

Z. Śloderbach · J. Pajak

Thermodynamic potential of free energy for thermo-elastic-plastic body

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Abstract The procedure of derivation of thermodynamic potential of free energy (Helmholtz free energy) for a thermo-elastic-plastic body is presented. This procedure concerns a special thermodynamic model of a thermo-elastic-plastic body with isotropic hardening characteristics. The classical thermodynamics of irreversible processes for material characterized by macroscopic internal parameters is used in the derivation. Thermodynamic potential of free energy may be used for practical determination of the level of stored energy accumulated in material during plastic processing applied, e.g., for industry components and other machinery parts received by plastic deformation processing. In this paper the stored energy for the simple stretching of austenitic steel will be presented.

Keywords Free energy · Thermo-elastic-plastic body · Enthalpy · Exergy · Stored energy of plastic deformations · Mechanical energy dissipation · Thermodynamic reference state

1 Introduction

The derivation of thermodynamic potential of free energy for a model of the thermo-elastic-plastic body with the isotropic hardening is presented. The additive form of the thermodynamic potential is assumed, see, e.g., [1–4]. Such assumption causes that the influence of plastic strains on thermo-elastic properties of the bodies is not taken into account, meaning the lack of “elastic–plastic” coupling effects, see Śloderbach [4–10]. Many processes of plastic forming of metals (for example tube bending) are not cyclic, but have the nature of quasi-static loadings. In such cases the effect of kinematic hardening can be neglected, since most metallic materials do not exhibit “elastic–plastic” coupling effects.

A fully developed form of the free energy potential and its subset—the enthalpy of plastic deformation which is equal to the stored energy of plastic deformation (SEPD), see, e.g., [11–32]—is presented in this paper.

We believe that this thermodynamic potential will be of interest for people dealing with industrial processes of plastic working applied to power plant components, since it enables to estimate the amount of cold work energy stored during a given forming process of plastic deformation. The formula of thermodynamic potential was derived using the classical thermodynamics of irreversible processes, see [1–9, 33–36] applying macroscopic internal parameters for characterization of materials.

Recent papers [37–40] on classical and non-classical thermodynamic potentials using new investigation methods have largely contributed to analysis of behavior of elastic and inelastic (plastic) materials during deformations. Green and Naghdi [37] deal with thermo-elastic material behavior without energy dissipation

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Z. Śloderbach (✉) · J. Pajak
Faculty of Applications of Chemistry and Mechanics, Opole University of Technology, ul. Luboszycka 7, 45-036 Opole, Poland
E-mail: z.słoderbach@po.opole.pl

applying both nonlinear and linear theories. The linearized theory that they discussed differs from the classical theory since it does not sustain energy dissipation and permits transmission of heat flow as thermal waves of finite speed. A general uniqueness theorem appropriate for linear thermo-elasticity without energy dissipation is also proved. Papers [38,39] are devoted to elastic materials and paper [40] to inelastic (elasto-plastic) materials. In paper [38] the authors addressed thermo-elasticity without energy dissipation, using a procedure based on convex/concave functions and Legendre transformations. Starting from the free energy of Green and Naghdi, the thermodynamic potentials, i.e., internal energy, enthalpy and Gibbs free energy, were derived. In [39] the structural boundary-value problem in the context of nonlocal (integral) elasticity and quasi-static loads in a geometrically linear range was considered. The nonlocal elastic behavior was described by the so called Eringen model in which the nonlocality lies in the constitutive relation. The diffusion processes of the nonlocality are governed by an integral relation containing symmetric spatial weight function expressed in terms of an attenuation function. In [40] a thermodynamically consistent formulation of nonlocal plasticity in the framework of the internal variable theories of inelastic (plastic) behaviors of associative type is presented. It is shown that a suitable minimum principles provides a rational basis to exploit the iterative elastic predictor–plastic corrector algorithm in terms of the dissipation functional. The nonlinear stability analysis of the nonlocal problem is carried out following the concept of nonexpansivity proposed in local plasticity.

However, earlier first author's studies have shown that the SEPD may be a useful tool of characterization of structural integrity of processed materials and of controlling the working process itself. The results presented in [7,9,30,41,42] show that cold-worked pipes exhibit severely damaged and fragmented crystalline structure, especially within areas that were subjected to tensile stresses. It means that a model should take into account the physical changes occurring in deformed materials. Higher dislocation density, smaller grains with larger area of total grain boundary surface and stored energy of deformation being a driving force of diffusion – all these factors make processed material susceptible to high-temperature creep, which is the prime cause of failures in power station components.

It will be shown that the SEPD is equal to the change in the internal energy of a body due to induced plastic deformation, and further, that it is equal to the enthalpy of plastic deformation. A quantitative model of the SEPD may be helpful in many areas of power generation technology where both designers and in-field operators frequently deal with components that have built-in histories of plastic deformation. There is also a great demand for workable models of the SEPD in power plant-related forensic engineering since utility companies all over the world are plagued by shutdowns linked to component integrity failures.

The energy balance of a process of plastic processing of material is given by the first law of thermodynamics. Due to the irreversibility of plastic processing, the second law of thermodynamics (the dissipation law) should also be obeyed.

The final part of this paper will show computations resulting from real estimates of the SEPD for the austenitic steel strengthened in simple tension [24,25].

1.1 The main notation and abbreviations

Latin symbols:

A	Helmholtz free energy,
A^e, A^p	“Elastic” and “plastic” part of the free energy A , respectively,
A_0	Free energy value corresponding to the TRS,
B_0	Material constant in Pa^{-1} ,
C	Material constant,
c_e	Specific heat capacity measured at a constant elastic deformation,
D	Amount of mechanical energy dissipated within a unit volume and unit time,
ϵ^p	Deviatoric part of the plastic strain tensor ϵ^p ,
k	Symmetric tensor of thermal conductivity coefficients, such that $k\delta_{ij} = \text{const}$,
\mathbf{M}, \mathbf{L}	Tensor of isothermal elastic moduli and tensor of elastic compliance, respectively, $\mathbf{M} = (\mathbf{L}^{-1}) _{\sigma=\sigma(T, \epsilon^e)}$ where $M_{ijmn} = M_{mnni} = M_{jimn} = M_{ijnm}$ and $L_{ijmn} = L_{mnij} = L_{jimn} = L_{ijnm}$,
n	Strain-hardening exponent,
s	Specific entropy (referred to unit mass),
s_0	Entropy value corresponding to the TRS,
s^e, s^p	Change in entropy due to elastic and plastic deformation and to the already accomplished plastic deformation (if applicable), respectively,

$E_s \equiv$	SEPD stored energy of plastic deformation,
T_0	Reference temperature corresponding to the TRS – it may be for example the ambient temperature,
T	Thermodynamic temperature in [K],
TRS	“Thermodynamic reference state”, in which: $T = T_0, K = 0$ and $\boldsymbol{\varepsilon}^e = 0$,
Y, \bar{Y}	Yield stress in uniaxial tension as dependent on (κ, T) and (π, T) , respectively,
Y_0	Initial yield stress,
Y_1	Yield stress in uniaxial tension as dependent on (π, κ, T) ;

Greek symbols:

α	Symmetric tensor of thermal expansion coefficients, such that $\alpha \delta_{ij} = \text{const}$,
ε_0	Strain corresponding to the initial yield stress,
ε_1	True axial strain, $\varepsilon_1 = \ln \frac{l}{l_0}$, where l_0 and l are the initial and current total gauge lengths, respectively,
ε_1^p	True plastic axial strain, $\varepsilon_1^p = \ln \frac{l_p}{l_0} = \ln \frac{l - l_e}{l_0}$, where l_e, l_p current elastic and plastic gauge lengths, respectively,
$\boldsymbol{\varepsilon}^e$	Tensor of small elastic deformations, $\boldsymbol{\varepsilon}^p$ plastic strain tensor,
$\varepsilon_{(i)}^p$	Effective plastic strain,
Γ^p	Enthalpy of the plastic deformation,
κ, κ	Symmetric second rank tensor and scalar internal parameter, respectively,
K	Pair of internal parameters, $K \Leftrightarrow \{\kappa, \kappa\}$,
Π	Pair of symmetric second rank tensor and scalar internal thermodynamic forces, $\Pi \Leftrightarrow \{\pi, \pi\}$, associated with a pair of internal parameters K , respectively,
π_0 and π_0	Initial value of an internal thermodynamic forces for $\boldsymbol{\varepsilon}^p = 0$,
ρ_0, ρ	Body density in a thermodynamic reference state and in the current state, respectively,
σ	Bauchy stress tensor,
σ_p	Yield stress value obtained in the uniaxial tension test,
ξ, ζ	Integral variables.

Tensors will be printed in bold. The summation convention is assumed along with the following detailed notation:

$$\begin{aligned}
 \mathbf{AB} &= A_{ij}B_j \quad \text{or} \quad A_{ijkl}B_{kl} \quad (i, j, k, l, m, n, \dots = 1, \dots, 3) \\
 \text{tr}\mathbf{A} &= A_{kk}, \text{tr}(\mathbf{AB}) = A_{ij}B_{ji} \\
 \mathbf{A} : \mathbf{B} &= A_i B_i \quad \text{or} \quad A_{ij} B_{ij} \\
 \mathbf{1} &- \text{identity tensor, } \delta_{ij} - \text{Kronecker delta and } 0 - \text{null tensor.} \\
 (\text{sym}A)_{ij} &\stackrel{\text{def}}{=} \frac{1}{2}(A_{ij} + A_{ji}). \\
 \text{dev}\mathbf{A} &= \mathbf{A} - \frac{1}{3}(\text{tr}\mathbf{A})\mathbf{1} - \text{deviatoric part of tensor } \mathbf{A}. \\
 A_{i,j} &= \frac{\partial A_i}{\partial x_j}, \quad \text{where: } x_j - \text{coordinates of a material particle.} \\
 \dot{A} &= \frac{\partial A}{\partial t}, \quad t - \text{time.}
 \end{aligned}$$

If Π and K denote pairs of tensors of the second and the zeroth order, then the operation $\Pi \cdot K$ produces a scalar, see [4–10]:

$$\Pi \cdot K = \boldsymbol{\pi} : \kappa + \pi \kappa = \pi_{ij} \kappa_{ij} + \pi \kappa.$$

If the function F is relative to Π and K , then $F(\Pi) = F(\boldsymbol{\pi}, \pi)$ and $F(K) = F(\kappa, \kappa)$. The derivative of function F with respect to a pair Π is defined as follows:

$$\frac{\partial F}{\partial \Pi} = \left\{ \frac{\partial F}{\partial \boldsymbol{\pi}}, \frac{\partial F}{\partial \pi} \right\}.$$

The differential of function F with respect to the pair K of tensors of the second and the zeroth order produces a sum [4–9], such that:

$$\frac{\partial F}{\partial K} \cdot \dot{K} \stackrel{\text{def}}{=} \left\{ \frac{\partial F}{\partial K_{ij}} \dot{k}_{ij} + \frac{\partial F}{\partial \kappa} \dot{\kappa} \right\}.$$

Application of pairs results in more condensed notation throughout the paper.

2 Generalized description and basic assumptions

We assume that local thermodynamic state of the thermo-elastic-plastic body is characterized by the following state variables [1–4]:

- $\boldsymbol{\varepsilon}^e$ Tensor of small elastic deformations,
- T Thermodynamic temperature in [K], where: $T \in (0; T_m)$,
- T_m Melting temperature in [K],
- κ and κ Symmetric second order tensorial and scalar internal parameter, respectively.

Further, let us assume after [5–7] pairs K and Π as defined in the nomenclature.

Let $A = A(T, \boldsymbol{\varepsilon}^e, K)$ stand for the free energy. A generally acknowledged simplification adopted here results from the lack of any elastic–plastic couplings:

$$A = A(T, \boldsymbol{\varepsilon}^e, K) = A^e(T, \boldsymbol{\varepsilon}^e) + A^p(T, K). \quad (2.1)$$

Such separation of the terms has been assumed for polycrystalline metals and alloys by many authors, e.g., [1–4]. Physically, it means that tensors \mathbf{M} , \mathbf{L} and $\boldsymbol{\alpha}$ are assumed to be not affected by prior plastic deformation.

From (2.1) we obtain the following state equations:

$$-s = \frac{\partial A(T, \boldsymbol{\varepsilon}^e, K)}{\partial T}, \quad \frac{1}{\rho_0} \boldsymbol{\sigma} = \frac{\partial A(T, \boldsymbol{\varepsilon}^e, K)}{\partial \boldsymbol{\varepsilon}^e}, \quad \frac{1}{\rho_0} \Pi = \frac{\partial A(T, \boldsymbol{\varepsilon}^e, K)}{\partial K}. \quad (2.2)$$

With this potential function we will be able to take into account the missing couplings present in the coupling diagram, see [1–3, 5, 7–9], such as changes in internal parameters K due to internal thermodynamic forces Π , temperature T and elastic strain $\boldsymbol{\varepsilon}^e$.

Since function s^p represents the variation of entropy due to internal changes and possible past plastic deformation, then it is reasonable to assume that s^p is zero in a natural state, i.e., $-s^p = 0$, for $K = 0$, see [1–9].

Amount of mechanical energy dissipated in unit time and volume has the form [4–9]:

$$D \geq 0 \quad \text{or} \quad \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p - \Pi \cdot \dot{K} \geq 0 \quad \text{and} \quad \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p - \boldsymbol{\pi} : \dot{\boldsymbol{\kappa}} - \pi \kappa \geq 0. \quad (2.3)$$

The following properties were assigned to the thermoplastic material model:

- material is isotropic with regard to its properties,
- material is isotropic with regard to work-hardening according to the HMH (Huber–Mises–Hencky) yield condition,
- thermo-elastic constants and specific heat do not depend on thermodynamic state and plastic properties; thermal conductivities tensor \mathbf{k} is constant and symmetric ($k_{ij} = k_{ji}$),
- yield stress value for simple tension is given and is such that:

$$Y = Y(\kappa, T) = \bar{Y}(\pi, T), \quad \text{and} \quad Y|_{\kappa=0} = \bar{Y}|_{\pi=\pi_0} = Y_0(T) - \text{initial yield stress.}$$

3 Procedure of deriving the thermodynamic potential

Derivation of the thermodynamic potential of free energy A is carried out using assumptions presented in Sect. 2. The total differential of A has the following form, see [1–4]:

$$dA = -sdT + \frac{1}{\rho_0} \boldsymbol{\sigma} : d\boldsymbol{\varepsilon}^e + \frac{1}{\rho_0} \boldsymbol{\Pi} \cdot d\boldsymbol{K}. \quad (3.1)$$

The following definitions can be easily inferred from thermostatics:

$$c_\varepsilon = T \frac{\partial s(T, \boldsymbol{\varepsilon}^e, K)}{\partial T}, \text{ hence } : s - s_0 = c_\varepsilon \ln \frac{T}{T_0} + a(\boldsymbol{\varepsilon}^e, K) \quad (3.2_1)$$

$$\mathbf{M} = \frac{\partial \boldsymbol{\sigma}(T, \boldsymbol{\varepsilon}^e, K)}{\partial \boldsymbol{\varepsilon}^e}, \text{ hence } : \boldsymbol{\sigma} = \mathbf{M}\boldsymbol{\varepsilon}^e + \mathbf{b}(T, K) \quad (3.2_2)$$

$$\boldsymbol{\alpha} = \frac{\partial \boldsymbol{\varepsilon}^e(\boldsymbol{\sigma}, T, K)}{\partial T}, \text{ hence } : \Rightarrow \boldsymbol{\varepsilon}^e = \boldsymbol{\alpha}(T - T_0) + \mathbf{c}(\boldsymbol{\sigma}, K). \quad (3.2_3)$$

Properties of the thermodynamic potential yield the following identities:

$$\mathbf{M} = (\mathbf{L}^{-1})|_{\boldsymbol{\sigma}=\boldsymbol{\sigma}(T, \boldsymbol{\varepsilon}^e)}, \text{ hence } : 2(M_{ijmn}L_{mnrs})|_{\boldsymbol{\sigma}=\boldsymbol{\sigma}(T, \boldsymbol{\varepsilon}^e)} = \delta_{is}\delta_{jr} + \delta_{ir}\delta_{js}. \quad (3.3)$$

In view of assumption (2.1) concerning additive properties of potential free energy A , the right-hand equations (3.2)_{2–3} may be rewritten as:

$$\begin{cases} \boldsymbol{\sigma} = \mathbf{M}\boldsymbol{\varepsilon}^e + \mathbf{b}_1(T) \mid \mathbf{L} \\ \boldsymbol{\varepsilon}^e = \boldsymbol{\alpha}(T - T_0) + \mathbf{c}_1(\boldsymbol{\sigma}) \mid \mathbf{M} \end{cases} \Rightarrow \begin{aligned} \boldsymbol{\varepsilon}^e &= \mathbf{L}\boldsymbol{\sigma} - \mathbf{L}\mathbf{b}_1(T), \\ \mathbf{M}\boldsymbol{\varepsilon}^e &= \mathbf{M}\boldsymbol{\alpha}(T - T_0) + \mathbf{M}\mathbf{c}_1(\boldsymbol{\sigma}). \end{aligned} \quad (3.4)$$

It follows from (3.4) that:

$$\mathbf{L}\boldsymbol{\sigma} - \mathbf{L}\mathbf{b}_1(T) = \boldsymbol{\alpha}(T - T_0) + \mathbf{c}_1(\boldsymbol{\sigma}). \quad (3.5)$$

Hence:

$$\begin{cases} \mathbf{L}\boldsymbol{\sigma} = \mathbf{c}_1(\boldsymbol{\sigma}), \\ \boldsymbol{\alpha}(T - T_0) = -\mathbf{L}\mathbf{b}_1(T) \mid \mathbf{M} \end{cases} \Rightarrow \mathbf{b}_1(T) = -\mathbf{M}\boldsymbol{\alpha}(T - T_0). \quad (3.6)$$

By substituting suitably expressions (3.6) into (3.4) we obtain:

$$\begin{cases} \boldsymbol{\sigma} = \mathbf{M}\boldsymbol{\varepsilon}^e - \mathbf{M}\boldsymbol{\alpha}(T - T_0), \\ \boldsymbol{\varepsilon}^e = \mathbf{L}\boldsymbol{\sigma} + \boldsymbol{\alpha}(T - T_0). \end{cases} \quad (3.7)$$

General properties of thermodynamic potentials [1–9] enable to formulate the following identity:

$$\frac{\partial s(T, \boldsymbol{\varepsilon}^e)}{\partial \boldsymbol{\varepsilon}^e} = -\frac{1}{\rho_0} \frac{\partial \boldsymbol{\sigma}(T, \boldsymbol{\varepsilon}^e)}{\partial T} = \frac{1}{\rho_0} \boldsymbol{\alpha} \mathbf{M} \Big|_{\boldsymbol{\sigma}=\boldsymbol{\sigma}(T, \boldsymbol{\varepsilon}^e)}. \quad (3.8)$$

By comparing (3.8) to the right-hand equation (3.2)₁ we obtain:

$$\frac{\partial a(\boldsymbol{\varepsilon}^e, K)}{\partial \boldsymbol{\varepsilon}^e} = \frac{1}{\rho_0} \mathbf{M}\boldsymbol{\alpha} = \text{const and thus } : a = a_1(K) + \frac{1}{\rho_0} \mathbf{M}\boldsymbol{\alpha} : \boldsymbol{\varepsilon}^e. \quad (3.9)$$

Substituting the second term from formula (3.9) for “ a ” into the right-hand side of (3.2), we obtain:

$$s - s_0 = c_\varepsilon \ln \frac{T}{T_0} + \frac{1}{\rho_0} \mathbf{M}\boldsymbol{\alpha} : \boldsymbol{\varepsilon}^e + a_1(K), \quad (3.10)$$

where: s_0 —amount of entropy in the thermodynamic reference state (TRS).

Taking into account the following thermodynamic identity resulting from assumption (2.1)

$$-\rho_0 \frac{\partial s(T, \boldsymbol{\varepsilon}^e, K)}{\partial K} = -\rho_0 \frac{\partial s^P(T, K)}{\partial K} = \frac{\partial \boldsymbol{\Pi}(T, K)}{\partial T}, \quad (3.11)$$

we obtain from (3.10), after performing suitable transformations, the following relationship:

$$-\rho_0 \frac{da_1(K)}{dK} = \frac{\partial \Pi(T, K)}{\partial T} \quad (3.12)$$

and hence:

$$a_1(K) = -\frac{1}{\rho_0} \int_0^K \frac{\partial \Pi(T, K)}{\partial T} \cdot dK. \quad (3.13)$$

It follows from formulas (3.12) and (3.13) that the pair of thermodynamic forces Π (namely the tensor force π and the scalar force π) depends linearly on temperature $T \cdot K$ is the pair of internal parameters, $K \Leftrightarrow \{\kappa, \kappa\}$, see subplot 1.1. *The Main Notation and Abbreviations*.

Substituting expressions (3.7)₁ and (3.10) into (3.1), we obtain:

$$dA = - \left[c_\varepsilon \ln \frac{T}{T_0} + \frac{1}{\rho_0} \alpha \mathbf{M} : \boldsymbol{\varepsilon}^e + a_1(K) + s_0 \right] dT + \frac{1}{\rho_0} [\mathbf{M} \boldsymbol{\varepsilon}^e - \alpha \mathbf{M}(T - T_0)] : d\boldsymbol{\varepsilon}^e + \frac{1}{\rho_0} \Pi(K, T) \cdot dK \quad (3.14)$$

It is fully developed form of (3.1)—the total differential of thermodynamic potential of free energy A . Properties of potential A enable the following set of differential equations to be established:

$$\frac{\partial A}{\partial T} = -s = - \left[c_\varepsilon \ln \frac{T}{T_0} + \frac{1}{\rho_0} (\alpha \mathbf{M}) : \boldsymbol{\varepsilon}^e + a_1(K) + s_0 \right], \quad (3.15)$$

$$\frac{\partial A}{\partial \boldsymbol{\varepsilon}^e} = \frac{1}{\rho_0} \sigma = \frac{1}{\rho_0} [\mathbf{M} \boldsymbol{\varepsilon}^e - \alpha \mathbf{M}(T - T_0)], \quad (3.16)$$

$$\frac{\partial A}{\partial K} = \frac{1}{\rho_0} \Pi(K, T). \quad (3.17)$$

For the TRS we have: $T = T_0, \boldsymbol{\varepsilon}^e = \mathbf{0}, K = 0$, and thus from (3.13) we can infer that $a_1(K) = 0$, and from (3.10)—that $s = s_0$.

Making use of the above result we obtain from (3.15):

$$A - A_0 = -c_\varepsilon \ln \frac{T}{T_0} - \frac{1}{\rho_0} (\alpha \mathbf{M}) : \boldsymbol{\varepsilon}^e (T - T_0) - a_1(K)(T - T_0) - s_0(T - T_0) + \varphi(\boldsymbol{\varepsilon}^e, K), \quad (3.18)$$

where:

$$A_0 = A|_{T=T_0, \boldsymbol{\varepsilon}^e=\mathbf{0}, K=0}, \quad \varphi(0, 0) = 0, \quad A|_{T=T_0, \boldsymbol{\varepsilon}^e=0, K=0} = 0, \quad (3.19)$$

and $A_0 = -c_\varepsilon \int_0^{T_0} \ln \frac{\xi}{T_0} d\xi - s_0 T_0$.

After performing suitable calculations, we obtain from (3.16) and (3.18):

$$\frac{\partial A}{\partial \boldsymbol{\varepsilon}^e} = \frac{1}{\rho_0} \sigma = \frac{1}{\rho_0} [\mathbf{M} \boldsymbol{\varepsilon}^e - \alpha \mathbf{M}(T - T_0)] = -\frac{1}{\rho_0} \alpha \mathbf{M}(T - T_0) + \frac{\partial \varphi(\boldsymbol{\varepsilon}^e, K)}{\partial \boldsymbol{\varepsilon}^e} \quad (3.20)$$

and hence:

$$\frac{\partial \varphi(\boldsymbol{\varepsilon}^e, K)}{\partial \boldsymbol{\varepsilon}^e} = \frac{1}{\rho_0} \mathbf{M} \boldsymbol{\varepsilon}^e. \quad (3.21)$$

Formula (3.21) gives on integration:

$$\varphi = \frac{1}{2\rho_0} \boldsymbol{\varepsilon}^e : (\mathbf{M} \boldsymbol{\varepsilon}^e) + \psi(K), \quad \text{where } \psi(0) = 0. \quad (3.22)$$

Taking now (3.17) and formulas (3.18) and (3.22), we obtain:

$$\frac{\partial A}{\partial K} = \frac{1}{\rho_0} \Pi = \frac{d\psi(K)}{dK} - \frac{da_1(K)}{dK}(T - T_0). \quad (3.23)$$

This equation gives after integration:

$$\psi = (T - T_0) \int_0^K \frac{da_1(\xi)}{d\xi} d\xi + \frac{1}{\rho_0} \int_0^K \boldsymbol{\Pi} \cdot d\xi. \quad (3.24)$$

Substituting (3.24) into (3.22) yields:

$$\varphi = \frac{1}{2\rho_0} \boldsymbol{\epsilon}^e : \mathbf{M} \boldsymbol{\epsilon}^e + (T - T_0) \int_0^K \frac{da_1(\xi)}{d\xi} d\xi + \frac{1}{\rho_0} \int_0^K \boldsymbol{\Pi} \cdot d\xi. \quad (3.25)$$

On inserting (3.25) into (3.18) and on carrying out suitable calculations, we finally obtain:

$$A = -c_e \int_{T_0}^T \ln \frac{\zeta}{T_0} d\xi - \frac{1}{\rho_0} \boldsymbol{\alpha} : \mathbf{M} \boldsymbol{\epsilon}^e (T - T_0) + \frac{1}{2\rho_0} \boldsymbol{\epsilon}^e : \mathbf{M} \boldsymbol{\epsilon}^e + \frac{1}{\rho_0} \int_0^K \boldsymbol{\Pi} \cdot d\xi + s_0(T - T_0) + A_0. \quad (3.26)$$

A term $\frac{1}{\rho_0} \int_0^K \boldsymbol{\Pi} \cdot d\xi$ represents energy of inner changes resulting from irreversible processes during plastic deformations. When ($\kappa = \mathbf{0} \Rightarrow K = \kappa$ and $\boldsymbol{\pi} = \mathbf{0} \Rightarrow \boldsymbol{\Pi} = \boldsymbol{\pi}$ —description without effects of kinematic hardening—Bauschinger's effect is neglected), then this term is equal to

$$A^P = \frac{1}{\rho_0} \int_0^K \boldsymbol{\pi}(T, \xi) d\xi. \quad (3.27)$$

A^P —exergy of the enthalpy of plastic deformation. Term A^P represents the part of plastic deformation enthalpy Γ^P linked to the isotropic hardening effect, which is almost equal to the stored energy of plastic deformation E_s . On the basis of thermostatic equations (2.2), Eqs. (3.26) and (9)₃ we obtain that (see [4,8,9])

$$\Gamma^P = U^P = \frac{1}{\rho_0} \int_0^K \boldsymbol{\pi}(T, \xi) d\xi + \frac{1}{\rho_0} T s^P = A^P + \frac{1}{\rho_0} T s^P, \quad (3.28)$$

therefore,

$$A^P = \Gamma^P - \frac{1}{\rho_0} T s^P.$$

Expression $\frac{1}{\rho_0} T s^P = -\frac{T}{\rho_0} \frac{\partial}{\partial T} \left(\int_0^K \boldsymbol{\pi}(T, \xi) d\xi \right)$ stands for the thermal part of the plastic deformations enthalpy Γ^P (or in other words—the heat of internal changes). If the description is restricted to such changes in internal structure that are always accompanied by plastic deformation on a macroscopic scale, then the above quantity may also be termed “the reversible heat of plastic deformation,” see [3,4,9,43]. The term $T s^P$ is according to [3,4,9,43] likely to be small when compared to U^P (plastic part of internal energy) or A^P and the assumption $s^P \cong 0$ may be regarded as considerably small from the point of view of applied thermoplasticity. In physical terms it means that entropy of elasto-plastic body is comparable to that of purely elastic body [3,4,9,43]. In general, the higher the term $T s^P$, the higher is sensitivity of internal thermodynamic forces $\boldsymbol{\pi}$ to temperature, see (3.11). For isothermal processes, the term $T s^P$ is equal to zero, since $s^P = 0$ and $A^P = E_s$ [2,4,7,9].

Other parts of potential A represent, respectively:

- a) term: $c_e \int_{T_0}^T \ln \frac{\zeta}{T_0} d\xi$ —thermal capacity of the body,
- b) term: $\frac{1}{\rho_0} \boldsymbol{\epsilon}^e : \mathbf{M} \boldsymbol{\epsilon}^e (T - T_0)$ —energy of thermo-elastic dilatation,
- c) term: $\frac{1}{2\rho_0} \boldsymbol{\epsilon}^e : \mathbf{M} \boldsymbol{\epsilon}^e$ —energy of the isothermal elastic deformation,
- d) term: $s_0(T - T_0)$ —thermal energy of the body in the temperature T ,
- e) term: A_0 —internal energy of body in the TRS state.

4 Discussion of experimental data for the austenitic steel

In a series of papers concerned with problems of energy storage [21–25], austenitic steel 00H19N17Pr with grain sizes: 8 μm and 80 μm during the simple tension was studied. In the present paper we will limit our discussion to the coarser-grain grade (80 μm) since steel of this grade was found to obey a linear dependence of the SEPD value on the squared yield stress, see [12, 24, 25].

The evolution equation for stored energy E_s assumed here is quadratic with respect to the yield stress, i.e., $E_s \propto [Y^2(e_{(i)}^p, T) - Y_0^2(T)]$, see [4, 12, 24, 25]. This type of dependence was described in [24, 25] for austenitic steel 00H19N17Pr (grain size 80 μm and $T \cong \text{const}$) tested in simple stretching. As before, we additionally assume that the yield stress Y is explicitly dependent on the thermodynamic internal parameter k (equal effective plastic strain $e_{(i)}^p$) and temperature T . We have then:

$$\rho_0 E_s = B_0 [Y^2(e_{(i)}^p, T) - Y_0^2(T)] [\text{J/m}^3], \quad (4.1)$$

where B_0 —material constant expressed in Pa^{-1} .

The experimental hardening curve obtained in tensile testing of the steel is approximated by means of the following formulas:

$$\sigma_p = Y = C \times (\varepsilon_0 + e_{(i)}^p)^n \quad (4.2)$$

where ε_0 —strain corresponding to the initial yield stress and $e_{(i)}^p$ —effective true plastic strain.

For steel of grains of 80 μm it has the detailed form:

$$Y = 1275.1 \times (0.0214 + e_1^p)^{0.5359} [\text{MPa}] \quad (4.3)$$

where e_1^p —true plastic axial strain, $e_1^p = \ln \frac{l_p}{l_0} = \ln \frac{l - l_e}{l_0}$, where l , l_e and l_p —actual total, elastic and plastic gauge lengths, respectively. Parabolic regression coefficient $\hat{R}^2 \approx 99.75\%$, see [4, 9]. Following the method used in [14, 28], from (2.1)–(2.3) and associated laws of plastic flow [2–9] and (4.1), we obtain

$$\pi = \pi_0 \exp \left(\frac{1}{2B_0} \int_0^{e_{(i)}^p} \frac{d\xi}{Y(\xi, T)} \right), \quad (4.4)$$

where: $\pi_0 = \pi|_{e_{(i)}^p=0}$ —a constant thermodynamic force in the TRS and such that $\pi_0 \geq 0$, $T \cong \text{const}$, and

$$\dot{\kappa} = \frac{2B_0}{\pi_0} Y(e_{(i)}^p, T) \exp \left(-\frac{1}{2B_0} \int_0^{e_{(i)}^p} \frac{d\xi}{Y(\xi, T)} \right) \frac{\partial Y(e_{(i)}^p, T)}{\partial e_{(i)}^p} \dot{e}_{(i)}^p. \quad (4.5)$$

Based on expression (3.27) and the data obtained from [24, 25], we can write for the case when ($\xi = \kappa = e_{(i)}^p$ and $T = \text{const}$, where $\dot{e}_{(i)}^p = (\frac{2}{3}\dot{\epsilon}^p : \dot{\epsilon}^p)^{\frac{1}{2}}$ —effective deviatoric plastic strain rate, and $e_{(i)}^p = \int_0^t \dot{e}_{(i)}^p d\tau$ —effective deviatoric plastic strain (Odqvist parameter)), the following expressions

$$\begin{cases} E_s \cong A^p = \frac{2B_0}{\rho_0} \int_0^{e_{(i)}^p} Y(\xi, T) \frac{\partial Y(\xi, T)}{\partial \xi} d\xi = \frac{2B_0}{\rho_0} \int_0^{e_{(i)}^p} Y(\xi, T) dY, \\ \pi = 2B_0 Y(\xi, T) \frac{\partial Y(\xi, T)}{\partial \xi} \quad \text{and} \quad \pi_0 = \pi|_{e_{(i)}^p=0} \geq 0. \end{cases}. \quad (4.6)$$

With assumptions usually adopted for the TRS, it is possible to state that the force π_0 is a measure of internal stresses of the third kind, see [7, 9, 11, 25, 29, 44–46], which cannot be removed by annealing and so π_0 never becomes equal to zero.

Amount of mechanical energy dissipated in unit time and volume has the following form, see [1–9]:

$$D \geq 0 \quad \text{or} \quad \sigma : \dot{\epsilon}^p - \Pi \cdot \dot{K} \geq 0 \quad \text{and} \quad \sigma : \dot{\epsilon}^p - \pi : \dot{\kappa} - \pi \kappa \geq 0. \quad (4.7)$$

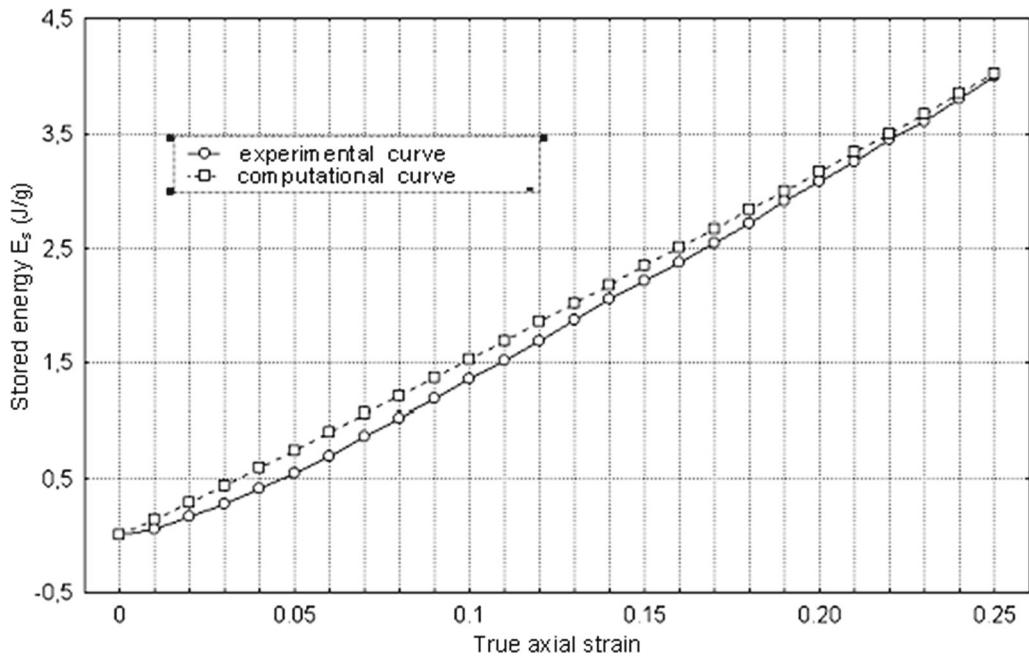


Fig. 1 Plots of energy stored vs true axial strain for austenitic steel. Compared are actual experimental curve [25] and the approximated results obtained using formula (4.6)₁

The condition satisfying the dissipation law (4.7) takes in this case the form:

$$Y(e_{(i)}^p, T) - Y_0(T) \leq \frac{1}{2B_0} e_{(i)}^p \quad \text{or} \quad Y - Y_0 \leq \frac{1}{2B_0} e_{(i)}^p \quad (4.8)$$

where: Y —yield stress in simple tension at temperature T [K],

Y_0 —initial yield stress ($e_{(i)}^p = 0$) at temperature T [K].

For the austenitic steel investigated by Oliferuk *et al* [24,25] it has been found (beyond the initial stage of plastic flow) that value of material constant $B_0 \approx 88 \times 10^{-12}$ [Pa⁻¹], and therefore the dissipation law (second law of thermodynamics for this kind of plastic deformation process) (4.7) is satisfied. Density ρ_0 for this steel is 7.8×10^3 [kg/m³]. Based on (4.3), $Y_0 \cong 1275.1 \times 0.0214^{0.5359} \cong 162.5$ [MPa]. Material constant B_0 was obtained from linear relation (beyond the initial stage of plastic flow) between E_s and the square of the strain of plastic flow Y during uniaxial stretching of austenitic steel 00H19N17Pr. The measurements were performed on 10 samples of 80 μm grain size and presented as plots in [24,25]. Then, material constant B_0 was calculated from Eq. (4.1). Experimental curve shown in Fig. 1 was obtained in [25] from stretching of samples on Instron 1251 apparatus at constant velocity of 10 mm/min of move of machine jaws, which is equivalent to $\dot{\varepsilon} = 2 \times 10^{-3} \text{ s}^{-1}$. Infrared radiation emitted by the deformed samples was measured by means of AGA-680 thermography system [25].

Results of the experimental and computed values of stored energy E_s vs true axial strain ε_1 are presented in Fig. 1, where $\varepsilon_1 = \ln \frac{l}{l_0}$.

As can be seen in Fig. 1, the greatest difference between the computational value and the experimental one appears for the deformation ε_1 about 0.05 and the relative error amounts about 36%.

The first reason for that error may be that for small deformations the relation (4.1) of stored energy vs square of flow stress is not linear how is assumed in our work.

Experimental material constant B_0 according to [24,25] is not constant and is smaller than 88×10^{-12} [Pa⁻¹] for small deformations when $\varepsilon_1 < 0.05$. The dashed line linearization is upper estimation of experimental results, see graphs in [24,25]. The value of material constant $B_0 \approx 88 \times 10^{-12}$ [Pa⁻¹] assumed in this work, which is correct for $\varepsilon_1 > 0.05$, enlarged the computational value of E_s . The assumption of constant value of B_0 for the entire range of deformations (0.00 \div 0.25) results from the fact that such thermodynamic model of the body, adopted in this work, was used for calculating of SEPD in pipes during bending processes for knees of energy pipelines. Stored energy plays an important role during their exploitation [4,7,30], where activated

processes are occurring, e.g., diffusion of atoms through border of grains (Fick's law) and phenomena of creeping in accordance with the law of Arrhenius or Weertman [46]. SEPD reduces the activation energy causing faster occurrence of these processes. The deformations during pipe bending are great ($\varepsilon_1 \gg 0.05$) and then the errors in calculations are reduced almost to zero for $\varepsilon_1 \approx 0.25$, see Fig. 1. During pipes bending with small radius, the bending deformations ε_1 are higher than 0.25 [7,30] and theoretically can take maximum value of about 0.5.

The second reason of the error can be that we neglected the kinematical hardening in our thermodynamic model of body used for calculations. According to [16,47] the kinematical hardening model assures the decrease in the stored energy. The kinematic hardening plays important role for small deformations, but for greater deformations the isotropic hardening plays an important role. Authors of this work do not have the experimental data which could let them determine the tensor parameters κ and π for examined austenitic steel 00H19N17Pr. The tensor parameters (κ and π) are used for a description of the kinematic hardening in the H-M-H hypothesis [3,7,9,16].

5 Conclusions

1. The thermodynamic potential of free energy for the thermo-elastic-plastic body, derived in this work, can serve to estimate the energy stored during cold plastic processing of materials, for example in tube bending [7,30]. Cold tube bending and many other processes of plastic forming of metals are quasi-static and quasi-isothermal ($T \approx \text{const}$). The results presented in this paper obtained for calculated and measured [24,25] values of stored energy for austenitic steel are in good agreement

2. The computed values of SEPD were compared with experimental data obtained under plain tension conditions. Good agreement was found provided that material constant B_0 varied appropriately. The advantage of the presented method is such that it permits the E_s value to be accurately evaluated in processes of plastic forming of metals (beyond the initial stage of plastic flow) [4,7,9,30] by having only the yield curves in simple tension (Y) and material constants B_0 , cf [4,7,9].

3. A practical example of how to determine the internal thermodynamic forces π linked to the value of material constant B_0 (using equation (4.6)₂) and of internal thermodynamic forces π_0 (using Eq. (4.4)) can be found in [4,7,9].

4. In the future, it would be interesting to determine the value of energy stored in materials that have undergone operations of plastic forming at high temperature or with preheating under non-isothermal conditions [7,9,20,30]. Even moderate temperature (600 – 950°C) of plastic deformations results in lowering levels of stored energy and in favorable changes in material microstructure, see [4,7,9,30]. A fundamental solution of this problem will be the derivation of equations of evolution of value of SEPD during inner transformations of materials depending on temperature and time.

5. The estimation of the value of element of enthalpy, called sometimes “anergy” $\frac{1}{\rho_0} T s^P = -\frac{T}{\rho_0} \frac{\partial}{\partial T} \left(\int_0^\kappa \pi(T, \xi) d\xi \right)$, can be important. This value, according to [3,4,9,43], is likely to be small when compared to U^P (plastic part of internal energy) or A^P . To calculate anergy, the change in the yield point for austenitic steel 00H19N17Pr with the temperature, see expression (4.4), is required and the authors unfortunately do not have such data.

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