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ERRATUM

Erratum to: Deformation-obstruction theory for complexes via Atiyah and Kodaira–Spencer classes

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Fabian Langholf and Pierrick Bousseau pointed out that Lemma 2.2 in the Original publication requires X and A to be flat over the base B. For instance the sequence (2.4) in Proof of Lemma 2.2 is not in general exact without this assumption.

After inserting the assumption of flatness of X_0, X and A over B, Sects. 2 and 3 can be left unchanged. Applied to $B = \operatorname{Spec} k$ with k a field, this is enough to produce virtual cycles in Sect. 4. But to prove their deformation invariance, we need to work with a relative moduli space $\mathcal{M}/B$ whose flatness over B cannot be assumed. In fact, in the Behrend–Fantechi relative obstruction theory of Sect. 4, flatness of X and X_0 over the base does not hold in general, so we prefer to proceed as follows.[1]

– Throughout Sects. 2 and 3 we set $B = \operatorname{Spec} k$ with k a field. All necessary flatness assumptions are then automatic. Thus, without further modifications, the paper defines the absolute, rather than relative, truncated Atiyah class $A(E_0)$ and the absolute truncated Kodaira–Spencer class $\kappa(X_0/X)$. Then Corollary 3.4 proves that their product

[1] Alternatively the reader could consult arXiv:0805.3527v2, which is a corrected version of the paper in which these modifications have been enacted.

The online version of the original article can be found under doi:10.1007/s00208-009-0397-6.

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$$\big(\mathrm{id}_{E_0} \otimes \kappa(X_0/X)\big) \circ A(E_0) \in \mathrm{Ext}^2_{X_0}(E_0, E_0 \otimes I) \tag{1.1}$$

is the obstruction to deforming a perfect complex E_0 from X_0 to its thickening X.

– To define relative versions of these objects we first need to define the relative truncated cotangent complex. To do this we embed $X_0 \subset X$ as follows:

$$\begin{array}{ccccccc} X_0 & \hookrightarrow & X & \hookrightarrow & A_B & \hookrightarrow & A \\ & & & \searrow & \downarrow & & \downarrow \\ & & & & B & \hookrightarrow & \widetilde{B}. \end{array}$$

Here $\widetilde{B}$ and $A \to \widetilde{B}$ are both smooth, and the square is Cartesian (i.e. $A_B = A \times_B \widetilde{B}$). Thus A and $A_B \to B$ are also smooth. Letting J_{0B} denote the ideal sheaf of $X_0 \subset A_B$ we get the natural commutative diagram

$$\begin{array}{ccc} J_0/J_0^2 & \longrightarrow & J_{0B}/J_{0B}^2 \\ \downarrow & & \downarrow \\ \Omega_A|_{X_0} & \longrightarrow & \Omega_{A/\widetilde{B}}|_{X_0} \cong \Omega_{A_B/B}|_{X_0}. \end{array}$$

The vertical 2-complex on the left is $\mathbb{L}_{X_0}$ and we define $\mathbb{L}_{X_0/B}$ to be the vertical 2-term complex on the right, giving the projection

$$\mathbb{L}_{X_0} \to \mathbb{L}_{X_0/B}. \tag{1.2}$$

Applied to the Atiyah class $A(E_0) \in \mathrm{Ext}^1_{X_0}(E_0, E_0 \otimes \mathbb{L}_{X_0})$ this defines a relative Atiyah class $A(E_0/B) \in \mathrm{Ext}^1_{X_0}(E_0, E_0 \otimes \mathbb{L}_{X_0/B})$.

The natural map $J_0/J_0^2 \to I$ factors through $J_{0B}/J_{0B}^2 \to I$. The former defines the Kodaira–Spencer class $\kappa(X_0/X)\colon \mathbb{L}_{X_0} \to I[1]$, so using the latter to define the relative Kodaira–Spencer class $\kappa(X_0/X/B)\colon \mathbb{L}_{X_0/B} \to I[1]$ we see they commute with the projection (1.2).

Thus the product of the relative Atiyah class and the relative Kodaira–Spencer class equals the product (1.1) of their absolute versions:

$$\big(\mathrm{id}_{E_0} \otimes \kappa(X_0/X/B)\big) \circ A(E_0/B) = \big(\mathrm{id}_{E_0} \otimes \kappa(X_0/X)\big) \circ A(E_0) \in \mathrm{Ext}^2_{X_0}(E_0, E_0 \otimes I). \tag{1.3}$$

– Finally, in Sect. 4, we project the Atiyah class $A(\mathbb{E}) \in \mathrm{Ext}^1_{X \times_B \mathcal{M}}(\mathbb{E}, \mathbb{E} \otimes \mathbb{L}_{X \times_B \mathcal{M}})$ of the universal sheaf $\mathbb{E}$ by $\mathbb{L}_{X \times_B \mathcal{M}} \to \mathbb{L}_{X \times_B \mathcal{M}/X} = \pi^*_{\mathcal{M}} \mathbb{L}_{\mathcal{M}/B}$ to give Eq. 4.2:

$$\pi_{\mathcal{M}*}\Big(\mathcal{H}om(\mathbb{E}, \mathbb{E})_0 \otimes \pi_X^* \omega_{X/B}\Big)[n-1] \longrightarrow \mathbb{L}_{\mathcal{M}/B}.$$

The proof that this is a perfect obstruction theory proceeds without change using the relative classes defined above. As in Eq. 4.5 we end up with an obstruction class expressed as a product of a relative truncated Atiyah class $A(\bar{f}^*\mathbb{E}/X)$ and a relative truncated Kodaira–Spencer class $\kappa(X \times_B S_0/X \times_B S/X) = \bar{\pi}^*\kappa(S_0/S/B)$.

This equals the product of the corresponding absolute classes by (1.3). By Corollary 3.4, this equals the obstruction to deforming $\bar{f}^*\mathbb{E}$ from $X \times_B S_0$ to $X \times_B S$. Since such deformations are in one-to-one correspondence with extensions from S_0 to S of the B-map f, the proof concludes just as in Original publication.