



# Markov's inequality and polynomial mappings

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**Abstract** Markov's inequality is a certain estimate for the norm of the derivative of a polynomial in terms of the degree and the norm of this polynomial. It has many interesting applications in approximation theory, constructive function theory and in analysis (for instance, to Sobolev inequalities or Whitney-type extension problems). One of the purposes of this paper is to give a solution to an old problem, studied among others by Baran and Pleśniak, and concerning the invariance of Markov's inequality under polynomial mappings (polynomial images). We also address the issue of preserving Markov's inequality when taking polynomial preimages. Lastly, we give a sufficient condition for a subset of a Markov set to be a Markov set.

## 1 Introduction

Throughout the paper,  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$  and  $\mathbb{R}^N$  will be treated as a subspace of  $\mathbb{C}^N$ . If  $\emptyset \neq A \subset \mathbb{C}^N$  and  $f : A \rightarrow \mathbb{C}^{N'}$ , then we put  $\|f\|_A := \sup_{z \in A} |f(z)|$ , where  $|\cdot|$  denotes the maximum norm. Moreover,  $\mathbb{N} := \{1, 2, 3, \dots\}$  and  $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ . We will also use the following notation: for each set  $\emptyset \neq A \subset \mathbb{C}^N$  and each  $\lambda > 0$ , we put

$$A_{(\lambda)} := \left\{ z \in \mathbb{C}^N : \text{dist}(z, A) \leq \lambda \right\}, \quad A_\lambda := \left\{ z \in \mathbb{C}^N : \text{dist}(z, A) < \lambda \right\}.$$

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One of the most important polynomial inequalities is the following Markov's inequality (cf. [42]).

**Theorem 1.1** (Markov) *If  $P$  is a polynomial of one variable, then*

$$\|P'\|_{[-1, 1]} \leq (\deg P)^2 \|P\|_{[-1, 1]}.$$

*Moreover, this inequality is optimal, because for the Chebyshev polynomials  $T_n$  ( $n \in \mathbb{N}_0$ ), we have  $T_n'(1) = n^2$  and  $\|T_n\|_{[-1, 1]} = 1$ .*

Recall that

$$T_n(u) = \frac{1}{2} \left[ (u + \sqrt{u^2 - 1})^n + (u - \sqrt{u^2 - 1})^n \right].$$

In fact, the above inequality for quadratic polynomials was discovered by the celebrated chemist Mendeleev. Markov's inequality and its various generalizations found many applications in approximation theory, analysis, constructive function theory, but also in other branches of science (for example, in physics or chemistry). There is now such extensive literature on Markov type inequalities that it is beyond the scope of this paper to give a complete bibliography. Let us mention only certain works which are most closely related to our paper (with emphasis on those dealing with generalizations of Markov's inequality on sets admitting cusps), for example [1–7, 11–31, 35, 36, 39, 43, 44, 46, 48, 50, 51, 57, 58]. We should stress here that the present paper owes a great debt particularly to Pawłucki and Pleśniak's work, because in [43] they laid the foundations for the theory of polynomial inequalities on “tame” (for example, semialgebraic) sets with cusps.

From the point of view of applications, it is important that the constant  $(\deg P)^2$  in Markov's inequality grows not too fast (that is, polynomially) with respect to the degree of the polynomial  $P$ . This is the reason why the concept of a Markov set is widely investigated.

**Definition 1.2** We say that a compact set  $\emptyset \neq E \subset \mathbb{C}^N$  satisfies Markov's inequality (or: is a Markov set) if there exist  $\varepsilon, C > 0$  such that, for each polynomial  $P \in \mathbb{C}[z_1, \dots, z_N]$  and each  $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}_0^N$ ,

$$\|D^\alpha P\|_E \leq (C(\deg P)^\varepsilon)^{|\alpha|} \|P\|_E,$$

where  $D^\alpha P := \frac{\partial^{|\alpha|} P}{\partial z_1^{\alpha_1} \dots \partial z_N^{\alpha_N}}$  and  $|\alpha| := \alpha_1 + \dots + \alpha_N$ .

Clearly, by iteration, it is enough to consider in the above definition multi-indices  $\alpha$  with  $|\alpha| = 1$ . We begin by giving some examples.

- Obviously, if  $\emptyset \neq E_1, \dots, E_p \subset \mathbb{C}^N$  are compact sets satisfying Markov's inequality, then the union  $E_1 \cup \dots \cup E_p$  satisfies Markov's inequality as well. In general, this is no longer so for the intersection  $E_1 \cap \dots \cap E_p$ .

- It is straightforward to show that the Cartesian product of Markov sets is a Markov set. More precisely, if  $\emptyset \neq E_j \subset \mathbb{C}^{N_j}$  ( $N_j \in \mathbb{N}$ ) is a compact set satisfying Definition 1.2 with  $\varepsilon_j, C_j > 0$  ( $j = 1, \dots, p$ ), then  $E_1 \times \dots \times E_p \subset \mathbb{C}^{N_1 + \dots + N_p}$  satisfies this definition with  $\varepsilon := \max\{\varepsilon_1, \dots, \varepsilon_p\}$  and  $C := \max\{C_1, \dots, C_p\}$ .
- In Sect. 5, we give a sufficient condition for a subset of a Markov set to be a Markov set—see Theorem 5.1 and Corollary 5.2.
- Let  $\emptyset \neq E \subset \mathbb{C}$  be a compact set such that, for each connected component  $K$  of  $E$ , we have  $\text{diam}(K) \geq \eta$  with some  $\eta > 0$  being independent of  $K$ . Then  $E$  is a Markov set—see Lemma 3.1 in [56] and Sect. 3.
- By Theorem 3.1 in [43], each compact UPC set satisfies Markov’s inequality. Recall that a set  $E \subset \mathbb{R}^N$  is UPC (uniformly polynomially cuspidal) if there exist  $\nu, \theta > 0$  and  $d \in \mathbb{N}$  such that, for each  $x \in \overline{E}$ , we can choose a polynomial map  $S_x : \mathbb{R} \rightarrow \mathbb{R}^N$  with  $\text{deg } S_x \leq d$  satisfying the following conditions:
  - $S_x(0) = x$ ,
  - $\text{dist}(S_x(t), \mathbb{R}^N \setminus E) \geq \theta t^\nu$  for each  $t \in [0, 1]$ .

Note that a UPC set is in particular fat, that is  $\overline{E} = \overline{\text{Int}E}$ . In [43,44,46], some large classes of UPC sets (and hence Markov sets) are given. These classes include for example all compact, fat and semialgebraic subsets of  $\mathbb{R}^N$  (see Sect. 2 for the definition).

The following result is due to Baran and Pleśniak (cf. [3]).

**Theorem 1.3** (Baran, Pleśniak) *Let  $E \subset \mathbb{R}^N$  be a compact UPC set. Suppose that  $h : \mathbb{R}^N \rightarrow \mathbb{R}^N$  is a polynomial map such that  $\text{Jac } h(\zeta) \neq 0$  for each  $\zeta \in \text{Int}E$ . Then  $h(E)$  satisfies Markov’s inequality.*

Since each compact UPC set satisfies Markov’s inequality, the Baran–Pleśniak theorem says that, under a certain assumption on a Markov set  $E \subset \mathbb{R}^N$  and under a certain assumption on a polynomial map  $h : \mathbb{R}^N \rightarrow \mathbb{R}^N$ , the image  $h(E)$  also satisfies Markov’s inequality.

Our aim is among others to show that in Theorem 1.3:

- Very strong UPC assumption on the Markov set  $E$  is superfluous.
- The assumption that  $\text{Jac } h(\zeta) \neq 0$  for each  $\zeta \in \text{Int}E$  can be replaced by much weaker assumption that  $h : \mathbb{R}^N \rightarrow \mathbb{R}^{N'}$  and

$$\text{rank } h := \max \left\{ \text{rank } d_\zeta h : \zeta \in \mathbb{R}^N \right\} = N'.$$

Moreover, the latter assumption is the weakest possible condition on the polynomial map  $h$  that must be assumed (see Lemma 2.3).

More precisely, we will prove the following result in Sect. 2.

**Theorem 1.4** *Suppose that  $\emptyset \neq E \subset \mathbb{K}^N$  is a compact set satisfying Markov’s inequality and  $h : \mathbb{K}^N \rightarrow \mathbb{K}^{N'}$  is a polynomial map such that*

$$\text{rank } h := \max \left\{ \text{rank } d_\zeta h : \zeta \in \mathbb{K}^N \right\} = N'$$

( $N, N' \in \mathbb{N}$ ). Then  $h(E)$  also satisfies Markov’s inequality.

It is worth noting that there is a holomorphic version of Theorem 1.3 in [3], which reads as follows. Suppose that  $E \subset \mathbb{C}^N$  is a compact, polynomially convex set satisfying Markov's inequality. If  $h : U \rightarrow \mathbb{C}^N$  is a holomorphic map in a neighbourhood  $U$  of  $E$  such that  $h(E)$  is nonpluripolar and  $\text{Jac } h(\zeta) \neq 0$  for each  $\zeta \in E$ , then  $h(E)$  also satisfies Markov's inequality. (The notion of a polynomially convex set and the notion of a nonpluripolar set are defined in Sect. 3.)

In connection with Theorem 1.4, the following question naturally arises.

**Question 1.5** *Suppose that  $\emptyset \neq E \subset \mathbb{K}^{N'}$  is a compact set satisfying Markov's inequality and  $g : \mathbb{K}^N \rightarrow \mathbb{K}^{N'}$  is a polynomial map ( $N, N' \in \mathbb{N}$ ). Under what conditions is it true that  $g^{-1}(E)$  satisfies Markov's inequality?*

The precise answer is not known to us. However, we will address this issue in Sects. 3 and 4. In particular, we will give some specific examples to show a variety of situations that we encounter exploring this problem. Eventually, we will give a result (Theorem 3.8) being a partial answer to Question 1.5.

## 2 A proof of Theorem 1.4

We will need the notion of a semialgebraic set and the notion of a semialgebraic map.

**Definition 2.1** A subset of  $\mathbb{R}^N$  is said to be semialgebraic if it is a finite union of sets of the form

$$\{x \in \mathbb{R}^N : \xi(x) = 0, \xi_1(x) > 0, \dots, \xi_q(x) > 0\},$$

where  $\xi, \xi_1, \dots, \xi_q \in \mathbb{R}[x_1, \dots, x_N]$  (cf. [9, 59]).

**Definition 2.2** A map  $f : A \rightarrow \mathbb{R}^{N'}$ , where  $A \subset \mathbb{R}^N$ , is said to be semialgebraic if its graph is a semialgebraic subset of  $\mathbb{R}^{N+N'}$ .

All semialgebraic sets constitute the simplest polynomially bounded o-minimal structure (see [59, 60] for the definition and properties of o-minimal structures). However, the knowledge of o-minimal structures is not necessary to follow the present paper. Whenever we say "a set (a map) definable in a polynomially bounded o-minimal structure", the reader who is not familiar with the basic notions of o-minimality can just think of a semialgebraic set (map).

Before going to the proof of Theorem 1.4, it is worth noting that the assumption that  $\text{rank } h = N'$  is necessary in this theorem, as is seen by the following lemma.

**Lemma 2.3** *Suppose that  $h : \mathbb{K}^N \rightarrow \mathbb{K}^{N'}$  is a polynomial map such that  $\text{rank } h < N'$  ( $N, N' \in \mathbb{N}$ ). Then for each compact set  $\emptyset \neq E \subset \mathbb{K}^N$  the image  $h(E)$  does not satisfy Markov's inequality.*

*Proof* By Sard's theorem, the set  $h(\mathbb{K}^N)$  has Lebesgue measure zero.

CASE 1:  $\mathbb{K} = \mathbb{R}$ . By the Tarski–Seidenberg theorem (cf. [8,9,40]), the set  $h(\mathbb{R}^N)$  is semialgebraic. Therefore  $h(\mathbb{R}^N) = \bigcup_{j=1}^s H_j$ , where  $s \in \mathbb{N}$  and

$$H_j := \left\{ w \in \mathbb{R}^{N'} : P_j(w) = 0, P_{1,j}(w) > 0, \dots, P_{q_j,j}(w) > 0 \right\}$$

with some  $P_j, P_{i,j} \in \mathbb{R}[w_1, \dots, w_{N'}]$ . We can clearly assume that each  $H_j$  is non-empty. Put  $P := P_1 \cdot \dots \cdot P_s$ . Note that  $P \not\equiv 0^1$  and  $P|_{h(\mathbb{R}^N)} \equiv 0$ . Take a point  $a \in h(E)$ . For each  $w \in \mathbb{R}^{N'}$ , we have

$$P(w) = \sum_{\alpha \in \mathbb{N}_0^{N'}} \frac{D^\alpha P(a)}{\alpha!} (w - a)^\alpha$$

and therefore  $D^\alpha P(a) \neq 0$  for some  $\alpha \in \mathbb{N}_0^{N'}$ . Since  $\|P\|_{h(E)} = 0$ , it follows that  $h(E)$  does not satisfy Markov’s inequality.

CASE 2:  $\mathbb{K} = \mathbb{C}$ . By Chevalley’s theorem, the set  $h(\mathbb{C}^N)$  is constructible (see [41, pp. 393–396], for the definition and details). Moreover,  $\overline{h(\mathbb{C}^N)} \neq \mathbb{C}^{N/2}$  and  $\overline{h(\mathbb{C}^N)}$  is a complex algebraic set (see [41, p. 394]), that is the set of common zeros of some collection of complex polynomials. In particular, there exists  $P \in \mathbb{C}[w_1, \dots, w_{N'}]$  such that  $P \not\equiv 0$  and  $P|_{h(\mathbb{C}^N)} \equiv 0$ . Arguing as in Case 1 we see that  $h(E)$  does not satisfy Markov’s inequality.<sup>3</sup> □

We will try to keep the exposition as self-contained as possible. It should be stressed, however, that our proof of Theorem 1.4 is influenced by ideas from the original proof of Theorem 1.3 by Baran and Pleśniak.

*Proof of Theorem 1.4.* Clearly, it suffices to consider the case  $\mathbb{K} = \mathbb{C}$ . Put

$$\chi : \mathbb{R}^{2N} \ni (u_1, v_1, \dots, u_N, v_N) \mapsto (u_1 + iv_1, \dots, u_N + iv_N) \in \mathbb{C}^N.$$

Take an open and bounded set  $I \subset \mathbb{C}^N$  such that  $E \subset \bar{I}$  and  $\chi^{-1}(I)$  is semialgebraic (for example, a sufficiently large open polydisk).

Put  $A := I \setminus T$ , where

$$T := \left\{ \zeta \in \mathbb{C}^N : \text{rank } d_\zeta h < \text{rank } h \right\} = \left\{ \zeta \in \mathbb{C}^N : \text{rank } d_\zeta h < N' \right\}.$$

Since the set  $T$  is (complex) algebraic and nowhere dense (see [41, p. 158]), it follows that  $\chi^{-1}(A) \subset \mathbb{R}^{2N}$  is open, semialgebraic and  $\overline{\chi^{-1}(A)} = \chi^{-1}(\bar{I})$ . Consequently, by Corollary 6.6 in [43],  $\chi^{-1}(A)$  is UPC. Therefore there exist  $\nu, \theta > 0$  and  $d \in \mathbb{N}$

<sup>1</sup> Because otherwise  $\emptyset \neq H_j \subset \text{Int } h(\mathbb{R}^N)$  for some  $j \leq s$ .

<sup>2</sup> We have  $\dim \overline{h(\mathbb{C}^N)} = \dim h(\mathbb{C}^N) < N'$ —see [41, pp. 393–394]. The complex dimension of a set  $A \subset \mathbb{C}^n$  ( $\dim A$ ) is defined in Sect. 4.

<sup>3</sup> In fact, we could first consider the case  $\mathbb{K} = \mathbb{C}$  and then notice that the real case follows from the complex case.

such that, for each  $x \in \overline{\chi^{-1}(A)}$ , we can choose a polynomial map  $S_x : \mathbb{R} \rightarrow \mathbb{R}^{2N}$  satisfying the following conditions:

- (i)  $\deg S_x \leq d$ ,
- (ii)  $S_x(0) = x$ ,
- (iii)  $\text{dist}(S_x(t), \mathbb{R}^{2N} \setminus \chi^{-1}(A)) \geq \theta t^u$  for each  $t \in [0, 1]$ .

By Lemma 3.1 in [44], the maps  $G_0, G_1, \dots, G_d : \overline{\chi^{-1}(A)} \rightarrow \mathbb{R}^{2N}$ , defined by

$$S_x(t) = G_0(x) + G_1(x)t + \dots + G_d(x)t^d,$$

are bounded. Thus there exists  $C_1 > 0$  such that, for each  $z \in \overline{A}$  and  $t \in [0, 1]$ ,

$$|P_z(t) - z| \leq C_1 t, \quad (1)$$

where

$$P_z(t) := \chi(S_{\chi^{-1}(z)}(t)). \quad (2)$$

(Use the fact that  $S_{\chi^{-1}(z)}(0) = \chi^{-1}(z)$ .)

By [41, p. 243], there exist  $C_2, \kappa > 0$  such that, for each  $\zeta \in \overline{I}$ ,

$$\max \left\{ \left| \frac{\partial(h_1, \dots, h_{N'})}{\partial(z_{j_1}, \dots, z_{j_{N'}})}(\zeta) \right| : 1 \leq j_1 < \dots < j_{N'} \leq N \right\} \geq C_2 \left( \text{dist}(\zeta, \mathbb{C}^N \setminus A) \right)^\kappa. \quad (3)$$

(Use the fact that  $T \subset \mathbb{C}^N \setminus A$  and consider two cases:  $T = \emptyset$  and  $T \neq \emptyset$ .)

Take  $\varepsilon, C > 0$  such that, for each polynomial  $P \in \mathbb{C}[z_1, \dots, z_N]$  and each  $\alpha \in \mathbb{N}_0^N$ ,

$$\|D^\alpha P\|_E \leq (C(\deg P)^\varepsilon)^{|\alpha|} \|P\|_E \quad (4)$$

(see Definition 1.2). Put

$$\begin{aligned} \sigma &:= \max \left\{ \frac{1}{2}, \kappa v \right\}, \\ C_3 &:= \max \left\{ \left\| \frac{\partial h_\nu}{\partial z_j} \right\|_A : 1 \leq \nu \leq N', 1 \leq j \leq N \right\}, \\ C_4 &:= (N')^{\frac{N'}{2}} \frac{C_3^{N'-1} C k^\varepsilon}{C_2}, \\ C_5 &:= \frac{4^\sigma}{\theta^\kappa} C_4 \exp(N C C_1 k^\varepsilon), \end{aligned}$$

where  $k := \deg h \geq 1$ .

Let  $w_1, \dots, w_{N'}$  denote the variables in  $\mathbb{C}^{N'}$ . We will show that, for each polynomial  $Q \in \mathbb{C}[w_1, \dots, w_{N'}]$  with  $\deg Q \leq n$  ( $n \in \mathbb{N}$ ), each  $l \in \{1, \dots, N'\}$  and each  $a \in E$ ,

$$\left| \frac{\partial Q}{\partial w_l}(h(a)) \right| \leq C_5 n^{\varepsilon(\sigma+1)} (dk(n-1) + 1)^{2\sigma} \|Q\|_{h(E)}. \tag{5}$$

Obviously, the above estimate proves the required assertion that  $h(E)$  satisfies Markov’s inequality.

Fix therefore  $Q, l, a$  as above. First, we will show that, for each  $\zeta \in \mathbb{C}^N$  and each  $j \in \{1, \dots, N\}$ ,

$$\left| \frac{\partial(Q \circ h)}{\partial z_j}(\zeta) \right| \leq C(kn)^\varepsilon \exp(NC(kn)^\varepsilon |\zeta - a|) \|Q\|_{h(E)}. \tag{6}$$

By Taylor’s formula and (4),

$$\begin{aligned} \left| \frac{\partial(Q \circ h)}{\partial z_j}(\zeta) \right| &= \left| \sum_{\alpha \in \mathbb{N}_0^N} \frac{D^\alpha \left( \frac{\partial(Q \circ h)}{\partial z_j} \right)(a)}{\alpha!} (\zeta - a)^\alpha \right| \\ &\leq \sum_{\alpha \in \mathbb{N}_0^N} \frac{(C(kn)^\varepsilon)^{|\alpha|+1} \|Q \circ h\|_E}{\alpha!} |\zeta - a|^{|\alpha|} \\ &= \sum_{\alpha \in \mathbb{N}_0^N} C(kn)^\varepsilon \frac{(C(kn)^\varepsilon |\zeta - a|)^{|\alpha|}}{\alpha!} \|Q \circ h\|_E \\ &= C(kn)^\varepsilon \exp(NC(kn)^\varepsilon |\zeta - a|) \|Q\|_{h(E)}, \end{aligned}$$

which completes the proof of (6).

We will show moreover that, for each  $\zeta \in A$ ,

$$\left| \frac{\partial Q}{\partial w_l}(h(\zeta)) \right| \leq C_4 n^\varepsilon \frac{\exp(NC(kn)^\varepsilon |\zeta - a|)}{(\text{dist}(\zeta, \mathbb{C}^N \setminus A))^\kappa} \|Q\|_{h(E)}. \tag{7}$$

To this end, take the integers  $j_1 = j_1(\zeta), \dots, j_{N'} = j_{N'}(\zeta)$  such that  $1 \leq j_1 < \dots < j_{N'} \leq N$  and

$$\left| \frac{\partial(h_1, \dots, h_{N'})}{\partial(z_{j_1}, \dots, z_{j_{N'}})}(\zeta) \right| \geq C_2 \left( \text{dist}(\zeta, \mathbb{C}^N \setminus A) \right)^\kappa \tag{8}$$

(see (3)). Consider the system of equations:

$$\begin{cases} \frac{\partial(Q \circ h)}{\partial z_{j_1}}(\zeta) = \frac{\partial Q}{\partial w_1}(h(\zeta)) \cdot \frac{\partial h_1}{\partial z_{j_1}}(\zeta) + \dots + \frac{\partial Q}{\partial w_{N'}}(h(\zeta)) \cdot \frac{\partial h_{N'}}{\partial z_{j_1}}(\zeta) \\ \vdots \\ \frac{\partial(Q \circ h)}{\partial z_{j_{N'}}}(\zeta) = \frac{\partial Q}{\partial w_1}(h(\zeta)) \cdot \frac{\partial h_1}{\partial z_{j_{N'}}}(\zeta) + \dots + \frac{\partial Q}{\partial w_{N'}}(h(\zeta)) \cdot \frac{\partial h_{N'}}{\partial z_{j_{N'}}}(\zeta). \end{cases}$$

Now it is enough to apply Cramer’s rule, Hadamard’s inequality<sup>4</sup> and the estimates (6) and (8).

For each  $t \in [0, 1]$ , we have by (iii) and (2)

$$\text{dist}(P_a(t), \mathbb{C}^N \setminus A) \geq \theta t^\nu.$$

Combining this with (1) and (7), we get for each  $t \in (0, 1]$  the following estimate:

$$\left| \frac{\partial Q}{\partial w_l}(h(P_a(t))) \right| \leq \frac{C_4}{\theta \kappa t^{\kappa \nu}} n^\varepsilon \exp(NCC_1(kn)^\varepsilon t) \|Q\|_{h(E)}. \tag{9}$$

Note that  $\frac{\partial Q}{\partial w_l} \circ h \circ P_a$  is the restriction to  $\mathbb{R}$  of a polynomial  $\Upsilon : \mathbb{C} \rightarrow \mathbb{C}$  of degree  $\leq dk(n - 1)$ . Put  $\delta := n^{-\varepsilon}$  and

$$\varphi : [-1, 1] \ni \tau \mapsto \frac{\delta(\tau + 1)}{2} \in [0, \delta].$$

By Schur’s inequality,<sup>5</sup>

$$\|\Upsilon \circ \varphi\|_{[-1, 1]} \leq (dk(n - 1) + 1) \|\sqrt{1 - \tau^2} \Upsilon(\varphi(\tau))\|_{[-1, 1]}. \tag{10}$$

<sup>4</sup> If  $B = [b_{ij}]$  is a  $q \times q$  matrix of complex numbers, then

$$|\det B|^2 \leq \prod_{i=1}^q \sum_{j=1}^q |b_{ij}|^2.$$

<sup>5</sup> *Schur’s inequality*: For each polynomial  $R$  of one variable,

$$\|R\|_{[-1, 1]} \leq (\deg R + 1) \|\sqrt{1 - \tau^2} R(\tau)\|_{[-1, 1]} \tag{\diamond}$$

—see [10, p. 233], where this inequality is stated for real polynomials. If however  $R \in \mathbb{C}[\tau]$  and  $R = R_1 + iR_2$  with  $R_1, R_2 \in \mathbb{R}[\tau]$ , then for  $\tau_0 \in [-1, 1]$  such that  $\|R\|_{[-1, 1]} = |R(\tau_0)|$ , we have

$$\begin{aligned} |R(\tau_0)|^2 &\leq \|R_1(\tau_0)R_1 + R_2(\tau_0)R_2\|_{[-1, 1]} \\ &\leq (\deg R + 1) \|\sqrt{1 - \tau^2}(R_1(\tau_0)R_1(\tau) + R_2(\tau_0)R_2(\tau))\|_{[-1, 1]} \\ &\leq |R(\tau_0)| (\deg R + 1) \|\sqrt{1 - \tau^2} R(\tau)\|_{[-1, 1]}, \end{aligned}$$

which proves  $(\diamond)$ .



Therefore either  $\left| \frac{\partial Q}{\partial w_l}(h(a)) \right| = 0$ , or  $\left| \frac{\partial Q}{\partial w_l}(h(a)) \right| > 0$  and then

$$\begin{aligned}
 \left| \frac{\partial Q}{\partial w_l}(h(a)) \right| &= \overbrace{\left| \frac{\partial Q}{\partial w_l}(h(P_a(0))) \right|}^{\text{by (ii) and (2)}} = |\Upsilon(0)| \leq \frac{\|\Upsilon \circ \varphi\|_{[-1, 1]}^{2\sigma}}{\|\Upsilon\|_{[0, \delta]}^{2\sigma-1}} \\
 &\leq \frac{1}{\|\Upsilon\|_{[0, \delta]}^{2\sigma-1}} \overbrace{(dk(n-1) + 1)^{2\sigma} \|\sqrt{1 - \tau^2} \Upsilon(\varphi(\tau))\|_{[-1, 1]}^{2\sigma}}^{\text{by (10)}} \\
 &\leq \frac{1}{\|\Upsilon\|_{[0, \delta]}^{2\sigma-1}} (dk(n-1) + 1)^{2\sigma} \left(\frac{4}{\delta}\right)^\sigma \|\sqrt{\varphi(\tau)} \Upsilon(\varphi(\tau))\|_{[-1, 1]}^{2\sigma} \\
 &= \frac{1}{\|\Upsilon\|_{[0, \delta]}^{2\sigma-1}} (dk(n-1) + 1)^{2\sigma} \left(\frac{4}{\delta}\right)^\sigma \|\sqrt{t} \Upsilon(t)\|_{[0, \delta]}^{2\sigma} \\
 &\leq (dk(n-1) + 1)^{2\sigma} \left(\frac{4}{\delta}\right)^\sigma \|t^\sigma \Upsilon(t)\|_{[0, \delta]} \\
 &\leq (dk(n-1) + 1)^{2\sigma} \left(\frac{4}{\delta}\right)^\sigma \overbrace{\frac{C_4}{\theta^\kappa} n^\varepsilon \exp(NCC_1(kn)^\varepsilon \delta) \|Q\|_{h(E)}}^{\text{by (9)}} \\
 &= C_5 n^{\varepsilon(\sigma+1)} (dk(n-1) + 1)^{2\sigma} \|Q\|_{h(E)},
 \end{aligned}$$

which establishes the estimate (5) and hence completes the proof of the theorem.  $\square$

### 3 Markov’s inequality and polynomial preimages

In this section, we will look at Markov’s inequality from the point of view of polynomial preimages.

We begin by a brief discussion of another concept, called the HCP property, which is related to Markov’s inequality. For a compact set  $\emptyset \neq E \subset \mathbb{C}^N$ , the following function

$$\Phi_E(z) := \sup \left\{ |Q(z)|^{1/\deg Q} : Q \in \mathbb{C}[z_1, \dots, z_N], \deg Q > 0 \text{ and } \|Q\|_E \leq 1 \right\}$$

( $z \in \mathbb{C}^N$ ) is called the Siciak extremal function (cf. [34,37,38,49,54,55]). It is an elementary check that  $\Phi_E \geq 1$  in  $\mathbb{C}^N$ ,  $\Phi_E \equiv 1$  in  $E$  and  $\Phi_E \leq \Phi_K$  provided that  $\emptyset \neq K \subset E$  and  $K$  is compact. However, except for some very special cases, no explicit expression for  $\Phi_E$  is known.

We have a very simple formula (yet with nontrivial proof) connecting the function  $\Phi_E$  with potential and pluripotential theory:  $\log \Phi_E = V_E$ , where

$$V_E(z) := \sup \left\{ \phi(z) : \phi \in \mathcal{L}(\mathbb{C}^N), \phi \leq 0 \text{ on } E \right\}$$

and  $\mathcal{L}(\mathbb{C}^N)$  denotes the class of plurisubharmonic<sup>6</sup> functions  $\phi$  in  $\mathbb{C}^N$  satisfying the condition

$$\sup_{z \in \mathbb{C}^N} [\phi(z) - \log(1 + |z|)] < \infty$$

(cf. Theorem 4.12 in [55] or Theorem 5.1.7 in [34]). The upper semicontinuous regularization  $V_E^*$  of  $V_E$  is often called the pluricomplex Green function, because for a compact set  $E \subset \mathbb{C}$  with positive logarithmic capacity  $V_E^*$  is the Green function with pole at infinity of the unbounded component of  $\mathbb{C} \setminus E$ .

If  $\emptyset \neq E \subset \mathbb{C}^N$  is a compact set and  $\Phi_E$  is continuous at every point of  $E$ , then  $\Phi_E$  is continuous in  $\mathbb{C}^N$ , in other words, the set  $E$  is  $L$ -regular (cf. Proposition 6.1 in [55] or Corollary 5.1.4 in [34]).

**Definition 3.1** We say that a compact set  $\emptyset \neq E \subset \mathbb{C}^N$  has the HCP property if  $\Phi_E$  is Hölder continuous in the following sense: there exist  $\varpi, \mu > 0$  such that

$$\Phi_E(z) \leq 1 + \varpi (\text{dist}(z, E))^\mu \quad \text{as } z \in E_{(1)}.$$

We will also need the notion of a pluripolar set.

**Definition 3.2** (see [34]) A set  $A \subset \mathbb{C}^N$  is said to be pluripolar if one of the following two equivalent conditions holds:

- (i) For each point  $a \in A$ , there exists an open neighbourhood  $U$  of  $a$  such that  $A \cap U \subset \{z \in U : u(z) = -\infty\}$  for some plurisubharmonic function  $u : U \rightarrow [-\infty, +\infty)$ .
- (ii) There exists a plurisubharmonic function  $\psi$  in  $\mathbb{C}^N$  such that  $A \subset \{z \in \mathbb{C}^N : \psi(z) = -\infty\}$ .

Let us add that the implication (i)  $\implies$  (ii) is the content of Josefson’s theorem (saying that every locally pluripolar set in  $\mathbb{C}^N$  is globally pluripolar). We have moreover the following characterization of pluripolar sets in terms of the pluricomplex Green function: for each set  $\emptyset \neq A \subset \mathbb{C}^N$ ,

$$A \text{ is pluripolar} \iff V_A^* \equiv +\infty \iff V_A^* \notin \mathcal{L}(\mathbb{C}^N) \tag{11}$$

(cf. Corollary 3.9 and Theorem 3.10 in [55]). Recall also that pluripolar sets have Lebesgue measure zero (cf. Corollary 2.9.10 in [34]) and countable unions of pluripolar sets are pluripolar (cf. Corollary 4.7.7 in [34]).

There is a close relation between Markov’s inequality and HCP property. Namely,

- HCP  $\implies$  Markov’s inequality (see [43]).
- The validity of the reverse implication still remains open (see [48], where this problem is posed by Pleśniak).

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<sup>6</sup> See [34] for the definition and basic properties of plurisubharmonic functions.

Furthermore, it is worth noting that, for each compact subset of  $\mathbb{R}^N$ ,  $\text{UPC} \implies \text{HCP}$  (see [43]), yet the implication cannot be reversed.

It should come as no surprise that the inverse image of a Markov set under a polynomial map need not be a Markov set, even if it is a compact set. Consider for example the map

$$h : \mathbb{R} \ni w \longmapsto w^2 \in \mathbb{R}$$

and the set  $E := [-1, 0]$ . Then  $h^{-1}(E) = \{0\}$  does not satisfy Markov’s inequality. Clearly, the inverse image of an interval under any polynomial map  $h : \mathbb{R} \longrightarrow \mathbb{R}$  is a finite union of intervals (with infinite endpoints allowed) and points. Markov’s inequality for sets consisting of finitely many intervals was deeply investigated by Totik in [57].

The situation is quite different if we consider the complex case ( $\mathbb{K} = \mathbb{C}$ ). But also in this case the claim that the polynomial preimage of a Markov set is a Markov set is still far from being valid.

*Example 3.3* Consider the polynomial map

$$g : \mathbb{C}^2 \ni (w_1, w_2) \longmapsto (w_1, w_1 w_2) \in \mathbb{C}^2$$

and the set  $E := \Gamma \cup \{(\alpha, \beta)\}$ , where  $\Gamma := \{z \in \mathbb{C}^2 : |z_1| = |z_2| = 1\}$ ,  $\alpha, \beta \in \mathbb{C}$  and  $0 < |\alpha| < |\beta| \leq 1$ . Recall that:

- The Shilov boundary of the open polydisc  $\mathbb{D}_N := \{z \in \mathbb{C}^N : |z| < 1\}$ <sup>7</sup> is its skeleton, that is the set  $\{z \in \mathbb{C}^N : |z_1| = \dots = |z_N| = 1\}$  (see [53, p. 22]).
- The closed polydisk  $\overline{\mathbb{D}}_N$  satisfies Markov’s inequality. Indeed, for  $N = 1$ , this the content of Bernstein’s theorem: for each complex polynomial  $Q$  of one variable,

$$\|Q'\|_{\overline{\mathbb{D}}} \leq (\deg Q) \|Q\|_{\overline{\mathbb{D}}},$$

where  $\mathbb{D} := \mathbb{D}_1$  (see [10, p. 233]). For  $N > 1$ , it is enough to use the fact that the Cartesian product of Markov sets is a Markov set (see Sect.1).

It follows that the set  $E$  satisfies Markov’s inequality. However, the set  $g^{-1}(E) = \Gamma \cup \{(\alpha, \beta/\alpha)\}$  does not satisfy Markov’s inequality. Indeed, suppose otherwise and take  $\varepsilon, C$  of Definition 1.2. Then, for polynomials  $\Psi_n \in \mathbb{C}[w_1, w_2]$  ( $n \in \mathbb{N}$ ) defined by  $\Psi_n(w_1, w_2) := (\beta - \alpha w_2)w_1^n$ , we have

<sup>7</sup> That is, a closed set  $S \subset \partial\mathbb{D}_N$  such that:  
 (i) for each continuous function  $f : \overline{\mathbb{D}}_N \longrightarrow \mathbb{C}$ , holomorphic in  $\mathbb{D}_N$ ,

$$\|f\|_{\overline{\mathbb{D}}_N} = \|f\|_S,$$

(ii) any closed set  $\tilde{S} \subset \partial\mathbb{D}_N$  satisfying (i) contains  $S$ .

$$\begin{aligned}
 |\alpha| (|\beta|/|\alpha|)^n &= \left| \frac{\partial \Psi_n}{\partial w_2} (\alpha, \beta/\alpha) \right| \leq \left\| \frac{\partial \Psi_n}{\partial w_2} \right\|_{g^{-1}(E)} \leq C(n+1)^\varepsilon \|\Psi_n\|_{g^{-1}(E)} \\
 &= C(n+1)^\varepsilon \|\Psi_n\|_\Gamma \leq C(n+1)^\varepsilon (|\alpha| + |\beta|),
 \end{aligned}$$

which is impossible. □

The situation described in the above example is particular, because the set  $g^{-1}(E)$  is not  $L$ -regular.<sup>8</sup> This is no longer the case in the next example (Example 3.6).

It will be convenient to state beforehand, for easy reference, two results. The first one gives a sufficient and necessary condition for a bounded set  $A \subset \mathbb{R}^2$  definable in some polynomially bounded o-minimal structure to be UPC (cf. [44], Theorem B).

**Theorem 3.4** *Let  $A \subset \mathbb{R}^2$  be bounded and definable in some polynomially bounded o-minimal structure (for example, semialgebraic). Then the following two statements are equivalent:*

- $A$  is UPC.
- $A$  is fat and, for each  $a \in \overline{A}$ ,  $\rho > 0$  and any connected component  $S$  of the set  $\text{Int}A \cap B(a, \rho)$  such that  $a \in \overline{S}$ , there is a polynomial arc  $\gamma : (0, 1) \rightarrow S$  such that  $\lim_{t \rightarrow 0} \gamma(t) = a$ , where  $B(a, \rho) = \{x \in \mathbb{R}^2 : |x - a| < \rho\}$ .

The second result is a special case of the (semi)analytic accessibility criterion due to Pleśniak (cf. [47]).<sup>9</sup>

**Theorem 3.5** *Let  $K \subset \mathbb{K}^N$  be a compact set. Suppose that there exists a polynomial mapping  $\gamma : \mathbb{K} \rightarrow \mathbb{K}^N$  such that  $\gamma((0, 1]) \subset \text{Int}K$ . Then  $K$  is  $L$ -regular at  $\gamma(0)$ , i.e.,  $\Phi_K$  is continuous at  $\gamma(0)$ .*

*Example 3.6* Suppose that a continuous function  $f : [0, R] \rightarrow [0, +\infty)$ , where  $R > 0$ , has the following properties:

- $f > 0$  in  $(0, R]$ ,
- $\lim_{t \rightarrow 0} \frac{f(t)}{t^r} = 0$  for each  $r > 0$ ,
- there exists  $R_0 \in (0, R]$  such that  $f$  is nondecreasing in  $[0, R_0]$ ,
- there exists  $R_1 \in [0, R_0)$  such that  $f|_{[R_1, R]}$  is definable in a certain polynomially bounded o-minimal structure (for simplicity,  $f|_{[R_1, R]}$  can be thought of as a semialgebraic map).

Consider the map

$$F : \mathbb{K}^2 \ni (w_1, w_2) \mapsto (w_1, w_2^2) \in \mathbb{K}^2$$

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<sup>8</sup> For the polynomial  $Q(w_1, w_2) := w_2$ , we have  $\|Q\|_\Gamma = 1$  and  $|Q(\alpha, \beta/\alpha)| > 1$ . Consequently,  $\Phi_\Gamma(\alpha, \beta/\alpha) > 1$  and combining this with Corollary 5.2.5 in [34] we obtain

$$\Phi_{g^{-1}(E)}^*(\alpha, \beta/\alpha) = \Phi_\Gamma^*(\alpha, \beta/\alpha) \geq \Phi_\Gamma(\alpha, \beta/\alpha) > 1 = \Phi_{g^{-1}(E)}(\alpha, \beta/\alpha)$$

(recall that  $\phi^*$  denotes the upper semicontinuous regularization of  $\phi$ ). Since  $\Phi_{g^{-1}(E)}^*(\alpha, \beta/\alpha) > \Phi_{g^{-1}(E)}(\alpha, \beta/\alpha)$ , it follows that  $\Phi_{g^{-1}(E)}$  is not continuous at the point  $(\alpha, \beta/\alpha)$ .

<sup>9</sup> An alternative proof can also be found in [45, Corollary 2.8]

and the set

$$E := \{(x_1, x_2) \in [0, R] \times \mathbb{R} : -1 \leq x_2 \leq f(x_1)\} \subset \mathbb{K}^2.$$

Note that  $F^{-1}(E)$  is compact. We will show that:

- $E$  satisfies Markov’s inequality,
- $F^{-1}(E)$  is  $L$ -regular,
- $F^{-1}(E)$  does not satisfy Markov’s inequality for  $\mathbb{K} = \mathbb{R}$  but does satisfy Markov’s inequality for  $\mathbb{K} = \mathbb{C}$ .

To this end, put

$$E_1 := \{(x_1, x_2) \in [0, R_1] \times \mathbb{R} : 0 \leq x_2 \leq f(x_1)\},$$

$$E_2 := \{(x_1, x_2) \in [R_1, R] \times \mathbb{R} : 0 \leq x_2 \leq f(x_1)\} \cup ([0, R] \times [-1, 0]).$$

For each  $x = (x_1, x_2) \in E_1$ , denote by  $H_x$  the rectangle with the vertices at  $(x_1, x_2)$ ,  $(R_0, x_2)$ ,  $(x_1, -1)$ ,  $(R_0, -1)$ . Moreover, define  $S_x : \mathbb{R} \rightarrow \mathbb{R}^2$  by

$$S_x(t) := x + t \left( \frac{R_0 - x_1}{2}, \frac{-x_2 - 1}{2} \right).$$

Note that  $[0, 1] \ni t \mapsto S_x(t) \in \mathbb{R}^2$  is a parametrization of the line segment linking the point  $x$  and the midpoint of the diagonals of  $H_x$ . Since  $H_x \subset E$ , it follows that, for each  $t \in [0, 1]$ ,

$$\begin{aligned} \text{dist}(S_x(t), \mathbb{R}^2 \setminus E) &\geq \text{dist}(S_x(t), \mathbb{R}^2 \setminus H_x) = \min \left\{ \frac{(R_0 - x_1)t}{2}, \frac{(x_2 + 1)t}{2} \right\} \\ &\geq \frac{\min\{R_0 - R_1, 1\}}{2} t. \end{aligned}$$

On the other hand, by Theorem 3.4, the set  $E_2$  is UPC.<sup>10</sup> Therefore, there exist  $\nu, \theta > 0$  and  $d \in \mathbb{N}$  such that, for each  $x \in E_2$ , we can choose a polynomial map  $\tilde{S}_x : \mathbb{R} \rightarrow \mathbb{R}^2$  with  $\text{deg } \tilde{S}_x \leq d$  satisfying the following conditions:

- $\tilde{S}_x(0) = x$ ,
- $\text{dist}(\tilde{S}_x(t), \mathbb{R}^2 \setminus E_2) \geq \theta t^\nu$  for each  $t \in [0, 1]$ .

Note that, for each  $x \in E_2$  and each  $t \in [0, 1]$ ,

$$\text{dist}(\tilde{S}_x(t), \mathbb{R}^2 \setminus E) \geq \text{dist}(\tilde{S}_x(t), \mathbb{R}^2 \setminus E_2) \geq \theta t^\nu.$$

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<sup>10</sup> The assumption that  $f|_{[R_1, R]}$  is definable in a certain polynomially bounded o-minimal structure is used here to guarantee definability of  $E_2$  in a polynomially bounded o-minimal structure and to guarantee the existence of a polynomial arc  $\gamma : (0, 1) \rightarrow \text{Int}E$  such that  $\lim_{t \rightarrow 0} \gamma(t) = (R, f(R))$ . An explicit example of such an arc is  $\gamma : (0, 1) \ni t \mapsto (R - (\eta t)^m, f(R) - \eta t) \in \mathbb{R}^2$ , where  $\eta > 0$  is sufficiently small and  $m \in \mathbb{N}$  is sufficiently large, which follows from the definition of a polynomially bounded o-minimal structure.

Upon combining the above estimates for the sets  $E_1$  and  $E_2$ , it is straightforward to show that  $E = E_1 \cup E_2$  is UPC and hence, by Theorem 3.1 in [43], is a Markov set.

CASE 1:  $\mathbb{K} = \mathbb{R}$ . Note first that

$$F^{-1}(E) = \left\{ (w_1, w_2) \in [0, R] \times \mathbb{R} : |w_2| \leq \sqrt{f(w_1)} \right\}.$$

By Theorem 3.5, the set  $F^{-1}(E)$  is  $L$ -regular.<sup>11</sup> Suppose, to derive a contradiction, that  $F^{-1}(E)$  is a Markov set. In particular, there exist  $\varepsilon, C > 0$  such that, for each polynomial  $P \in \mathbb{C}[w_1, w_2]$ ,

$$\left\| \frac{\partial P}{\partial w_2} \right\|_{F^{-1}(E)} \leq C(\deg P)^\varepsilon \|P\|_{F^{-1}(E)}. \tag{12}$$

For each  $n \in \mathbb{N}$ , put

$$P_n(w_1, w_2) := w_2 \left( 1 - \frac{w_1}{R} \right)^n.$$

Moreover, take  $r > \varepsilon$  and set

$$C_r := \sup_{t \in (0, R]} \frac{\sqrt{f(t)}}{t^r} < +\infty.$$

Note that

$$\begin{aligned} \|P_n\|_{F^{-1}(E)} &= \max_{t \in [0, R]} \sqrt{f(t)} \left( 1 - \frac{t}{R} \right)^n \leq C_r \max_{t \in [0, R]} t^r \left( 1 - \frac{t}{R} \right)^n \\ &= C_r \left( \frac{rR}{r+n} \right)^r \left( \frac{n}{r+n} \right)^n. \end{aligned}$$

Combining this with (12), we get

$$1 = \left\| \frac{\partial P_n}{\partial w_2} \right\|_{F^{-1}(E)} \leq C(n+1)^\varepsilon C_r \left( \frac{rR}{r+n} \right)^r \left( \frac{n}{r+n} \right)^n,$$

which is impossible, because the right-hand side tends to zero as  $n \rightarrow \infty$ .

CASE 2:  $\mathbb{K} = \mathbb{C}$ . Note that, for each  $w \in \mathbb{C}^2$ ,

$$|w| \geq 1 \implies \frac{|F(w)|}{|w|} \geq 1.$$

<sup>11</sup> The only problem here is to see that there exists a polynomial arc  $\varphi : (0, 1) \rightarrow \text{Int}F^{-1}(E)$  such that  $\lim_{t \rightarrow 0} \varphi(t) = (R, \sqrt{f(R)})$ . However, this immediately follows from the assumption that  $f|_{[R_1, R]}$  is definable in a certain polynomially bounded o-minimal structure.

Therefore

$$\liminf_{|w| \rightarrow +\infty} \frac{|F(w)|}{|w|} > 0.$$

By Theorem 5.3.1 in [34], for each  $w \in \mathbb{C}^2$ ,

$$\Phi_{F^{-1}(E)}(w) \leq \Phi_E(F(w)). \tag{13}$$

Since  $E$  is UPC, it has the HCP property: there exist  $M_1, \mu > 0$  such that, for each  $z \in E_{(1)}$ ,

$$\Phi_E(z) \leq 1 + M_1 (\text{dist}(z, E))^\mu. \tag{14}$$

Put

$$M_2 := \max \left\{ M_1, \sup_{F(K)} \Phi_E - 1 \right\} < +\infty,$$

where  $K := F^{-1}(E)_{(1)}$ . By (14), for each  $z \in F(K)$ ,

$$\Phi_E(z) \leq 1 + M_2 (\text{dist}(z, E))^\mu. \tag{15}$$

Take  $M_3 > 0$  such that  $F|_K$  is Lipschitz with the constant  $M_3$ , that is

$$|F(w) - F(w')| \leq M_3 |w - w'| \tag{16}$$

for all  $w, w' \in K$ . For each  $w \in K$ , we have

$$\begin{aligned} \Phi_{F^{-1}(E)}(w) &\stackrel{\text{by (13)}}{\leq} \Phi_E(F(w)) \stackrel{\text{by (15)}}{\leq} 1 + M_2 (\text{dist}(F(w), E))^\mu \\ &\stackrel{\text{by (16)}}{\leq} 1 + M_2 M_3^\mu \left( \text{dist}(w, F^{-1}(E)) \right)^\mu, \end{aligned}$$

which yields the HCP property for the set  $F^{-1}(E)$ . Consequently,  $F^{-1}(E)$  is a Markov set and is  $L$ -regular. □

The previous examples may suggest that a compact,  $L$ -regular set, which is the inverse image of a Markov set under a complex (i.e., holomorphic) polynomial map, is also a Markov set. This claim is however not valid.

*Example 3.7* Let  $f : [0, R] \rightarrow [0, +\infty)$  be as in Example 3.6. Set  $D := D_1 \times D_2$ , where

$$\begin{aligned} D_1 &:= \{(x_1, x_2) \in [0, R] \times \mathbb{R} : -\max_{[0, R]} f \leq x_2 \leq f(x_1)\} \subset \mathbb{C}^2, \\ D_2 &:= \{(x_1, x_2) \in [0, R] \times \mathbb{R} : -f(x_1) \leq x_2 \leq \max_{[0, R]} f\} \subset \mathbb{C}^2. \end{aligned}$$

In the same way as we handled the set  $E$  of Example 3.6, we can show that  $D_1, D_2$  satisfy Markov's inequality. Moreover, put

$$G : \mathbb{C}^2 \ni w \mapsto (w, w) \in \mathbb{C}^4.$$

Note that

$$G^{-1}(D) = \{(w_1, w_2) \in [0, R] \times \mathbb{R} : |w_2| \leq f(w_1)\} \subset \mathbb{C}^2.$$

After repeating the argument from Case 1 of Example 3.6, we conclude that  $G^{-1}(D)$  is  $L$ -regular and does not satisfy Markov's inequality. On the other hand, the set  $D$ , as the Cartesian product of the Markov sets, is a Markov set.  $\square$

After giving the above examples illustrating various situations which occur naturally when we consider Markov's inequality in the context of polynomial preimages, we conclude this section with the statement of the following result, to be proved in the next section.

**Theorem 3.8** *Assume that  $g : U \rightarrow \mathbb{C}^{N'}$  is a holomorphic mapping, where  $U \subset \mathbb{C}^N$  is open ( $N, N' \in \mathbb{N}$ ). Suppose that a compact set  $\emptyset \neq E \subset \mathbb{C}^{N'}$  has the HCP property,  $\hat{E} \subset g(U)$  and  $g^{-1}(\hat{E})$  is compact. Then*

- $N = N'$ ,
- $g^{-1}(E)$  has the HCP property and, in particular, is a Markov set.

Recall that  $\hat{E}$  denotes the polynomially convex hull of  $E$ :

$$\hat{E} := \{z \in \mathbb{C}^{N'} : |Q(z)| \leq \|Q\|_E \text{ for each } Q \in \mathbb{C}[z_1, \dots, z_{N'}]\}.$$

If  $\hat{E} = E$ , then we say that  $E$  is polynomially convex. For example, each compact subset of  $\mathbb{R}^N$  is polynomially convex in  $\mathbb{C}^N$  (cf. Lemma 5.4.1 in [34]).

## 4 A proof of Theorem 3.8

For the convenience of the reader we recall first the relevant notions and results from [41].

**Definition 4.1** For a set  $A \subset \mathbb{C}^N$ , we define its complex dimension by the formula

$$\dim A := \max \left\{ \dim \Gamma : \Gamma \subset A, \Gamma \text{ is a submanifold of } \mathbb{C}^N \right\}.$$

(We assume here that the maximum on the empty set is equal to  $-\infty$ .)

**Definition 4.2** Let  $\Omega \subset \mathbb{C}^N$  be an open set. A set  $A \subset \Omega$  is called an analytic subset (of  $\Omega$  or in  $\Omega$ ) if, for each point  $a \in \Omega$ , there is an open neighbourhood  $U$  of  $a$  and there exist holomorphic functions  $\xi_1, \dots, \xi_k : U \rightarrow \mathbb{C}$  such that

$$A \cap U = \{z \in U : \xi_1(z) = \dots = \xi_k(z) = 0\}.$$



**Definition 4.3** A set  $A \subset \mathbb{C}^N$  is called a locally analytic set (in  $\mathbb{C}^N$ ) if, for each point  $a \in A$ , there is an open neighbourhood  $U$  of  $a$  and there exist holomorphic functions  $\xi_1, \dots, \xi_k : U \rightarrow \mathbb{C}$  such that

$$A \cap U = \{z \in U : \xi_1(z) = \dots = \xi_k(z) = 0\}.$$

The subsequent proofs make use of the following two results.

**Theorem 4.4** Assume that  $f : W \rightarrow \mathbb{C}^{N'}$  is a holomorphic mapping, where  $W \subset \mathbb{C}^N$  is open ( $N, N' \in \mathbb{N}$ ). Suppose that  $B \subset W$  is a locally analytic set such that, for some  $m \in \mathbb{N}$ ,

$$\text{rank } d_z f \leq m \text{ for } z \in B.$$

Then  $f(B)$  is a countable union of submanifolds of dimension  $\leq m$ .

*Proof* See [41, p. 254]. □

**Theorem 4.5** Every compact analytic subset of  $\mathbb{C}^N$  is finite.

*Proof* See [41, p. 235]. □

Before proceeding with the proof of Theorem 3.8, let us also state the following lemma.

**Lemma 4.6** Assume that  $f : W \rightarrow \mathbb{C}^{N'}$  is a holomorphic mapping, where  $W \subset \mathbb{C}^N$  is open ( $N, N' \in \mathbb{N}$ ). Suppose that a set  $A \subset f(W)$  is nonpluripolar. Then  $f^{-1}(A)$  is nonpluripolar as well.

*Proof* We will consider two cases.

CASE 1:  $N < N'$ . Obviously,  $\text{rank } d_w f \leq N$  for each  $w \in W$ . By Theorem 4.4,  $f(W)$  is a countable union of submanifolds of dimension  $\leq N$ . In particular, the set  $f(W)$  (and hence  $A$ ) is pluripolar, which is a contradiction. The case  $N < N'$  cannot therefore occur.

CASE 2:  $N \geq N'$ . We have  $W = B \cup W_0$ , where

$$B := \{w \in W : \text{rank } d_w f \leq N' - 1\},$$

$$W_0 := \{w \in W : \text{rank } d_w f = N'\}.$$

Clearly, the set  $B$  is an analytic subset of  $W$ . As in Case 1, we show via Theorem 4.4 that  $f(B)$  is pluripolar. In particular, the set  $A \cap f(W_0)$  is nonpluripolar.

By the rank theorem, for each  $a \in W_0$ , there exists an open set  $U_a$  such that  $a \in U_a \subset W_0$ ,  $f(U_a)$  is open, and there exist biholomorphic mappings  $\varphi_a : U_a \rightarrow \Delta_a \times \Omega_a$ ,  $\psi_a : f(U_a) \rightarrow \Delta_a$ , where  $\Delta_a \subset \mathbb{C}^{N'}$ ,  $\Omega_a \subset \mathbb{C}^{N-N'}$  are open sets, such that the mapping

$$\psi_a \circ f \circ \varphi_a^{-1} : \Delta_a \times \Omega_a \rightarrow \Delta_a$$

is the natural projection. Clearly, there is a sequence  $a_j \in W_0$  ( $j \in \mathbb{N}$ ) such that

$$W_0 = U_{a_1} \cup U_{a_2} \cup \dots$$

Take  $l \in \mathbb{N}$  such that  $A \cap f(U_{a_l})$  is nonpluripolar. Then the set  $\psi_{a_l}(A \cap f(U_{a_l}))$  is also nonpluripolar.

Suppose, to derive a contradiction, that  $f^{-1}(A)$  is pluripolar. Then  $\varphi_{a_l}(f^{-1}(A) \cap U_{a_l})$  is also pluripolar and therefore

$$\varphi_{a_l}(f^{-1}(A) \cap U_{a_l}) \subset \{\zeta \in \mathbb{C}^N : u(\zeta) = -\infty\}$$

for some plurisubharmonic function  $u$  in  $\mathbb{C}^N$ . Note that, for each  $y \in \Omega_{a_l}$ ,

$$\psi_{a_l}(A \cap f(U_{a_l})) \subset \{x \in \mathbb{C}^{N'} : u(x, y) = -\infty\}.$$

Since  $\psi_{a_l}(A \cap f(U_{a_l}))$  is nonpluripolar, it follows that  $u \equiv -\infty$  in  $\mathbb{C}^{N'} \times \Omega_{a_l}$ , which is impossible.<sup>12</sup> □

In the proof of Theorem 3.8, we will use the notion of the relative extremal function. Suppose that  $\Omega \subset \mathbb{C}^N$  is an open set and  $A \subset \Omega$ . The relative extremal function for  $A$  in  $\Omega$  is defined as follows:

$$u_{A,\Omega}(z) := \sup \{\varphi(z) : \varphi \in \text{PSH}(\Omega), \varphi \leq 0, \varphi|_A \leq -1\}$$

( $z \in \Omega$ ), where  $\text{PSH}(\Omega)$  denotes the plurisubharmonic functions in  $\Omega$ .

*Proof of Theorem 3.8.* We will consider three cases.

CASE 1:  $N > N'$ . Take  $b = (b_1, \dots, b_{N'}) \in \hat{E} \subset g(U)$ . By the formula on p. 169 of [41],

$$\dim g^{-1}(b) = \dim \{w \in U : g_1(w) - b_1 = 0, \dots, g_{N'}(w) - b_{N'} = 0\} \geq N - N' > 0. \tag{17}$$

On the other hand,  $g^{-1}(b) = g^{-1}(b) \cap g^{-1}(\hat{E})$  is compact, analytic and hence finite (cf. Theorem 4.5), in contradiction with (17). This means that the case  $N > N'$  cannot occur.

CASE 2:  $N < N'$ . It follows from Theorem 4.4 that  $g(U)$  is a countable union of submanifolds of dimension  $\leq N$ . In particular,  $g(U)$  (and hence  $E$ ) is pluripolar, which is a contradiction. The case  $N < N'$  cannot therefore occur.

CASE 3:  $N = N'$ . Put  $K := g^{-1}(\hat{E})$  and take  $\lambda > 0$  such that  $K_{(\lambda)} \subset U$ .<sup>13</sup> Note that there exists  $\epsilon > 0$  such that

$$g^{-1}(\hat{E}_\epsilon) \cap \left\{ w \in \mathbb{C}^N : \text{dist}(w, K) = \lambda \right\} = \emptyset. \tag{18}$$

<sup>12</sup> Recall that pluripolar sets have Lebesgue measure zero.

<sup>13</sup> Recall that  $K_{(\lambda)} := \{w \in \mathbb{C}^N : \text{dist}(w, K) \leq \lambda\}$  and  $K_\lambda := \{w \in \mathbb{C}^N : \text{dist}(w, K) < \lambda\}$ .

Suppose, towards a contradiction, that this is not the case and take a sequence  $a_j \in U$  ( $j \in \mathbb{N}$ ) such that

$$\text{dist}(a_j, K) = \lambda \quad \text{and} \quad \text{dist}(g(a_j), \hat{E}) \rightarrow 0.$$

Passing to a subsequence if necessary, we can assume that  $a_j \rightarrow a \in K_{(\lambda)} \setminus K_\lambda$ . Consequently,  $\text{dist}(a, K) = \lambda$  and  $\text{dist}(g(a), \hat{E}) = 0$ , which means that  $a \notin K$  and  $a \in K$ , a contradiction.

Put  $\Omega := K_\lambda \cap g^{-1}(\hat{E}_\epsilon)$ , where  $\epsilon > 0$  is of (18). For each compact set  $T \subset \hat{E}_\epsilon$ , we have

$$g^{-1}(T) \cap \Omega = g^{-1}(T) \cap K_\lambda \stackrel{\text{by (18)}}{=} g^{-1}(T) \cap K_{(\lambda)},$$

and therefore the set  $g^{-1}(T) \cap \Omega$  is compact. It follows that

$$\Omega \ni w \longmapsto g(w) \in \hat{E}_\epsilon$$

is a proper holomorphic map. Since the set  $\Omega' := g(\Omega)$  is open (cf. [52], Theorem 15.1.6), it follows that

$$\Omega \ni w \longmapsto g(w) \in \Omega'$$

is also a proper holomorphic map. Note moreover that  $g^{-1}(E) \subset K \subset \Omega$ ,  $\hat{E} = g(K) \subset \Omega'$  and  $\Omega, \Omega'$  are bounded. (Use the fact that  $\Omega' \subset \hat{E}_\epsilon$ .)

By Lemma 4.6, the set  $g^{-1}(E)$  is nonpluripolar and hence  $V_{g^{-1}(E)}^* \in \mathcal{L}(\mathbb{C}^N)$  (see (11)). In particular,

$$V_{g^{-1}(E)}^* \leq M_1 \quad \text{in} \quad \Omega \cup g^{-1}(E)_{(1)}$$

for some  $M_1 > 0$ . It is clear that

$$V_{g^{-1}(E)} \leq M_1(u_{g^{-1}(E), \Omega} + 1) \quad \text{in} \quad \Omega. \tag{19}$$

Moreover, by Proposition 5.3.3 in [34], there exists  $M_2 > 0$  such that

$$u_{E, \Omega'} + 1 \leq M_2 V_E \quad \text{in} \quad \Omega'. \tag{20}$$

Let  $M_3, \mu > 0$  be such that, for each  $z \in E_{(1)}$ ,

$$\Phi_E(z) \leq 1 + M_3 (\text{dist}(z, E))^\mu$$

(see Definition 3.1). Obviously, for each  $z \in \Omega'$ ,

$$V_E(z) \leq M_4 (\text{dist}(z, E))^\mu, \tag{21}$$

where  $M_4 := \max \{M_3, \sup_{\Omega' \setminus E(1)} V_E\}$ . (If  $\Omega' \subset E(1)$ , then we set  $M_4 := M_3$ .)  
 By Proposition 4.5.14 in [34],

$$u_{E, \Omega'} \circ g = u_{g^{-1}(E), \Omega} \quad \text{in } \Omega. \tag{22}$$

Consequently, for each  $w \in \Omega$ ,

$$\begin{aligned} V_{g^{-1}(E)}(w) &\stackrel{\text{by (19)}}{\leq} M_1(u_{g^{-1}(E), \Omega}(w) + 1) \stackrel{\text{by (22)}}{=} M_1(u_{E, \Omega'}(g(w)) + 1) \\ &\stackrel{\text{by (20)}}{\leq} M_1 M_2 V_E(g(w)) \stackrel{\text{by (21)}}{\leq} M_1 M_2 M_4 (\text{dist}(g(w), E))^\mu \end{aligned}$$

and therefore

$$V_{g^{-1}(E)}(w) \leq M_1 M_2 M_4 M_5^\mu \left(\text{dist}(w, g^{-1}(E))\right)^\mu, \tag{23}$$

where  $M_5 > 0$  is such that  $|g(\zeta) - g(\zeta')| \leq M_5 |\zeta - \zeta'|$  for all  $\zeta, \zeta' \in \Omega$ . Put

$$M_6 := \max \left\{ M_1 M_2 M_4 M_5^\mu, M_1 M_7^{-1} \right\},$$

where

$$M_7 := \inf_{\mathbb{C}^N \setminus \Omega} \left(\text{dist}(\cdot, g^{-1}(E))\right)^\mu > 0.$$

The estimate (23) implies that, for each  $w \in g^{-1}(E)_{(1)}$ ,

$$V_{g^{-1}(E)}(w) \leq M_6 \left(\text{dist}(w, g^{-1}(E))\right)^\mu.$$

Hence

$$\Phi_{g^{-1}(E)}(w) \leq 1 + M \left(\text{dist}(w, g^{-1}(E))\right)^\mu,$$

where  $M > 0$  is such that  $e^{M_6 t} \leq 1 + M t$  for  $t \in [0, 1]$ . The case  $N = N'$  is therefore settled, and the proof of the theorem is complete.  $\square$

With regard to Theorem 3.8, we have the following remark.

*Remark 4.7* In Theorem 3.8, even if  $U = \mathbb{C}^N$  and  $g : \mathbb{C}^N \rightarrow \mathbb{C}^N$  is a polynomial map, the assumption that  $\hat{E} \subset g(U)$  and  $g^{-1}(\hat{E})$  is compact cannot be replaced by the assumption that  $E \subset g(U)$  and  $g^{-1}(E)$  is compact and  $L$ -regular.

*Proof* Set

$$g : \mathbb{C}^2 \ni (w_1, w_2) \mapsto (w_1, w_1 w_2) \in \mathbb{C}^2.$$

Take a compact set  $K \subset \mathbb{R}^2 \subset \mathbb{C}^2$  such that:

- $K \subset (0, \frac{1}{2}] \times (1, 2]$ ,
- $K$  is  $L$ -regular,
- $K$  does not satisfy Markov’s inequality.

(For instance, we can take a suitable translate of the set  $F^{-1}(E)$  from Case 1 of Example 3.6 with appropriately chosen function  $f$ .)

Put moreover  $E := \Gamma \cup g(K)$ , where  $\Gamma := \{z \in \mathbb{C}^2 : |z_1| = |z_2| = 1\}$ . Clearly,  $E \subset \overline{\mathbb{D}}_2$ ,  $E \subset g(\mathbb{C}^2) = ((\mathbb{C} \setminus \{0\}) \times \mathbb{C}) \cup \{(0, 0)\}$  and  $g^{-1}(E) = \Gamma \cup K$ .<sup>14</sup> It is well known that, for each  $u \in \mathbb{C}$ ,

$$\Phi_{\overline{\mathbb{D}}}(u) = \Phi_{\partial\mathbb{D}}(u) = \max\{1, |u|\},$$

where  $\mathbb{D} := \mathbb{D}_1$ . Combining this with Proposition 5.9 in [55] we get, for each  $z = (z_1, z_2) \in \mathbb{C}^2$ , the following estimates

$$\begin{aligned} \max\{1, |z|\} &= \max\{\Phi_{\overline{\mathbb{D}}}(z_1), \Phi_{\overline{\mathbb{D}}}(z_2)\} = \Phi_{\overline{\mathbb{D}}_2}(z) \leq \Phi_E(z) \leq \Phi_{\Gamma}(z) \\ &= \max\{\Phi_{\partial\mathbb{D}}(z_1), \Phi_{\partial\mathbb{D}}(z_2)\} = \max\{1, |z|\}. \end{aligned}$$

Therefore

$$\Phi_E(z) = \Phi_{\Gamma}(z) = \Phi_{\overline{\mathbb{D}}_2}(z) = \max\{1, |z|\}.$$

In particular,

- $E$  has the HCP property:  $\Phi_E(z) \leq 1 + \text{dist}(z, E)$  for each  $z \in \mathbb{C}^2$ ,
- $g^{-1}(E) = \Gamma \cup K$  is  $L$ -regular, because  $\Gamma$  and  $K$  are  $L$ -regular.

Suppose, towards a contradiction, that  $g^{-1}(E)$  is a Markov set. Since  $\Gamma$  is the Shilov boundary of  $\mathbb{D}_2$  (see Example 3.3), it follows that  $\overline{\mathbb{D}}_2 \cup K$  is a Markov set as well. Put  $\Pi : \mathbb{C}^2 \ni (w_1, w_2) \mapsto w_2 \in \mathbb{C}$ . Note that  $\Pi(\overline{\mathbb{D}}_2) = \overline{\mathbb{D}} \subset \mathbb{C}$  and  $\Pi(K) \subset (1, 2] \subset \mathbb{R} \subset \mathbb{C}$ . Consequently, the sets  $\Pi(\overline{\mathbb{D}}_2)$  and  $\Pi(K)$  are disjoint and polynomially convex. Clearly,  $\overline{\mathbb{D}}_2$  and  $K$  are also polynomially convex. Therefore, by Kallin’s separation lemma (cf. [33, p. 302]), we obtain the polynomial convexity of the set  $\overline{\mathbb{D}}_2 \cup K$ . On account of Corollary 5.2, we get a contradiction, because  $K$  is not a Markov set. □

### 5 Subsets of Markov sets

In this section, we will prove the following result announced in Introduction.

**Theorem 5.1** *Let  $E \subset \mathbb{C}^N$  be a compact and polynomially convex set satisfying Markov’s inequality. Assume that  $K \subset E$  is compact, nonpluripolar and open in  $E$ . Then  $K$  is a Markov set. Furthermore, if  $E$  satisfies Markov’s inequality with an exponent  $\varepsilon > 0$  (see Definition 1.2), then  $K$  satisfies Markov’s inequality with the exponent  $\varepsilon$  as well.*

---

<sup>14</sup> Recall that  $\mathbb{D}_N := \{z \in \mathbb{C}^N : |z| < 1\}$ .

*Proof* Choose  $\lambda > 0$  such that  $K_{(\lambda)} \cap (E \setminus K) = \emptyset$ . In particular, we have

$$E \subset K \cup \left( \mathbb{C}^N \setminus K_{(\lambda)} \right) \subset K_\lambda \cup \left( \mathbb{C}^N \setminus K_{(\lambda)} \right).$$

Take moreover a compact and polynomially convex set  $\mathcal{Z} \subset \mathbb{C}^N$  such that  $E \subset \text{Int} \mathcal{Z}$  and  $\mathcal{Z} \subset K_\lambda \cup (\mathbb{C}^N \setminus K_{(\lambda)})$ —see the proof of Lemma 2.7.4 in [32]. Define  $g : K_\lambda \cup (\mathbb{C}^N \setminus K_{(\lambda)}) \rightarrow \mathbb{C}$  by the formula

$$g(z) := \begin{cases} 1 & \text{if } z \in K_\lambda, \\ 0 & \text{if } z \in \mathbb{C}^N \setminus K_{(\lambda)}. \end{cases}$$

By Theorem 8.5(1) in [55], there exist  $M > 0, \rho \in (0, 1)$  with the following property: for each  $\mu \in \mathbb{N}$ , we can choose a polynomial  $R_\mu \in \mathbb{C}[z_1, \dots, z_N]$  with  $\deg R_\mu \leq \mu$  and such that

$$\|g - R_\mu\|_{\mathcal{Z}} \leq M\rho^\mu. \quad (24)$$

We can clearly assume that  $M \geq 1$ .

By (11),  $V_K$  (and hence  $\Phi_K$ ) is bounded on each compact subset of  $\mathbb{C}^N$ . Thus we may choose  $k \in \mathbb{N}$  such that

$$1 - M\rho^k > 0 \quad \text{and} \quad M\rho^k \sup_E \Phi_K \leq 1. \quad (25)$$

Moreover, let  $\varepsilon, C > 0$  be of Definition 1.2 for the set  $E$ . Therefore, for each polynomial  $P \in \mathbb{C}[z_1, \dots, z_N]$  and each  $\alpha \in \mathbb{N}_0^N$ ,

$$\|D^\alpha P\|_E \leq (C(\deg P)^\varepsilon)^{|\alpha|} \|P\|_E. \quad (26)$$

Take also  $C_1 > 0$  such that

$$C_1 \leq C^{-1}, \quad C_1(k+1)^{-\varepsilon} < \lambda, \quad K_{(C_1(k+1)^{-\varepsilon})} \subset \mathcal{Z}$$

and put

$$C_2 := (k+1)^\varepsilon \frac{e^N(1 + M\rho^k)}{C_1(1 - M\rho^k)}.$$

We will show that, for each polynomial  $Q \in \mathbb{C}[z_1, \dots, z_N]$  with  $\deg Q \leq n$  ( $n \in \mathbb{N}$ ) and each  $a \in K_{(C_1(k+1)^{-\varepsilon}n^{-\varepsilon})}$ , we have

$$|Q(a)| \leq e^N \frac{1 + M\rho^k}{1 - M\rho^k} \|Q\|_K. \quad (27)$$

To this end, fix  $Q, a$  as above and take  $b \in K$  such that  $|a - b| = \text{dist}(a, K)$ . Clearly,  $a \in \mathcal{Z} \cap K_\lambda$  and

$$|a - b| \leq \frac{C_1}{(k+1)^\varepsilon n^\varepsilon} \leq \frac{1}{C(k+1)^\varepsilon n^\varepsilon}. \quad (28)$$

Put  $P := R_{nk} \cdot Q$ . Then

$$|P(a)| = \left| \sum_{\alpha \in \mathbb{N}_0^N} \frac{D^\alpha P(b)}{\alpha!} (a - b)^\alpha \right| \stackrel{\text{by (26)}}{\leq} \sum_{\alpha \in \mathbb{N}_0^N} \frac{(C(k + 1)^\varepsilon n^\varepsilon)^{|\alpha|}}{\alpha!} |a - b|^{|\alpha|} \|P\|_E$$

$$\stackrel{\text{by (28)}}{\leq} \sum_{\alpha \in \mathbb{N}_0^N} \frac{\|P\|_E}{\alpha!} = e^N \|P\|_E$$

and hence

$$|P(a)| \leq e^N \|P\|_E. \tag{29}$$

We will check now that

$$\|P\|_E \leq (1 + M\rho^{nk}) \|Q\|_K. \tag{30}$$

To this end, fix  $y \in E$ .

CASE 1:  $y \in K$ . Then

$$|P(y)| = |R_{nk}(y) \cdot Q(y)| \stackrel{\text{by (24)}}{\leq} (|g(y)| + M\rho^{nk}) \|Q\|_K = (1 + M\rho^{nk}) \|Q\|_K.$$

CASE 2:  $y \in E \setminus K$ . Then  $y \in \mathcal{Z} \setminus K(\lambda)$  and

$$|P(y)| = |R_{nk}(y) \cdot Q(y)| \stackrel{\text{by (24)}}{\leq} (|g(y)| + M\rho^{nk}) |Q(y)| = M\rho^{nk} |Q(y)|$$

$$\leq M\rho^{nk} \overbrace{(\Phi_K(y))^n \|Q\|_K}^{\text{by the definition of } \Phi_K} \leq \left( M\rho^k \sup_E \Phi_K \right)^n \|Q\|_K \stackrel{\text{by (25)}}{\leq} \|Q\|_K,$$

which completes the proof of (30).

Consequently,

$$(1 - M\rho^{nk}) |Q(a)| = (|g(a)| - M\rho^{nk}) |Q(a)| \stackrel{\text{by (24)}}{\leq} |P(a)| \stackrel{\text{by (29)}}{\leq} e^N \|P\|_E$$

$$\stackrel{\text{by (30)}}{\leq} e^N (1 + M\rho^{nk}) \|Q\|_K$$

and hence

$$|Q(a)| \leq e^N \frac{1 + M\rho^{nk}}{1 - M\rho^{nk}} \|Q\|_K \leq e^N \frac{1 + M\rho^k}{1 - M\rho^k} \|Q\|_K,$$

which completes the proof of (27).

By Cauchy's inequalities, for each  $Q \in \mathbb{C}[z_1, \dots, z_N]$ ,  $\alpha \in \mathbb{N}_0^N$ ,  $z \in \mathbb{C}^N$  and each  $r > 0$ ,

$$\frac{|D^\alpha Q(z)|}{\alpha!} \leq \frac{\|Q\|_{\mathbb{B}(z,r)}}{r^{|\alpha|}},$$

where  $\mathbb{B}(z, r) := \{w \in \mathbb{C}^N : |w - z| \leq r\}$ . Combining this with (27) we get, for each  $Q \in \mathbb{C}[z_1, \dots, z_N]$  and each  $\alpha \in \mathbb{N}_0^N$  with  $|\alpha| = 1$ , the following estimate

$$\|D^\alpha Q\|_K \leq C_2(\deg Q)^\varepsilon \|Q\|_K.$$

Consequently,  $K$  satisfies Markov's inequality with the exponent  $\varepsilon$ . □

**Corollary 5.2** *Assume that  $E_1, \dots, E_p \subset \mathbb{C}^N$  ( $p \in \mathbb{N}$ ) are compact, nonpluripolar and pairwise disjoint sets such that  $E := E_1 \cup \dots \cup E_p$  is polynomially convex. Let  $\varepsilon > 0$ . Then the following two statements are equivalent:*

1.  $E$  satisfies Markov's inequality with the exponent  $\varepsilon$ .
2. For each  $j \leq p$ , the set  $E_j$  satisfies Markov's inequality with the exponent  $\varepsilon$ .

We conclude this section with the following example concerning Corollary 5.2.

*Example 5.3* Set  $E_1 := \{0\} \cup [1/3, 2/3]$ ,  $E_2 := \partial\mathbb{D}$ , where  $\mathbb{D} := \mathbb{D}_1$ , and  $E := E_1 \cup E_2$ . Note that  $E_1$  does not satisfy Markov's inequality. Indeed, suppose otherwise and take  $\varepsilon, C$  of Definition 1.2. Then, for the polynomials  $P_n(z) := z(1-z)^n$  ( $n \in \mathbb{N}$ ), we have

$$1 = |P'_n(0)| \leq \|P'_n\|_{E_1} \leq C(n+1)^\varepsilon \|P_n\|_{E_1} \leq C(n+1)^\varepsilon \left(\frac{2}{3}\right)^{n+1},$$

which is impossible.

By Bernstein's theorem (see Example 3.3) and the maximum principle, for each complex polynomial  $Q$  of one variable,

$$\|Q'\|_E \leq \|Q'\|_{\mathbb{D}} \leq (\deg Q) \|Q\|_{\mathbb{D}} = (\deg Q) \|Q\|_{\partial\mathbb{D}} \leq (\deg Q) \|Q\|_E.$$

Hence,  $E$  satisfies Markov's inequality. Moreover,  $E$  is not polynomially convex. In Corollary 5.2, the assumption that  $E$  is polynomially convex is therefore relevant even if  $N = 1$ . □

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