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ORIGINAL ARTICLE

Squared $M_{\text{split}(q)}$ S-transformation of control network deformations

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Abstract
Identification of stable potential reference points (PRPs) is the most critical stage of computations in conventional deformation analysis of geodetic control networks. An appropriate matching of two adjusted networks at stable PRPs plays a key role in this task. Unfortunately, the geodetic control networks are free networks suffering from datum defect and can realize infinitely many possible matchings at PRPs. Therefore, accurate estimation of PRP displacements and later efficient identification of stable PRPs is a quite difficult task. This study makes some step forward in this field and presents a new approach to deformation analysis, including the identification of stable PRPs. The idea behind this approach is inspired by the theory of squared $M_{\text{split}(q)}$ estimation and lies in the non-conventional assumption that estimated displacements of PRPs can be the realizations—not of one but—of many congruence models, which simultaneously realize many different matchings. Displacements of unstable PRPs in such a multi-split congruence model do not have such a negative effect on expected matching at stable PRPs as in the conventional robust S-transformation. Here, these displacements can be realizations of other congruence models and their attention can be absorbed by other, unexpected, matchings. Thanks to this, the robustness of the suggested approach can be relatively high. To establish what the number of congruence models is in a given case, which model is the one expected and whether the chosen model is valid, the statistical hypothesis tests were proposed. The experiments performed on 1D and 2D simulated control networks showed that the presented approach can provide more accurate values of estimated displacements than conventional approaches, and in consequence, more efficient results of stable PRPs identification, especially when there exist more unstable PRPs than stable ones. In light of the above, the correct identification of stable PRPs and, in consequence, the correct final estimation of controlled object point displacements are possible in cases when it has not been possible so far.

Keywords Geodetic deformation analysis · Point displacement · Robust M-estimation · Squared $M_{\text{split}(q)}$ estimation · S-transformation

1 Introduction

One of the important functions of surveying and geodesy is the deformation monitoring of engineering structures and the surface of Earth's crust. The object or area under investigation is usually represented by a geodetic control network which is measured in two or more epochs of time, and the results of these measurements are then analyzed. The models used in this analysis may be categorized into two groups: descriptive models (Pelzer 1971; Niemeier 1981; Chen 1983; Caspary 2000), which are employed in conventional deformation analysis (CDA), and dynamic models (Papo and Perelmuter 1991, 1993; Shahar and Even-Tzur 2014). The descriptive models may also be divided into congruence models (also called static models) which geometrically describe the deformations by means of displacement vectors and kinematic models which temporally describe the deformations by means of displacement velocities and accelerations. Generally, the descriptive models exclusively examine deformations without regard to their influencing factors and the object's physical properties. While the dynamic models are the extension of the descriptive models which link deformations to their influencing factors (causative forces, internal and external loads) and the object's physical properties (material constants, extension coefficients, etc.) (Welsch and Heunecke 2001), the typical geodetic models are congruence models which are employed in CDA. In many geodetic applications, congruence models provide completely suffi-

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cient information about a deformable body, its change in shape and dimension, as well as rigid body movements and local deformations (Chrzanowski and Chen 1990; Caspary 2000).

The simultaneous analysis of observations from two epochs is performed in congruence models. It is usually done between any two consecutive epochs and, additionally, between the first and the current one. The matching of objects (represented by control network points) between both epochs is carried out on the group of mutually stable points (the group which has a congruent/rigid geometrical structure at both considered epochs). However, the main problem is the identification of such a group in the usually specified group of potential reference points (PRPs). The geodetic control networks are free networks suffering from datum defect (e.g., Chen 1983, ch.4), whereby correct identification of mutually stable points is rather difficult and it can be even impossible with a large number of unstable points. An erroneous identification leads to erroneous defining of datum for estimated object deformations and, in consequence, to disinformation. The identification of mutually stable points is the only serious problem in congruence models, and it is still the subject of interest for surveyors and geodesists (Baselga and García-Asenjo 2016; Amiri-Simkooei et al. 2016; Aydin 2017).

Today, mutually stable points can be identified iteratively with a global congruency test (GCT) or with a robust M-estimation with a high level of reliability. The first iterative step is the same for both approaches, and it involves the least squares estimation of PRPs displacements (the so-called raw displacements). A minimum trace datum is here defined for all PRPs. In the first approach, unstable PRPs are successively removed from the computational base until all the unstable points are identified. The test statistic can be based on estimated displacements of PRPs (Pelzer 1971, 1974; Niemeier 1981; Denli and Deniz 2003) or residuals (Kok 1982; Heck 1983; Gründig and Neureither 1985; Hekimoglu et al. 2010). A functional model forces zero displacements of PRPs in the second method; hence, any point displacements and measurement errors are "pushed out" in values of residuals. Despite the completely different algorithms, both methods give equivalent results. In a robust approach, which is especially useful in the deformation analysis of large and very large control networks, unstable PRPs are not removed from the computational base, but their participation in datum is iteratively suppressed. The PRPs displacements from the final S-transformation are then tested, one by one, to identify unstable points. This approach can be implemented by two methods: robust S-transformation of coordinates differences (Chen 1983; Caspary and Borutta 1987; Chen et al. 1990; Caspary et al. 1990; Caspary 2000, p.130–132) or robust M-estimation of observation differences (Nowel and Kamiński 2014; Nowel 2015, 2016a). The literature also provides new interesting approaches, such as the modern non-iterative GCT approach, where all considered subsets of non-congruent point patterns are tested (Velsink 2015, 2018; Lehmann and Lösler 2017), or the R-estimation approach (Duchnowski 2010, 2013). Furthermore, a completely different, non-conventional approach was recently presented by Zienkiewicz (2014), Zienkiewicz and Baryla (2015), Wiśniewski and Zienkiewicz (2016). The authors do not identify mutually stable points and start an estimation of an object deformation straight away. Datum is defined on all PRPs with the zero displacement condition (the so-called rigid datum instead of minimum trace datum). Any displacements of PRPs appear in residuals and are separated with M_{split} estimation. However, this approach is quite limited in practical applications because it does not tolerate a large number of displaced object points.

Generally, the GCT approach and the robust approach yield very comparable results (e.g., Caspary 2000, p. 149–154). These approaches are universal, most mature and commonly regarded as the most effective tools used to identify mutually stable points. It is noteworthy that these approaches dominate not only in a deformation analysis, but also in other similar issues of geodesy, e.g., in the identification of observations without outliers. Unfortunately, certain situations are known to exist in which even these methods fail. For example, if the number of outliers is much larger than that of the expected realizations in a set, then such methods—especially the robust approach—may fail (Hampel et al. 1986, p. 12; Koch 1996, 2010, p. 263). This also applies to identification of mutually stable points in congruence models, where displaced PRPs may be treated as outliers, and stable PRPs as expected realizations. Whereas this paper presents and pretests a completely different approach to the identification of mutually stable points, unlike in the conventional congruence model, unstable points are not suppressed or deleted from a computational base. This approach is based on the concept of the $M_{\text{split}(q)}$ estimation and allows for the simultaneous existence of many congruence models which differ by the datum parameters. The unstable points do not have such a negative effect on datum parameters of expected congruence model as in conventional approaches because these points can be the realizations of other congruence models and their attention can be absorbed by other datum parameters. Thanks to this, the robustness of the suggested approach can be relatively high. The presented idea of applying $M_{\text{split}(q)}$ estimation in deformation analysis is completely different from the one presented in the literature and referenced earlier in this section. Most of all, that simple concept, based on the M_{split} estimation, does not deal with the identification of stable PRPs and focuses only on observation residuals disclosing some information about unstable points, which can be treated as outliers.

The next part of the paper is organized as follows: Sect. 2 describes the conventional robust approach to the S-

transformation of deformations. Particular attention is paid to a mathematical congruence model, an optimization problem which corresponds with this model and a solution for it. Section 3 demonstrates the general idea of $M_{split(q)}$ estimation. Section 4 presents, with reference to Sects. 2 and 3, the suggested approach to the S-transformation of deformations. A research motivation is presented in Sect. 4.1. The next subsections focus on a suggested multi-split mathematical congruence model, an optimization problem which corresponds with this model and a suggested strategy for its solution. A validation method for the solution is presented as well. Section 5 is devoted to numerical experiments to demonstrate the suggested concept against conventional ones. Section 6 gives the conclusions from the study.

2 Robust S-transformation of deformations

The approach proposed will be described in Sect. 4 in some reference to robust S-transformation; therefore, this section will be devoted to this approach.

Robust S-transformation, especially the iterative weighted similarity transformation (IWST) (Chen 1983; Caspary and Borutta 1987), was used for deformation analysis of the Tevatron atomic particle accelerator complex at the Fermilab laboratory in the USA (Bocean et al. 2006). This method was also implemented in the automated ALERT monitoring system developed by the Canadian Centre for Geodetic Engineering (Wilkins et al. 2003) and in the universal GeoLab geodetic computation software (Chrzanowski et al. 2011). A robust S-transformation is especially useful in the deformation analysis of large and very large control networks, such as the network of Tevatron accelerator, which comprised nearly 2000 control points. With such networks, calculations by this method are much more convenient than with GCT, and the quality of results is at a similar, satisfying level.

2.1 Congruence model

Separate adjusted networks—represented by the least squares (LS) estimator of PRP coordinate vector at an initial epoch, $\hat{\mathbf{x}}_1(u \times 1)$, and a current epoch, $\hat{\mathbf{x}}_2(u \times 1)$—are the input data for the robust S-transformation approach. These data may have different datums at both epochs—due to, e.g., unstable points being used in a computational base in epoch 2 or different computational bases being used at both epochs—hence, the raw displacement vector:

$$\boldsymbol{\Delta}_{\hat{x}} = \hat{\mathbf{x}}_2 - \hat{\mathbf{x}}_1 \tag{1}$$

may show biased values of possible single point displacements. As a result, the network adjusted at epoch 2 may be freely shifted, rotated and rescaled in relation to the network adjusted at epoch 1. Assuming small values of $\boldsymbol{\Delta}_{\hat{x}}$, a linear model of a similarity transformation that retains the geometry of the network is employed:

$$\hat{\mathbf{x}}_2 = \hat{\mathbf{x}}_1 + \mathbf{H}\mathbf{t} + \mathbf{d} \;\Rightarrow\; \boldsymbol{\Delta}_{\hat{x}} = \mathbf{H}\mathbf{t} + \mathbf{d} \tag{2}$$

where $\mathbf{t}(h \times 1)$ is the vector of translation, rotation, scale distortion (datum parameters), $\mathbf{H}(u \times h)$ is the design matrix of similarity transformation (e.g., Chen 1983 p. 55; Chen et al. 1990, p. 142; Caspary et al. 1990, p. 51; Caspary 2000, p. 31–33,131), and $\mathbf{d}(u \times 1)$ is the vector of discrepancies. The vector of raw displacements (1) is considered as a vector of observations of a deformation field which can be modeled in some way.

In a geometrical sense, the idea behind the robust S-transformation approach is based on the matching of both adjusted networks at mutually stable points to disclose the unbiased values of possible single point displacements. It means that one estimates such a vector of datum parameters for displacement vector, $\hat{\mathbf{t}}$, which realizes a network congruency at mutually stable points and, in consequence, clearly discloses possible single point displacements in the residual vector, $\hat{\mathbf{d}}$. In other words, the raw displacement vector, $\boldsymbol{\Delta}_x$, is here transformed to such datum (by means of estimated datum parameters, $\hat{\mathbf{t}}$) which most clearly shows a deformation pattern, $\hat{\mathbf{d}}$. However, the question remains: What estimation/matching method should be used?

The choice of the appropriate estimation method is here justified in some probabilistic assumption. Namely, one assumes that the majority of points in the PRP group are stable, with only individual points which may be unstable. With regard to the functional model (2) and assuming that the estimated coordinates are normally distributed, one assumes that the raw displacement vector consists mainly of regular and normally distributed displacements, which come from different datums at both epochs and from a measurement noise, and only some irregular outlying displacements, which additionally come from single point displacements (analogously as, e.g., a vector of observations with suspected gross errors). In accordance with this assumption and in regard to the contemporary robust statistics (Huber 1981), one can formulate a probabilistic model of the raw displacement vector:

$$\boldsymbol{\Delta}_{\hat{x}} \sim \underbrace{(1-\tau)\cdot P_n\left(\mathbf{H}\mathbf{t}, \hat{\mathbf{C}}_{\Delta_{\hat{x}}}\right)}_{\text{stable points}} + \underbrace{\tau \cdot P_d\left(\mathbf{H}\mathbf{t}, c\cdot\hat{\mathbf{C}}_{\Delta_{\hat{x}}}\right)}_{\text{displaced points}} \tag{3}$$

where τ is a small number ($0<\tau<0.5$), $\hat{\mathbf{C}}_{\Delta_{\hat{x}}}$ is the estimated covariance matrix of raw displacements, $P_n(\cdot)$ means accepted normal distribution, $P_d(\cdot)$ means some other unaccepted probability distribution, and $c>1$. Stable PRPs have the probability of $(1-\tau)$, and displaced PRPs have the probability of τ. Such τ-contaminated normal distribution is

also known as the variance inflation (VI) model and can be interpreted as a probabilistic foundation of the robust S-transformation approach.

2.2 Estimation

The optimization condition for the residual vector results from the congruence model (2), (3) and has the following form:

$$\min_{\mathbf{t}} \sum_{i=1}^{u} \rho\left(\Delta_{\hat{x}_i}; \mathbf{h}_i \mathbf{t}\right) \tag{4}$$

where $\rho(\cdot)$ is some objective function from the robust M-estimation class, also taking into account the L_1-norm objective function:

$$\min_{\mathbf{t}} \sum_{i=1}^{u} \left|\Delta_{\hat{x}_i} - \mathbf{h}_i \mathbf{t}\right| \Leftrightarrow \min_{\mathbf{t}} \sum_{i=1}^{u} |d_i| \tag{5}$$

which is implemented in the most popular method of the robust S-transformation approach, i.e., the IWST method, and $\mathbf{h}_i$ is the ith row of the matrix $\mathbf{H}$. According to Huber (1981), to find the minimum of the multiple differentiable and convex objective function (4) for the linear model (2), one can determine the gradient of this function and equate this gradient to zero. Then, the numerical solution for the M-estimator of datum parameter vector and discrepancy vector has the following form:

$$\hat{\mathbf{t}}^k = \left(\mathbf{H}^T \mathbf{W}^k \mathbf{H}\right)^{-1} \mathbf{H}^T \mathbf{W}^k \boldsymbol{\Delta}_{\hat{x}}, \ \mathbf{W}^k = \operatorname{diag}\left(w_i^k\right) \tag{6.1}$$

$$\hat{\mathbf{d}}^k = \boldsymbol{\Delta}_{\hat{x}} - \mathbf{H}\hat{\mathbf{t}} = \left(\mathbf{I} - \mathbf{H}\left(\mathbf{H}^T \mathbf{W}^k \mathbf{H}\right)^{-1} \mathbf{H}^T \mathbf{W}^k\right) \boldsymbol{\Delta}_{\hat{x}} \tag{6.2}$$

where $k = 1, 2, \ldots$ is an iterative step, $\mathbf{I}(u \times u)$ is the identity matrix, and w_i is any weight function of the selected method from the robust M-estimation class. For example, the L_1-norm weight function will have the following form: $w_i^k = 1/\left|\hat{d}_i^k\right|$ (IWST method), where $\hat{d}_i^k$ is the given component of vector $\hat{\mathbf{d}}^k$ for single PRP ($\hat{d}_{x_i}^k$, $\hat{d}_{y_i}^k$ or $\hat{d}_{z_i}^k$). The weights are formulated only for PRPs, and zero values should be indicated for controlled object points. The computation is initialized, $k = 0$, with the weights $\mathbf{W} = \mathbf{I}$ (LS solution), and the iteration process is repeated until convergence is achieved. The final residual vector (6.2) is the estimated vector of single point displacements. Unfortunately, the covariance matrix of this vector cannot be computed in a familiar way since the relation between $\boldsymbol{\Delta}_{\hat{x}}$ and $\hat{\mathbf{d}}$ is nonlinear; thus, the law of variance propagation does not apply. However, an approximate covariance matrix may be here the one which realizes the minimum trace or the one obtained from the law of variance propagation according to (6.2). Note that the covariance

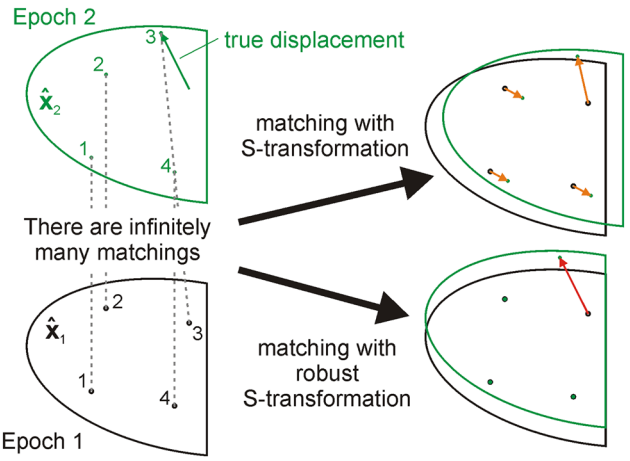

Fig. 1 Matching of PRP networks "adjusted" in two epochs; point 3 is displaced

information is not involved in the original criterion of the IWST method (5). However, nothing stands in the way to involve the one of the approximate covariance matrices in this criterion.

The estimation method is robust in the sense that large displacements of single points do not affect the estimated vector of datum parameters, $\hat{\mathbf{t}}$. The displacements emerge as residuals with a full magnitude and do not contaminate the residuals of the stable points. To demonstrate the behavior of the robust S-transformation approach and to compare its performance with the ordinary (LS) S-transformation, a simple example of two-dimensional subnetwork consisting of four PRPs is presented below (Fig. 1). For simplicity, it was assumed that the measurement errors did not exist.

Obviously, the matching which is based on the ordinary S-transformation fails to detect the single point displacements. With the robust S-transformation, no problems arise as long as the majority of points conform to the congruence model (2), i.e., are mutually stable. However, when the majority of points will not conform to model (2) the robust S-transformation also might fail.

2.3 Statistical testing and final S-transformation

Of course, the measurement errors exist in the real world and the separation of stable and unstable PRPs is necessary by means of statistical tests. The simple null hypothesis: "all PRPs are stable" against the composite alternative hypothesis: "there is at least one displaced PRP":

$$H_0 : E\left(\hat{\mathbf{d}}\right) = \mathbf{0} \text{ vs } H_A : E\left(\hat{\mathbf{d}}\right) \neq \mathbf{0} \tag{7}$$

is here tested by global F-test (Caspary 2000; Aydin 2012) for the significance level:

$$P\{\text{accept } H_A | H_0\} = \alpha \tag{8}$$

If the test does not pass, the simple null hypotheses (point i is stable) and the composite alternative hypotheses (point i is unstable):

$$H_{0,i} : E\left(\hat{\mathbf{d}}_i\right) = \mathbf{0} \text{ vs } H_{A,i} : E\left(\hat{\mathbf{d}}_i\right) \neq \mathbf{0}, \quad i = 1, \ldots, m \tag{9}$$

are sequentially tested for each PRP with approximate local F-tests for the significance level:

$$P\left\{\text{accept } H_{\mathrm{A},i} \middle| H_{0,i}\right\} = \alpha_i = 1 - (1-\alpha)^{1/m} \approx \alpha/m \tag{10}$$

where m is the number of all PRPs (Caspary 2000, p. 132). Since the local F-tests are stochastically dependent it is not possible to develop a statistically rigorous procedure. It is also worth noting that these tests are not S-system (reference system) invariant.

After stable PRPs are identified a final S-transformation for all the network points and their covariance matrices should be conducted to the minimum trace datum defined on the previously identified stable PRPs. The estimator of displacement vector still has the form of (6.2), but the weight matrix now fulfills the role of datum selector matrix and has the form $\mathbf{W} = \text{diag}(\mathbf{I}, \mathbf{0})$, where ones concern the datum points and zeros other points. The deformations of the object points (including also the unstable PRPs), as computed from final S-transformation, form the basis for all further deformation analyses which may concern rigid body displacement, deformation tensor and/or polynomial deformation model. More information on the robust S-transformation approach can be found in the papers: Chen (1983), Chen et al. (1990), Caspary and Borutta (1987), Caspary (2000, p. 130–132).

3 The general idea of $M_{\text{split}(q)}$ estimation

Recently, Wiśniewski (2009, 2010) proposed certain generalizations of M-estimation, called M_{split} and $M_{\text{split}(q)}$ estimation. The idea behind this development is based on the assumption that a random sample can be an unsteady mixture of realizations—not of one, but of many (q)—competitive random variables whose probability distributions differ, e.g., by the location parameter (expected/mean value). However, it is not known a priori which random variable is proper for a particular value of random sample. Figure 2 shows the results of the robust M-estimation, the $M_{\text{split}(q)}$ estimation and the LS estimation on a certain split sample.

Let us assume that the above random sample contains "good" values: the points in the middle, and two extreme groups of outliers: the other points. The robust M-estimation treats two extreme groups as outliers and the middle group as "good." However, the estimator of location parameter is not fitted well into the "good" values because it is slightly

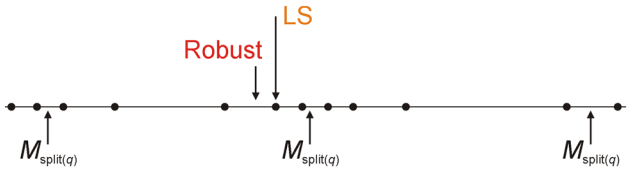

Fig. 2 Results of the $M_{\text{split}(q)}$ estimation for $q = 3$, robust M-estimation and LS estimation (based on Wiśniewski 2010)

absorbed by the left outliers. A similar solution comes from the LS estimation, whereas the $M_{\text{split}(q)}$ estimation allows to treat the random sample as the realizations of three random variables. In a simultaneous and joint optimization process, each estimator finds its values. Other values do not attract the estimator, because the other estimators absorb their attention. Note that the $M_{\text{split}(q)}$ estimation for two random variables ($q = 2$) is treated as a special case of the presented theory and is called the M_{split} estimation (Wiśniewski 2009).

If a functional model is linear(ized) and concerns, for example, some observations, $\mathbf{y}(n \times 1)$, which are realizations of q random variables differing only by the unknown parameter, $\mathbf{x}_{(j)}(u \times 1)$, then the multi-split mathematical model in the $M_{\text{split}(q)}$ estimation has the following form:

$$\mathbf{y} = \mathbf{A}\mathbf{x}_{(j)} + \mathbf{e}_{(j)}, \ \mathbf{y} \sim P\left(\mathbf{A}\mathbf{x}_{(j)}, \mathbf{C}_y\right); \quad j = 1, 2, \ldots, q \leq n \tag{11}$$

where $\mathbf{A}(n \times u)$ is design matrix, $\mathbf{e}_{(j)}(n \times 1)$ are error vectors, $P(\cdot)$ are some accepted distributions which differ by the location parameters, $\mathbf{A}\mathbf{x}_{(j)}$, and $\mathbf{C}_y$ is the covariance matrix of observations. Since the random sample (here, $\mathbf{y}$) is mixed in an unsteady way, it is a priori assumed that each value of the sample, y_i, can be a realization of each random variable. Next, individual values of the sample find their random variable in the simultaneous and joint estimation process. The $M_{\text{split}(q)}$ estimators of parameters of a multi-split model (11) are such vectors $\hat{\mathbf{x}}_{(1)}, \ldots, \hat{\mathbf{x}}_{(q)}$, which realize an optimal—in some sense—fitting of the q competitive models in the whole observation set.

The optimization condition of $M_{\text{split}(q)}$ estimation is based on the assumption that a single observation, y_i, can be assigned with a certain quantity which determines its multi-split potential, i.e., the "willingness" to belong to any of the q competitive models (11). The interest is focused on searching for such estimators, $\hat{\mathbf{x}}_{(1)}, \ldots, \hat{\mathbf{x}}_{(q)}$, for which the multi-split potential contained in the whole random sample has the highest value. According to Wiśniewski (2010), the condition of maximization of this global multi-split potential can also be written in the convenient probabilistic form:

$$\max_{\mathbf{x}_{(1)},\ldots,\mathbf{x}_{(q)}} \sum_{i=1}^{n} \prod_{j=1}^{q} \ln p\left(y_i; \mathbf{a}_i \mathbf{x}_{(j)}\right) \tag{12}$$

which is equivalent to:

$$\min_{\mathbf{x}_{(1)},\ldots,\mathbf{x}_{(q)}} \left[\Phi\left(\mathbf{y}; \mathbf{A}\mathbf{x}_{(1)}, \ldots, \mathbf{A}\mathbf{x}_{(q)}\right) = \sum_{i=1}^{n} \prod_{j=1}^{q} \rho\left(y_i; \mathbf{a}_i \mathbf{x}_{(j)}\right) \right] \tag{13}$$

where $p(\cdot)$ means some accepted probability density function (PDF), $p(\cdot)$ is at least twice differentiable convex objective function, and $\mathbf{a}_i$ is the ith row of the matrix $\mathbf{A}$. One can note that for $q = 1$, condition (13) is the optimization condition of M-estimation and condition (12)—of the maximum likelihood (ML) estimation. Therefore, $M_{\text{split}(q)}$ estimation can be treated as the generalization of M-estimation and of the ML estimation. From both theoretical and practical points of view, it is often the most convenient to assume that the random sample values have normal distributions, $\mathbf{y} \sim P_n\left(\mathbf{A}\mathbf{x}_{(j)}, \mathbf{C}_y\right)$. Condition (13) is then, in fact, a multi-split LS optimization condition of the squared $M_{\text{split}(q)}$ estimation:

$$\min_{\mathbf{x}_{(1)},\ldots,\mathbf{x}_{(q)}} \left[\Phi\left(\mathbf{y}; \mathbf{A}\mathbf{x}_{(1)}, \ldots, \mathbf{A}\mathbf{x}_{(q)}\right) = \sum_{i=1}^{n} \prod_{j=1}^{q} e_{i(j)}^2 \right] \tag{14}$$

More information, e.g., the solutions of the above optimization problems using Newton's iterative procedure, can be found in Wiśniewski (2009, 2010).

4 Squared $M_{\text{split}(q)}$ S-transformation of deformations

4.1 Motivation

Methods of the robust M-estimation class yield misinforming results when the so-called breakdown point for a given M-estimator is exceeded (Hampel et al. 1986, p. 12; Koch 1996, 2010, p. 263). A robust S-transformation of deformations may fail when the number of unstable PRPs is larger than that of stable points and always fails when these displacements have the same sign. The strength of unstable PRPs is then larger than that of the stable ones. In consequence, the vector of datum parameters for displacement vector, $\mathbf{t}$, does not realize the expected matching of both adjusted networks (the matching at stable points); it realizes some other matching. In effect, the raw displacement vector, $\boldsymbol{\Delta}_x$, is transformed into the datum which shows a biased deformation pattern. Figure 3 presents the extension of the scenario from Fig. 1; three unstable PRPs have been added (points 5, 6, 7) and,

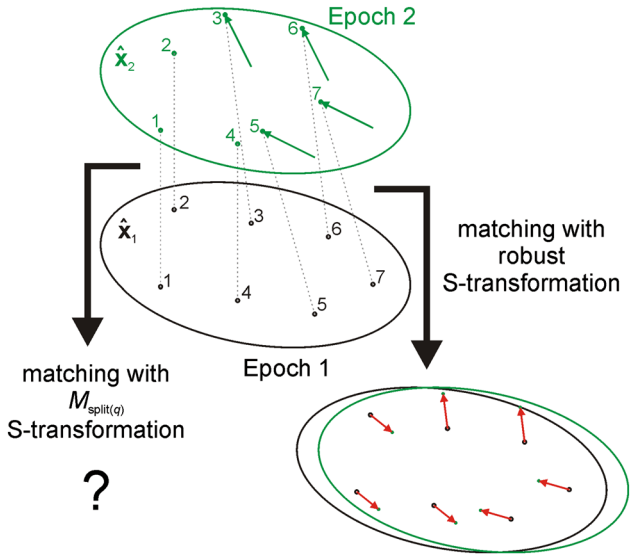

Fig. 3 Matching of PRP networks "adjusted" in two epochs; four points are displaced

in effect, the robust S-transformation has given the biased deformation pattern.

However, a question arises: How would the $M_{\text{split}(q)}$ estimation behave in such critical situations?

Keep in mind that the raw displacement vector forms a random sample in deformation congruence model (2). It has been pointed out that the attention of a given $M_{\text{split}(q)}$ estimator is absorbed only by the values of a random sample which are realizations of a given random variable (values of random sample which well—in some sense—fit to this estimator) and other values do not have a negative effect on this estimator. Therefore, it can be expected that when the right calculation strategy is followed, one of the $M_{\text{split}(q)}$ estimators should identify good datum parameters for displacement vector (matching both adjusted networks at stable points), even in critical situations. The thought experiments results suggest that this hypothesis may be plausible, and they inspire this research.

4.2 Congruence model

In the robust S-transformation approach one assumes that the majority of points in the PRP group are stable and only individual points may be unstable. This is a sufficient condition to obtain an expected datum for displacement vector (expected matching). In a probabilistic sense, one assumes that the raw displacement vector of PRPs consists mainly of values which realize the random variable with accepted normal distribution (stable points) and there are only some outlying values which realize the random variable with some other unaccepted probability distributions (unstable points), (2) and (3).

Now, let us consider a different model. Let us assume that the raw displacement vector of PRPs contains realizations of q random variables with accepted normal distributions which differ by the location parameters (expected/mean value). In the geometric sense, one assumes that a group of PRPs may have q competitive congruences between both epochs represented by the q competitive datum parameters $\mathbf{t}_{(j)}$—one realizing the expected matching at stable points and the others realizing the matchings at displaced points. Unlike in the conventional congruence model, all the PRPs are—in some sense—good; there are no outlying points in the model. This way of considering the raw displacement vector of PRPs leads to a multi-split congruence model, split into q potential congruence models:

$$\boldsymbol{\Delta}_{\hat{x}} = \mathbf{H}\mathbf{t}_{(j)} + \mathbf{d}_{(j)}, \quad \boldsymbol{\Delta}_{\hat{x}} \sim P_n\left(\mathbf{H}\mathbf{t}_{(j)}, \hat{\mathbf{C}}_{\Delta_{\hat{x}}}\right); \quad j = 1, 2, \ldots, q \le u \tag{15}$$

In deformation reality, the multi-split model (15) might describe the scenario where several subgroups of PRPs may regularly displace each other and the points are mutually stable inside each subgroup. Then, each j local model realizes the congruency of one subgroup of such points. It is the most intuitive scenario for model (15). However, this model may also describe the extended scenario where additionally single point displacements exist. Then, the congruences of single points may also be realized by means of additional models. For example, if the group of PRPs consists of two different displaced subgroups with mutually stable points inside each subgroup and three unstable single points, model (15) should be theoretically split into five local congruence models ($q=5$).

It is also worth noting that the multi-split model (15) can be interpreted as the collection of mean shift (MS) models which are considered in the statistical testing theory which is also widely applied in geodetic deformation analysis. Of course, these models play a quite different role in the suggested approach.

4.3 Estimation

Let us assume that the multi-split model (15) consists of q local congruence models. It is a priori not known which model is proper for the particular point(s)—raw displacement(s); therefore, each value of the raw displacement vector can be a priori a realization of each model. However, according to Wiśniewski (2010), theoretically, each raw displacement can be assigned a certain measure, known as elementary split potential, providing an opportunity to assign it to any of the congruence models (15). A given raw displacement realizes better some congruence and worse the other. Hence, our interest is focused on seeking such local congruences—represented by datum parameters, $\mathbf{t}_{(1)}, \ldots, \mathbf{t}_{(q)}$—for which the elementary split potential in the whole vector of raw displacements has the highest value. According to the cited paper, the condition of maximization of this global split potential for model (15) can be written in an equivalent probabilistic form:

$$\max_{\mathbf{t}_{(1)},\ldots,\mathbf{t}_{(q)}} \sum_{i=1}^{u} \prod_{j=1}^{q} \ln p_n\left(\Delta_{\hat{x}_i}; \mathbf{h}_i \mathbf{t}_{(j)}\right) \tag{16}$$

which is equivalent to:

$$\min_{\mathbf{t}_{(1)},\ldots,\mathbf{t}_{(q)}} \left[\Phi\left(\boldsymbol{\Delta}_{\hat{x}}; \mathbf{H}\mathbf{t}_{(1)}, \ldots, \mathbf{H}\mathbf{t}_{(q)}\right) = \sum_{i=1}^{u} \prod_{j=1}^{q} \left(\Delta_{\hat{x}_i} - \mathbf{h}_i \mathbf{t}_{(j)}\right)^2 \right] \tag{17}$$

where $p_n(\cdot)$ means the PDF of accepted normal distribution (15). For example, if $q=3$ our interest is focused on seeking such local congruences—represented by datum parameters, $\mathbf{t}_{(1)}, \ldots, \mathbf{t}_{(q)}$—for which:

$$\begin{aligned} &\min_{\mathbf{t}_{(1)},\mathbf{t}_{(2)},\mathbf{t}_{(3)}} \sum_{i=1}^{u} \left(\Delta_{\hat{x}_i} - \mathbf{h}_i \mathbf{t}_{(1)}\right)^2 \left(\Delta_{\hat{x}_i} - \mathbf{h}_i \mathbf{t}_{(2)}\right)^2 \left(\Delta_{\hat{x}_i} - \mathbf{h}_i \mathbf{t}_{(3)}\right)^2 \\ &= \min_{\mathbf{t}_{(1)},\mathbf{t}_{(2)},\mathbf{t}_{(3)}} \sum_{i=1}^{u} d_{i(1)}^2 \cdot d_{i(2)}^2 \cdot d_{i(3)}^2 \end{aligned} \tag{18}$$

For simplification and avoiding a significant loss of results quality, the covariance information of model (15) may be neglected in the optimization condition (17), like in the IWST method (5). The optimization problems (15) and (17) can be solved by such values of $\hat{\mathbf{t}}_{(j)}, j = 1,\ldots, q$, for which gradients:

$$\begin{aligned} \mathbf{g}_{(j)}\left(\mathbf{t}_{(1)}, \ldots, \mathbf{t}_{(q)}\right) &= \frac{\partial}{\partial \mathbf{t}_{(j)}} \Phi\left(\boldsymbol{\Delta}_{\hat{x}}; \mathbf{H}\mathbf{t}_{(1)}, \ldots, \mathbf{H}\mathbf{t}_{(q)}\right) \\ &= \frac{\partial \mathbf{d}_{(j)}}{\partial \mathbf{t}_{(j)}} \frac{\partial}{\partial \mathbf{d}_{(j)}} \Phi\left(\boldsymbol{\Delta}_{\hat{x}}; \mathbf{H}\mathbf{t}_{(1)}, \ldots, \mathbf{H}\mathbf{t}_{(q)}\right) \end{aligned} \tag{19}$$

are zero vectors. The components of objective function (17) in relation to $\mathbf{t}_{(j)}$ are quadratic functions and the Hessian of such component takes the form of an identity matrix; hence, Newton's method may be reduced to the method of zeroing gradient for this particular case (approximate Newtonian process). Since for each $j = 1,\ldots, q$ is obtained $\partial \mathbf{d}_{(j)} / \partial \mathbf{t}_{(j)} = -\mathbf{H}^T$, the necessary conditions for the existence of a minimum of the objective function (17) have the following form:

$$\mathbf{g}_{(j)}\left(\mathbf{t}_{(1)},\ldots,\mathbf{t}_{(q)}\right) = -2\mathbf{H}^T\mathbf{W}_{(j)}\left(\boldsymbol{\Delta}_{\hat{x}} - \mathbf{H}\hat{\mathbf{t}}_{(j)}\right)$$
$$= 2\mathbf{H}^T\mathbf{W}_{(j)}\mathbf{H}\hat{\mathbf{t}}_{(j)} - 2\mathbf{H}^T\mathbf{W}_{(j)}\boldsymbol{\Delta}_{\hat{x}} = \mathbf{0} \tag{20}$$

where:

$$\mathbf{W}_{(j)} = \operatorname{diag}\left[\prod_{l=1,\,l\neq j}^{q}\hat{d}^2_{1(l)},\ldots,\prod_{l=1,\,l\neq j}^{q}\hat{d}^2_{u(l)}\right] \tag{21}$$

Of course, weights (21) are formulated only for PRPs and zero values should be indicated for object points. The solution can be carried out in an iterative cycle with the following formula ($k=1,\ldots$), ($j=1,\ldots,q$):

$$\hat{\mathbf{t}}^k_{(j)} = \left(\mathbf{H}^T\mathbf{W}^k_{(j)}\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{W}^k_{(j)}\boldsymbol{\Delta}_{\hat{x}} \tag{22.1}$$

$$\hat{\mathbf{d}}^k_{(j)} = \boldsymbol{\Delta}_{\hat{x}} - \mathbf{H}\hat{\mathbf{t}}^k_{(j)} = \left(\mathbf{I} - \mathbf{H}\left(\mathbf{H}^T\mathbf{W}^k_{(j)}\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{W}^k_{(j)}\right)\boldsymbol{\Delta}_{\hat{x}} \tag{22.2}$$

The iterative process is relatively simple. The weight matrix of the j model is a function of recently calculated deformations of all the other models, i.e., $\mathbf{W}_{(j)} = f\left(\hat{\mathbf{d}}_{(l)}\right)$, $\forall l \neq j$. Of course, there are no calculated deformations in the starting iterative step ($k=0$), for weight matrices of the first model ($j=1$) and later—partly—for consecutive models. However, missing values of deformations in the starting iterative step can be, for example, their LS estimators:

$$\hat{\mathbf{d}}^{LS}_{(j)} = \left(\mathbf{I} - \mathbf{H}\left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\right)\boldsymbol{\Delta}_{\hat{x}}, \quad \forall j \neq 1 \tag{23}$$

analogously as in the squared $M_{\text{split}(q)}$ estimation (Wiśniewski 2010). The iteration process (22) is repeated until convergence is achieved; usually, several steps are enough.

The algorithm for the displacement vectors (22.2), in a more detailed version, has the following form ($k=1,\ldots$).

For $q=2$:

$$\left.\begin{aligned}\hat{\mathbf{d}}^k_{(1)} &= \left(\mathbf{I} - \mathbf{H}\left(\mathbf{H}^T\mathbf{W}^k_{(1)}\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{W}^k_{(1)}\right)\boldsymbol{\Delta}_{\hat{x}},\\ \mathbf{W}^k_{(1)} &= \operatorname{diag}\left[\left(\hat{d}^{k-1}_{1(2)}\right)^2,\ldots,\left(\hat{d}^{k-1}_{u(2)}\right)^2\right]\\ \hat{\mathbf{d}}^k_{(2)} &= \left(\mathbf{I} - \mathbf{H}\left(\mathbf{H}^T\mathbf{W}^k_{(2)}\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{W}^k_{(2)}\right)\boldsymbol{\Delta}_{\hat{x}},\\ \mathbf{W}^k_{(2)} &= \operatorname{diag}\left[\left(\hat{d}^{k}_{1(1)}\right)^2,\ldots,\left(\hat{d}^{k}_{u(1)}\right)^2\right]\end{aligned}\right\} \tag{24}$$

For $q=3$:

$$\left.\begin{aligned}\hat{\mathbf{d}}^k_{(1)} &= \left(\mathbf{I} - \mathbf{H}\left(\mathbf{H}^T\mathbf{W}^k_{(1)}\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{W}^k_{(1)}\right)\boldsymbol{\Delta}_{\hat{x}},\\ \mathbf{W}^k_{(1)} &= \operatorname{diag}\left[\left(\hat{d}^{k-1}_{1(2)}\right)^2\left(\hat{d}^{k-1}_{1(3)}\right)^2,\ldots,\left(\hat{d}^{k-1}_{u(2)}\right)^2\left(\hat{d}^{k-1}_{u(3)}\right)^2\right]\\ \hat{\mathbf{d}}^k_{(2)} &= \left(\mathbf{I} - \mathbf{H}\left(\mathbf{H}^T\mathbf{W}^k_{(2)}\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{W}^k_{(2)}\right)\boldsymbol{\Delta}_{\hat{x}},\\ \mathbf{W}^k_{(2)} &= \operatorname{diag}\left[\left(\hat{d}^{k}_{1(1)}\right)^2\left(\hat{d}^{k-1}_{1(3)}\right)^2,\ldots,\left(\hat{d}^{k}_{u(1)}\right)^2\left(\hat{d}^{k-1}_{u(3)}\right)^2\right]\\ \hat{\mathbf{d}}^k_{(3)} &= \left(\mathbf{I} - \mathbf{H}\left(\mathbf{H}^T\mathbf{W}^k_{(3)}\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{W}^k_{(3)}\right)\boldsymbol{\Delta}_{\hat{x}},\\ \mathbf{W}^k_{(3)} &= \operatorname{diag}\left[\left(\hat{d}^{k}_{1(1)}\right)^2\left(\hat{d}^{k}_{1(2)}\right)^2,\ldots,\left(\hat{d}^{k}_{u(1)}\right)^2\left(\hat{d}^{k}_{u(2)}\right)^2\right]\end{aligned}\right\} \tag{25}$$

and analogously for $q>3$. It should be emphasized that for $q=1$ the squared $M_{\text{split}(q)}$ S-transformation is the ordinary S-transformation.

4.4 Number of models

In general, it is assumed in the theory of $M_{\text{split}(q)}$ estimation that the number of competitive mathematical models, q, is known a priori (Wiśniewski 2010). However, the number of competitive models, differing by datum parameters, $\mathbf{t}_{(j)}$, is not known a priori in our multi-split congruence model (15). Although stable PRPs realize one congruence model, displaced PRPs can realize many models. Theoretically, each displacement may even realize a different congruence model. Now, the crucial question is: How many congruence models should a split model (15) contain in a given case?

Establishing—not known a priori—the appropriate number of potential mathematical models is still one of not properly solved problems of $M_{\text{split}(q)}$ estimation. Only one solution to this problem has been presented so far (Wiśniewski and Zienkiewicz 2016). In most general terms, the authors propose that the appropriate number of models should be determined with a certain control value, added to values of the original random sample. However, this solution does not yield satisfying results for a congruence model (15); therefore, a different solution is proposed in this study, based on hypothesis testing.

Let us note that if the null hypothesis: "all PRPs are stable" passes against the alternative hypothesis: "there is at least one displaced PRP" (7), it means that all PRPs may be regarded as stable. It may be assumed that all estimated PRP displacements realize one congruence model, $q=1$, and no other absorbing models are needed. Otherwise, it may be assumed that the estimated PRPs displacements realize at least two

congruence models, $q \geq 2$. Such a scenario can correspond to the multi-split null hypothesis ($j = 1,\ldots, q$):

$$\forall i, \exists j : \left\{ H_{0,i(j)} : E\left(\hat{\mathbf{d}}_{i(j)}\right) = 0 \text{ vs } H_{A,i(j)} : E\left(\hat{\mathbf{d}}_{i(j)}\right) \neq 0 \right\} \tag{26}$$

If the multi-split null hypothesis (26) passes for $q = 2$, it may be assumed that all estimated PRP displacements realize two congruence models and additional absorbing models are not needed. Otherwise, it may be assumed that estimated PRP displacements realize at least three congruence models, $q \geq 3$, and now three models should be tested. If the multi-split null hypothesis (26) passes for $q = 3$, it may be assumed that all estimated PRP displacements realize three congruence models and additional absorbing models are not needed. Otherwise, the splitting into four congruence models should be tested, etc. Finally, if the multi-split null hypothesis (26) passes, it may be assumed that all the estimated PRP displacements have found their absorbing congruence models. Splitting the congruence model (15) can now be completed.

Global and local F-tests (Caspary 2000, p. 132; Aydin 2012) can be suggested to test hypotheses (7) and (26):

$$T = \frac{\hat{\mathbf{d}}^T \mathbf{Q}_{\hat{d}}^{+} \hat{\mathbf{d}}}{r \hat{\sigma}_0^2} \leq F_\alpha(r, f) \tag{27}$$

and

$$T_{i(j)} = \frac{\hat{\mathbf{d}}_{i(j)}^T \mathbf{Q}_{\hat{d}_{i(j)}}^{-1} \hat{\mathbf{d}}_{i(j)}}{r_i \hat{\sigma}_0^2} \leq F_{\alpha_i}(r_i, f) \tag{28}$$

respectively, where $(\cdot)^+$ denotes the pseudo-inverse, e.g., $\mathbf{Q}_{\hat{d}}^{+} = (\mathbf{Q}_{\hat{d}} + \mathbf{H}\mathbf{H}^T)^{-1}\mathbf{Q}_{\hat{d}}(\mathbf{Q}_{\hat{d}} + \mathbf{H}\mathbf{H}^T)^{-1}$, $\mathbf{Q}_{\hat{d}}$ is the cofactor matrix of estimated displacement vector of all PRPs [a satisfactory cofactor matrix may be here the one which realizes the minimum trace or the one obtained from the law of variance propagation according to (22.2)], $\mathbf{Q}_{\hat{d}_{i(j)}}$ is the cofactor matrix of estimated displacement vector of point i for model j, $r = \text{rank}(\mathbf{Q}_{\hat{d}})$, $r_i = \text{rank}(\mathbf{Q}_{\hat{d}_i})$, $\hat{\sigma}_0^2 = (\hat{\sigma}_{01}^2 + \hat{\sigma}_{02}^2)/2$ is the variance factor estimator, pooled for two epochs, $f = f_1 + f_2$ is the number of degrees of freedom at two epochs, α, α_i are the significance levels (8), (10), and F_α, F_{α_i} are the critical values from Fisher's cumulative distribution function. It is noteworthy that the above test statistics, T, $T_{i(j)}$, do not have an exact Fisher's distribution for the squared $M_{\text{split}(q)}$ estimators (22.2) like for the robust M-estimators (Caspary 2000, p. 132; Nowel 2016b). Exact distributions of such statistics are not known, which is why the method of testing hypotheses (7), (9), as well as hypotheses (7), (26), does not have rigorous mathematical foundations, and the above tests may only be treated as approximate solutions. As in robust S-transformation approach, these tests are not S-system invariant.

4.5 Choice of the best model and its validation

If it turns out that there are unstable points in the group of PRPs and, in consequence, the number of congruence models is greater than one, $q > 1$, then one more question emerges: Which congruence model is expected, i.e., which realizes the matching at mutually stable PRPs?

The answer to this question can be trivial if it is assumed that a subgroup of stable PRPs is the most populated subgroup of mutually stable points in the whole group of PRPs. With this assumption, the congruence model with the largest number of statistically insignificant estimated displacements (28) should be the one which realizes the matching at mutually stable PRPs. It must be noted that the above sufficient condition for the proper solution of the squared $M_{\text{split}(q)}$ S-transformation is less restrictive than in the robust S-transformation. This is because the conventional approach requires that the number of stable points should be greater than unstable ones. However, the approach presented here only requires that stable points should make up the most populated subgroup of mutually stable points. If this condition is met, then the presented approach should be effective, even if the number of unstable points is larger than stable ones.

As it has been mentioned in the previous section, the local F-tests (28) used here to the choice of the model with the largest number of statistically insignificant estimated displacements do not have rigorous mathematical foundations and are not invariant for a change of S-system in which the point displacements are defined. Hence, the above concept can only give a preliminary identification of stable PRPs and the final validation has to be done by means of the testing method that is rigorous and independent of the S-system. Such final test can be derived from the generalized likelihood ratio test (GLRT) theory, in which some discrepancies between the observations and their functional model can be considered (Teunissen 2006). In relation to this theory, the linear(ized) null hypothesis: "all PRPs which have preliminarily been identified as stable are mutually stable":

$$H_0 : E\left(\begin{bmatrix} \mathbf{y}_1 \\ \hline \mathbf{y}_2 \end{bmatrix}\right) = \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \hline \mathbf{A}_2 & \mathbf{A}_{2,0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{d}_0 \end{bmatrix} \tag{29}$$

where $E(\cdot)$ is expectation operator, against the most relaxed linear(ized) alternative hypothesis: "there is at least one displaced point in the subgroup of PRPs which were preliminarily identified as stable":

$$H_A : E\left(\begin{bmatrix} \mathbf{y}_1 \\ \hline \mathbf{y}_2 \end{bmatrix}\right) = \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{A}_2 & \underbrace{\mathbf{A}_{2,0} \quad \mathbf{A}_{2,A}}_{\equiv \mathbf{A}_2} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{d}_0 \\ \mathbf{d}_A \end{bmatrix} \tag{30}$$

may be formulated, where $\mathbf{y}_i$ is the vector of observations (or observed minus computed values of observations) at epoch i,

$\mathbf{A}_i$ is design matrix at epoch i concerning all network points, $\mathbf{A}_{2,0}$ is design matrix at epoch 2 concerning PRPs identified as unstable and object points, $\mathbf{A}_{2,A}$ is design matrix at epoch 2 concerning PRPs identified as stable, $\mathbf{x}$ is the vector of coordinates (or increments) of all network points (the same at both epochs), $\mathbf{d}_0$ is the displacement vector of PRPs identified as unstable and object points, $\mathbf{d}_A$ is the displacement vector of PRPs identified as stable, $\mathbf{y} = [\mathbf{y}_1^T\ \mathbf{y}_2^T]^T$ is the observation vector from both epochs and $\mathbf{y} \sim P_n(E(\mathbf{y}), \mathbf{C}_y = \sigma_0^2\mathbf{Q}_y)$. According to Teunissen (2006, p. 53), the appropriate GLRT can be computed from the probability density function of $\mathbf{y}$ under H_0 and H_A. The normal distribution of $\mathbf{y}$ under H_0 reads:

$$p_n(\mathbf{y}|\mathbf{x}) = (2\pi)^{-\frac{n}{2}}\left|\sigma_0^2\mathbf{Q}_y\right|^{-\frac{1}{2}}\exp\left(-\frac{1}{2}\frac{\mathbf{e}_0^T\mathbf{Q}_y^{-1}\mathbf{e}_0}{\sigma_0^2}\right) \tag{31}$$

and under H_A it reads analogously, where n is the number of observations, and

$$\begin{aligned}\mathbf{e}_0 &= \mathbf{y} - \left[\begin{array}{cc}\mathbf{A}_1 & \mathbf{0}\\ \mathbf{A}_2 & \mathbf{A}_{2,0}\end{array}\right]\left[\begin{array}{c}\mathbf{x}\\ \mathbf{d}_0\end{array}\right],\\ \mathbf{e}_A &= \mathbf{y} - \underbrace{\left[\begin{array}{ccc}\mathbf{A}_1 & \mathbf{0} & \mathbf{0}\\ \mathbf{A}_2 & \mathbf{A}_{2,0} & \mathbf{A}_{2,A}\end{array}\right]}_{\mathbf{B}}\left[\begin{array}{c}\mathbf{x}\\ \mathbf{d}_0\\ \mathbf{d}_A\end{array}\right]\end{aligned} \tag{32}$$

According to Teunissen (2006, p. 73), the GLRT statistic reads:

$$\begin{aligned}\frac{\max\limits_{\mathbf{x},\mathbf{d}_0} p_n(\mathbf{y}|\mathbf{x},\mathbf{d}_0)}{\max\limits_{\mathbf{x},\mathbf{d}_0,\mathbf{d}_A} p_n(\mathbf{y}|\mathbf{x},\mathbf{d}_0,\mathbf{d}_A)} &= \frac{p_n\left(\mathbf{y}|\hat{\mathbf{x}},\hat{\mathbf{d}}_0\right)}{p_n\left(\mathbf{y}|\hat{\mathbf{x}},\hat{\mathbf{d}}_0,\hat{\mathbf{d}}_A\right)}\\ &= \frac{\cdots\exp\left(-0.5\left(\hat{\mathbf{e}}_0^T\mathbf{Q}_y^{-1}\hat{\mathbf{e}}_0/\hat{\sigma}_0^2\right)\right)}{\cdots\exp\left(-0.5\left(\hat{\mathbf{e}}_A^T\mathbf{Q}_y^{-1}\hat{\mathbf{e}}_A/\hat{\sigma}_0^2\right)\right)}\\ = \exp\left(-\frac{1}{2}\left(\frac{\left(\hat{\mathbf{e}}_0^T\mathbf{Q}_y^{-1}\hat{\mathbf{e}}_0 - \hat{\mathbf{e}}_A^T\mathbf{Q}_y^{-1}\hat{\mathbf{e}}_A\right) \sim \chi^2(r_A)}{\hat{\sigma}_0^2 \sim F(f_A,\infty)}\right)\right)\end{aligned} \tag{33}$$

where $\hat{\mathbf{x}}$, $\hat{\mathbf{d}}_0$, $\hat{\mathbf{d}}_A$ are the maximum likelihood estimates (for any S-system) which are identical to the LS estimates in case of a normal distribution, $\hat{\mathbf{e}}_0$, $\hat{\mathbf{e}}_A$ are the vectors of residuals under H_0, H_A, respectively, $r_A = \mathrm{rank}\left(\mathbf{A}_{2,A}\right)$ is the number of independent estimates of $\mathbf{d}_A$, $f_A = n - \mathrm{rank}(\mathbf{B})$ is the degrees of freedom, and $\hat{\sigma}_0^2 = \hat{\mathbf{e}}_A^T\mathbf{Q}_y^{-1}\hat{\mathbf{e}}_A/f_A$. Finally, the GLRT for testing H_0 against H_A can be written as:

$$T = \frac{\hat{\mathbf{e}}_0^T\mathbf{Q}_y^{-1}\hat{\mathbf{e}}_0 - \hat{\mathbf{e}}_A^T\mathbf{Q}_y^{-1}\hat{\mathbf{e}}_A}{\hat{\sigma}_0^2 r_A} \le F_\alpha(r_A, f_A) \tag{34}$$

A detailed explanation on the aforementioned derivations can be found in Teunissen (2006, ch.3-4).

The acceptance of the null hypothesis (29) indicates that the preliminary solution of squared $M_{\mathrm{split}(q)}$ S-transformation is valid, i.e., preliminarily identified stable PRPs can be treated as mutually stable. Otherwise, the preliminary solution cannot be valid and it should be rejected. It can be due to the collapse of the squared $M_{\mathrm{split}(q)}$ S-transformation, e.g., there can exist more congruence models than necessary. In this case, also the second-best congruence model and other models can give invalid solutions. Hence, for example, the solution of robust S-transformation may be recommended in such cases. However, the problem of collapse of squared $M_{\mathrm{split}(q)}$ S-transformation may be treated as an open issue and it deserves further research.

4.6 Final S-transformation

After a stable reference base is identified, it is suggested here that—like in the conventional robust S-transformation— the final S-transformation of all the network points and their covariance matrix should be conducted to the minimum trace datum, defined on the reference base, as described in Sect. 2. This will result in a better matching of both networks at stable PRPs than the original matching from the best congruence model of the squared $M_{\mathrm{split}(q)}$ S-transformation.

Finally, to better understand the presented approach, the following are the key stages of calculations with a block diagram against the conventional approach (Fig. 4).

5 Numerical experiments

The following hypothesis was put forward in an earlier study: "If the subgroup of stable PRPs is the most populated subgroup of mutually stable points, then the squared $M_{\mathrm{split}(q)}$ S-transformation will identify stable PRPs." This hypothesis was examined numerically in the paper based on simulated and real control networks. All sets of simulated observations were free of outliers. The displacements of PRPs were estimated with the squared $M_{\mathrm{split}(q)}$ S-transformation (22.2) and, additionally, with the robust S-transformation (6.2). The minimal L_1-norm of the vector of estimated displacements (5) was taken as an objective function in the robust S-transformation. The cofactor matrices of estimated displacements were always the one which realizes the minimum trace. For simplicity and without loss of generality, the significance levels for global and local hypothesis testing were assumed as $\alpha = 0.05$ and $\alpha_i = 0.001$, respectively, in each experiment.

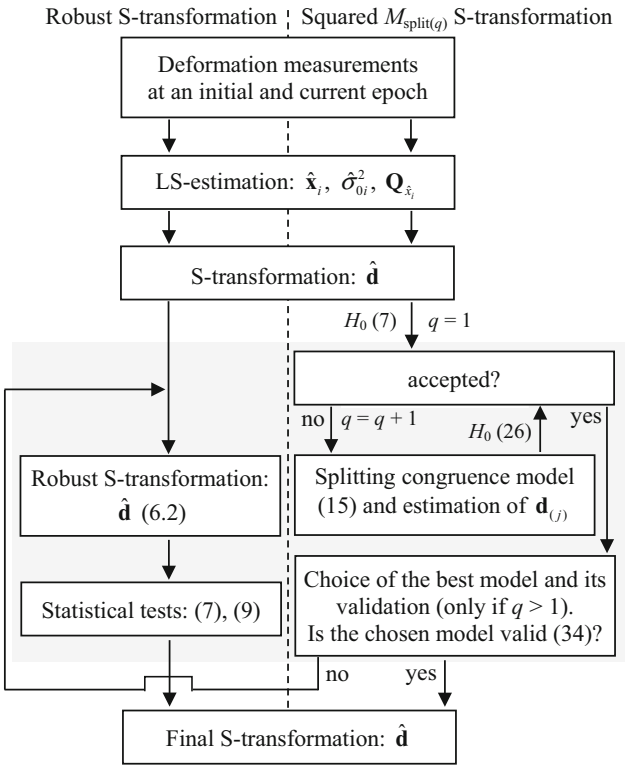

Fig. 4 Block diagram of the conventional and presented approach

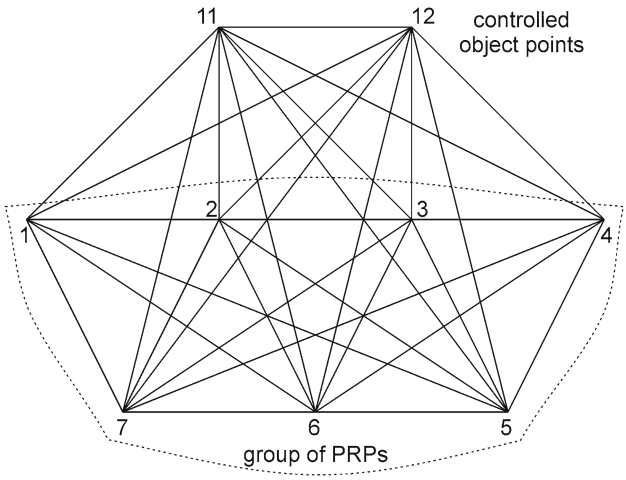

Fig. 5 Leveling control network

5.1 Experiment 1

The first experiment concerns a simulated leveling network, consisting of seven PRPs, 1–7, and two object points, 11–12 (Fig. 5).

It was assumed here that all the height differences ($n = 32$) were measured in two epochs and that measurement errors were realizations of a random variable with a normal distribution with a mean of zero and standard deviation $\sigma_{\Delta h} = 1$ mm. The pseudo-random numbers were generated from https://www.random.org/gaussian-distributions (64 numbers from a Gaussian distribution in two columns with mean 0.0, standard deviation 1.0 and using 2 significant digits, based on persistent identifier: example1). The following measurement error vectors: $\mathbf{e}_{l_1^{obs}}(32 \times 1) = [-0.35, 0.28, \ldots, -0.77]^T$ mm and $\mathbf{e}_{l_2^{obs}}(32 \times 1) = [-0.67, 0.33, \ldots, 1.70]^T$ mm, in epochs 1 and 2, respectively, were obtained in this way. The order of height differences, corresponding to the above order of measurement error values, is the following in both epochs: 1–2, 2–3, 3–4, 4–5, 5–6, 6–7, 1–7, 1–5, 1–6, 2–7, 2–6, 2–5, 3–7, 3–6, 3–5, 4–7, 4–6, 1–11, 2–11, 3–11, 4–11, 5–11, 6–11, 7–11, 1–12, 2–12, 3–13, 4–12, 5–12, 6–12, 7–12, 11–12. It means that the error values -0.35, -0.67 correspond to the height differences between points 1, 2; where point 1 defines the beginning of the leveling line and point 2 its end; next, the error values 0.28, 0.33 correspond to the height differences between points 2, 3, and so on. To simplify computations, it was assumed that before a deformation occurred, all points had the same heights. However, after a deformation in epoch 2, the height of selected PRPs was disturbed by the value of displacements. Two scenarios of displacements were considered; they had different signs in the first scenario and the same in the other. Four variants of displacements were considered in each scenario. The values of simulated, $\mathbf{d}$, and estimated displacements, $\hat{\mathbf{d}}$, together with the results of their significance tests, are shown in Table 1. The iterative processes (6.2), (22.2) were stopped after several iterative steps in both approaches. The validation test for the best congruence model in the squared $M_{split(q)}$ S-transformation (34) passed for each variant, i.e., the value of test statistic, T, was smaller than the critical value, $F_{\alpha=0.05}(r_A, 48)$, for each the best congruence model.

For the displacements with different signs, very similar values of estimated displacements and always correct results of the identification of stable PRPs were obtained in both approaches. Both approaches yielded good results even in the critical variants 3 and 4, when there were 4/7 and 5/7 unstable PRPs, respectively. It was different for displacements with the same sign. When there existed more displaced PRPs than stable ones (variants 3 and 4), the robust S-transformation failed. Estimated and simulated values of displacements were considerably different and, in consequence, the identification of stable points gave completely wrong results. However, the squared $M_{split(q)}$ S-transformation was reliable in each variant.

To illustrate some empirical properties of the squared $M_{split(q)}$ S-transformation, the graphical interpretation for the critical variant 3 from scenario 2 (4/7 PRPs are displaced with the same sign) is presented in Fig. 6. The raw displacement vector was obtained from ordinary (LS) S-transformation and has the value: $\boldsymbol{\Delta}_{\hat{x}}(9 \times 1) = [-5.7, -5.7, -5.4, -1.5, 0.6, 2.3, 15.2, -5.7, -4.8]^T$ mm. For

Table 1 Results of the stable PRPs identification

Point	Simulated displacements, **d**[mm]				Estimated displacements, $\hat{\mathbf{d}}$[mm]							
					Robust S-transformation				Squared $M_{\text{split}(q)}$ S-transformation (results of the best model)			
Variant												
	1	2	3	4	1	2	3	4	1	2	3	4
Scenario 1: Displacements have different signs												
1	0	0	0	0	*−0.2*	*−0.2*	*−0.1*	*−0.1*	*−0.2*	*−0.1*	*−0.1*	*0.0*
2	0	0	0	0	*−0.2*	*−0.2*	*0.0*	*−0.1*	*−0.2*	*−0.1*	*−0.1*	*0.0*
3	0	0	0	2	*0.1*	*0.1*	*0.2*	2.2	*0.1*	*0.2*	*0.2*	2.3
4	0	0	−4	−4	*0.0*	*0.0*	−3.9	−4.0	*0.0*	*0.1*	−3.9	−3.8
5	0	6	6	6	*0.3*	6.3	6.4	6.4	*0.3*	6.4	6.4	6.5
6	−8	−8	−8	−8	−8.2	−8.2	−8.1	−8.2	−8.2	−8.2	−8.1	−8.0
7	20	20	20	20	20.7	20.7	20.8	20.7	20.6	20.7	20.7	20.8
11	0	0	0	0	−0.2	−0.2	−0.1	−0.2	−0.3	−0.2	−0.2	−0.1
12	0	0	0	0	0.7	0.7	0.8	0.7	0.7	0.8	0.8	0.9
									Number of congruence models, q			
									3	4	5	6
Scenario 2: All displacements have positive sign												
1	0	0	0	0	*−0.4*	*−1.0*	−4.2	−4.2	*−0.2*	*−0.1*	*−0.1*	*0.0*
2	0	0	0	0	*−0.3*	*−1.0*	−4.2	−4.2	*−0.2*	*−0.1*	*−0.1*	*0.0*
3	0	0	0	2	*−0.1*	*−0.7*	−3.9	−1.9	*0.1*	*0.2*	*0.2*	2.3
4	0	0	4	4	*−0.2*	*−0.8*	*0.0*	*0.0*	*0.0*	*0.0*	4.1	4.2
5	0	6	6	6	*0.1*	5.5	2.3	2.3	*0.3*	6.4	6.4	6.5
6	8	8	8	8	7.6	7.0	3.8	3.8	7.8	7.8	7.9	7.9
7	20	20	20	20	20.5	19.9	16.7	16.7	20.7	20.7	20.8	20.8
11	0	0	0	0	−0.4	−1.0	−4.2	−4.2	−0.2	−0.2	−0.1	−0.1
12	0	0	0	0	0.5	−0.1	−3.3	−3.3	0.7	0.8	0.8	0.9
									Number of congruence models, q			
									3	3	4	5

Italic value—the PRP can be regarded as stable; the local F-test is accepted; $\hat{d}_i^2/\hat{\sigma}_0^2\sigma_{\hat{d}_i}^2 \leq F_{\alpha_i=0.001}(1, 48)$

cognitive purposes, the results of all splitting (for $q=2$, $q=3$ and $q=4$) of congruence model (15) in the squared $M_{\text{split}(q)}$ S-transformation and the results of the robust S-transformation, and ordinary S-transformation are presented. The datum parameter for displacements (here, only shift) of the best congruence model (here, $j=2$) from final splitting (here, $q=4$) in the squared $M_{\text{split}(q)}$ S-transformation is shown by means of the green line, and the datum parameters in the robust S-transformation and ordinary S-transformation are shown by means of the red and orange lines, respectively. The black points represent the PRP displacement values which can be related to the different datums (also Table 1). The confidence intervals which were derived from accepted local F-tests (28) were added for the results of squared $M_{\text{split}(q)}$ S-transformation by means of the vertical bars, $c_i = \pm\sqrt{\hat{\sigma}_0^2 \cdot \sigma_{\hat{d}_i}^2 \cdot F_{\alpha_i=0.001}(1, 48)}$.

One can note that—unlike the conventional approach (red line)—the best congruence model in the squared $M_{\text{split}(q)}$ S-transformation (green line) located the datum correctly, at actually stable PRPs (1, 2, 3) and these points were also identified as stable by means of local F-tests (Fig. 6 or Table 1). Thanks to this, the values of estimated displacements (4.1, 6.4, 7.9, 20.8 [mm], Fig. 6 or Table 1) of actually unstable PRPs (4, 5, 6, 7), i.e., the distances between these points and the green line (datum) in the above figure, were very similar to the simulated ones (4.0, 6.0, 8.0, 20.0 [mm], Table 1). Theoretically, the results would be even better for the final S-transformation to the minimum trace datum defined on the previously identified stable PRPs.

Furthermore, it is worth noting that only four congruence models turned out sufficient in the squared $M_{\text{split}(q)}$ S-transformation, instead of five expected. (One subgroup of mutually stable points and four different displaced points

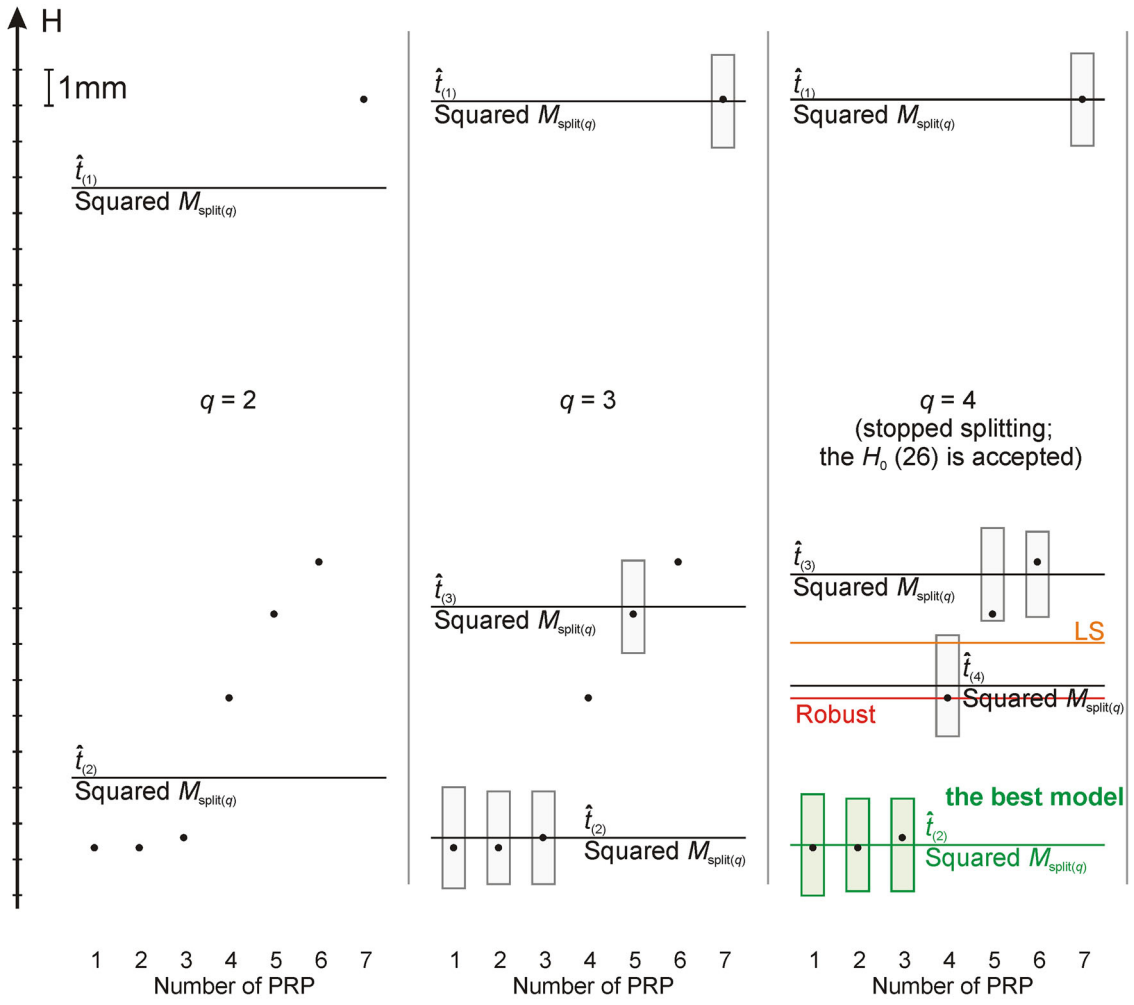

Fig. 6 Graphical interpretation of stable PRPs identification for variant 3 from scenario 2; points 1–3 are stable and points 4–7 are unstable

give together five congruence models.) However, the values of simulated displacements are quite similar in the experiment under discussion; hence, the estimated displacements of points 5, 6 can be assigned to one congruence model ($j = 3$), and this is why the null hypothesis (26) has been accepted for already four congruence models in this variant.

5.2 Experiment 2

Experiment 1 is transparent and can easily be repeated by the reader. However, since the values of simulated displacements in individual variants were established arbitrarily, the conclusions drawn from this experiment may not be convincing. Therefore, another experiment was conducted for the same network. This time, values of measurement errors and, additionally, displacements were generated independently 1000 times, with the MATLAB software. The errors were generated in accordance with the same probabilistic model as before, whereas displacements were selected randomly from the uniform interval of $d_i \in \langle 2, 25 \rangle$ mm. It was assumed two variants of displacements: Three (variant 1) and four (variant 2) randomly selected PRPs were stable, and the other PRPs were unstable. As before, two scenarios of displacements were considered; they had different signs in the first scenario and the same in the other. Table 2 presents the average values of mean absolute true errors of estimated PRP displacements, $\bar{e}_{\hat{d}} = \sum_{s=1}^{1000} e_{\hat{d},s}/1000$, where $e_{\hat{d},s} = \|\hat{\mathbf{d}} - \mathbf{d}\|_1/7$ is the mean absolute true error of estimated PRP displacements for s simulation. The invalid solution of squared $M_{\text{split}(q)}$ S-transformation was replaced/equal to the solution of robust S-transformation, as recommended in Sect. 4.5.

Additionally, Fig. 7 shows empirical distributions of mean absolute true errors of estimated PRP displacements, for the more critical scenario 2.

Generally, the experiment results are similar to those of experiment 1. For displacements with different signs, both approaches gave similar, satisfying values of errors; only for the critical case, the robust S-transformation gave significantly higher errors, $\bar{e}_{\hat{d}} = 1.35$ mm. However, the robust S-transformation—more or less—failed in the case

Table 2 Average values of mean absolute true errors of estimated PRP displacements

Number of stable PRPs	Value of error, $\bar{e}_{\hat{d}}$ [mm]		Number of valid solutions of squared $M_{\text{split}(q)}$
	Robust	Squared $M_{\text{split}(q)}$	
Scenario 1: Displacements have different signs			
4/7	0.52	0.41	964
3/7	1.35	0.52	956
Scenario 2: Displacements have positive sign			
4/7	0.98	0.43	989
3/7	6.89	0.74	987

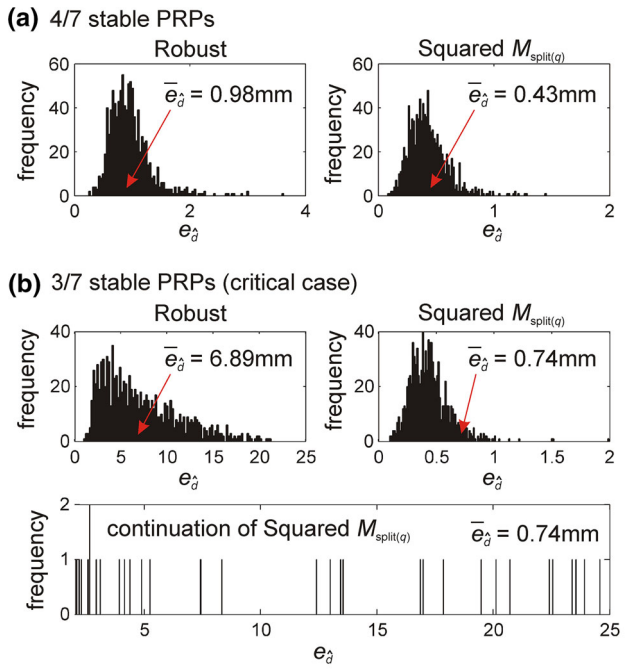

Fig. 7 Scenario 2: The empirical distributions of the average values of mean absolute true errors of PRPs estimated displacements

of displacements with the same sign. Average value of mean absolute true errors of all simulations was relatively large: $\bar{e}_{\hat{d}} = 6.89$ mm. Since the number of unstable PRPs was always greater than stable ones, the conventional approach always failed. However, the squared $M_{\text{split}(q)}$ S-transformation failed only in 39/1000 cases (Fig. 7; the last part); hence, the average value of mean absolute true errors of all simulations was only slightly larger than earlier, in scenario 1, $\bar{e}_{\hat{d}} = 0.74$ mm. Those 39 cases, when the solution breaks down, can be explained by the cases not meeting the sufficient condition for the correct solution. Let us recall that the squared $M_{\text{split}(q)}$ S-transformation requires that stable points should make up the most populated subgroup of mutually stable points. It is conceivable that in 1000 simulations,

Table 3 Results of the stable PRPs identification

Method	How many times was identified as stable:			
	4 points	3 points	2 points	1 point
Scenario 1: Displacements have different signs				
4/7 points are stable				
Robust	921	64	9	5
Squared $M_{\text{split}(q)}$	982	16	2	0
GCT	967	16	0	0
3/7 points are stable				
Robust	–	834	46	12
Squared $M_{\text{split}(q)}$	–	962	28	3
GCT	–	934	18	0
Scenario 2: Displacements have positive sign				
4/7 points are stable				
Robust	712	215	47	14
Squared $M_{\text{split}(q)}$	964	32	4	0
GCT	958	28	1	0
3/7 points are stable				
Robust	–	4	7	33
Squared $M_{\text{split}(q)}$	–	940	29	5
GCT	–	835	15	0

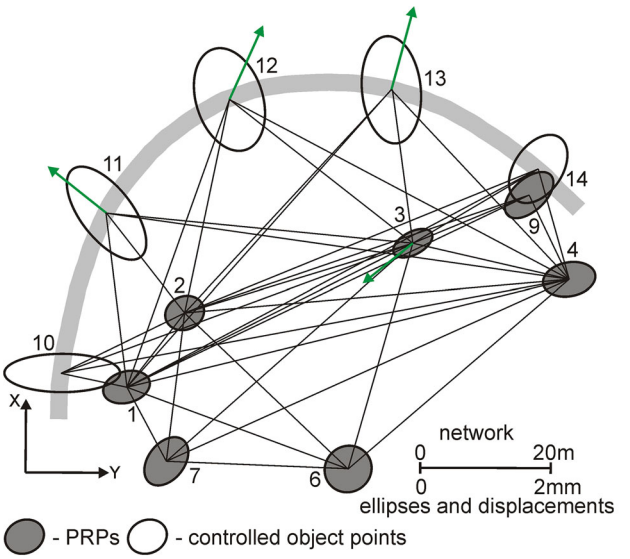

Fig. 8 Dam control network with a priori 95% confidence ellipses and simulated displacements (based on Caspary 2000, p. 146)

there were about a several dozen times when displacements of at least three PRPs were similar enough to be statistically regarded as a subgroup of mutually stable points and, in consequence, the datum parameter (here, only shift) chose this unstable location.

Additionally, as part of this experiment, Table 3 shows how many times a given method identified four, three, two and one stable PRP. For cognitive purposes, the same com-

Table 4 Results of the stable PRPs identification

Number of the point	Simulated	Estimated					
		Robust S-transformation		Squared $M_{split(q)}$ S-transformation			
				Model $j=1$		Model $j=2$ (the best model)	
	$\mathbf{d}$ [mm]	$\hat{\mathbf{d}}$[mm]	T_i	$\hat{\mathbf{d}}$[mm]	T_i	$\hat{\mathbf{d}}$[mm]	T_i
PRPs							
1		0.10	*1.22*	0.72	45.67	0.09	*0.64*
		0.16		0.57		0.03	
2		−0.08	*0.25*	0.47	13.77	−0.13	*0.78*
		−0.01		0.48		−0.10	
3	−0.50	−0.39	13.48	−0.09	*0.50*	−0.57	22.08
	−0.60	−0.60		−0.03		−0.65	
4		0.16	*0.75*	0.29	8.54	−0.11	*0.35*
		0.03		0.56		−0.04	
6		0.18	*2.54*	0.56	7.35	0.04	*3.61*
		−0.31		0.01		−0.49	
7		0.00	*1.66*	0.58	11.72	−0.03	*0.25*
		0.26		0.59		0.09	
9		−0.04	*0.03*	0.13	7.32	−0.28	*1.86*
		0.00		0.62		−0.02	
						Validation test (34) is accepted: $T=1.13 \le F_{\alpha=0.05}(9, 58)=2.05$	
Object points							
10		0.04	–	0.74	–	0.07	–
		0.28		0.71		0.16	
11	0.60	0.31	–	0.95	–	0.31	–
	−0.75	−0.73		−0.13		−0.76	
12	1.10	1.57	–	2.08	–	1.50	–
	0.50	0.41		1.14		0.44	
13	1.00	1.34	–	1.67	–	1.18	–
	0.30	0.33		1.07		0.37	
14		0.10	–	0.26	–	−0.15	–
		0.08		0.74		0.08	

Italic value—the PRP is identified as stable; the local F-test is accepted; $T_i \le F_{\alpha_i=0.01}(2, 58) = 4.99$

putations were carried out with the application of the global congruency test (GCT) for the same global significance level, $\alpha = 0.05$.

It can be claimed that the results are as expected. The results of identification of stable points in scenario 1 are satisfying and quite similar for the robust and squared $M_{split(q)}$ S-transformation; the conventional approach is only slightly less effective. However, the robust S-transformation proved completely ineffective in the critical case from scenario 2. This approach identified all the three stable points in 4/1000 cases, and it did not identify any stable points in as many as 956/1000 cases. In these cases, the minimum trace datum in final S-transformation would be surely defined on an unstable reference base; in consequence, deformations of object points would be completely misinforming. However, the efficiency of squared $M_{split(q)}$ S-transformation is still satisfying and only slightly lower than in scenario 1. Despite a critical displacement scenario, this approach identified all three stable points in 940/1000 cases.

5.3 Experiment 3

This experiment concerns the control network for the dam in Montsalvens, Switzerland, which has been considered for decades in the literature (Fig. 8).

In the 1980s, this network (in fact, its subnetwork of PRPs) was used to analyze the efficiency of various identification methods of stable PRPs at the International Federation

Table 5 Average values of mean absolute true errors of estimated PRP displacements

Number of stable PRPs	Value of error, $\bar{e}_{\hat{d}}$ [mm]		Number of valid solutions of squared $M_{\text{split}(q)}$
	Robust	Squared $M_{\text{split}(q)}$	
Scenario 1: Displacements have different signs			
5/7	0.14	0.13	951
4/7	0.50	0.30	845
3/7	1.36	0.93	514
Scenario 2: Displacements have positive sign			
5/7	0.13	0.13	948
4/7	1.48	0.45	815
3/7	2.62	1.81	446

of Surveyors (FIG) (Chrzanowski 1981). Caspary (2000, p. 149–154) also used this network to compare the results of stable PRPs identification obtained by the robust S-transformation and the global congruency test. This is why it was decided to use this network in this study to compare the results of stable PRPs identification obtained by the robust and squared $M_{\text{split}(q)}$ S-transformation, in the same way as Caspary (2000). Since the analysis concerned only PRPs, the other points were, therefore, equal to zero in optimization weight matrices, $\mathbf{W}$, (6.1), (21), in the conventional and presented approach, respectively. All input data can be found in Caspary (2000, p. 147–148). Here it is only worth noting that the considered network consists of two measurement epochs, and the two observation sets consist of 49 directions and 6 horizontal distances. The measurement errors were generated from the normal distribution with a mean of zero and standard deviation $\sigma_k = 3^{cc}$ and $\sigma_k = 0.3$ mm, for direction and distance, respectively. The considered data do not concern any critical displacement scenario; they concern the case with only one unstable PRP (point 3). The results of stable PRPs identification are shown in Table 4.

It can be noted that very similar values of estimated displacements and correct results of identification of stable PRPs were obtained in both approaches. It means that both approaches identified points 1, 2, 4, 6, 7, 9 as reference base. Since for both approaches the final S-transformation to the minimum trace datum defined on the reference base is still recommended, therefore, the final results of the analysis of deformation of all the network points would be exactly the same for both approaches.

5.4 Experiment 4

The previous experiment was based on the data which are very popular, and that experiment can easily be repeated by the reader. However, since the dataset does not concern any critical displacement scenario and it was established arbitrarily by the FIG working group, the conclusions drawn from the previous experiment may not be convincing. Therefore, the additional experiment was conducted for the control network of Montsalvens dam, analogously to the concept of experiment 2. Thus, the values of measurement errors and, additionally, displacements were generated independently 1000 times, with the MATLAB software. The measurement errors were generated in accordance with the same probabilistic model as previously, whereas displacements were selected randomly from the uniform interval of $d_{x_i}, d_{y_i} \in \langle 1, 10 \rangle$ mm. It was assumed three variants of displacements: Three (variant 1), four (variant 2) and five (variant 3) randomly selected PRPs were stable, and the other PRPs were unstable. As before, two scenarios of displacements were considered; they had different signs in the first scenario and the same in the other. Table 5 presents the average values of mean absolute true errors of estimated PRP displacements, $\bar{e}_{\hat{d}} = \sum_{s=1}^{1000} e_{\hat{d},s}/1000$, where $e_{\hat{d},s} = \|\hat{\mathbf{d}} - \mathbf{d}\|_1/14$ is the mean absolute true error of estimated PRP displacements for s simulation. The invalid solutions of squared $M_{\text{split}(q)}$ S-transformation were replaced/equal to the solution of robust S-transformation.

Additionally, Fig. 9 shows empirical distributions of mean absolute true errors of estimated PRP displacements, for the more critical scenario 2.

Generally, the results are similar to those of the previous experiments. One can see that for the non-critical cases (5/7 stable PRPs) both approaches gave similar, satisfying values of errors $\bar{e}_{\hat{d}} \approx 0.13$ mm. Then, the more the PRPs were unstable, the higher the errors obtained were in both approaches. The squared $M_{\text{split}(q)}$ S-transformation gave significantly lower errors; however, the advantage of this approach here was not as large as for 1D network (experiments 1 and 2). This could be explained by the fact that the validation test for the best congruence model (34) passed less frequently here (Table 6) than for 1D network (Table 3). In consequence, the invalid solutions of squared $M_{\text{split}(q)}$ S-transformation were replaced/equal to the solutions of robust S-transformation (Fig. 9; the green part) more frequently than before.

Additionally, as before, Table 6 shows how many times a given method identified five, four, three and two stable PRPs.

Generally, it can be concluded from the above table that the results are consistent with previous ones. Hence, a detailed study of stable PRPs identification (Table 6) is left to the reader.

Finally, for better understanding, the graphical interpretation of one critical case is presented in Fig. 10, analogously as for 1D network (Fig. 6). The 2D network of Montsalvens dam was considered where only 3/7 PRPs were stable (points 1, 2, 3) and displacements had a positive sign. The following simulated displacement vector: $\mathbf{d} = [0, 0, 0, 0, 0, 0, 1, 2, 3,$

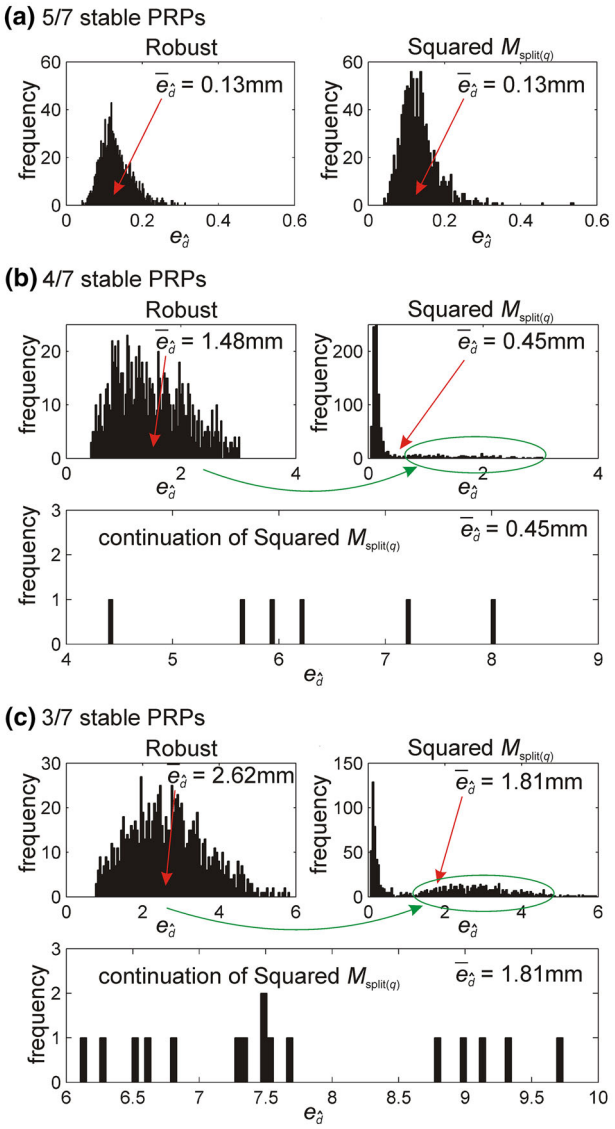

Fig. 9 Scenario 2: The empirical distributions of the average values of mean absolute true errors of PRPs estimated displacements

2, 1, 2, 4, 1,…]^T mm was used (green arrows). The measurement errors were generated in accordance with the same probabilistic model as previously. The raw displacement vector was obtained from the ordinary S-transformation (orange lines). The results of all splitting ($q=2$ and $q=3$) of congruence model (15) in the squared $M_{\text{split}(q)}$ S-transformation (black lines) and the results of the robust S-transformation (red lines) are presented as well. The confidence ellipses which were derived from local F-tests (28) are also depicted.

One can see that only three congruence models ($q=3$) turned out sufficient in the squared $M_{\text{split}(q)}$ S-transformation. It means the null hypothesis (26) has already passed for three models. The estimated displacements of all seven PRPs are located inside their confidence regions here; model $j=1$ contains two statistically insignificant estimated displacements (points 4, 7), model $j=2$ contains three such displacements (points 1, 2, 3), and model $j=3$ contains two such displacements (points 6, 9). The model $j=2$ is the best model because it contains the largest number of statistically insignificant estimated PRP displacements. This model turned out also valid (34).

Table 6 Results of the stable PRPs identification

Method	How many times was identified as stable:			
	5 points	4 points	3 points	2 points
Scenario 1: Displacements have different signs				
5/7 points are stable				
Robust	791	125	61	21
Squared $M_{\text{split}(q)}$	874	80	35	11
4/7 points are stable				
Robust	–	441	108	179
Squared $M_{\text{split}(q)}$	–	665	131	143
3/7 points are stable				
Robust	–	–	185	136
Squared $M_{\text{split}(q)}$	–	–	412	210
Scenario 2: Displacements have positive sign				
5/7 points are stable				
Robust	934	49	16	1
Squared $M_{\text{split}(q)}$	912	59	23	6
4/7 points are stable				
Robust	–	0	0	24
Squared $M_{\text{split}(q)}$	–	622	106	82
3/7 points are stable				
Robust	–	–	0	0
Squared $M_{\text{split}(q)}$	–	–	278	123

As can be seen from the right graphs—unlike the conventional approach (red line)—the best congruence model in the squared $M_{\text{split}(q)}$ S-transformation realizes the expected matching at actually stable PRPs. In consequence, these points were identified as stable by means of local F-tests (28); the estimated displacements of these points are inside their confidence regions (green ellipses). The actually unstable points do not have such a negative effect on matching/datum parameters of expected congruence model as in robust S-transformation because these points are realizations of other congruence models ($j=1$ and $j=3$) and their attention is absorbed by those matching/datum parameters. Thanks to this, the vectors of estimated displacements of actually unstable points are very similar to the simulated ones. Theoretically, the results would be even better for the final

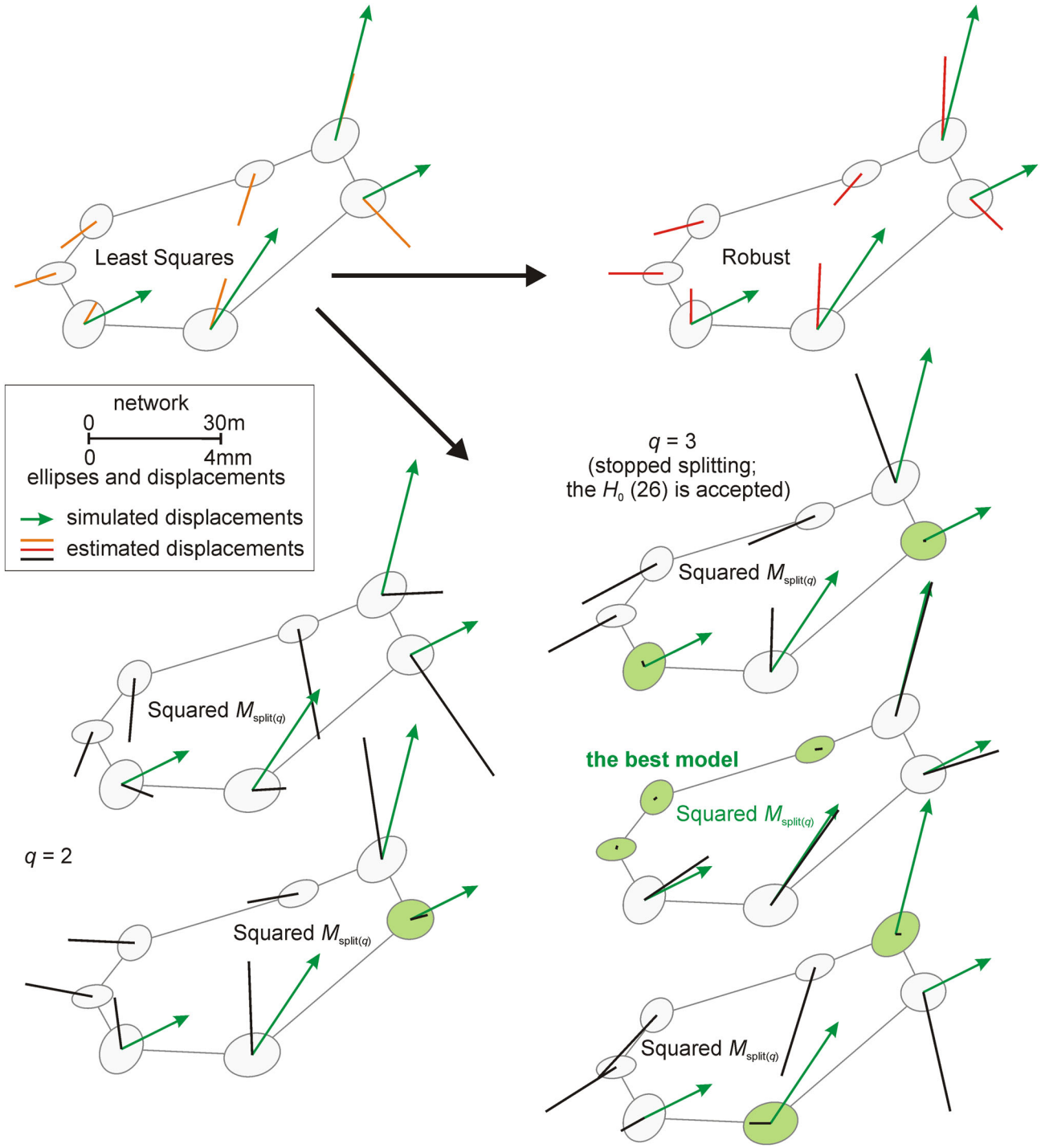

Fig. 10 Graphical interpretation of stable PRPs identification; 3/7 points are stable

S-transformation to the minimum trace datum defined on the previously identified stable points.

6 Summary and conclusions

The identification of stable PRPs is a key issue in conventional deformation analysis. Since geodetic control networks have a datum defect (free networks), the accurate estimation of PRP displacements and later efficient identification of stable PRPs is a quite difficult task. For example, when there are more unstable PRPs than stable ones this task is not often possible by means of the conventional robust S-transformation. It is still a challenge for surveyors and geodesists.

This paper presents and pretests a new approach to the S-transformation of control network deformations. The idea behind this approach comes from the theory of squared $M_{\text{split}(q)}$ estimation and lies in the non-conventional assumption that between the control networks adjusted in two considered epochs can simultaneously exist—not one as in the conventional robust S-transformation but—many congruences (matchings) which differ by the datum/transformation parameters. It is assumed that one model realizes the expected congruence, i.e., the congruence at a subgroup of stable PRPs, and other models can realize different congruences at unstable PRPs. Thanks to this, the robustness of the presented approach can be very high, because the unstable PRPs can be absorbed by other models. To establish what the number of congruence models is in a given case and whether the chosen/best model is valid, the statistical hypothesis tests were suggested.

The paper proves the hypothesis that if stable points make up the most populated subgroup of mutually stable points in the group of PRPs and measurement errors do not mask or generate displacements, then the presented approach can transform the estimated raw displacements to an expected datum (in a geometrical sense, can realize the expected matching at stable PRPs) and the results of identification of stable PRPs can be correct. It is noteworthy that the above sufficient condition is much less restrictive than the sufficient condition in the conventional robust S-transformation. Numerical experiments showed that the suggested S-transformation—unlike the conventional approach—can be effective even in critical cases, when there are more unstable PRPs than stable ones, and the sign of all displacements is the same. Owing to this, correct identification of stable PRPs and, in consequence, the correct final estimation of controlled object point displacements are possible in cases when it has not been possible so far. Hence, the squared $M_{\text{split}(q)}$ S-transformation presented here seems to be an interesting and useful alternative to the more conventional robust S-transformation and, as such, deserves further research. For example, since the presented method allows to identify many subgroups of mutually stable points (there are no outlying points in the multi-split congruence model), it can have a wider application in geodetic deformation analysis.

Finally, it is advisable to know that the presented approach to deformation analysis, by means of M_{split} estimation, is completely different from the one presented in the literature and referenced in Introduction section. Most of all, that simple concept, based on the M_{split} estimation, does not deal with the identification of stable PRPs and focuses only on observation residuals which disclose some information about unstable points, which can be treated as outliers.

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