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News on Baer–Nunziato-type model at pressure equilibrium

Abstract A six-equation Baer–Nunziato model at pressure equilibrium for two ideal gases is derived from a full non-equilibrium model by applying an asymptotic pressure expansion. Conditions on the interfacial pressure are provided that ensure hyperbolicity of the reduced model. Closure conditions for the relaxation terms are given that ensure consistency of the model with the second law of thermodynamics.

1 Introduction

Compressible two-fluid flows where the fluid is a mixture of two components possibly at different phase have a wide range of applications, for instance, a mixture of reacting gases or a mixture of a liquid and a gas. In the literature, several approaches for the modeling of such flows are available. Our interest is in so-called Baer–Nunziato-type models. These can be derived by the ensemble averaging procedure of Drew [4]. Note that ensemble averaging differs from volume averaging, time averaging or statistical averaging. It accounts for the uncertainty in the exact location of particular constituents at a particular time and, thus, allows the interpretation of flow phenomena in terms of repeatability of flows. A comprehensive introduction to these models can be found in the classical book of Drew and Passman [5].

Originally, Baer and Nunziato [1] proposed a two-fluid model for detonation waves in granular explosives. This model is a full non-equilibrium model where each component has its own pressure, velocity and temperature and is governed by its own set of fluid equations. In contrast to other two-fluid models separating the fluids, here the fluids are assumed to coexist in each location due to an averaging procedure. The presence of more than one component in the mixture is modeled by the concept of volume fractions α_k describing the ratio of volume of component k to the total volume in a ball in the limit when the volume of the ball tends to zero, see [5]. It was modified and generalized by several authors. For instance, Saurel and Abgrall [18] included relaxation terms for the pressure and the velocities of the components. By instantaneous relaxation procedures,

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equilibrium values for the pressure and the velocity can be found. Using further relaxation procedures to drive the temperatures and the Gibbs free energies into equilibrium, the mass transfer between the two fluids can be modeled, see Abgrall et al. [16,20] or Zein et al. [30].

According to [14,19,20] mechanical relaxation, thermal relaxation and relaxation of the chemical potentials proceed on different time scales. Therefore, reduced models have been derived assuming zero relaxation times for some of the non-equilibrium quantities relaxing much faster than the remaining quantities, see [10]. Thus, the stiffness inherent in the non-equilibrium model is avoided that allows for a much faster numerical simulation of the reduced model. Typically, the reduced models are classified by the number of equations in case of *two* fluids in *one* space dimension. Usually reduced models suffer from some short-comings. For instance, conservation of energy might be violated or the system loses its hyperbolicity. A detailed discussion of these models is beyond the scope of this work. For this purpose, the interested reader is referred to [29] and the references cited therein.

In [6,12], a hierarchy of two-fluid models derived from the Baer–Nunziato model [1] is investigated where mechanical, thermal and chemical relaxation is assumed to proceed in different order. This hierarchy splits into two branches of models distinguished by the assumption of dynamical velocity equilibrium and local velocity equilibrium where either both fluids have the *same* velocity or the fluids have *different* velocities that coincide for a particular state.

Since Baer–Nunziato-type models are averaged models, the resulting balance laws are underdetermined. To compensate for the loss in information *closure conditions* have to be imposed additionally. Therefore, there exists a rich literature on Baer–Nunziato-type models, see, for instance, [3,7–9,13,17] investigating their properties, in particular, *hyperbolicity* and *thermodynamical consistency*. These properties are essential from an analytical, numerical and physical point of view. For instance, hyperbolicity is needed in the analysis of the Riemann problem and the construction of (approximate) Riemann solvers whereas a nonnegative entropy production ensures consistency with the second law of thermodynamics and also might ensure well-posedness so not yet mathematically rigorous verified. The analysis helps to identify physically admissible closure conditions for the interfacial pressure and the interfacial velocity that occur as model parameters and cannot be closed due to the averaging procedure.

We are particularly interested in a model where both fluids are assumed to have the *same* pressure, while other quantities may be in non-equilibrium. This model may be applied whenever the pressure relaxation time is much faster than the relaxation terms for the velocities, temperatures and chemical potentials. Such a model of Baer–Nunziato type with highest level of detail consists of six balance equations where each fluid has its own density and its own velocity. Further variables may be the common pressure and the volume fraction of one of the fluids.

In the literature, it is often claimed that the so-called classical six-equation model, which denotes the six-equation pressure-equilibrium Baer–Nunziato model, is ill-posed, see, for instance, [15,21].

The classical six-equation model is discussed in detail in the review article of Stewart and Wendroff [21]. There it is derived by some averaging procedure. This model differs from the model one obtains by taking the asymptotic limit for the pressure relaxation procedure in the full Baer–Nunziato model as we will discuss below. Moreover, source terms corresponding to relaxation processes for velocity, temperature and chemical potentials are missing. In fact, this model is ill-posed. It may have complex eigenvalues if the velocities are in non-equilibrium.

In the literature, several results can be found how to enforce hyperbolicity in the model. For instance, Toumi and Raymond, cf. [24,25], consider a two-fluid model for mass, momentum and total enthalpy without source terms but non-conservative product in the momentum equation with equal pressures. They claim that by choosing a particular interfacial pressure hyperbolicity can be achieved. Unfortunately, a proof as well as a physical explanation is missing.

In [22], Tiselj and Petelin introduce a virtual mass term that contains derivatives of the velocities to enforce their model to be hyperbolic. However, this model is not asymptotically correct, i.e., it cannot be derived from the full non-equilibrium model by a Chapman–Enskog-like asymptotic expansion. Moreover, also here a physical motivation is missing.

Another model is discussed by Toro [23] who investigates ignition and combustion of reactive solid particles in an expanding combustion chamber. The system also consists of six equations, two balances of mass and two balances of momentum for both the solid and the gas. In addition, an energy balance equation for the gas is considered while an evolution equation for the number of solid particles is used. The model exhibits relaxation terms and is not hyperbolic as well.

Nowadays, pressure-equilibrium Baer–Nunziato-type models seem not to be considered anymore in the literature. Probably, it is common sense that models of this type without modifications are not hyperbolic. In this work, we will show that providing the correct asymptotic limit from a Chapman–Enskog-like asymptotic expansion and using an appropriate closure for the interfacial velocity and the interfacial pressure the Baer–Nunziato model leads to a novel hyperbolic pressure-equilibrium model. To the best of our knowledge, it has not yet been discussed in the literature. In particular, up to now this model was not derived. This is surprising because a similar asymptotic procedure has been performed in [10] to derive asymptotically correct models at velocity equilibrium and velocity–pressure equilibrium.

Our main objective is to verify that the pressure-equilibrium Baer–Nunziato model is *not* generally ill-posed. We exemplify this by means of two ideal gases. At first glance, this might be considered a contradiction to the underlying *immiscibility* assumption of Baer–Nunziato models because gases are known to mix perfectly at equilibrium, i.e., they are *miscible*. We will comment on this in more detail below. Note that using general equations of state makes the analysis much more cumbersome without any appreciable benefit. Moreover, since we are using specific equations of state, we are able to derive explicit expressions for the relaxation terms as well as for the constraints on the closure terms.

The paper is organized as follows. First of all, in Sect. 2 we derive the pressure-equilibrium Baer–Nunziato model starting from a full non-equilibrium Baer–Nunziato model by investigating the limit of an asymptotic pressure expansion. In Sect. 3, we investigate hyperbolicity of the resulting six-equation-model. In particular, we derive constraints on the closure of the interfacial pressure ensuring hyperbolicity. Several choices for the interfacial pressure and their influence on the hyperbolicity regime. Furthermore, in Sect. 4 we derive constraints on the relaxation terms that ensure nonnegativity of the entropy production, i.e., the model is in agreement with the second law of thermodynamics. We conclude with a summary of our findings in Sect. 5.

2 Derivation of the pressure-equilibrium model

To derive the pressure-equilibrium Baer–Nunziato model, we start with the full non-equilibrium model. Since this model is known to be Galilean invariant, see, for instance, [13], the flux function is the same for all directions $\boldsymbol{\omega}$ when expanding the velocity vector with respect to a particular direction $\boldsymbol{\omega}$ and its normal directions. For the investigation of hyperbolicity, it is thus sufficient to consider only a quasi-one-dimensional flow, for instance, in the x-direction. Since the evolution equations for all the $d-1$ normal momentum components (with d denoting the spatial dimension) correspond to linearly degenerated fields we may confine ourselves only to the genuinely one-dimensional case as is typically done in the literature. Therefore, we consider the original model introduced by Baer and Nunziato in 1986, see [1], which is given by

$$\frac{\partial \left(\alpha_k \rho_k \right)}{\partial t} + \frac{\partial \left(\alpha_k \rho_k v_k \right)}{\partial x} = (-1)^{k+1} \mathcal{C}, \tag{2.1a}$$

$$\frac{\partial \left(\alpha_k \rho_k v_k \right)}{\partial t} + \frac{\partial \left(\alpha_k \rho_k v_k^2 + \alpha_k p_k \right)}{\partial x} = (-1)^{k+1} \left(P_{\mathrm{I}} \frac{\partial \alpha_1}{\partial x} + \mathcal{M} \right), \tag{2.1b}$$

$$\frac{\partial \left(\alpha_k \rho_k E_k \right)}{\partial t} + \frac{\partial \left(\alpha_k (\rho_k E_k + p_k) v_k \right)}{\partial x} = (-1)^k \left(P_{\mathrm{I}} \mathcal{F} - P_{\mathrm{I}} V_{\mathrm{I}} \frac{\partial \alpha_1}{\partial x} - \mathcal{E} \right), \tag{2.1c}$$

$$\frac{\partial \alpha_1}{\partial t} + V_{\mathrm{I}} \frac{\partial \alpha_1}{\partial x} = \frac{\mathcal{C}}{\varrho} + \mathcal{F}. \tag{2.1d}$$

Here $\alpha_k, \rho_k, v_k, p_k, E_k$ denote the *volume fractions*, the *densities*, the *velocities*, the *pressures* and the *specific total energies* of the two components $k = 1, 2$. The volume fractions satisfy the saturation condition

$$\alpha_1 + \alpha_2 = 1. \tag{2.2}$$

The specific total energies E_k are related to the *specific internal energies* e_k via

$$E_k = e_k + \frac{1}{2} v_k^2.$$

The pressures p_k as well as the *temperatures* T_k are related to the densities and specific internal energies by the equation of state. In the following, we will consider the ideal gas law which is given by the relations

$$e_k = c_{v,k} T_k \quad \text{and} \quad p_k = \rho_k e_k (\gamma_k - 1) \equiv \rho_k e_k \kappa_k. \tag{2.3}$$

The quantities $c_{v,k}$ and γ_k are material parameters, the *specific heat capacity at constant volume* and *the adiabatic exponent*, respectively. Moreover, the interfacial velocity V_I and the interfacial pressure P_I are model parameters. For more details on this, we refer to [13].

The expressions $\mathcal{C}, \mathcal{M}, \mathcal{F}, \mathcal{E}$ describe the exchange of mass, momentum and energy between the components. Here, $\mathcal{C} = \mathcal{C}(\mu_2 - \mu_1)$ is a function of the difference of the *chemical potentials* μ_k of the components, see [10]. Furthermore, we have

$$\mathcal{F} = \alpha_1\alpha_2\theta(p_1 - p_2), \quad \mathcal{M} = \mathcal{C}V_I + \left(\nu + \frac{1}{2}\mathcal{C}\right)(v_2 - v_1), \tag{2.4a}$$

$$\mathcal{E} = \mathbb{E}\mathcal{C} + (\mathcal{M} - \mathcal{C}V_I)V_I + \mathcal{H}(T_2 - T_1) \tag{2.4b}$$

with the nonnegative relaxation parameters θ, ν and $\mathcal{H}$ for pressure, velocity and temperature, respectively. The quantities $\mathbb{E}$ and ϱ are additional model parameters. Choosing $P_I = p_2$, $V_I = v_1$, $\varrho = \rho_1$ and $\mathbb{E} = E_1$ the model coincides with the model given in [10].

For the derivation of the pressure-equilibrium model we perform a Chapman–Enskog-like analysis where we proceed as follows: Based on the balance equations for total energy, we determine balance equations for the internal energies as an intermediate step and afterward balance equations for the pressures and the pressure difference. Using an asymptotic expansion for the pressures where the pressure relaxation parameter θ tends to infinity, while the pressures p_1 and p_2 tend to the equilibrium pressure p, we find an expression for $\mathcal{F}$ which has to be applied in the energy balance equations (2.1c) as well as in the transport equation (2.1d). As a consequence, one of these three equations becomes redundant and we conclude with the six-equation pressure-equilibrium model.

First we derive

$$\frac{\partial}{\partial t}(\alpha_k\rho_k e_k) + \frac{\partial}{\partial x}(\alpha_k\rho_k e_k v_k) + \alpha_k p_k \frac{\partial}{\partial x}v_k$$
$$+ (-1)^k\left((V_I - v_k)P_I\frac{\partial \alpha_1}{\partial x} - P_I\mathcal{F}\right) = \mathbb{E}_{0k}, \tag{2.5}$$

with

$$\mathbb{E}_{0k} = (-1)^{k+1}\left((V_I - v_k)\mathcal{M} + (T_2 - T_1)\mathcal{H} + \mathcal{C}\left(\mathbb{E} - V_I^2 + \frac{v_k^2}{2}\right)\right). \tag{2.6}$$

From the equation of state (2.3), we then deduce

$$\frac{\partial p_k}{\partial t} + \gamma_k p_k \frac{\partial v_k}{\partial x} + v_k \frac{\partial p_k}{\partial x} + \frac{(-1)^k}{\alpha_k}\left((p_k + \kappa_k P_k)(V_I - v_k)\frac{\partial \alpha_1}{\partial x} - (p_k + \kappa_k P_I)\mathcal{F}\right)$$
$$= \frac{\kappa_k}{\alpha_k}\mathbb{E}_{0k} + (-1)^k\frac{p_k}{\alpha_k}\frac{\mathcal{C}}{\varrho}. \tag{2.7a}$$

Finally, we obtain a balance equation for the pressure difference that is given by

$$\frac{\partial}{\partial t}(p_1 - p_2) + D_1 + D_2 + G\mathcal{F} = \mathbb{E}_p \quad \text{with} \tag{2.8a}$$

$$D_1 = \left(\gamma_1 p_1 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial p_1}{\partial x}\right) - \left(\gamma_2 p_2 \frac{\partial v_2}{\partial x} + v_2 \frac{\partial p_2}{\partial x}\right), \tag{2.8b}$$

$$D_2 = \frac{1}{\alpha_1\alpha_2}\left(\alpha_2\left(p_1 + \kappa_1 P_I\right)\left(v_1 - V_I\right) - \alpha_1\left(p_2 + \kappa_2 P_I\right)\left(V_I - v_2\right)\right)\frac{\partial \alpha_1}{\partial x}, \tag{2.8c}$$

$$G = \left(\frac{1}{\alpha_1}\left(p_1 + \kappa_1 P_I\right) + \frac{1}{\alpha_2}\left(p_2 + \kappa_2 P_I\right)\right), \tag{2.8d}$$

$$\mathbb{E}_p = \left(\frac{\kappa_1}{\alpha_1}\mathbb{E}_{01} - \frac{p_1}{\alpha_1}\frac{\mathcal{C}}{\varrho}\right) - \left(\frac{\kappa_2}{\alpha_2}\mathbb{E}_{02} + \frac{p_2}{\alpha_2}\frac{\mathcal{C}}{\varrho}\right). \tag{2.8e}$$

For the asymptotic expansion, we use the following notation

$$p_k \sim p_k^0 + \frac{1}{\theta} p_k^1 + \frac{1}{\theta^2} p_k^2 + \cdots, \tag{2.9a}$$

$$P_{\mathrm{I}} \sim P_{\mathrm{I}}^0 + \frac{1}{\theta} P_{\mathrm{I}}^1 + \frac{1}{\theta^2} P_{\mathrm{I}}^2 + \cdots. \tag{2.9b}$$

From this, we infer

$$\mathcal{F} \sim \theta \mathcal{F}^0 + \mathcal{F}^1 + \frac{1}{\theta} \mathcal{F}^2 + \cdots \tag{2.10a}$$

$$\mathcal{F}^0 := \alpha_1 \alpha_2 (p_1^0 - p_2^0), \tag{2.10b}$$

$$\mathcal{F}^1 := \alpha_1 \alpha_2 (p_1^1 - p_2^1). \tag{2.10c}$$

Inserting these expressions into Eq. (2.8a) and ordering the terms by their orders of θ leads to the following relation for the pressure difference $\Delta p := p_1 - p_2$

$$\frac{\partial \Delta p^0}{\partial t} + \frac{1}{\theta} \frac{\partial \Delta p^1}{\partial t} + D_1^0 + \frac{1}{\theta} D_1^1 + D_2^0 + \frac{1}{\theta} D_2^1 + G^0 \mathcal{F}^0 + G^0 \mathcal{F}^1 + \frac{1}{\theta} G^1 \mathcal{F}^0$$
$$= \mathbb{E}_p^0 + \frac{1}{\theta} \mathbb{E}_p^1 + \mathcal{O}(\theta^{-2}) \tag{2.11}$$

with

$$D_1^0 = \left(\gamma_1 p_1^0 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial p_1^0}{\partial x} \right) - \left(\gamma_2 p_2^0 \frac{\partial v_2}{\partial x} + v_2 \frac{\partial p_2^0}{\partial x} \right), \tag{2.12a}$$

$$D_2^0 = \frac{1}{\alpha_1 \alpha_2} \left((\alpha_2 (p_1^0 + \kappa_1 P_{\mathrm{I}}^0) (v_1 - V_{\mathrm{I}}) - \alpha_1 \left(p_2^0 + \kappa_2 P_{\mathrm{I}}^0 \right) (V_{\mathrm{I}} - v_2) \right) \frac{\partial \alpha_1}{\partial x}, \tag{2.12b}$$

$$G^0 = \left(\frac{1}{\alpha_1} \left(p_1^0 + \kappa_1 P_{\mathrm{I}}^0 \right) + \frac{1}{\alpha_2} \left(p_2^0 + \kappa_2 P_{\mathrm{I}}^0 \right) \right), \tag{2.12c}$$

$$\mathbb{E}_p^0 = \left(\frac{\kappa_1}{\alpha_1} \mathbb{E}_{01}^0 + \frac{p_1^0}{\alpha_1} \frac{\mathcal{C}}{\varrho} \right) - \left(\frac{\kappa_2}{\alpha_2} \mathbb{E}_{02}^0 - \frac{p_2^0}{\alpha_2} \frac{\mathcal{C}}{\varrho} \right), \tag{2.12d}$$

$$\mathbb{E}_{0k}^0 = (-1)^{k+1} \left((V_{\mathrm{I}} - v_k)\mathcal{M} + (T_2 - T_1)\mathcal{H} + (\mathbb{E} - V_{\mathrm{I}}^2 + v_k^2/2)\mathcal{C} \right), \tag{2.12e}$$

$$\mathcal{F}^1 = \frac{\mathbb{E}_p^0 - D_1^0 - D_2^0}{G^0}. \tag{2.12f}$$

The relaxation parameter θ is assumed to be large. Thus, elimination of the leading order terms corresponding to θ^1 and θ^0 in (2.11) provides us with conditions on the pressures p_k^0 and $\mathcal{F}^1$. From the highest-order terms, we obtain

$$\mathcal{F}^0 = 0. \tag{2.13}$$

This implies

$$p_1^0 = p_2^0 = p \quad \text{and} \quad P_{\mathrm{I}}^0 = P, \tag{2.14}$$

where p and P are referred to as the equilibrium pressure and the equilibrium interfacial pressure, respectively. Note that $P = p$ if P_{I} is chosen as a convex combination of p_1 and p_2; otherwise, also different equilibrium states are possible.

From the next order terms in (2.11), we then conclude

$$D_1^0 = \left(\gamma_1 p \frac{\partial v_1}{\partial x} + v_1 \frac{\partial p}{\partial x}\right) - \left(\gamma_2 p \frac{\partial v_2}{\partial x} + v_2 \frac{\partial p}{\partial x}\right), \tag{2.15a}$$

$$D_2^0 = \frac{1}{\alpha_1 \alpha_2} \left(\alpha_2 \left(p + \kappa_1 P\right) \left(v_1 - V_I\right) - \alpha_1 \left(p + \kappa_2 P\right) \left(V_I - v_2\right)\right) \frac{\partial \alpha_1}{\partial x}, \tag{2.15b}$$

$$G^0 = \frac{1}{\alpha_1 \alpha_2} \left(p + \left(\alpha_2 \kappa_1 + \alpha_1 \kappa\right) P\right), \tag{2.15c}$$

$$\mathbb{E}_p^0 = \frac{1}{\alpha_1 \alpha_2} \left(\alpha_2 \kappa_1 \mathbb{E}_{01} - \alpha_1 \kappa_2 \mathbb{E}_{02}\right) - \frac{1}{\alpha_1 \alpha_2} p \frac{\mathcal{C}}{\varrho}, \tag{2.15d}$$

$$\mathbb{E}_{0k}^0 = (-1)^{k+1} \left(\left(V_I - v_k\right) \mathcal{M} + \left(T_2 - T_1\right) \mathcal{H} + \left(\mathbb{E} - V_I^2 + v_k^2/2\right) \mathcal{C}\right), \tag{2.15e}$$

$$\mathcal{F}^1 = \frac{\mathbb{E}_p^0 - D_1^0 - D_2^0}{G^0}. \tag{2.15f}$$

The last Eq. (2.15f) has to be inserted into system (2.1). This yields seven equations. The balances of mass and momentum are independent. One of the remaining three equations is redundant. Accordingly, there are several possibilities to select six equations for the pressure-equilibrium Baer–Nunziato model. Of course, all admissible choices are equivalent due to the correct asymptotic limit.

We prefer to use two energy balance equations due to the symmetry of the resulting system and the similarity to models discussed in the literature, see, for instance, [21]:

$$\frac{\partial \left(\alpha_k \rho_k\right)}{\partial t} + \frac{\partial \left(\alpha_k \rho_k v_k\right)}{\partial x} = (-1)^{k+1} \mathcal{C}, \tag{2.16a}$$

$$\frac{\partial \left(\alpha_k \rho_k v_k\right)}{\partial t} + \frac{\partial \left(\alpha_k \rho_k v_k^2 + \alpha_k p\right)}{\partial x} + (-1)^k P \frac{\partial \alpha_1}{\partial x} = (-1)^{k+1} \mathcal{M}, \tag{2.16b}$$

$$\frac{\partial \left(\alpha_k \rho_k E_k\right)}{\partial t} + \frac{\partial \left(\alpha_k \left(\rho_k E_k + p\right) v_k\right)}{\partial x} + (-1)^k P \left(V_I \frac{\partial \alpha_1}{\partial x} + \frac{D_1^0 + D_2^0}{G^0}\right)$$
$$= (-1)^k \left(P \frac{\mathbb{E}_p^0}{G^0} - \mathcal{E}\right). \tag{2.16c}$$

The system (2.16) needs to be closed by appropriate models for p, P and V_I. The equilibrium pressure is determined by the mixture pressure that for an ideal gas reads

$$p = (\gamma_1 - 1)\alpha_1 \rho_1 e_1 + (\gamma_2 - 1)\alpha_2 \rho_2 e_2, \tag{2.17}$$

Note that in the system (2.16) the volume fraction α_1 is a dependent variable because for an ideal gas it holds $p_1 = p$ at pressure equilibrium. Then, we infer from (2.3)

$$\alpha_1 = \frac{p}{(\gamma - 1)\rho_1 e_1} \tag{2.18}$$

satisfying the balance law for the volume fraction

$$\frac{\partial \alpha_1}{\partial t} + V_I \frac{\partial \alpha_1}{\partial x} + \frac{D_1^0 + D_2^0}{G^0} = \frac{\mathcal{C}}{\varrho} + \frac{\mathbb{E}_p^0}{G^0}. \tag{2.19}$$

Here, we employ the balances for internal energy

$$\frac{\partial \left(\alpha_k \rho_k e_k\right)}{\partial t} + \frac{\partial \left(\alpha_k \rho_k e_k v_k\right)}{\partial x} + \alpha_k p \frac{\partial v_k}{\partial x} + (-1)^k P \left(\left(V_I - v_k\right) \frac{\partial \alpha_1}{\partial x} + \frac{D_1^0 + D_2^0}{G^0}\right)$$
$$= (-1)^k \left(P \frac{\mathbb{E}_p^0}{G^0} - \mathcal{E} - \frac{v_k^2}{2} \mathcal{C} + v_k \mathcal{M}\right). \tag{2.20a}$$

The closure of the equilibrium interfacial pressure and the interfacial velocity will be discussed in the subsequent sections.

Remark 1 (Other Baer–Nunziato-type models at pressure equilibrium) When comparing the 6-equation model (2.16) with other well-known models at pressure equilibrium, cf. [21,24,25,28], we note that besides the relaxation terms on the right-hand side the term $(D_1^0 + D_2^0)/G^0$ is missing in the energy equation (2.16c). Thus, these models are not asymptotically correct derived from the non-equilibrium model (2.1), i.e., they are not the limit of an Chapman–Enskog-like asymptotic expansion. Therefore, the system (2.16) closed by (2.17) and (2.18) is a new model.

Remark 2 (Immiscibility and miscibility) The Baer–Nunziato model describes the multi-component flow of *immiscible* fluids. Thus, at equilibrium each component fills a different portion of the accessible volume separated by interfaces. For our analysis, we deliberately have chosen a mixture of ideal gases. This seems to contradict the model since gases are known to mix perfectly, i.e., at equilibrium it must hold $p = p_1 + p_2$ instead of $p = p_1 = p_2$. To our opinion, this contradiction can be resolved by reinterpreting the notions of pressures, densities and volume fractions. We recall that in the equilibrium system (2.16) it holds $e_k = c_{v,k} T_k$ and $p = \rho_k e_k (\gamma_k - 1)$. Introducing the notations $\hat{\rho}_k := \alpha_k \rho_k$ and $\hat{p}_k := \hat{\rho}_k e_k (\gamma_k - 1) = \alpha_k p$ the equilibrium system (2.16) can equivalently be rewritten as

$$\frac{\partial \hat{\rho}_k}{\partial t} + \frac{\partial (\hat{\rho}_k v_k)}{\partial x} = (-1)^{k+1} \mathcal{C}, \tag{2.21a}$$

$$\frac{\partial (\hat{\rho}_k v_k)}{\partial t} + \frac{\partial (\hat{\rho}_k v_k^2 + \hat{p}_k)}{\partial x} + (-1)^k P \frac{\partial \alpha_1}{\partial x} = (-1)^{k+1} \mathcal{M}, \tag{2.21b}$$

$$\frac{\partial (\hat{\rho}_k E_k)}{\partial t} + \frac{\partial (\hat{\rho}_k E_k + \hat{p}_k) v_k)}{\partial x} + (-1)^k P \left(V_{\mathrm{I}} \frac{\partial \alpha_1}{\partial x} + \frac{D_1^0 + D_2^0}{G^0} \right)$$

$$= (-1)^k \left(P \frac{\mathbb{E}_p^0}{G^0} - \mathcal{E} \right), \tag{2.21c}$$

where in the equations of state for p and $\hat{p}_k$ we use the **same** material parameters. For the pressure of the mixture p_{mix}, we have

$$p_{mix} = \hat{p}_1 + \hat{p}_2 = \alpha_1 p + \alpha_2 p = p,$$

whereas the volume fractions α_k satisfy

$$\alpha_k = \frac{\hat{p}_k}{\hat{p}_1 + \hat{p}_2}.$$

Obviously, the miscible and the immiscible notation of the model are equivalent.

We prefer to use the notation of system (2.16) because this notation is more convenient for several reasons: (i) comparability to the full model, (ii) implementation and (iii) extension to the general case of non-ideal fluids.

3 Hyperbolicity

To investigate hyperbolicity of the six-equation model (2.16) at pressure equilibrium closed by (2.17) and (2.19), we first rewrite the system in terms of the primitive variables volume fraction α_1, equilibrium pressure p, velocities v_k and densities ρ_k. Let

$$\mathbf{u}^{\mathrm{T}} = (u_1, u_2, u_3, u_4, u_5, u_6) = (\alpha_1 \rho_1, \alpha_1 \rho_1 v_1, \alpha_1 \rho_1 E_1, \alpha_2 \rho_2, \alpha_2 \rho_2 v_2, \alpha_2 \rho_2 E_2)$$

denote the conserved quantities of the system (2.16). Then, the primitive variables can be obtained as follows.

$$v_1 = \frac{u_2}{u_1}, \quad v_2 = \frac{u_5}{u_4}, \quad \alpha_1 \rho_1 e_1 = u_3 - \frac{v_1 u_2}{2}, \quad \alpha_2 \rho_2 e_2 = u_6 - \frac{v_2 u_5}{2}, \tag{3.1a}$$

$$p = (\gamma_1 - 1)\alpha_1 \rho_1 e_1 + (\gamma_2 - 1)\alpha_2 \rho_2 e_2, \quad \alpha_1 = \frac{p}{(\gamma_1 - 1)\rho_1 e_1}. \tag{3.1b}$$

From (2.1), we derive the evolution equations for the primitive variables:

$$\frac{\partial \alpha_1}{\partial t} + \frac{1}{G^0}\left(\left(\frac{z_2 v_2}{\alpha_2} + \frac{z_1 v_1}{\alpha_1}\right)\frac{\partial \alpha_1}{\partial x} + (v_1 - v_2)\frac{\partial p}{\partial x} + \rho_1 c_1^2 \frac{\partial v_1}{\partial x} - \rho_2 c_2^2 \frac{\partial v_2}{\partial x}\right) = \mathcal{A}, \tag{3.2a}$$

$$\frac{\partial p}{\partial t} + \frac{1}{\alpha_1 \alpha_2 G^0}\left(z_1 z_2 (v_1 - v_2)\frac{\partial \alpha_1}{\partial x} + (\alpha_1 z_2 v_1 + \alpha_2 z_1 v_2)\frac{\partial p}{\partial x}\right.$$

$$\left. + \alpha_1 z_2 \rho_1 c_1^2 \frac{\partial v_1}{\partial x} + \alpha_2 z_1 \rho_2 c_2^2 \frac{\partial v_2}{\partial x}\right) = \mathcal{P}, \tag{3.2b}$$

$$\frac{\partial v_1}{\partial t} - \frac{P - p}{\alpha_1 \rho_1}\frac{\partial \alpha_1}{\partial x} + \tau_1 \frac{\partial p}{\partial x} + v_1 \frac{\partial v_1}{\partial x} = \mathcal{V}_1 \tag{3.2c}$$

$$\frac{\partial v_2}{\partial t} + \frac{P - p}{\alpha_2 \rho_2}\frac{\partial \alpha_1}{\partial x} + \tau_2 \frac{\partial p}{\partial x} + v_2 \frac{\partial v_2}{\partial x} = \mathcal{V}_2 \tag{3.2d}$$

$$\frac{\partial \rho_1}{\partial t} + \frac{1}{\alpha_1 \alpha_2 G^0}\left(\rho_1 z_2 (v_1 - v_2)\frac{\partial \alpha_1}{\partial x} - \rho_1 \alpha_2 (v_1 - v_2)\frac{\partial p}{\partial x}\right. \tag{3.2e}$$

$$\left. + \rho_1 \left(\alpha_1 \rho_2 c_2^2 + (P - p)\overline{\kappa}\right)\frac{\partial v_1}{\partial x} + \alpha_2 \rho_1 c_1^2 \frac{\partial v_2}{\partial x}\right) + v_1 \frac{\partial \rho_1}{\partial x} = \mathcal{R}_1,$$

$$\frac{\partial \rho_2}{\partial t} + \frac{1}{\alpha_1 \alpha_2 G^0}\left(\rho_2 z_1 (v_1 - v_2)\frac{\partial \alpha_1}{\partial x} + \rho_2 \alpha_1 (v_1 - v_2)\frac{\partial p}{\partial x}\right.$$

$$\left. + \alpha_1 \rho_2 c_2^2 \frac{\partial v_1}{\partial x} + \rho_2 \left(\alpha_2 \rho_1 c_1^2 + (P - p)\overline{\kappa}\right)\frac{\partial v_2}{\partial x}\right) + v_2 \frac{\partial \rho_2}{\partial x} = \mathcal{R}_2 \tag{3.2f}$$

with relaxation terms

$$\mathcal{A} = \frac{\mathcal{C}}{\varrho} + \frac{\mathbb{E}_p^0}{G^0}, \tag{3.3a}$$

$$\mathcal{P} = \frac{1}{\alpha_1}\left(\kappa_1 \mathbb{E}_{01}^0 - \frac{p}{\varrho}\mathcal{C}\right) + \frac{1}{\alpha_2}\left(\kappa_2 \mathbb{E}_{02}^0 + \frac{p}{\varrho}\mathcal{C}\right), \tag{3.3b}$$

$$\mathcal{V}_k = (-1)^k \frac{1}{\alpha_k \rho_k}\left(v_k \mathcal{C} - \mathcal{M}\right), \tag{3.3c}$$

$$\mathcal{R}_k = (-1)^k \frac{\rho_k}{\alpha_k}\left(\frac{\mathbb{E}_p^0}{G^0} - \left(\frac{1}{\rho_k} - \frac{1}{\varrho}\right)\mathcal{C}\right). \tag{3.3d}$$

Here, we use the notation

$$\overline{\gamma} := \alpha_2 \gamma_1 + \alpha_1 \gamma_2, \tag{3.4a}$$

$$\overline{\kappa} := \alpha_2 \kappa_1 + \alpha_1 \kappa_2, \tag{3.4b}$$

$$\Delta := P - p, \tag{3.4c}$$

$$c_k^2 := \frac{\gamma_k p}{\rho_k}, \tag{3.4d}$$

$$z_k := \gamma_k p + \Delta \kappa_k. \tag{3.4e}$$

Then, we can rewrite G^0 as

$$G^0 = \frac{p\overline{\gamma} + \Delta\overline{\kappa}}{\alpha_1 \alpha_2}.$$

In compact form, the system of primitive variables can be written in quasi-conservative form as

$$\frac{\partial \boldsymbol{w}}{\partial t} + \boldsymbol{J}(\boldsymbol{w})\frac{\partial \boldsymbol{w}}{\partial x} = \boldsymbol{S}(\boldsymbol{w})$$

with $\boldsymbol{w} := (\alpha_1, p, v_1, v_2, \rho_1, \rho_2)^{\mathrm{T}}$ and $\boldsymbol{S} := (\mathcal{A}, \mathcal{P}, \mathcal{V}_1, \mathcal{V}_2, \mathcal{R}_1, \mathcal{R}_2)^{\mathrm{T}}$ the vector of primitive variables and relaxation terms, respectively. The matrix $\boldsymbol{J}$ is represented by a 2×2-block matrix

$$\boldsymbol{J} = \begin{pmatrix} \boldsymbol{J}_4 & \boldsymbol{*}_{4\times 2} \\ \boldsymbol{0}_{2\times 4} & \mathrm{diag}(v_1, v_2) \end{pmatrix}, \tag{3.5}$$

where the principal part $\boldsymbol{J}_4$ can be split into two parts separating terms depending and not depending on Δ

$$\boldsymbol{J}_4 = \begin{pmatrix} j_{11} & j_{12} & j_{13} & j_{14} \\ j_{21} & j_{22} & j_{23} & j_{24} \\ 0 & j_{32} & v_1 & 0 \\ 0 & j_{42} & 0 & v_2 \end{pmatrix} + f(\Delta) \begin{pmatrix} \tilde{j}_{11} & \tilde{j}_{12} & \tilde{j}_{13} & \tilde{j}_{14} \\ \tilde{j}_{21} & \tilde{j}_{22} & \tilde{j}_{23} & \tilde{j}_{24} \\ \tilde{j}_{31} & 0 & 0 & 0 \\ \tilde{j}_{41} & 0 & 0 & 0 \end{pmatrix}$$

with

$$f(\Delta) := \frac{\Delta}{\Delta \overline{\kappa} + p\overline{\gamma}}.$$

Introducing the convex parameters

$$\beta_1 := \frac{\alpha_1 \gamma_2}{\overline{\gamma}}, \quad \beta_2 := \frac{\alpha_2 \gamma_1}{\overline{\gamma}}, \quad \beta_1 + \beta_2 = 1, \quad \beta_i \in [0, 1], \quad i = 1, 2, \tag{3.6}$$

the matrix entries can be written as

$$j_{11} = \beta_2 v_1 + \beta_1 v_2, \quad j_{12} = \frac{\overline{\gamma}}{p\gamma_1\gamma_2}\beta_1\beta_2(v_1 - v_2), \quad j_{13} = \frac{\overline{\gamma}}{\gamma_2}\beta_1\beta_2, \quad j_{14} = -\frac{\overline{\gamma}}{\gamma_1}\beta_1\beta_2,$$

$$j_{21} = \frac{p\gamma_1\gamma_2}{\overline{\gamma}}(v_1 - v_2), \quad j_{22} = \beta_1 v_1 + \beta_2 v_2, \quad j_{23} = \beta_1\gamma_1 p, \quad j_{24} = \beta_2\gamma_2 p,$$

$$j_{32} = \frac{1}{\rho_1}, \quad j_{42} = \frac{1}{\rho_2},$$

$$\tilde{j}_{11} = (\alpha_1\beta_2 - \alpha_2\beta_1)(v_1 - v_2), \quad \tilde{j}_{12} = -\overline{\kappa}\frac{\overline{\gamma}}{\gamma_1\gamma_2}\beta_1\beta_2(v_1 - v_2), \quad \tilde{j}_{13} = -\overline{\kappa}\frac{\overline{\gamma}}{\gamma_2}\beta_1\beta_2,$$

$$\tilde{j}_{14} = \overline{\kappa}\frac{\overline{\gamma}}{\gamma_1}\beta_1\beta_2, \quad \tilde{j}_{21} = (v_1 - v_2)(p(\beta_2\gamma_1\kappa_2 + \beta_1\gamma_2\kappa_1) + \Delta\kappa_1\kappa_2),$$

$$\tilde{j}_{22} = -\frac{\overline{\gamma}}{\gamma_1\gamma_2}\beta_1\beta_2(\kappa_1 - \kappa_2)(v_1 - v_2), \quad \tilde{j}_{23} = -(\kappa_1 - \kappa_2)\alpha_1\beta_2 p,$$

$$\tilde{j}_{24} = (\kappa_1 - \kappa_2)\alpha_2\beta_1 p, \quad \tilde{j}_{31} = -\frac{(\Delta\overline{\kappa} + p\overline{\gamma})}{\alpha_1\rho_1}, \quad \tilde{j}_{41} = \frac{(\Delta\overline{\kappa} + p\overline{\gamma})}{\alpha_2\rho_2}.$$

Obviously, the matrix is well defined if and only if

$$\Delta \neq -\frac{p\overline{\gamma}}{\overline{\kappa}} = -p - \frac{p}{\overline{\kappa}} \quad \text{or, equivalently,} \quad P \neq -\frac{p}{\overline{\kappa}}. \tag{3.7}$$

This is a first constraint on the choice of the interfacial pressure at pressure equilibrium.

From the 2×2-block matrix (3.5), we conclude for the corresponding characteristic polynomial

$$\det(\boldsymbol{J} - \lambda \boldsymbol{I}_6) = (\lambda - v_1)(\lambda - v_2)\det(\boldsymbol{J}_4 - \lambda \boldsymbol{I}_4),$$

where the characteristic polynomial of $\boldsymbol{J}_4$ reads

$$\begin{aligned}
\chi(\lambda) := {}& \det(\boldsymbol{J}_4 - \lambda \boldsymbol{I}_4) \\
= {}& (v_1 - \lambda)^2 (v_2 - \lambda)^2 \\
& + (v_1 - \lambda)^2 (f(\Delta)\alpha_1 - \beta_2) c_2^2 + (v_2 - \lambda)^2 (f(\Delta)\alpha_2 - \beta_1) c_1^2 \\
& + (v_1 - \lambda)(v_2 - \lambda) f(\Delta)g (c_1^2 - c_2^2) \\
& - (v_1 - \lambda)(v_1 - v_2) f(\Delta)gc_2^2 \\
& - (v_2 - \lambda)(v_1 - v_2) f(\Delta)gc_1^2 \\
& - f(\Delta) c_1^2 c_2^2,
\end{aligned}$$

with

$$g := \frac{(\gamma_1 - \gamma_2)\,\alpha_1\alpha_2}{\overline{\gamma}} = \alpha_1\beta_2 - \alpha_2\beta_1 = \alpha_1 - \beta_1 = \beta_2 - \alpha_2$$

and the single-fluid sound speeds c_k defined by Eq. (3.4d).

Obviously, the velocities v_1 and v_2 are eigenvalues of $\boldsymbol{J}$. In the following, we investigate the roots of the characteristic polynomial χ. For this purpose, we introduce the transformation

$$x := \left(\lambda - \frac{v_1 + v_2}{2}\right), \quad \delta := \frac{v_1 - v_2}{2} \tag{3.8}$$

also applied in [21]. Then, the characteristic polynomial can be written in the form

$$P_4(x) = x^4 + a_2 x^2 + a_1 x + a_0 \tag{3.9}$$

with coefficients

$$a_2 := -(A - f(\Delta)D + 2\delta^2), \tag{3.10a}$$

$$a_1 := -2\delta(B - f(\Delta)C), \tag{3.10b}$$

$$a_0 := \delta^4 - \delta^2(A - f(\Delta)D) - f(\Delta)c_1^2 c_2^2 \tag{3.10c}$$

and linear combinations of the squares of the single-fluid sound speeds

$$A := \beta_1 c_1^2 + \beta_2 c_2^2, \quad B := \beta_1 c_1^2 - \beta_2 c_2^2, \quad C := \beta_2 c_1^2 - \beta_1 c_2^2, \quad D := \beta_1 c_2^2 + \beta_2 c_1^2. \tag{3.11}$$

We emphasize that the sign of the terms A and D is independent of the enumeration of the fluids, whereas the terms B and C flip sign.

To ensure hyperbolicity of our model, we have to investigate the roots of the polynomial P_4. In the following, we will derive constraints on $f(\Delta)$ and, equivalently, Δ that ensure the existence of real roots. This problem has already been considered in the context of ideal gases for $f(\Delta) = 0$ or, equivalently, $\Delta = 0$, see [15,28]. We start with a characterization for polynomials of the particular type (3.9) that can be found in [26].

Proposition 1 (Real roots of P_4; Viher [26]) *Let be*

$$D_1 := \frac{q^2}{4} + \frac{r^3}{27}, \quad r := -4a_0 - \frac{1}{3}a_2^2, \quad q := \frac{8}{3}a_0 a_2 - a_1^2 - \frac{2}{27}a_2^3. \tag{3.12}$$

Then, P_4 has only real roots if and only if

$$D_1 \leq 0 \text{ and} \tag{3.13a}$$

$$a_2^2 - 4a_0 \geq 0 \text{ and} \tag{3.13b}$$

$$a_2 \leq 0. \tag{3.13c}$$

This theorem does not distinguish between simple and multiple roots. The latter are characterized in [27].

In the following, we derive sufficient and necessary conditions on $f(\Delta)$ or, equivalently, Δ for which the conditions (3.13) hold. First of all, we verify that

$$f(\Delta) \leq 0 \text{ or, equiv., } -\frac{\overline{\gamma}}{\kappa}p \leq \Delta \leq 0 \text{ or, equiv., } -\frac{p}{\kappa} \leq P \leq p. \tag{3.14}$$

is a necessary condition on $f(\Delta)$ to ensure the existence of real roots of the characteristic polynomial (3.9) for *all* admissible physical states

$$\mathcal{D} := \{(\alpha_1, \rho_1, \rho_2, v_1, v_2, p) : \alpha_1 \in [0, 1].\ v_k \in \mathbb{R},\ \rho_k \in \mathbb{R}_+,\ \rho \in \mathbb{R}_+\}.$$

Besides condition (3.7), this is another constraint on the choice of the interfacial pressure P at pressure equilibrium.

Theorem 1 (Necessary condition) *To ensure the existence of real roots of the characteristic polynomial (3.9) for **all** physical admissible states in $\mathcal{D}$, the condition (3.14) is necessary.*

Proof We consider each of the conditions in (3.13) separately:

Condition (3.13a) This condition only holds if r defined by (3.12) is not positive. We now derive a constraint on $f(\Delta)$ that ensures the correct sign of r for all physical admissible states in $\mathcal{D}$. For this purpose, we consider the term $-3r$ as a quadratic polynomial in $f \equiv f(\Delta)$ depending on the parameter $y \equiv \delta^2$:

$$G(f; y) := 12a_0 + a_2^2 = D^2 f^2 - 2 \left(AD - 4Dy + 6c_1^2 c_2^2 \right) f + (A - 4y)^2,$$

where we suppress the dependency on the other physical quantities α_1, ρ_k, v_k and p. We now verify for which f this polynomial is nonnegative for *all* velocity differences δ^2.

Since $D > 0$ according to (3.10), we may factorize the polynomial

$$G(f; y) = D^2 (f - \hat{f}_-)(f - \hat{f}_+) \tag{3.15}$$

with

$$\hat{f}_\pm := \frac{1}{D^2} \left(6c_1^2 c_2^2 + AD - 4Dy \pm \sqrt{(6c_1^2 c_2^2 + AD - 4Dy)^2 - D^2(A - 4y)^2} \right).$$

Herein, the discriminant can be written as

$$\hat{g}(y) = 12c_1^2 c_2^2 \left(3c_1^2 c_2^2 + AD - 4Dy \right).$$

Case 1a Obviously, if

$$\delta^2 > \frac{AD + 3c_1^2 c_2^2}{4D} =: \hat{\delta}_{crit}^2, \tag{3.16}$$

the discriminant is negative and, thus, $G(f; \delta^2)$ must be positive because $D^2 > 0$, i.e., in this case there is no constraint on $f(\Delta)$.

Case 1b On the other hand, if $0 \le \delta^2 \le \hat{\delta}_{crit}^2$, i.e., $-3c_1^2 c_2^2 \le AD$, then the roots $\hat{f}_\pm$ are real and we conclude from the factorization (3.15) that $G(f; \delta^2)$ is nonnegative if

$$f \le \hat{f}_- \quad \text{or} \quad f \ge \hat{f}_+. \tag{3.17}$$

By definition of $\hat{f}_\pm$ and the positivity of the single-fluid sound speeds we check that

$$0 \le \hat{f}_- \le \frac{(A - 4\delta^2)D + 6c_1^2 c_2^2}{D^2} \le \hat{f}_+ \le 2\frac{(A - 4\delta^2)D + 6c_1^2 c_2^2}{D^2} = \hat{f}_- + \hat{f}_+. \tag{3.18}$$

Note that for $0 < \delta^2 = A/4 < \hat{\delta}_{crit}^2$ the minimum and the maximum are attained, i.e., $\hat{f}_- = 0$ and $\hat{f}_+ = 12c_1^2 c_2^2$.

To summarize the findings of the above investigations for the different cases, we conclude that G is nonnegative for all physical states if the condition (3.14) holds.

Condition (3.13b) For the investigation of this condition, we consider the term $a_2^2 - 4a_0$ as a quadratic polynomial in $f \equiv f(\Delta)$ depending on the parameter $y \equiv \delta^2$:

$$F(f; y) := a_2^2 - 4a_0 = D^2 f^2 - 2 \left(AD + 4Dy - 2c_1^2 c_2^2 \right) f + A(A + 8y),$$

where we suppress the dependency on the other physical quantities α_1, ρ_k, v_k and p. We now verify for which f this polynomial is nonnegative for *all* velocity differences δ^2.

Since $D > 0$ according to (3.10), we may factorize the polynomial

$$F(f; y) = D^2 \left(f(\Delta) - f_- \right) \left(f(\Delta) - f_+ \right) \tag{3.19}$$

with

$$f_\pm := \frac{1}{D^2} \left(AD + 4Dy - 2c_1^2 c_2^2 \pm 2\sqrt{(2Dy - c_1^2 c_2^2)^2 - c_1^2 c_2^2 AD} \right).$$

Herein, the discriminant can be factorized by

$$g(y) = 4D^2 (y - y_-)(y - y+), \quad y_\pm := \frac{c_1 c_2 \left(c_1 c_2 \pm \sqrt{AD}\right)}{2D}.$$

Since the product AD is positive due to the positivity of the single-fluid sound speeds and (3.11), the roots $y_\pm$ are real numbers.

Case 1 Obviously, the discriminant g becomes negative if $y_- < y < y_+$. Since $y \equiv \delta^2$, the admissible regime reduces to

$$\max(0, \delta^2_{crit,-}) < \delta^2 < \delta^2_{crit,+}, \quad \delta^2_{crit,\pm} := y_\pm.$$

Thus, the polynomial $F(f; \delta^2)$ must be positive because $D^2 > 0$, i.e., in this case there is no constraint on $f(\Delta)$.

Case 2 On the other hand, if $\delta^2 \geq \delta^2_{crit,+}$ or $\delta^2 \leq \max(0, \delta^2_{crit,-})$, then the discriminant g is nonnegative and the roots $f_\pm$ are real. According to the factorization (3.19) the polynomial $F(f; \delta^2)$ is nonnegative if and only if

$$f \leq f_- \quad \text{or} \quad f \geq f_+. \tag{3.20}$$

If $\delta^2 \geq \delta^2_{crit,+}$, then we conclude from the definition of the roots $f_\pm$:

$$0 \leq f_- \leq \frac{A + 4\delta^2 D - 2c_1^2 c_2^2}{D^2} \leq f_+ \leq 2\frac{A + 4\delta^2 D - 2c_1^2 c_2^2}{D^2} = f_- + f_+.$$

For the other option, we first note that the interval $\delta^2 \leq \max(0, \delta^2_{crit,-})$ is non-empty if and only if $AD \leq c_1^2 c_2^2$. Since by definition (3.11) of A and D it holds $AD \geq c_1^2 c_2^2$, the only choice is $AD = c_1^2 c_2^2$. This is possible only for a pure fluid, i.e., $\alpha_1 = 1$ or $\alpha_2 = 1$. Then, the roots coincide, i.e., $f_- = f_+ = -c_1^2 c_2^2/D^2$. Thus, for this case the polynomial $F(f; \delta^2)$ is nonnegative, i.e., no constraint is imposed on $f(\Delta)$.

Condition (3.13c) Obviously, this condition holds by positivity of A and D due to the positivity of the single-fluid sound speeds. $\qquad\square$

To derive sufficient conditions on $f(\Delta)$ in the non-equilibrium case, the sign of D_1 needs to be further investigated. For this purpose, we first note that by rescaling of D_1 we may equivalently consider the sign of $\tilde{D}_1 := \tilde{q}^2 - 4\tilde{r}^3$ with $\tilde{r} := -3r = 12a_0 + a_2^2$ and $\tilde{q} := -27q = 12a_2^3 + 27a_1^2 - 72a_0 a_2$. To investigate the sign of $\tilde{D}_1$, we split this term into two parts

$$\tilde{D}_1 = 432c_1^2 c_2^2 \left(\tilde{D}_1^s + \tilde{D}_1^t\right) \tag{3.21}$$

with

$$\tilde{D}_1^s = f(\Delta)\left(256\tilde{r}_8^s \delta^8 + 64\tilde{r}_6^s \delta^6 + 16\tilde{r}_4^s \delta^4 + 4\tilde{r}_2^s \delta^2 + \tilde{r}_0^s\right), \tag{3.22a}$$

$$\tilde{D}_1^t = -\beta_1 \beta_2 (f(\Delta) + 1)^2 \delta^2 \left(256\tilde{r}_8^t \delta^6 + 64\tilde{r}_6^t \delta^4 + 16\tilde{r}_4^t \delta^2 + 4\tilde{r}_2^t\right) \tag{3.22b}$$

and coefficients

$$\tilde{r}_8^s := 1, \tag{3.23a}$$

$$\tilde{r}_6^s := 4\left(Df(\Delta) - A\right), \tag{3.23b}$$

$$\tilde{r}_4^s := 8c_1^2 c_2^2 f(\Delta) + 6(Df(\Delta) - A)^2, \tag{3.23c}$$

$$\tilde{r}_2^s := 4(Df(\Delta) - A)\left((Df(\Delta) - A)^2 + 4c_1^2 c_2^2 f(\Delta)\right), \tag{3.23d}$$

$$\tilde{r}_0^s := \left(4f(\Delta)c_1^2 c_2^2 + (Df(\Delta) - A)^2\right)^2 \tag{3.23e}$$

and

$$\tilde{r}_8^t := 1, \tag{3.24a}$$

$$\tilde{r}_6^t := 3\,(Df(\Delta) - A), \tag{3.24b}$$

$$\tilde{r}_4^t := 3(Df(\Delta) - A)^2 + 9c_1^2 c_2^2 \left(4f(\Delta) - 3\beta_1\beta_2(f(\Delta) + 1)^2\right), \tag{3.24c}$$

$$\tilde{r}_2^t := (Df(\Delta) - A)\left((Df(\Delta) - A)^2 + 36c_1^2 c_2^2 f(\Delta)\right). \tag{3.24d}$$

Here, we have applied the relations

$$C^2 - D^2 = B^2 - A^2 = -4\beta_1\beta_2 c_1^2 c_2^2, \quad AD - CB = B^2 - A^2 + 2c_1^2 c_2^2$$

that hold by (3.11). The derivation of this particular splitting is tedious work collecting appropriate terms. It is motivated by the observation that for local single-fluid the sign of $\tilde{D}_1$ can be easily checked, see Remark 4.

In case of a genuine two-fluid flow, the representation (3.21) of $\tilde{D}_1$ is too complex to explicitly determine the roots. The best we may hope for is to find another constraint on $f(\Delta)$ that ensures the existence of real roots of the characteristic polynomial.

Theorem 2 (Sufficient condition in case of $f(\Delta) \not\equiv 0$) *Let the necessary condition* (3.14) *hold true for all admissible states in* $\mathcal{D}$. *If either* Δ *is independent of* δ *or in case of* $\Delta = \Delta(\delta)$, *there exists* δ *such that*

$$\delta^2 = \frac{1}{4}\left(A - Df(\Delta(\delta)) + 2|f(\Delta(\delta))|^{1/2} c_1 c_2\right), \tag{3.25}$$

then the roots of the characteristic polynomial (3.9) *are real for **all** $\delta \in \mathbb{R}$ if and only if either the flow locally degenerates to a single-fluid flow, i.e.,*

$$\beta_1\beta_2 = 0 \quad \text{or, equivalently,} \quad \alpha_1\alpha_2 = 0, \tag{3.26}$$

or

$$f(\Delta) = -1 \quad \text{or, equivalently,} \quad \Delta = -p \quad \text{or, equivalently,} \quad P = 0. \tag{3.27}$$

Proof We now factorize the term $\tilde{D}_1^s$ as follows

$$\tilde{D}_1^s = 256 f(\Delta)\left(\delta^2 - y_+\right)^2 \left(\delta^2 - y_-\right)^2, \quad y_\pm := \frac{1}{4}\left(A - Df(\Delta) \pm 2|f(\Delta)|^{1/2} c_1 c_2\right). \tag{3.28}$$

Since $f(\Delta) \le 0$, this term is non-positive and vanishes if and only if $f(\Delta) = 0$ or δ^2 coincides with $y_\pm$, i.e., $\delta^2 = y_\pm$. Note that whenever Δ depends on δ the latter requires to solve a nonlinear problem to determine δ^2. That is why we need assumption (3.25). Obviously, $y_+ > 0$ is always real, whereas y_- may become negative. We now plug $\delta^2 = y_+$ into $\tilde{D}_1^t$ and check the sign of this term. For this purpose, we first note that

$$\tilde{D}_1^t = -\beta_1\beta_2(f(\Delta) + 1)^2 \delta^2 \left(4(4\delta^2 + Df(\Delta) - A)^3 \right.$$
$$\left. + 144 c_1^2 c_2^2 \left((4f(\Delta) - 3\beta_1\beta_2(f(\Delta) + 1)^2)\delta^2 + f(\Delta)(Df(\Delta) - A)\right)\right). \tag{3.29}$$

For $\delta^2 = y_+$, we then obtain

$$\tilde{D}_1^t = \beta_1\beta_2(f(\Delta) + 1)^2 y_+$$
$$\left(256 c_1^3 c_2^3 |f(\Delta)|^{3/2} + 108\beta_1\beta_2(f(\Delta) + 1)^2(A - Df(\Delta) + 2|f(\Delta)|^{1/2} c_1 c_2)\right).$$

This term vanishes if and only if $f(\Delta) = -1$ or $\beta_1\beta_2 = 0$, i.e., the flow degenerates locally to a single-fluid flow. Otherwise, this term is positive and we conclude for $\delta^2 = y_+$

$$\tilde{D}_1 = 432 c_1^2 c_2^2 \left(\tilde{D}_1^s + \tilde{D}_1^t\right) = 432 c_1^2 c_2^2 \tilde{D}_1^t > 0,$$

and, thus, by Proposition 1 our characteristic polynomial (3.9) has complex roots and the flow model is not hyperbolic. From this, we finally conclude that the model has only real roots for **all** δ, if (3.26) or (3.27) hold true. $\qquad\square$

Remark 3 (Local velocity equilibrium) For states at local velocity equilibrium, i.e., $\delta = 0$, we immediately conclude from (3.21), (3.22) and (3.23)

$$\tilde{D}_1 = 432 c_1^2 c_2^2 \tilde{D}_1^s = 432 c_1^2 c_2^2 f(\Delta) \tilde{r}_0^s = 432 c_1^2 c_2^2 f(\Delta) \left(4 f(\Delta) c_1^2 c_2^2 + (Df(\Delta) - A)^2 \right)^2 .$$

Obviously, $\tilde{D}_1$ is non-positive if and only if the condition (3.14) holds. Thus, this condition is both necessary and sufficient to ensure the existence of real roots due to Proposition 1.

The above result can be extended to weak velocity non-equilibrium, i.e., $|\delta| \ll 1$, provided that $\Delta \neq 0$ and $f(\Delta) < 0$ for $\delta = 0$. If Δ is either independent of δ or depends smoothly on δ, then $\Delta \neq 0$ and $f(\Delta) < 0$ in a neighborhood of local velocity equilibrium. Thus, it holds $\tilde{D}_1 < 0$ for sufficiently small $|\delta|$.

Remark 4 (Local single-fluid flow) If the flow locally degenerates to a single-fluid flow, i.e., $\alpha_1 \alpha_2 = 0$ or, equivalently, $\beta_1 \beta_2 = 0$, then the roots of characteristic polynomial (3.9) are real if and only if the condition (3.14) holds for all admissible states in $\mathcal{D}$. In particular, there exist one real double root and two single real roots if

$$f(\Delta) = 0 \quad \text{or} \quad f(\Delta) = -\frac{1}{c_j^2} \left(c_i^2 + 4\delta^2 \mp 4|\delta| c_i \right), \quad i, j = 1, 2, , i \neq j.$$

This can be concluded from the simplification of (3.21)

$$\tilde{D}_1 = 432 c_1^2 c_2^2 c_j^4 f(\Delta) \left(f(\Delta) - \overline{f}_- \right) \left(f(\Delta) - \overline{f}_+ \right), \quad \overline{f}_\pm := -\frac{1}{c_j^2} \left(c_i \mp 2|\delta| \right)^2$$

and applying Theorem 1, Proposition 1 and the characterization of multiple roots in [27].

Combining Theorems 1 and 2 we conclude with the main result.

Theorem 3 (Global hyperbolicity) *Let Δ, and, equivalently, P be independent of δ. Then, the pressure-equilibrium system (3.2) is hyperbolic for **all** admissible states in $\mathcal{D}$ if and only if the interfacial pressure at pressure equilibrium vanishes, i.e., $P = 0$.*

Note that there might exist $P = P(\delta)$ such that for all δ the condition is (3.25) is not satisfied and the system (3.2) is globally hyperbolic.

We will now discuss three particular choices for Δ and, equivalently, P, and their influence on the hyperbolicity.

Remark 5 (Special case: $\Delta = 0$, i.e., $P = p$) For this particular choice $f(\Delta)$ vanishes and, thus, the term $\tilde{D}_1^s \equiv 0$ vanishes according to (3.22) and for $\tilde{D}_1^t$ we derive from (3.22) and (3.24)

$$\tilde{D}_1^t = -4\beta_1 \beta_2 \delta^2 \left((4\delta^2 - A)^3 - 108 \beta_1 \beta_2 c_1^2 c_2^2 \delta^2 \right)$$
$$= -4\beta_1 \beta_2 \delta^2 \left(\delta^{2/3} - \delta_0^{2/3} \right) \left(\left(\delta^{2/3} - \delta_0^{2/3}/2 \right)^2 + 3 \right)$$

with

$$4\delta_0^2 = \left((\beta_1 c_1^2)^{1/3} + (\beta_2 c_2^2)^{1/3} \right)^3 .$$

Then, we conclude from (3.21) that $\tilde{D}_1$ is positive if and only if

$$0 < \delta^2 < \delta_0^2.$$

Therefore, the roots of the characteristic polynomial (3.9) are real if and only if

$$\delta = 0 \quad \text{or} \quad \delta^2 \geq \delta_0^2,$$

i.e., the pressure-equilibrium system (3.2) is hyperbolic for admissible states in $\mathcal{D}$ either at local velocity equilibrium ($v_1 = v_2$) or at sufficiently strong velocity non-equilibrium ($|v_1 - v_2| \geq \delta_0$. This confirms Wendroff's W-inequality [15,28].

Remark 6 (Special case: $\Delta = \Delta_B$, i.e., $P = p + \Delta_B$) Kumbaro et al. [11] suggested the following non-trivial choice

$$\Delta_B = -4\overline{\theta}M\delta^2, \quad M := \frac{\alpha_1\alpha_2\rho_1\rho_2}{\alpha_1\rho_2 + \alpha_2\rho_1}, \quad \overline{\theta} \geq 1. \tag{3.30}$$

The parameter $\overline{\theta}$ was introduced by Kumbaro et al. [11], whereas in Bestion's original work $\overline{\theta} = 1$, cf. [2]. Obviously, the condition (3.14) that is necessary to ensure the existence of real roots due to Theorem 1 is satisfied if it holds

$$0 \leq \delta^2 < \frac{1}{4\overline{\theta}} \frac{\alpha_1\rho_2 + \alpha_2\rho_1}{\alpha_1\alpha_2\rho_1\rho_2} \frac{\overline{\gamma}}{\kappa}p, \tag{3.31}$$

i.e., for δ^2 exceeding the bound on the right-hand side the characteristic polynomial (3.9) has complex roots. On the other hand, for δ^2 close to zero we may expand $f(\Delta_B)$ in powers of δ^2 using Taylor expansion as

$$f(\Delta_B) = -\frac{4\overline{\theta}M}{\overline{\gamma}p}\delta^2 + \mathcal{O}\left(\delta^4\right).$$

Applying this expansion in (3.22), we obtain

$$\tilde{D}_1^s = -\frac{4M\overline{\theta}A^4\delta^2}{\overline{\gamma}p} + \mathcal{O}\left(\delta^4\right), \quad \tilde{D}_1^t = 4\beta_1\beta_2A^3\delta^2 + \mathcal{O}\left(\delta^4\right).$$

Then, the term $\tilde{D}_1$ defined by (3.21) is non-positive, if

$$\overline{\theta} \geq \frac{\beta_1\beta_2\overline{\gamma}p}{MA} \tag{3.32}$$

provided that δ^2 is sufficiently small. The equality follows by the definitions of c_i^2 and β_i. In [11], it is mentioned that "we find, using a perturbation method Toumi [24], that the model is hyperbolic provided that the interface pressure coefficient $\overline{\theta}$ is greater than a minimum value $\overline{\theta}_0 = 1$. However, there is no guarantee that this value will lead to a hyperbolic system for any flow." In fact, we have verified this statement by the above observations.

Indeed, the above result can be extended to

$$\Delta = \overline{M}|\delta|^q + \mathcal{O}\left(|\delta|^{q+\varepsilon}\right), \; q \in (0, 2], \; \varepsilon > 0,$$

where $\overline{M}$ may depend on the physical state except for the velocities. Performing a similar perturbation analysis, we obtain

$$\tilde{D}_1 = 432c_1^2c_2^2\left(\frac{\overline{M}}{\overline{\gamma}p}A^4|\delta|^q + 4\beta_1\beta_2A^3\delta^2 + \mathcal{O}\left(|\delta|^{q+\min(2,q)}\right)\right).$$

For $|\delta|$ sufficiently small, we then conclude that $\tilde{D}_1$ is non-positive if and only if

$$\begin{cases} \overline{M} \leq 0, & 0 < q < 2 \\ \overline{M} \leq -\frac{4\beta_1\beta_2}{A}\overline{\gamma}p, & q = 2 \end{cases}.$$

Obviously, for $q > 2$ the term $\tilde{D}_1$ is positive.

In summary, choosing the interfacial pressure at pressure equilibrium as $P = p + \Delta_B$, the hyperbolicity region for the pressure equilibrium system (3.2) is extended near to local velocity equilibrium in comparison with the choice $P = p$ in Remark 5.

Remark 7 (Special case: $\Delta = -p$, i.e., $P = 0$) Since in this case we have $f(\Delta) = -1$, we conclude from Theorem 2 that the characteristic polynomial (3.9) has only real roots independent of δ. In general, these roots are all distinct. To see this, we once more consider $\tilde{D}_1$. Here, we obtain for the coefficients (3.23)

$$
\begin{aligned}
\tilde{r}_8^s &:= 1, \\
\tilde{r}_6^s &:= -4(D + A) = -4\left(c_1^2 + c_2^2\right), \\
\tilde{r}_4^s &:= -8c_1^2 c_2^2 + 6(D + A)^2 = -8c_1^2 c_2^2 + 6\left(c_1^2 + c_2^2\right)^2, \\
\tilde{r}_2^s &:= -4(D + A)\left((D + A)^2 - 4c_1^2 c_2^2\right) = -4\left(c_1^2 + c_2^2\right)\left(\left(c_1^2 + c_2^2\right)^2 - 4c_1^2 c_2^2\right), \\
\tilde{r}_0^s &:= \left(-4c_1^2 c_2^2 + (D + A)^2\right)^2 = \left(-4c_1^2 c_2^2 + \left(c_1^2 + c_2^2\right)^2\right)^2
\end{aligned}
$$

and we may rewrite (3.21) as

$$
\tilde{D}_1 = 432 c_1^2 c_2^2 \tilde{D}_1^s = -432 c_1^2 c_2^2 \left(\left(4\delta^2 - c_1^2 - c_2^2\right)^2 - 4c_1^2 c_2^2\right)^2.
$$

Obviously, $\tilde{D}_1$ is non-positive and it only vanishes if the non-resonance condition holds, i.e.,

$$
4\delta^2 = (c_1 \pm c_2)^2,
$$

where the characteristic polynomial has one double root and two single roots according to Viher, cf. [27]. Since it is reasonable to assume that there is some symmetry in the distribution of the roots, we can determine the roots by the factorization of polynomial (3.9)

$$
P_4(x) = \left(x - x_1^-\right)\left(x - x_1^+\right)\left(x - x_2^-\right)\left(x - x_2^+\right)
$$

with $x_1^\pm := \delta \pm c_1$ and $x_2^\pm := \delta \pm c_2$. By the transformation (3.8), the roots of the polynomial $\chi_B(\lambda)$ and, thus, the eigenvalues of the matrix B are determined by

$$
\lambda_i^\pm = v_i \pm c_i, \quad i = 1, 2.
$$

Note that these eigenvalues are the same as in case of the full non-equilibrium Baer–Nunziato model, cf. [13]. In the non-resonance case, we have four distinct real roots and, thus, there exists a basis of eigenvectors. Hence, the model is hyperbolic and we can explicitly determine left and right eigenvectors as well as Riemann invariants and check whether the corresponding fields are genuinely nonlinear or linearly degenerated.

Finally, we would like to point out that for this particular choice of Δ, i.e., $P = 0$, the pressure-equilibrium Baer–Nunziato model (2.16) is conservative, i.e., all non-conservative products vanish. Therefore, this choice seems to be optimal. However, there are problems when verifying that the entropy production is nonnegative, see Sect. 4.

Remark 8 (Baer–Nunziato model at full non-equilibrium) Although the above investigation has been performed for the Baer–Nunziato model at pressure equilibrium, the results have a direct consequence for the full non-equilibrium model. Typically, the models are closed by interfacial pressure P_I and interfacial velocity V_I where P_I does not tend to zero when approaching pressure equilibrium. For instance, Baer and Nunziato [1] use

$$
P_I = p_1, \quad V_I = v_2. \tag{3.33}
$$

In [18], Saurel and Abgrall suggest

$$
P_I = \alpha_1 p_1 + \alpha_2 p_2, \quad V_I = \frac{\alpha_1 \rho_1 v_1 + \alpha_2 \rho_2 v_2}{\alpha_1 \rho_1 + \alpha_2 \rho_2}. \tag{3.34}
$$

Furthermore, in [20] Saurel et al. apply

$$
P_I = (Z_1 p_2 + Z_2 p_1 + \text{sign}(\partial_x \alpha_1)(v_2 - v_1) Z_1 Z_2) / (Z_1 + Z_2), \quad Z_k := \rho_k c_k, \tag{3.35a}
$$

$$
V_I = (Z_1 v_1 + Z_2 v_2 + \text{sign}(\partial_x \alpha_1)(p_2 - p_1)) / (Z_1 + Z_2). \tag{3.35b}
$$

Another approach chooses a convex combination of the velocities for the interfacial velocity and then a unique interfacial pressure is derived such that the entropy production resulting from the interfacial pressure and velocity vanishes, cf. [7,9,13]

$$V_I = \overline{\beta}_1 v_1 + \overline{\beta}_2 v_2, \overline{\beta}_1, \overline{\beta}_2 \in [0, 1], \ \overline{\beta}_1 + \overline{\beta}_2 = 1, \tag{3.36a}$$

$$P_I = \frac{\overline{\beta}_1 T_1 p_2 + \overline{\beta}_2 T_2 p_1}{\overline{\beta}_1 T_1 + \overline{\beta}_2 T_2}. \tag{3.36b}$$

4 Second law of thermodynamics

From a physical point of view, a model is admissible if it is in agreement with the principles of thermodynamics. For this purpose, we briefly summarize the entropy law for the non-equilibrium model and discuss the entropy production terms providing us with admissibility criteria for the interfacial pressure and the interfacial velocity as well as the relaxation terms. Applying the pressure asymptotic to the entropy law, we then derive admissibility criteria for the pressure-equilibrium model.

4.1 Non-equilibrium model

In order to investigate thermodynamical properties of the non-equilibrium model (2.1), we assume that the entropy $s_k = s_k(\tau_k, e_k)$ of each component satisfies

$$T_k ds_k = de_k + p_k d\tau_k = de_k - \frac{p_k}{\rho_k^2} d\rho_k \tag{4.1}$$

with partial derivatives

$$\frac{\partial s_k}{\partial \tau_k}(\tau_k, e_k) = \frac{p_k}{T_k} > 0, \quad \frac{\partial s_k}{\partial e_k}(\tau_k, e_k) = \frac{1}{T_k} > 0. \tag{4.2}$$

Furthermore, to ensure thermodynamic stability we assume that the Hessian of s_k is negative-definite, i.e.,

$$\frac{\partial^2 s_k}{\partial^2 \tau_k}(\tau_k, e_k) \leq 0, \quad \frac{\partial^2 s_k}{\partial^2 e_k}(\tau_k, e_k) \leq 0,$$
$$\frac{\partial^2 s_k}{\partial^2 \tau_k}(\tau_k, e_k) \frac{\partial^2 s_k}{\partial^2 e_k}(\tau_k, e_k) \geq \left(\frac{\partial^2 e_k}{\partial \tau_k \partial e_k}(\tau_k, e_k) \right)^2. \tag{4.3}$$

By means of the evolution equations for the masses (2.1a), the volume fractions (2.1d), (2.2) and the internal energies (2.5), we obtain the evolution equations for the volume specific entropies

$$\partial_t (\alpha_k \rho_k s_k) + \frac{\partial (\alpha_k \rho_k s_k v_k)}{\partial x} = \Pi_k + \mathcal{S}_k \tag{4.4}$$

with the entropy production due to the interfacial pressure and the interfacial velocity as well as the relaxation process

$$\mathcal{S}_k := (-1)^{k+1} \frac{1}{T_k} \left(\left(\mathbb{E} - V_I^2 + \frac{v_k^2}{2} - \mu_k + \frac{p_k}{\varrho} \right) \mathcal{C} \right.$$
$$+ (V_I - v_k) \mathcal{M} + (T_2 - T_1) \mathcal{H} + (p_k - P_I)\mathcal{F} \big)$$
$$= \frac{1}{T_k} \left(\mathbb{E}_{0k} + (-1)^{k+1} \left(\left(\frac{p_k}{\varrho} - \mu_k \right) \mathcal{C} + (p_k - P_I)\mathcal{F} \right) \right) \tag{4.5a}$$

$$\Pi_k := (-1)^{k+1} \frac{1}{T_k} (p_k - P_I)(v_k - V_I) \frac{\partial \alpha_1}{\partial x} \tag{4.5b}$$

with the chemical potentials

$$\mu_k := e_k + \tau_k p_k - T_k s_k.$$

Introducing the entropy of the components of the mixture

$$\rho s := \alpha_1 \rho_1 s_1 + \alpha_2 \rho_2 s_2 \tag{4.6}$$

we finally obtain the entropy law of the mixture

$$\partial_t (\rho s) + \frac{\partial (\alpha_1 \rho_1 s_1 v_1 + \alpha_2 \rho_2 s_2 v_2)}{\partial x} = \Pi_1 + \Pi_2 + \mathcal{S}_1 + \mathcal{S}_2. \tag{4.7}$$

4.2 Pressure-equilibrium model

To derive the entropy equation at pressure equilibrium, we apply the pressure expansion (2.9) in (4.4). Since the entropies s_k and the temperatures T_k depend on the specific volume τ_k and the internal energy e_k, we perform a change of variables according to (2.3)

$$s_k(\tau_k, e_k) = s_k(\tau_k, p_k \tau_k / \kappa_k) =: \bar{s}_k(\tau_k, p_k), \tag{4.8a}$$

$$T_k(\tau_k, e_k) = T_k(\tau_k, p_k \tau_k / \kappa_k) =: \overline{T}_k(\tau_k, p_k). \tag{4.8b}$$

Then, by the asymptotic expansion (2.9) we conclude in the limit $\theta \to \infty$ with

$$\partial_t (\alpha_k \rho_k \bar{s}_k^0) + \frac{\partial (\alpha_k \rho_k \bar{s}_k^0 v_k)}{\partial x} = \overline{\Pi}_k^0 + \overline{\mathcal{S}}_k^0, \tag{4.9}$$

where the production terms (4.5) corresponding to the interfacial pressure and the interfacial velocity as well as the relaxation process tend to

$$\overline{\mathcal{S}}_k^0 := \frac{1}{\overline{T}_k^0} \left(\mathbb{E}_{0k}^0 + (-1)^{k+1} \frac{1}{\overline{T}_k^0} \left(\left(\frac{p}{\varrho} - \overline{\mu}_k^0 \right) \mathcal{C} + (p - P) \mathcal{F}^1 \right) \right)$$

$$= (-1)^{k+1} \frac{1}{\overline{T}_k^0} \left(\left(\mathbb{E} - V_I^2 + \frac{v_k^2}{2} - \overline{\mu}_k^0 + \frac{p}{\varrho} \right) \mathcal{C} + (V_I - v_k) \mathcal{M} \right.$$

$$\left. + \left(\overline{T}_2^0 - \overline{T}_1^0 \right) \mathcal{H} + (p - P) \mathcal{F}^1 \right), \tag{4.10a}$$

$$\overline{\Pi}_k^0 := (-1)^{k+1} \frac{1}{\overline{T}_k^0} (p - P) (v_k - V_I) \frac{\partial \alpha_1}{\partial x}. \tag{4.10b}$$

Here, $\mathbb{E}_{0k}^0$ is determined by (2.12e). In particular, we note that due to the asymptotic expansions (2.9) and (2.10)

$$(p_k - P_I)\mathcal{F} = \left((p_k^0 - P_I^0) + \mathcal{O}(\theta^{-1}) \right) \left(\theta \mathcal{F}^0 + \mathcal{F}^1 + \mathcal{O}(\theta^{-1}) \right) = (p - P_I^0)\mathcal{F}^1 + \mathcal{O}(\theta^{-1}),$$

because it holds $\mathcal{F}^0 = 0$ in the limit $\theta \to \infty$. For the mixture, we then obtain

$$\frac{\partial (\rho \bar{s}^0)}{\partial t} + \frac{\partial (\alpha_1 \rho_1 \bar{s}_1^0 v_1 + \alpha_2 \rho_2 \bar{s}_2^0 v_2)}{\partial x} = \overline{\Pi}_1^0 + \overline{\Pi}_2^0 + \overline{\mathcal{S}}_1^0 + \overline{\mathcal{S}}_2^0 =: \mathcal{S}^0. \tag{4.11}$$

To investigate the sign of the entropy production $\mathcal{S}^0$ we first split the term E_p^0 into the contributions of the mechanical, thermal and chemical relaxation processes, i.e.,

$$E_p^0 := E_{p,\mathcal{M}}^0 \mathcal{M} + E_{p,\mathcal{H}}^0 \mathcal{H} + E_{p,\mathcal{C}}^0 \mathcal{C} \quad \text{with} \tag{4.12a}$$

$$E_{p,\mathcal{M}}^0 := \frac{1}{\alpha_1 \alpha_2} \left(\alpha_2 \kappa_1 (V_I - v_1) + \alpha_1 \kappa_2 (V_I - v_2) \right), \tag{4.12b}$$

$$E_{p,\mathcal{H}}^0 := \frac{1}{\alpha_1 \alpha_2} \bar{\kappa} \left(\overline{T}_2^0 - \overline{T}_1^0 \right), \tag{4.12c}$$

$$E_{p,\mathcal{C}}^0 := \frac{1}{\alpha_1 \alpha_2} \left(\bar{\kappa} \left(\mathbb{E} - V_I^2 \right) - \frac{p}{\varrho} + \frac{1}{2} \left(\alpha_2 \kappa_1 v_1^2 + \alpha_1 \kappa_2 v_2^2 \right) \right). \tag{4.12d}$$

Then, we can split the entropy production term similarly

$$S^0 := S_D^0 + S_{\mathcal{M}}^0 \mathcal{M} + S_{\mathcal{H}}^0 \mathcal{H} + S_{\mathcal{C}}^0 \mathcal{C} \quad \text{with} \tag{4.13a}$$

$$S_D^0 := \frac{p - P}{\overline{T}_1^0 \overline{T}_2^0} \left(\left(\overline{T}_2^0 v_1 - \overline{T}_1^0 v_2 \right) \frac{\partial \alpha_1}{\partial x} + \left(\overline{T}_1^0 - \overline{T}_2^0 \right) \left(V_{\mathrm{I}} \frac{\partial \alpha_1}{\partial x} + \frac{D_1^0 + D_2^0}{G^0} \right) \right), \tag{4.13b}$$

$$S_{\mathcal{M}}^0 := \frac{1}{\overline{T}_1^0 \overline{T}_2^0} \left(\overline{T}_1^0 v_2 - \overline{T}_2^0 v_1 + \left(\overline{T}_2^0 - \overline{T}_1^0 \right) \left(V_{\mathrm{I}} + (p - P) \frac{E_{p,\mathcal{M}}^0}{G^0} \right) \right), \tag{4.13c}$$

$$S_{\mathcal{H}}^0 := \frac{1}{\overline{T}_1^0 \overline{T}_2^0} \left(\overline{T}_2^0 - \overline{T}_1^0 \right)^2 \left(1 + \frac{\overline{\kappa}(p - P)}{\alpha_1 \alpha_2 G^0} \right), \tag{4.13d}$$

$$S_{\mathcal{C}}^0 := \frac{1}{\overline{T}_1^0 \overline{T}_2^0} \left(\left(\overline{T}_2^0 - \overline{T}_1^0 \right) \left(\mathbb{E} - V_{\mathrm{I}}^2 + \frac{p}{\varrho} + (p - P) \frac{E_{p,\mathcal{C}}^0}{G^0} \right) + \right.$$
$$\left. \overline{T}_2^0 \left(\frac{1}{2} v_1^2 - \overline{\mu}_1^0 \right) - \overline{T}_1^0 \left(\frac{1}{2} v_2^2 - \overline{\mu}_2^0 \right) \right). \tag{4.13e}$$

Here, it is important to note that the term S_D^0 only depends on spatial derivatives of the volume fraction, the velocities and the pressure but is independent of the relaxation processes.

In the following investigations, we will now neglect relaxation due to chemical reactions, i.e., $\mathcal{C} = 0$. This case becomes significant for frozen flows. Then, the entropy production reduces to

$$\mathcal{S}^0 = S_D^0 + S_{\mathcal{M}}^0 \mathcal{M} + S_{\mathcal{H}}^0 \mathcal{H}$$
$$= \frac{1}{\overline{T}_1^0 \overline{T}_2^0} \left(\left(\overline{T}_2^0 v_1 - \overline{T}_1^0 v_2 + V_{\mathrm{I}} \left(\overline{T}_1^0 - \overline{T}_2^0 \right) \right) \left((p - P) \frac{\partial \alpha_1}{\partial x} - \mathcal{M} \right) \right.$$
$$- \left(\overline{T}_1^0 - \overline{T}_2^0 \right) (P - p) \frac{D_1^0 + D_2^0 - E_{p,\mathcal{M}}^0 \mathcal{M}}{G^0}$$
$$\left. + \left(\overline{T}_2^0 - \overline{T}_1^0 \right)^2 \left(1 + \frac{\overline{\kappa}(p - P)}{\alpha_1 \alpha_2 G^0} \right) \mathcal{H} \right). \tag{4.14}$$

To enforce nonnegativity of the entropy production, we have to impose constraints on the mechanical and thermal relaxation terms $\mathcal{M}$ and $\mathcal{H}$ as well as the interfacial velocity V_{I}. For this purpose, we draw the following conclusions from (4.14).

1. In case of local thermal equilibrium, i.e., $\overline{T}_1^0 = \overline{T}_2^0 = \overline{T}$, the entropy production reduces to

$$\mathcal{S}^0 = \frac{1}{\overline{T}} (v_1 - v_2) \left((P - p) \frac{\partial \alpha_1}{\partial x} - \mathcal{M} \right).$$

Obviously, in this case the interfacial velocity does not enter. The sign of the entropy production can be controlled by the term

$$(v_2 - v_1) \mathcal{M}$$

using a sufficiently large velocity relaxation parameter ν, see (2.4a).

2. Also in case of thermal non-equilibrium the sign of term S_D^0 can be controlled by a sufficiently large velocity relaxation parameter ν, if the interfacial velocity V_{I} is chosen as a convex combination of the single fluid velocities v_1 and v_2, i.e., $V_{\mathrm{I}} = \overline{\beta}_1 v_1 + \overline{\beta}_2 v_2$, $\overline{\beta}_1 + \overline{\beta}_2 = 1$, $\overline{\beta}_1, \overline{\beta}_2 \in [0, 1]$. Then, we have

$$\left(\overline{T}_2^0 v_1 - \overline{T}_1^0 v_2 + V_{\mathrm{I}} \left(\overline{T}_1^0 - \overline{T}_2^0 \right) \right) = (\overline{\beta}_1 \overline{T}_1^0 + \overline{\beta}_2 \overline{T}_2^0)(v_1 - v_2).$$

3. To ensure at least hyperbolicity in the neighborhood of local velocity equilibrium the pressure difference $\Delta = P - p$ must be non-positive. Since $\overline{\kappa}$ and G^0 are positive, the entropy production due to thermal relaxation $S_{\mathcal{H}}^0 \mathcal{H}$ is nonnegative due to $\mathcal{H} \geq 0$, see (2.4b). This implies that for sufficiently large temperature relaxation parameter $\mathcal{H}$ the third term $S_{\mathcal{H}}^0 \mathcal{H}$ dominates the second term $S_{\mathcal{M}}^0 \mathcal{M}$.

The above observations are summarized in the following

Theorem 4 (Entropy production) *Let the interfacial velocity V_I be a convex combination of the single-fluid velocities v_1 and v_2. The flow is assumed to be chemically frozen, i.e., $C = 0$. Then, the entropy production is nonnegative if the velocity relaxation parameter v and the temperature relaxation parameter $\mathcal{H}$ are sufficiently large.*

This theorem applies to both a small pressure difference $\Delta = P - p$ choosing for instance the interfacial pressure $P = p$ or $P = p + \Delta_B$ near to local velocity equilibrium according to Remark 5 and 6 , respectively, and strong pressure difference when choosing $P = 0$, see Remark 7.

Remark 9 (Interfacial pressure) From the above observations, we conclude that at pressure equilibrium the entropy production is always nonnegative if the flow is at local velocity and local temperature equilibrium. Otherwise the local flow must locally tend to velocity and temperature equilibrium. The faster the local relaxation must be, the larger the pressure difference $\Delta = P - p$.

Remark 10 (Interfacial velocity) It is worthwhile mentioning that the second law of thermodynamics does not provide a closure for the interfacial velocity. In the literature, the choice of a convex combination is frequently used.

In the literature, several closures for the interfacial pressure and the interfacial velocity have been discussed in the context of the full non-equilibrium Baer–Nunziato model. In the following, we check for some of these closures whether they are also admissible at pressure equilibrium.

Remark 11 (Baer–Nunziato model at full non-equilibrium) In [7,9,13], a convex combination of the velocities for the interfacial velocity is considered and then a unique interfacial pressure is derived such that the entropy production due to $\overline{\Pi}_1 + \overline{\Pi}_2$ vanishes, see Equation (3.36). At pressure equilibrium $p_1 = p_2 = p$, these equations are determined by

$$V_I = \overline{\beta}_1 v_1 + \overline{\beta}_2 v_2, \ \overline{\beta}_1, \ \overline{\beta}_2 \in [0, 1], \ \overline{\beta}_1 + \overline{\beta}_2 = 1,$$

$$P_I = \frac{\overline{\beta}_1 \ p_2 + \overline{\beta}_2 T_2 p_1}{\overline{\beta}_1 T_1 + \overline{\beta}_2 T_2} = p.$$

Then, the entropy production terms read

$$S_D^0 = 0, \quad S_{\mathcal{M}}^0 = \frac{\overline{\beta}_1 \overline{T}_1^0 + \overline{\beta}_2 \overline{T}_2^0}{\overline{T}_1^0 \overline{T}_2^0} (v_2 - v_1), \quad S_{\mathcal{H}}^0 = \frac{\left(\overline{T}_1^0 - \overline{T}_2^0\right)^2}{\overline{T}_1^0 \overline{T}_2^0}.$$

From this, we conclude for the mechanical relaxation process that $\mathcal{M}$ must have the same sign as the velocity difference $v_1 - v_2$ and the thermal relaxation process $\mathcal{H}$ must be nonnegative. This is the case, see (2.4a) and (2.4b). Note that the closure proposed by Baer and Nunziato [1], see Eq. (3.33), fits into this class of closures when choosing $\overline{\beta}_1 = 0$ and $\overline{\beta}_2 = 1$.

The closure of Saurel and Abgrall [18], see Equation (3.34), at pressure equilibrium reads

$$P_I = p, \quad V_I = \frac{\alpha_1 \rho_1 v_1 + \alpha_2 \rho_2 v_2}{\alpha_1 \rho_1 + \alpha_2 \rho_2} \tag{4.15}$$

results in similar entropy production terms

$$S_D^0 = 0, \quad S_{\mathcal{M}}^0 = \frac{v_2 - v_1}{\overline{T}_1^0 \overline{T}_2^0} \frac{\alpha_1 \rho_1 \overline{T}_1^0 + \alpha_2 \rho_2 \overline{T}_2^0}{\alpha_1 \rho_1 + \alpha_2 \rho_2}, \quad S_{\mathcal{H}}^0 = \frac{\left(\overline{T}_1^0 - \overline{T}_2^0\right)^2}{\overline{T}_1^0 \overline{T}_2^0}.$$

The conclusion for the mechanical and thermal relaxation process is the same as before.
Furthermore, in [20] Saurel et al. introduce a closure, see Eq. (3.35), that at pressure equilibrium reads

$$P_I = p + \frac{\text{sign}(\partial_x \alpha_1)(v_2 - v_1) \overline{Z}_1^0 \overline{Z}_2^0}{\overline{Z}_1^0 + \overline{Z}_2^0}, \quad V_I = \frac{\overline{Z}_1^0 v_1 + \overline{Z}_2^0 v_2}{\overline{Z}_1^0 + \overline{Z}_2^0}, \quad \overline{Z}_k^0 := \rho_k \overline{c}_k^0.$$

For this closure, we derive

$$S_D^0 = \frac{\overline{Z}_1^0 \overline{Z}_2^0}{\overline{T}_1^0 \overline{T}_2^0} \frac{\overline{Z}_1^0 \overline{T}_1^0 + \overline{Z}_2^0 \overline{T}_2^0}{(\overline{Z}_1^0 + \overline{Z}_2^0)^2} \left| \frac{\partial \alpha_1}{\partial x} \right| (v_2 - v_1)^2$$
$$+ \frac{\overline{Z}_1^0 \overline{Z}_2^0}{\overline{T}_1^0 \overline{T}_2^0} \frac{\overline{T}_1^0 - \overline{T}_2^0}{\overline{Z}_1^0 + \overline{Z}_2^0} (v_1 - v_2) \frac{D_1^0 + D_2^0}{G^0} \mathrm{sign}\left(\frac{\partial \alpha_1}{\partial x} \right).$$

While the first term is nonnegative, we cannot control the sign of the second term except at local velocity equilibrium. We emphasize that the second term arises due the asymptotic derivation of the pressure equilibrium model. It is not present in the full non-equilibrium model for which the closure was originally derived. Thus, we conclude that not all closures derived at non-equilibrium are admissible for the equilibrium model.

5 Conclusion

We have derived a six-equation model at pressure equilibrium where we confine ourselves to two ideal gases. For this purpose, we have incorporated an asymptotic expansion for the single-fluid pressures and the interfacial pressure into the full non-equilibrium model. Considering the asymptotic limit yields the reduced model. In contrast to well-known six-equation models, an additional term occurs verifying that these models are not asymptotically correct.

Furthermore, we do not insist that the interfacial pressure tends to the equilibrium pressure when the single-fluid pressures are approaching equilibrium. Numerous available closures for the interfacial pressure satisfy this modeling assumption although there is no physical evidence for this assumption. Here, we derive constraints on the equilibrium interfacial pressure that ensure hyperbolicity of the pressure-equilibrium model.

Finally, we have presented constraints on the closures for the interfacial velocity as well as the mechanical and thermal relaxation terms in case of a chemically frozen flow.

So far, we considered only the case of two ideal gases as is common practice in the literature on pressure-equilibrium models. Currently, we are extending the investigations to non-ideal fluids. Preliminary results show that choosing the equilibrium interfacial pressure *smaller* than the equilibrium pressure ensures hyperbolicity at least in a local neighborhood of local velocity equilibrium.

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