{-1}(x_0)$ is one-dimensional. Indeed, let $n_1, \dots, n_k \in N$ be the corners of the polygon P and write $\Pi_1, \dots, \Pi_k$ for the corresponding perfect matchings. Let $m \in M$ correspond to the loop ℓ in Q at vertex i. After reordering the corner perfect matchings if necessary, there exists $1 \leq l \leq k$ such that $\ell \in \Pi_j$ if and only if $1 \leq j \leq l$. Then $\langle n_i, m \rangle = \deg_{\Pi_i} \ell = 1$ for $1 \leq i \leq l$, whereas $\langle n_j, m \rangle = 0$ for $l + 1 \leq j \leq k$. Choose a $\mathbb{Z}$-basis of N such that the affine span of P is the plane $\{(x, y, 1) \in N \otimes \mathbb{R} \mid x, y \in \mathbb{R}\}$. If we write $m := (u, v, w) \in M$ in the dual coordinates, then the polygon P is sandwiched between the parallel lines $ux + vy = -w$ and $ux + vy = -w + 1$ in the plane. Thus, P contains no internal lattice points, so $\tau^{-1}(x_0)$ contains no surfaces. The proves the assertion.

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✉ Alastair Craw
a.craw@bath.ac.uk

Raf Bocklandt
raf.bocklandt@gmail.com

Alexander Quintero Vélez
aquinte2@unal.edu.co

1 Korteweg de Vries Instituut voor Wiskunde, Universiteit van Amsterdam, P.O. Box 94248, 1090 GE Amsterdam, The Netherlands

2 Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK

3 Escuela de Matematicas, Universidad Nacional de Colombia Sede Medellin, Carrera 65 # 59A-110, Medellin, Colombia

A-module V_y are shown

$$\circlearrowleft \rho_0 \rightrightarrows \rho_1 \circlearrowright$$

where ρ_0 and ρ_1 are the trivial and nontrivial representations of $\mathbb{Z}/2$ respectively. The submodule $W_y \subset V_y$ of dimension vector S_1 destabilises V_y as $\vartheta \in C$ moves into the wall $\overline{C} \cap S_i^{\perp}$ of the GIT chamber C, so y lies in the unstable locus of this wall. However, $W_y \not\cong S_1$, so $S_i \nsubseteq \mathrm{soc}(V_y)$.

This example shows that even when $\overline{C} \cap S_i^{\perp}$ is a wall of the chamber C, the unstable locus of the wall need not coincide with $Z_i := \{y \in Y \mid S_i \subseteq \mathrm{soc}(V_y)\}$. In such cases, the final sentence of Proposition 1.3 is false; that sentence should instead conclude that:

$$\text{...the locus } Z_i \text{ is } \textit{the intersection of } \tau^{-1}(x_0) \textit{ with} \\ \text{the unstable locus of the wall } \overline{C} \cap S_i^{\perp}. \tag{0.1}$$

Indeed, Z_i is a subset of the unstable locus of the wall, but this inclusion is equality if and only if the unstable locus is contained in $\tau^{-1}(x_0)$. Instead, for any point $y \in \tau^{-1}(x_0)$ that lies in the unstable locus for the wall $\overline{C} \cap S_i^{\perp}$, the destabilising submodule $W_y \subset V_y$ of dimension vector S_i is necessarily isomorphic to S_i since V_y is nilpotent, giving $y \in Z_i$. This proves the equality (0.1).

The error leads to the omission of a case from Lemma 4.8. We now correct that statement:

Lemma 2 (= Lemma 4.8) *Every wall of the chamber C that is of form $\overline{C} \cap S_i^{\perp}$ for some nonzero $i \in Q_0$ is either of type 0, type I, or it is a type III wall with unstable locus $\mathbb{P}^1 \times \mathbb{C}$. In particular, the support of $H^0(\Psi(S_i))$ is a single $(-1,-1)$-curve (in type I), a single $(0,-2)$-curve (in type III) or a connected union of compact torus-invariant divisors (in type 0).*

Proof The only walls that are excluded here are type III walls for which the unstable locus is $\mathbb{F}_n$ for some $n \geq 0$. Suppose that one such wall exists. The wall is of the form $\overline{C} \cap S_i^{\perp}$, so Proposition 4.7 implies that $\Psi(S_i) = L_i^{-1}|_{Z_i}$. Since $\mathbb{F}_n \subseteq \tau^{-1}(x_0)$, the locus Z_i coincides with the unstable locus $\mathbb{F}_n$, so the support of $\Psi(S_i)$ is of dimension two. To obtain a contradiction, let $\ell \subset Y$ be the fibre of the contraction $\mathbb{F}_n \to \mathbb{P}^1$ induced by the wall. For any $z \in \ell$, the sequence

$$0 \longrightarrow S_i \longrightarrow V_z \longrightarrow V_z/S_i \longrightarrow 0 \tag{0.2}$$

is the θ_0-destabilising sequence for V_z. In particular, the proof of Ishii–Ueda [2, Proposition 11.31] gives that $\Psi(S_i) = \mathcal{O}_\ell(-1)$, so the support of $\Psi(S_i)$ has dimension one, a contradiction. The second statement follows from (0.1) above, where in the type III case we compute Z_i to be the intersection of $\tau^{-1}(x_0)$ with the unstable locus $\mathbb{P}^1 \times \mathbb{C}$, i.e. Z_i is the torus-invariant $(0,-2)$-curve in $\mathbb{P}^1 \times \mathbb{C}$. □

The additional case of the type III wall in Lemma 2 should have been analysed in [1, Lemma 4.10, Proposition 4.11]. We now correct those omissions.

Lemma 3 (= Lemma 4.10) *Let ℓ be a $(-1,-1)$-curve or a $(0,-2)$-curve in Y that arises as the intersection of $\tau^{-1}(x_0)$ with the unstable locus for a wall of the form $\overline{C} \cap S_i^{\perp}$ for some nonzero $i \in Q_0$ that is of type I or type III respectively. Then $L_j|_\ell \cong \mathcal{O}_\ell$ for all $j \neq i$ and $L_i|_\ell \cong \mathcal{O}_\ell(1)$.*

Proof The proof from [1, Lemma 4.10] for a $(-1,-1)$-curve applies verbatim for a $(0,-2)$-curve, but the appropriate reference to the work of Ishii–Ueda in this latter case is [2, Lemma 11.32]. □

Proposition 4 (= Proposition 4.11) *Let $i \in Q_0$ be a nonzero vertex. If $H^0(\Psi(S_i)) \neq 0$, then $\Psi(S_i) \cong L_i^{-1}|_{Z_i}$, where Z_i is the intersection of $\tau^{-1}(x_0)$ with the unstable locus for the wall $\overline{C} \cap S_i^{\perp}$.*

Proof The additional case from Lemma 3 shows that the support of $H^0(\Psi(S_i))$ can be a single $(0,-2)$-curve ℓ_i equal to the locus Z_i for a type III wall $\overline{C} \cap S_i^{\perp}$. The proof from [1, Proposition 4.11] in the case where ℓ_i is a $(-1,-1)$-curve applies verbatim, except that the required isomorphisms $L_j|_{\ell_i} \cong \mathcal{O}_{\ell_i}$ for all $j \neq i$ and $L_i|_{\ell_i} \cong \mathcal{O}_{\ell_i}(1)$ are obtained from Lemma 3. □

The final correction is in [1, Proof of Theorem 1.1], where in describing the case $H^0(\Psi(S_i)) \neq 0$, the locus Z_i should equal *the intersection of* $\tau^{-1}(x_0)$ *with* the unstable locus of the wall $\overline{C} \cap S_i^{\perp}$. In particular, this locus Z_i can be either a single $(-1,-1)$-curve, a single $(0,-2)$-curve or a connected union of compact torus-invariant divisors according to the type of the wall as in Lemma 2.

It remains to note that [1, Conjecture 1.5] should refer to the intersection of $\tau^{-1}(x_0)$; in what follows, we take the determinant of $L_\rho^\vee$ before restricting to Z_ρ (this operation was omitted in [1]):

Conjecture 5 (= Conjecture 1.5) *The object $\Psi(S_\rho)$ is a pure sheaf in degree 0 if and only if $\overline{C} \cap S_\rho^{\perp}$ is a wall of the chamber C defining G-Hilb$(\mathbb{C}^3)$, in which case $\Psi(S_\rho) \cong \det(L_\rho^\vee)|_{Z_\rho}$ where Z_ρ is the intersection of $\tau^{-1}(x_0)$ with the unstable locus of the wall $\overline{C} \cap S_\rho^{\perp}$.*

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