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ERRATUM

Basins of attraction and equilibrium selection under different learning rules

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As stated in the article, Theorems 2 and 4 are incorrect. They refer to strict equilibrium actions in the limit $\vec{P} \to \hat{\vec{P}}$ when in fact the results only hold for pure, evolutionarily stable strategies (ESS) that are uniform in this limit. The corrected versions along with the definition of a uniformly ESS are included below.

Definition An equilibrium $\mathbf{s}$ is a *uniformly ESS* in the limit as $\vec{P} \to \hat{\vec{P}}$ if there is a punctured neighborhood $\dot{U}(\mathbf{s})$ of $\mathbf{s}$ such that for all $\mathbf{s}' \in \dot{U}(\mathbf{s})$ and all $\vec{P} \neq \hat{\vec{P}}$ in some neighborhood of $\hat{\vec{P}}$,

$$\mathbf{s} \cdot \vec{\pi}(\mathbf{s}') > \mathbf{s}' \cdot \vec{\pi}(\mathbf{s}'),$$

where the payoff vector $\vec{\pi} = (\pi_1, \ldots, \pi_n)$.

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Theorem 2 *Suppose*

$$\lim_{\vec{P} \to \hat{\vec{P}}} \sum_a m\left(B(\mathbf{R}, a, \vec{P}) \cap B(\mathbf{B}, a, \vec{P})\right) = 0.$$

Then every pure, uniformly ESS satisfies the Never an Initial Best Response Property at $\hat{\vec{P}}$.[1]

Proof Suppose that **s** is a pure, uniformly ESS such that $m\left(\mathrm{BR}^{-1}(s)\right)$ remains strictly positive in the limit $\vec{P} \to \hat{\vec{P}}$. We will identify a nonvanishing region inside the basins of attraction of **s** for both replicator dynamics and best response dynamics.

As a pure strategy equilibrium, **s** can be written as $(x_s = 1, x_{-s} = 0)$ where s is the action always taken in this equilibrium. Let $U(\mathbf{s})$ be a neighborhood of **s** such that $\dot{U}(\mathbf{s}) = U(\mathbf{s})\backslash\{\mathbf{s}\}$ satisfies the condition for **s** to be a uniformly ESS. Let $\nu = \sup_{\mathbf{x} \notin U(\mathbf{s})} x_s$. Define the neighborhood $W(\mathbf{s}) \subseteq U(\mathbf{s})$ of all points satisfying $x_s > \nu$. We have constructed $W(\mathbf{s})$ such that $\mathbf{x} \in \dot{W}(\mathbf{s})$ implies that $\dot{x}_s > 0$ under the replicator dynamics (because by the ESS condition, action s has better than average payoff here) and in turn, $\dot{x}_s > 0$ implies that **x** remains in $W(\mathbf{s})$.

We now observe that $\mathrm{BR}^{-1}(s)$ is a convex set because of the linearity of payoffs. Additionally, since **s** is a pure Nash Equilibrium, $\mathbf{s} \in \mathrm{BR}^{-1}(s)$. Thus, $\mathrm{BR}^{-1}(s)$ and $W(\mathbf{s})$ have positive intersection. By the fact that $W(\mathbf{s})$ is independent of $\vec{P}$ and our hypothesis that $\mathrm{BR}^{-1}(s)$ is nonvanishing, we conclude that $m\left(W(\mathbf{s}) \cap \mathrm{BR}^{-1}(s)\right)$ remains strictly positive in the limit $\vec{P} \to \hat{\vec{P}}$. Note that by the ESS condition and the linearity of payoffs, we can rule out the possibility that there are multiple best responses anywhere in the interior of $\mathrm{BR}^{-1}(s)$. For points **x** in the interior of $W(\mathbf{s}) \cap \mathrm{BR}^{-1}(s)$, best response dynamics flows to **s** because $\mathrm{BR}(\mathbf{x}) = \{\mathbf{s}\}$ and replicator dynamics flows to **s** because $\mathbf{x} \in W(\mathbf{s})$. □

Theorem 4 *Suppose*

$$\lim_{\vec{P} \to \hat{\vec{P}}} \sum_a m\left(B(\mathbf{OSPP}, a, \vec{P}) \cap B(\mathbf{TD}, a, \vec{P})\right) = 0.$$

Then every pure, uniformly ESS satisfies the Never an Initial Best Response Property at $\hat{\vec{P}}$.

Proof The proof here mirrors the one for Theorem 2. We construct the neighborhood $W(\mathbf{s})$ in the same way, but with the additional condition that

[1] We would like to thank Andreas Blass for pointing out that this theorem, and Theorem 4 as well, would also hold for a particular action and not just for the entire action set, in the sense that if there is vanishing overlap in the basins of attraction of a particular pure equilibrium **s**, then either **s** is not a uniformly ESS or **s** is almost never an initial best response.

$x_s > 1 - \hat{K}$. We need only show that for $\mathbf{x} \in \text{int}\big(W(\mathbf{s}) \cap \text{BR}^{-1}(s)\big)$, both classes of dynamics flow to $\mathbf{s}$. Under one-sided payoff positive dynamics, $\dot{x}_s > 0$ for $\mathbf{x} \in W(\mathbf{s})$ because action s has an above average payoff, and such a flow cannot leave $W(\mathbf{s})$. Under threshold dynamics, when $\mathbf{x} \in \text{int}\big(W(\mathbf{s}) \cap \text{BR}^{-1}(s)\big)$, Eq. 5 applies to all actions other than s because they have payoffs below the $\hat{K}^{\text{th}}$ percentile. All other actions must have the same negative growth rate, so $\dot{\mathbf{x}} = \alpha(\mathbf{s} - \mathbf{x})$ for some positive constant α. □