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ERRATUM

Erratum to: The Frölicher spectral sequence can be arbitrarily non-degenerate

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1 Introduction

In the original publication the second author constructed for $n \geq 2$ a series of compact complex manifolds X_n and an element $[\beta_1]$ in the E_n-term of the Frölicher spectral sequence claiming that $d_n([\beta_1]) \neq 0$ (Lemma 2 in loc.cit.). This claim is incorrect: we explain in Remark 2 that on the contrary β_1 induces a class in E_∞.

However, the main result of the original publication remains true (up to a change in the dimension of the examples).

Theorem 1 *For every $n \geq 2$ there exist a complex $4n-2$-dimensional compact complex manifold X_n such that the Frölicher spectral sequence does not degenerate at the E_n term, i.e., $d_n \neq 0$.*

The method of construction has remained the same, but we needed to introduce some extra counting variables.

We believe that in every dimension there are examples of nilmanifolds with left-invariant complex structure where the maximal possible non-degeneracy occurs, but the structure equations might be quite complicated.

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2 Construction of the example

Consider the space $G_n := \mathbb{C}^{4n-2}$ with coordinates

$$x_1, \dots, x_{n-1}, y_1, \dots, y_n, z_1, \dots, z_{n-1}, w_1, \dots, w_n.$$

Endow G_n with the structure of a real nilpotent Lie-group by identifying it with the subgroup of $\mathrm{Gl}(2n+2, \mathbb{C})$ consisting of upper triangular matrices of the form

$$A = \begin{pmatrix}
1 & 0 & & & & & \cdots & & 0 & \bar{y}_1 & w_1 \\
 & 1 & 0 & \dots & 0 & \bar{z}_1 & -x_1 & 0 & \dots & 0 & w_2 \\
 & & \ddots & & & & \ddots & & & \vdots & \vdots \\
 & & & 1 & 0 & \dots & 0 & \bar{z}_{n-1} & -x_{n-1} & 0 & w_n \\
 & & & & 1 & 0 & & \dots & & 0 & y_1 \\
 & & & & & \ddots & & & & \vdots & \vdots \\
 & & & & & & & & & & \\
 & & & & & & & \ddots & & \vdots & \vdots \\
 & & & & & & & & 1 & 0 & y_n \\
 & & & & & & & & & 1 & z_1 \\
 & & & & & & & & & & 1
\end{pmatrix}$$

Let $\Gamma = G_n \cap \mathrm{Gl}(2n+2, \mathbb{Z}[i])$, which is a lattice in the real Lie-group G. Note that if $g \in G_n$ is a fixed element then the action on the left, $g' \mapsto gg'$, is holomorphic with respect to the complex structure on $\mathbb{C}^{4n-2}$. The quotient

$$X_n = \Gamma / G_n$$

is a compact complex manifold; more precisely, it is a compact nilmanifold with left-invariant complex structure.

Remark 1 The manifold X_n admits a simple geometric description in terms of principal holomorphic torus bundles: the centre of G_n is given by the matrices for which all x_i, y_i and z_i vanish and hence isomorphic (as a Lie group) to $\mathbb{C}^n$. This yields an exact sequence of real Lie-groups

$$0 \to \mathbb{C}^n \to G_n \to \mathbb{C}^{3n-2} \to 0$$

which is compatible with the action of Γ. Denoting by T_k the quotient $\mathbb{C}^k/\mathbb{Z}[i]^k$ the exact sequence induces a T_n principal bundle structure on $X_n \to T_{3n-2}$.

The space of left-invariant 1-forms U is spanned by the components of $A^{-1}dA$ and their complex conjugates, so a basis for the forms of type $(1,0)$ is given by

$$dx_1, \dots, dx_{n-1}, dy_1, \dots, dy_n, dz_1, \dots, dz_{n-1}, \omega_1, \dots, \omega_n$$

where

$$\omega_1 = dw_1 - \bar{y}_1 dz_1,$$
$$\omega_k = dw_k - \bar{z}_{k-1} dy_{k-1} + x_{k-1} dy_k \quad (k = 2, \ldots, n).$$

For later reference we calculate the differentials of the above basis vectors:

$$\begin{aligned} d(dx_i) &= d(dz_i) = 0 && (i = 1 \cdots n-1) \\ d(dy_i) &= 0 && (i = 1 \cdots n) \\ d\omega_1 &= -d\bar{y}_1 \wedge dz_1 \\ d\omega_i &= dx_{i-1} \wedge dy_i + dy_{i-1} \wedge d\bar{z}_{i-1} \end{aligned}$$

The following lemma shows that the Frölicher spectral sequence of X_n has non-vanishing differential d_n thus proving our Theorem.

Lemma 1 *The differential form $\beta_1 = \bar{\omega}_1 \wedge d\bar{z}_2 \wedge \cdots \bar{d} z_{n-1}$ defines a class $[\beta_1]_n \in E_n^{0,n-1}$ and*

$$d_n([\beta_1]_n) = (-1)^{n-2}[dx_1 \wedge \cdots \wedge dx_{n-1} \wedge dy_n]_n \neq 0 \quad \text{in } E_n^{n,0}.$$

Proof By Remark 1 the projection to the (x, y, z)-coordinates endows X_n with the structure of holomorphic principal torus bundle over a complex torus. By the results of [2] the inclusion of left-invariant forms into the double complex $(\mathcal{A}^{p,q}(X_n), \partial, \bar{\partial})$ induces an isomorphism on the E_1-terms of the respective spectral sequences. Thus for our purpose we may work with left-invariant forms only, that is, start with the E_0-term

$$E_0^{p,q} = \Lambda^p U \otimes \Lambda^q \bar{U}.$$

A (p, q)-form α lives to E_r if it represents a class in $E_r^{p,q}$, which is a subquotient of $E_0^{p,q}$; the resulting class will be denoted by $[\alpha]_r$.

β_1 defines a class in E_n. As explained in [1, §14, p.161ff] this is equivalent to the existence of a *zig-zag* of length n, that is, a collection of elements $\beta_2, \ldots, \beta_n$ such that

$$\beta_i \in E_0^{p+i,q-i}, \quad \bar{\partial}\beta_1 = 0, \quad \partial\beta_{i-1} + \bar{\partial}\beta_i = 0 \quad (i = 2, \ldots n).$$

Consider the following differential forms β_k of bidegree $(k-1, n-k)$:

$$\begin{aligned} \beta_2 &= \omega_2 \wedge d\bar{z}_2 \wedge \cdots \wedge \bar{z}_{n-1} \\ \beta_k &= dx_1 \wedge \cdots \wedge dx_{k-2} \wedge \omega_k \wedge d\bar{z}_k \wedge \cdots \wedge \bar{z}_{n-1} \qquad (3 \leq k \leq n-1) \\ \beta_n &= dx_1 \wedge \cdots \wedge dx_{n-2} \wedge \omega_n \end{aligned}$$

A simple calculation shows that

$$\bar{\partial}\beta_1 = 0,$$
$$\partial\beta_1 = -dy_1 \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{n-1} = -\bar{\partial}\beta_2,$$

and for $2 \leq k \leq n-1$

$$\partial\beta_k = (-1)^{k-2} dx_1 \wedge \cdots \wedge dx_{k-1} \wedge dy_k \wedge d\bar{z}_k \wedge \cdots \wedge d\bar{z}_{n-1} = -\bar{\partial}\beta_{k+1}.$$

Therefore these elements define a zig-zag and β_1 defines a class in $E_n^{0,n-1}$.

It remains to prove that

$$d_n[\beta_1]_n = [\partial\beta_n]_n = (-1)^{n-2}[dx_1 \wedge \cdots \wedge dx_{n-1} \wedge dy_n]_n$$

defines a non-zero class in $E_n^{n,0}$, or equivalently, that β_1 does not live to E_{n+1}. In other words, we have to prove that does not exist a zig-zag of length $n+1$ for β_1. Since we are in a first quadrant double complex we have $E_0^{n,-1} = 0$ and there exists a zig-zag of length $n+1$ if and only if there exists a zig-zag $(\beta_1, \beta_2', \ldots, \beta_n')$ of length n such that $\partial\beta_n' = 0$.

To see that this cannot happen we put

$$U_1 = \langle dx_1, \ldots, dx_{n-1}, dy_1, \ldots, dy_n, dz_1, \ldots, dz_{n-1}\rangle_{\mathbb{C}} \quad \text{and} \quad U_2 = \langle \omega_1, \ldots, \omega_n\rangle_{\mathbb{C}},$$

such that $U = U_1 \oplus U_2$. The above basis of U and its complex conjugate induce a basis on each exterior power and a decomposition

$$\Lambda^n(U \oplus \bar{U}) = \Lambda^n(U_1 \oplus \bar{U}_1) \oplus S^n,$$

where S^n is spanned by wedge products of basis elements, at least one of which is in $U_2 \oplus \bar{U}_2$.

The elements β_k and $(-1)^k\partial\beta_k$ are basis vectors and we decompose

$$E_0^{k-1,n-k} = \beta_k\mathbb{C} \oplus V_k \quad \text{and} \quad E_0^{k,n-k} = \partial\beta_k\mathbb{C} \oplus W_k,$$

where V_k (resp. W_k) is spanned by all other basis elements of type $(k-1, n-k)$ (resp. $(k, n-k)$). Let ξ_k be the element of the dual basis such that $\xi_k \lrcorner \partial\beta_k = 1$ and the contraction with any other basis element is zero.

The differentials ∂ and $\bar{\partial}$ respect this decomposition, in the sense that

$$\partial(V_k) \subset W_k \quad \text{and} \quad \bar{\partial}(V_k) \subset W_{k-1}. \tag{1}$$

More precisely, let α be one of the forms in our chosen basis for $\Lambda^{n-1}(U \oplus \bar{U})$. Recall that α is a decomposable form. Then

$$\partial\alpha \notin W_k \iff \xi_k \lrcorner \partial\alpha \neq 0 \iff \alpha = \beta_k,$$
$$\bar{\partial}\alpha \notin W_k \iff \xi_k \lrcorner \bar{\partial}\alpha \neq 0 \iff \alpha = \beta_{k+1},$$

which implies (1). If $\alpha \in \Lambda^{n-1}(U_1 \oplus \bar{U}_1)$ then $d\alpha = 0$ and the claim is trivial. If the form α contains at least two basis elements of $U_2 \oplus \bar{U}_2$ then each summand of $d\alpha$ with respect to the basis contains at least one element of $U_2 \oplus \bar{U}_2$, in other words, $d\alpha \in S^n$. Since $\xi_k \lrcorner S^n = 0$ the claim is true also for those elements.

The remaining elements of the basis are of the form $\pm\alpha' \wedge \omega_i$ or $\pm\alpha' \wedge \bar{\omega}_i$ for some α' in our chosen basis for $\Lambda^{n-2}(U_1 \oplus \bar{U}_1)$. Paying special attention to the *counting variable* dy_k this case is easily checked by looking for solutions of the equation

$$\begin{aligned}\partial(\alpha' \wedge \omega_i) &= (-1)^{n-1}\alpha' \wedge \partial\omega_i \\ &= (-1)^{k-2} dx_1 \wedge \cdots \wedge dx_{k-1} \wedge dy_k \wedge d\bar{z}_k \wedge \cdots \wedge d\bar{z}_{n-1} = \partial\beta_k,\end{aligned}$$

and similarly in the other cases involving either $\bar{\partial}$ or $\bar{\omega}_i$.

Thus if $(\beta_1, \beta_2', \ldots, \beta_n')$ is any zig-zag of length n for β_1 then $\beta_k' \equiv \beta_k \mod V_k$ by (1) and, in particular,

$$\partial\beta_n' \equiv \partial\beta_n \not\equiv 0 \mod W_n.$$

Thus β_1 does not live to E_{n+1} and $d_n[\beta_1]_n$ is non-zero as claimed. □

Remark 2 In the original publication we constructed a compact complex manifold in a very similar way and an element $[\beta_1]_n \in E_n^{0,n-1}$. However, our claim that $d_n([\beta_1]_n) \neq 0$ was wrong: while the constructed zig-zag could not be extended, the sequence of elements (in the notation of [original publication, Lem. 2])

$$(\beta_1, dx_1 \wedge \bar{\omega}_2 \wedge d\bar{x}_3 \wedge \cdots \wedge d\bar{x}_{n-1}, 0, 0, \ldots)$$

gives an infinite zig-zag for β_1. In other words, the element considered gives an element of E_∞ and thus a de Rham cohomology class.

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