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Daubechies' Time–Frequency Localization Operator on Cantor Type Sets I

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Abstract

We study Daubechies' time–frequency localization operator, which is characterized by a window and weight function. We consider a Gaussian window and a spherically symmetric weight as this choice yields explicit formulas for the eigenvalues, with the Hermite functions as the associated eigenfunctions. Inspired by the fractal uncertainty principle in the separate time–frequency representation, we define the n-iterate mid-third spherically symmetric Cantor set in the joint representation. For the n-iterate Cantor set, precise asymptotic estimates for the operator norm are then derived up to a multiplicative constant.

Keywords Fractal Uncertainty Principle · Daubechies' localization operator · Cantor set

Mathematics Subject Classification 47A30 · 47A75

1 Introduction

The problem of localizing signals in time and frequency is an old and important one in signal analysis. In applications, we often wish to analyze signals on different time–frequency domains, and we would therefore attempt to concentrate signals on said domains. Different approaches for how to construct such *time–frequency localization operators* have been suggested, either based on a separate or joint time–frequency representation of the signal (see [4,13]). The localization operators, regardless of

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which we choose to work with, will however be limited by the fundamental barrier of time–frequency analysis, namely the *uncertainty principles*, which state that a signal cannot be highly localized simultaneously in *both* time and frequency. With regard to the localization operator, the limits posed by the uncertainty principles translate into the associated *operator norm*, as it measures the optimal efficiency of any given localization operator.

Many versions of the uncertainty principles exist (see [10]), and more recent versions start to take into account the *geometry* of the time–frequency domains. In particular, in [6], Dyatlov describes the development and applications of a *fractal uncertainty principle* (FUP) for the separate time–frequency representation, first introduced and developed in [3,7,8]. The relevant localization operator is the standard composition of projections $\pi_T Q_\Omega$, where π_T and Q_Ω project onto the sets T in time and Ω in frequency, respectively. In the context of the FUP, the sets T and Ω take the form of fractal sets. Here fractal sets are defined in terms of the general notion of δ-regularity (see [6, Definition 2.2.]), as families of sets $T(h)$, $\Omega(h) \subseteq [0, 1]$, dependent on a continuous parameter $0 < h \leq 1$. The FUP is then formulated for this general class of sets when $h \to 0$.

An illustrative example featured in [6] is the mid-third Cantor set, where both the time and frequency domain can be regarded as h-neighbourhoods of the Cantor set, say $C(h)$. Notice that the FUP is originally framed such that the parameter h is also encoded in the Fourier transform $\mathcal{F}_h$ (not unlike the normalization with Planck's constant in quantum mechanics) such that $\mathcal{F}_h f(\omega) = \frac{1}{\sqrt{h}} \hat{f}(\omega h^{-1})$, where $\hat{f}$ denotes the Fourier transform of f. If we instead formulate the FUP in terms of the regular Fourier transform, we now consider the family $C(h)/\sqrt{h}$. Further, if we translate the continuous h into discrete iterations n, we obtain a sequence based on the n-iterate Cantor set, defined in increasing intervals depending on n. More precisely, if $T = \Omega = C_n$ denotes the n-iterate defined in the interval $[0, M]$, then the interval length satisfies

$$3^n \sim M^2, \tag{1.1}$$

which means $|C_n| \sim \left(2/\sqrt{3}\right)^n \to \infty$ as $n \to \infty$. However, by Theorem 2.13 in [6], there exist constants $\alpha, \beta > 0$ such that

$$\|\pi_{C_n} Q_{C_n}\|_{\text{op}} \leq \alpha e^{-\beta n} \quad \forall\, n \geq 0. \tag{1.2}$$

We should expect some analogous result to the FUP when extending to the joint time–frequency representation (see Itinerary page 1 in [11]). Inspired by the Cantor set example in separate time and frequency, we derive a similar localization result for the Cantor set in the joint representation. In particular, we consider *Daubechies' localization operator*, first introduced in [4], based on the *Short-Time Fourier tranform*, with a *spherically symmetric* weight function and a Gaussian window. The reason for these restrictions is that, as was shown in the aforementioned paper, we obtain explicit expressions for the eigenvalues of the localization operator, with the Hermite functions as the associated eigenfunctions.

The remainder of the paper is organized as follows: In Sect. 2 we provide a more detailed introduction to the Daubechies operator (Sects. 2.1, 2.2), in addition to some necessary results in the spherically symmetric context (Sect. 2.3). We also make clear what we mean by a spherically symmetric Cantor set (Sect. 2.4). New results are found in Sects. 3 and 4, which contains several estimates for Daubechies' operator localizing on different spherically symmetric sets.

In particular, Sect. 3 contains two introductory examples of localization on spherically symmetric subsets, namely localization on a ring and a set of infinite measure. In Sect. 4, we finally consider the n-iterate spherically symmetric Cantor set. Here we derive *precise* asymptotic estimates (up to a multiplicative constant) for the operator norm (Theorem 4.2). A particular case of this two-parameter result, in terms of the radius R and iterate n, can be formulated as an estimate solely in terms of the parameter n or R. In the spherically symmetric context, we consider the condition

$$3^n \sim \left(\pi R^2\right)^2, \tag{1.3}$$

similar to condition (1.1). Hence, under the above condition, let P_n denote the Daubechies operator localizing on the n-iterate spherically symmetric Cantor set defined in the disk of radius $R > 0$. Then for some positive constants $c_1 \leq c_2$ the operator norm satisfies

$$c_1 \left(\frac{2}{3}\right)^{\frac{n}{2}} \leq \|P_n\|_{\text{op}} \leq c_2 \left(\frac{2}{3}\right)^{\frac{n}{2}}. \tag{1.4}$$

This result is analogous to knowing the exponential $\beta > 0$ in (1.2) precisely.

2 Preliminaries

2.1 Fourier and Short-Time Fourier Transform

For a function $f : \mathbb{R} \to \mathbb{C}$ the *Fourier transform* evaluated at point $\omega \in \mathbb{R}$ is given by

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(t)e^{-2\pi i\omega t}\mathrm{d}t.$$

If we interpret f as an amplitude signal depending on *time*, then its Fourier transform $\hat{f}$ corresponds to a *frequency* representation of the signal. The pair $(f, \hat{f})$ does not, however, offer a joint description with respect to both frequency and time. For this purpose, we consider the *Short-Time Fourier transform* (STFT) (see Chapter 3 in [11]).

The STFT is often referred to as the "windowed Fourier transform" as this transform relies on an additional fixed, non-zero function, $\phi : \mathbb{R} \to \mathbb{C}$, known as a *window function*. At point $(\omega, t) \in \mathbb{R} \times \mathbb{R}$ the STFT of f with respect to the window ϕ is then defined as

$$V_\phi f(\omega, t) = \int_{\mathbb{R}} f(x)\overline{\phi(x-t)}e^{-2\pi i \omega x}\mathrm{d}x.$$

The transformed signal now depends on both time t and frequency ω, and we refer to the (ω, t)-domain $\mathbb{R}^2$ as the *phase space* or the *time–frequency plane*.

We will restrict our attention to signals and windows in $L^2(\mathbb{R})$, which, by Cauchy–Schwarz' inequality, implies that $V_\phi f(\omega, t)$ is well-defined for all points $(\omega, t) \in \mathbb{R}^2$. Such restrictions also produce the following orthogonality relation

$$\langle V_{\phi_1} f_1, V_{\phi_2} f_2\rangle_{L^2(\mathbb{R}^2)} = \langle f_1, f_2\rangle\overline{\langle \phi_1, \phi_2\rangle} \;\; \forall\, f_1, f_2, \phi_1, \phi_2 \in L^2(\mathbb{R}). \tag{2.1}$$

Equipped with the standard L^2-norms, we deduce that the STFT is a bounded linear map, with target space $L^2(\mathbb{R}^2)$. If the window ϕ is normalized, i.e. $\|\phi\|_2 = 1$, then the STFT becomes, in fact, an *isometry* onto some subspace of $L^2(\mathbb{R}^2)$.

Further, by identity (2.1), the original signal f can be recovered from its phase space representation. Take any $\gamma \in L^2(\mathbb{R})$ such that $\langle \gamma, \phi\rangle \neq 0$, then the orthogonal projection of f onto any $g \in L^2(\mathbb{R})$ is given by

$$\langle f, g\rangle = \frac{1}{\langle \gamma, \phi\rangle} \iint_{\mathbb{R}^2} V_\phi f(\omega, t)\overline{V_\gamma g(\omega, t)}\mathrm{d}\omega\mathrm{d}t.$$

A canonical choice for γ is to set it equal to ϕ. Assuming ϕ is normalized, these projections then read

$$\langle f, g\rangle = \iint_{\mathbb{R}^2} V_\phi f(\omega, t)\overline{V_\phi g(\omega, t)}\mathrm{d}\omega\mathrm{d}t. \tag{2.2}$$

Since any signal $f \in L^2(\mathbb{R})$ is entirely determined by such inner products, the right-hand side of formula (2.2) provides a complete recovery from the STFT.

2.2 Daubechies' Localization Operator

One approach for how to construct operators that localize a signal f in both time and frequency was suggested by Daubechies in [4]. These operators can be summarized as modifying the STFT of f by multiplication of a weight function, say $F(\omega, t)$, before recovering a time-dependent signal. The weight function aims at enhancing certain features of the phase space while diminishing others. Based on formula (2.2), we consider the sesquilinear functional $\mathscr{P}_{F,\phi}$ on the product $L^2(\mathbb{R}) \times L^2(\mathbb{R})$, defined by

$$\mathscr{P}_{F,\phi}(f, g) = \iint_{\mathbb{R}^2} F(\omega, t)V_\phi f(\omega, t)\overline{V_\phi g(\omega, t)}\mathrm{d}\omega\mathrm{d}t.$$

Assuming $\mathscr{P}_{F,\phi}$ is a bounded functional, *Riesz' representation theorem* ensures the existence of a bounded, linear operator $P_{F,\phi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ such that

$$\mathscr{P}_{F,\phi}(f, g) = \langle P_{F,\phi} f, g\rangle.$$

The operator $P_{F,\phi}$ is our sought after time-frequency localization operator, which we will refer to as *Daubechies' localization operator*. From the above definition, $P_{F,\phi}$ is characterized by the choice of weight F and window function ϕ.

In particular, any real-valued, integrable weight F will produce self-adjoint, compact operators $P_{F,\phi}$ whose eigenfunctions form a complete basis for the space $L^2(\mathbb{R})$. Furthermore, the eigenvalues $\{\lambda_k\}_k$ satisfies $\sum_k |\lambda_k| \leq \|F\|_1$, in addition to $|\lambda_k| \leq \|F\|_\infty$ for all k.

2.3 Spherically Symmetric Weight

For an arbitrary weight F and window ϕ it remains a challenge to determine the eigenvalues of Daubechies' localization operator $P_{F,\phi}$. However, in [4], Daubechies narrows in her focus to operators with a normalized Gaussian window

$$\phi(x) = 2^{1/4} e^{-\pi x^2}, \tag{2.3}$$

and a spherically symmetric weight

$$F(\omega, t) = \mathscr{F}(r^2), \tag{2.4}$$

where $r^2 = \omega^2 + t^2$. For such operators, the Hermite functions[1]

$$H_k(t) = \frac{2^{1/4}}{\sqrt{k!}} \left(-\frac{1}{2\sqrt{\pi}}\right)^k e^{\pi t^2} \frac{\mathrm{d}^k}{\mathrm{d}t^k}(e^{-2\pi t^2}), \quad k = 0, 1, 2, \ldots \tag{2.5}$$

are shown to constitute the eigenfunctions. Further, explicit expressions for the associated eigenvalues $\{\lambda_k\}_k$ are derived.

Theorem 2.1 (Daubechies) *Let $P_{F,\phi}$ denote the localization operator with weight $F(\omega, t) = \mathscr{F}(r^2)$ and window ϕ equal to the normalized Gaussian in (2.3). Then the eigenvalues of $P_{F,\phi}$ are given by*

$$\lambda_k = \int_0^\infty \mathscr{F}\left(\frac{r}{\pi}\right) \frac{r^k}{k!} e^{-r} \mathrm{d}r, \quad \textit{for } k = 0, 1, 2, \ldots,$$

such that

$$P_{F,\phi} H_k = \lambda_k H_k,$$

where H_k denotes the k-th Hermite function.

[1] Due to the choice of normalization for the Fourier transform, both the Gaussian and the Hermite functions are normalized differently than in [4]. The normalization is chosen in accordance with Folland [9]. If h_k denotes the k-th Hermite function in [4], this relates to H_k in (2.5) by $H_k(x) = \frac{2^{1/4}}{\sqrt{2^k k!}} h_k(\sqrt{2\pi} x)$.

Observe that the normalized Gaussian in (2.3) coincides with H_0 in (2.5). It was shown recently in [2] that (for each j) the Hermite functions are also eigenfunctions of any localization operator with window H_j and a spherically symmetric weight. Nevertheless, we will always assume the window ϕ to be the normalized Gaussian.

We will consider the case when F projects onto a spherically symmetric subset $\mathcal{E} \subseteq \mathbb{R}^2$. This means $\mathscr{F}$ equals the characteristic function of some subset $E \subseteq \mathbb{R}_+$, i.e., $\mathscr{F}(r) = \chi_E(r)$, such that $\mathcal{E} = \{(\omega, t) \in \mathbb{R}^2 \mid \omega^2 + t^2 \in E\}$. As a matter of convenience, we will denote the associated Daubechies operator simply by $P_{\mathcal{E}}$. By Theorem 2.1, the eigenvalue corresponding to the k-th Hermite function is then given by

$$\lambda_k = \int_{\pi \cdot E} \frac{r^k}{k!} e^{-r} \mathrm{d}r, \quad \text{for } k = 0, 1, 2, \ldots, \tag{2.6}$$

where $\pi \cdot E := \{x \in \mathbb{R}_+ \mid x\pi^{-1} \in E\}$. Since the above integrands will appear frequently, we define, for simplicity, the functions

$$f_k(r) := \frac{r^k}{k!} e^{-r}, \ r \geq 0, \quad \text{for } k = 0, 1, 2, \ldots$$

In Sect. 4 we require two basic properties of the integrands $\{f_k\}_k$ (see Appendix A for additional details), namely

$$f_k(k - r) \leq f_k(k + r) \ \forall\, r \in [0, k] \quad \text{for } k = 1, 2, 3, \ldots \tag{2.7}$$

and

$$\int_E f_k(r) \mathrm{d}r \leq \int_0^{|E|} f_0(r) \mathrm{d}r = 1 - e^{-|E|} \quad \text{for } k = 0, 1, 2, \ldots, \tag{2.8}$$

where E is some measurable subset of $\mathbb{R}_+$.

2.4 Cantor Set

The *mid-third Cantor set* based in the interval $[0, R]$ is constructed as follows: Start with the interval $C_0(R) = [0, R]$. Each n-iterate $C_n(R)$ is the union of 2^n disjoint, closed intervals $\{I_{j,n}\}_j$. To obtain the next iterate $C_{n+1}(R)$ remove the open middle-third interval in every interval $I_{j,n}$. Such iterations yield a nested sequence $C_0 \supseteq C_1 \supseteq C_2 \supseteq \ldots$ The *mid-third Cantor* set $C(R)$ on the interval $[0, R]$ is then defined as the intersection of all the n-iterates, i.e.,

$$C(R) = \bigcap_{n=0}^{\infty} C_n(R).$$

For each n-iterate, we define a corresponding map $\mathcal{G}_{R,n} : \mathbb{R} \to [0, 1]$ by

$$\mathcal{G}_{R,n}(x) = \frac{1}{|C_n(R)|} \cdot \begin{cases} 0, & x \leq 0, \\ |C_n(R) \cap [0, x]|, & x > 0 \end{cases} \quad \text{for } n = 0, 1, 2, \ldots, \tag{2.9}$$

which we refer to as the n-iterate Cantor function. These functions will come into play in the latter part of Sect. 4, where we will utilize the fact that $\{\mathcal{G}_{R,n}\}_n$ are all *subadditive*, i.e.,

$$\mathcal{G}_{R,n}(a + b) \leq \mathcal{G}_{R,n}(a) + \mathcal{G}_{R,n}(b) \;\; \forall\, a, b \in \mathbb{R}, \tag{2.10}$$

which was shown by induction by Josef Doboš in [5].

In the spherically symmetric context, we consider the following Cantor set construction: For the disk of radius $R > 0$ centered at the orgin, we identify the n-iterate with the subset

$$\mathscr{C}_n(R) = \{(\omega, t) \in \mathbb{R}^2 \mid \omega^2 + t^2 \in C_n(R^2)\} \subseteq \mathbb{R}^2. \tag{2.11}$$

This means we consider weights of the form

$$\mathscr{F}(r) = \chi_{C_n(R^2)}(r), \quad \text{for } R > 0 \text{ and } n = 0, 1, 2, \ldots$$

Based on formula (2.6), the eigenvalues of $P_{\mathscr{C}_n(R)}$ can then be expressed as

$$\lambda_k(\mathscr{C}_n(R)) = \int_{\pi \cdot C_n(R^2)} f_k(r) \mathrm{d}r = \int_{C_n(\pi R^2)} f_k(r) \mathrm{d}r. \tag{2.12}$$

3 Examples of Localization on Spherically Symmetric Sets

In this section we present estimates for the operator norm of Daubechies' operator localizing on two different spherically symmetric sets. For this purpose, it would be sufficient to determine the largest eigenvalue of the operator and estimate said eigenvalue. Nonetheless, even with identity (2.6), it may prove difficult to determine which eigenvalue is the largest. Under such circumstances, we will instead attempt to derive a common upper bound for the eigenvalues.

3.1 Localization on a Ring: Asymptotic Estimate

The first example shows that any eigenvalue λ_k of Daubechies' localization operator can, in principle, be the largest eigenvalue. Consider localization on a ring of inner radius $R > 0$ in phase space of measure 1, that is, consider the subset

$$\mathcal{E}(R) := \{(\omega, t) \in \mathbb{R}^2 \mid \pi(\omega^2 + t^2) \in [\pi R^2, \pi R^2 + 1]\} \tag{3.1}$$

with the associated localization operator $P_{\mathcal{E}(R)}$. By (2.6), the eigenvalues of $P_{\mathcal{E}(R)}$ become

$$\lambda_k(R) := \lambda_k(\mathcal{E}(R)) = \int_{\pi R^2}^{\pi R^2+1} f_k(r)\mathrm{d}r \quad \text{for } k = 0, 1, 2, \ldots$$

Now, assume that $\pi R^2 \in [m, m+1]$ for some $m \in \mathbb{N} \cup \{0\}$. Since the difference $f_k(r) - f_{k+1}(r)$ is negative precisely when $r > k+1$, we obtain the ordering

$$\lambda_0(R) \leq \lambda_1(R) \leq \lambda_2(R) \leq \cdots \leq \lambda_m(R)$$

and

$$\lambda_{m+1}(R) \geq \lambda_{m+2}(R) \geq \lambda_{m+3}(R) \geq \cdots$$

Under these conditions, either $\lambda_m(R)$ or $\lambda_{m+1}(R)$ must be the largest eigenvalue. In particular, if $\pi R^2 = m$, then $\lambda_m(R)$ becomes the largest eigenvalue. In the next proposition we provide an estimate of the operator norm of $P_{\mathcal{E}(R)}$.

Proposition 3.1 *Let $\mathcal{E}(R) \subseteq \mathbb{R}^2$ be as in (3.1). For any fixed $\pi R^2 \geq 2$, there exists a positive, finite constant C such that the operator norm of $P_{\mathcal{E}(R)}$ satisfies the bounds*

$$\frac{1}{\pi\sqrt{2}}R^{-1} - CR^{-3} \leq \|P_{\mathcal{E}(R)}\|_{op} \leq \frac{1}{\pi\sqrt{2}}R^{-1} + CR^{-3}.$$

Proof Let $n := \lfloor \pi R^2 \rfloor$, where $\lfloor\cdot\rfloor$ denotes the floor function, rounding down to the nearest integer. Apply a max–min-approximation of the integrands $f_k(r)$ for $r \in [n, n+2] \supseteq [\pi R^2, \pi R^2 + 1]$. In particular, $f_n(n)$ serves as an upper bound and, by inequality (2.7), $f_{n+1}(n)$ serves as a lower bound for the operator norm. That is,

$$\frac{n^{n+1}}{(n+1)!}e^{-n} \leq \|P_{\mathcal{E}(R)}\|_{\mathrm{op}} \leq \frac{n^n}{n!}e^{-n}.$$

Once we combine this with *Stirling's approximation formula* for the factorial

$$\sqrt{2\pi} \cdot n^{n+1/2}e^{-n} \leq n! \leq e^{\frac{1}{12n}}\sqrt{2\pi} \cdot n^{n+1/2}e^{-n} \quad \text{for } n = 1, 2, 3, \ldots,$$

we obtain

$$\frac{1}{\sqrt{2\pi}}n^{-1/2}\left(1 + \frac{1}{n}\right)^{-1} e^{-\frac{1}{12n}} \leq \|P_{\mathcal{E}(R)}\|_{\mathrm{op}} \leq \frac{1}{\sqrt{2\pi}}n^{-1/2}.$$

Expressing the above inequality in terms of R, we use that $\pi R^2 - 1 \leq n \leq \pi R^2$ and factor out $1/\sqrt{\pi R^2}$. The error terms $\pm CR^{-3}$, follows by Taylor expansion of the remaining factors about $1/(\pi R^2) = 0$. □

Remark A careful reading of the Taylor series expansion reveals that for $\pi R^2 \geq 2$, the inequalities in Proposition 3.1 hold for constant $C = \pi^{-2}$.

3.2 Localization on Set of Infinite Measure

Next, we consider a non-trivial example of localization on a spherically symmetric set of *infinite* measure (see [12] for a similar example in the separate time-frequency representation). Define the subset

$$\mathcal{E}(s) := \left\{ (\omega, t) \in \mathbb{R}^2 \,\middle|\, \pi(\omega^2 + t^2) \in \bigcup_{n=0}^{\infty} [n, n+s] \right\}, \tag{3.2}$$

which we can identify as an infinite number of equidistant intervals in $\mathbb{R}_+$. Although the above set has infinite measure, we maintain good control over the operator norm of $P_{\mathcal{E}(s)}$ and can produce precise estimates in terms of the parameter s.

Theorem 3.1 *Let $\mathcal{E}(s) \subseteq \mathbb{R}^2$ be as in (3.2) with $s \in [0, 1]$. Then the operator norm of $P_{\mathcal{E}(s)}$ satisfies the bounds*

$$C(1 - e^{-s}) \leq \|P_{\mathcal{E}(s)}\|_{op} \leq \min\{Cs, 1\} \;\; \forall\, s \in [0, 1] \;\; \textit{with } C = \frac{e}{e-1}. \tag{3.3}$$

Further, there exists $s_0 > 0$ such that

$$\|P_{\mathcal{E}(s)}\|_{op} = C(1 - e^{-s}) \;\; \forall\, 0 < s < s_0. \tag{3.4}$$

Proof By formula (2.6), the eigenvalues read

$$\lambda_k(s) := \lambda_k(\mathcal{E}(s)) = \int_{\bigcup_n [n,n+s]} f_k(r) \mathrm{d}r = \sum_{n=0}^{\infty} \int_n^{n+s} f_k(r) \mathrm{d}r \qquad \text{for } k = 0, 1, 2, \ldots$$

For each integral over $[n, n+s]$, consider the maximum of $f_k(r)$ for $r \in [n, n+1]$ such that

$$\lambda_0(s) \leq s \sum_{n=0}^{\infty} f_0(n) = s \sum_{n=0}^{\infty} e^{-n} = \frac{s}{1 - e^{-1}} = Cs \tag{3.5}$$

and

$$\lambda_k(s) \leq s \left(f_k(k) + \sum_{n=0}^{\infty} f_k(n) \right) \qquad \text{for } k = 1, 2, 3, \ldots \tag{3.6}$$

We now claim that the following inequality holds

$$f_k(k) + \sum_{n=0}^{\infty} f_k(n) < \sum_{n=0}^{\infty} f_0(n) \quad \text{for } k = 1, 2, 3, \ldots \tag{3.7}$$

For $k = 1$, inequality (3.7) is verified by computing the series explicitly. While for $k > 1$, compare the series with the integral over $\mathbb{R}_+$, that is

$$\sum_{n \neq k} f_k(n) \leq \int_0^{\infty} f_k(r) \mathrm{d}r = 1.$$

Thus,

$$f_k(k) + \sum_{n=0}^{\infty} f_k(n) \leq 1 + 2 f_k(k) \leq 1 + 2 f_2(2) = 1 + 4e^{-2} \quad \text{for } k = 2, 3, 4, \ldots$$

Since $1 + 4e^{-2} < C$, claim (3.7) follows. Combining results (3.5)–(3.7) yields the upper bound in (3.3). In the lower bound case of (3.3), it is sufficient to observe

$$\lambda_0(s) = \sum_{n=0}^{\infty} \int_n^{n+s} e^{-r} \mathrm{d}r = (1 - e^{-s}) \sum_{n=0}^{\infty} e^{-n} = (1 - e^{-s}) C.$$

For the equality case (3.4), note that inequality (3.7) ensures that there exists a constant $0 < C_0 < C$ such that $\lambda_k(s) \leq C_0 s$ for any $k, s > 0$. Since $(1 - e^{-s}) s^{-1} \to 1$ from below as $s \to 0$, it follows that some $s_0 > 0$ with property (3.4) exists. □

Remark In [1] Theorem 3, a more general localization result is presented for signals $f \in M^p(\mathbb{R})$ with $p \geq 1$. The result is similar as it provides an upper bound when localizing on sparse sets in phase space. Applied to signals $f \in M^2 = L^2$ and the subset $\mathcal{E}(s)$, Theorem 3 yields a somewhat coarser upper bound, namely $\| P_{\mathcal{E}(s)} \|_{\mathrm{op}} \leq C' \sqrt{s}$ for some constant $C' > 0$.

4 Localization on Spherically Symmetric Cantor Set

In this section we consider localization on the n-iterate spherically symmetric Cantor set, i.e., the set $\mathscr{C}_n(R)$ in (2.11). Hence, we consider the localization operator $P_{\mathscr{C}_n(R)}$ and attempt to estimate its operator norm. Results are formulated in Sect. 4.1, with the proof strategy and formal proofs in the subsequent Sects. 4.2–4.4.

4.1 Results: Bounds for the Operator Norm

Below two theorems regarding the operator norm of $P_{\mathscr{C}_n(R)}$ are presented. The first theorem shows to what extent the operator norm is bounded by the first eigenvalue $\lambda_0(\mathscr{C}_n(R))$.

Theorem 4.1 *The operator norm of $P_{\mathscr{C}_n(R)}$ is bounded from above by*

$$\|P_{\mathscr{C}_n(R)}\|_{op} \leq 2\lambda_0(\mathscr{C}_n(R)) \quad \textit{for } n = 0, 1, 2, \ldots$$

The second theorem is a *precise* asymptotic estimate of the operator norm of $P_{\mathscr{C}_n(R)}$ (up to a multiplicative constant) based on the same asymptotic estimate for $\lambda_0(\mathscr{C}_n(R))$.

Theorem 4.2 *There exist positive, finite constants $c_1 \leq c_2$ such that for each $n = 0, 1, 2, \ldots$*

$$c_1 \leq \frac{\left(2\pi R^2 + 1\right)^{\frac{\ln 2}{\ln 3}}}{2^n\left(1 - e^{-\pi R^2/3^n}\right)} \cdot \|P_{\mathscr{C}_n(R)}\|_{op} \leq c_2 \;\; \forall\, \pi R^2 \in [0, 3^n/2].$$

Proofs of Theorem 4.1 and 4.2 can be found in Sects. 4.3 and 4.4 , respectively. At the end of Sect. 4.4, we also retrieve numerical estimates for the constants in Theorem 4.2.

If we now enforce condition (1.3), and note that $2^n \sim (\pi R^2)^{2\frac{\ln 2}{\ln 3}}$, we obtain the following corollary:

Corollary 4.1 *Suppose that the radius R depends on the iterate n such that $\pi R^2 \sim 3^{\frac{n}{2}}$. Then there exists positive, finite constants $c_1 \leq c_2$ such that*

$$c_1\left(\pi R^2\right)^{\frac{\ln 2}{\ln 3}-1} \leq \|P_{\mathscr{C}_n(R(n))}\|_{op} \leq c_2(\pi R^2)^{\frac{\ln 2}{\ln 3}-1}.$$

Note that the above corollary is the same as result (1.4), except that we have expressed the inequality in terms of the radius R rather than the iterate n. On this form we have been able to express bounds for the operator norm in terms of quantity $\delta = \frac{\ln 2}{\ln 3}$, which is the *fractal dimension* of the mid-third Cantor set. It would therefore be interesting to investigate whether the same or a similar statement holds when localizing on different Cantor sets, with a different fractal dimension $0 < \delta < 1$.

4.2 Main Strategy: Relative Areas

Both theorems are obtained from the integral formula (2.12) for the eigenvalues $\{\lambda_k(\mathscr{C}_n(R))\}_k$. However, as the number of intervals in the n-iterate Cantor set grows as 2^n, it soon becomes rather impractical to evaluate these integrals *directly*. Instead we consider the *local* effect on the integrals when increasing from one iterate to the next. In particular, this means we initially consider the integral of f_k over a *single* interval, say $[s, s + 3L]$ for $s \geq 0$ and $L > 0$. Then we attempt to determine the relative area left under the curve f_k once the mid-third of the interval is removed, i.e., we wish to understand the function

$$\mathcal{A}_k(s, 3L) := \left[\int_s^{s+L} f_k(r)\mathrm{d}r + \int_{s+2L}^{s+3L} f_k(r)\mathrm{d}r\right] \cdot \left[\int_s^{s+3L} f_k(r)\mathrm{d}r\right]^{-1}. \quad (4.1)$$

Computing the above integrals, $\mathcal{A}_k(s, 3L)$ can alternatively be expressed

$$\mathcal{A}_k(s, 3L) = \left[\sum_{n=0}^{k} \frac{1}{n!}\big(s^n - e^{-L}(s+L)^n + e^{-2L}(s+2L)^n - e^{-3L}(s+3L)^n\big)\right] \cdot \left[\sum_{n=0}^{k} \frac{1}{n!}\big(s^n - e^{-3L}(s+3L)^n\big)\right]^{-1}. \tag{4.2}$$

Observe that $\mathcal{A}_k(s, 3L)$ is independent of the starting point s precisely when $k = 0$. In particular,

$$\mathcal{A}_0(3L) := \mathcal{A}_0(s, 3L) = \frac{\big(1+e^{-2L}\big)\big(1-e^{-L}\big)}{1-e^{-3L}}. \tag{4.3}$$

For this reason, calculations with regard to $\lambda_0(\mathscr{C}_n(R))$ are significantly simpler than for the remaining eigenvalues. In particular, we have the recursive relation

$$\lambda_0(\mathscr{C}_{n+1}(R)) = \mathcal{A}_0(\pi R^2/3^n)\lambda_0(\mathscr{C}_n(R)) \quad \text{for } n = 0, 1, 2, \ldots,$$

which in return means

$$\lambda_0(\mathscr{C}_{n+1}(R)) = \lambda_0(\mathscr{C}_0(R)) \prod_{j=0}^{n} \mathcal{A}_0(\pi R^2/3^j) = \left(1 - e^{-\pi R^2}\right) \prod_{j=0}^{n} \mathcal{A}_0(\pi R^2/3^j). \tag{4.4}$$

Ideally, if all the relative areas $\mathcal{A}_k(s, L)$ were bounded by $\mathcal{A}_0(L)$ regardless of starting point $s > 0$ and interval length $L > 0$, we would conclude that $\lambda_0(\mathscr{C}_n(R))$ is always the largest eigenvalue. As it turns out, this is not the case, e.g.,

$$\lim_{L\to 0} \mathcal{A}_k(0, L) > \lim_{L\to 0} \mathcal{A}_0(L) \quad \text{for } k = 2, 3, 4, \ldots$$

Nonetheless, in Sect. 4.3, we are able to determine a common bound for the eigenvalues in terms of $\lambda_0(\mathscr{C}_n(R))$. Here, Lemma 4.4 is worth highlighting as it relies on the *subadditivity* of the Cantor function. Next, in Sect. 4.4, we compute the asymptotic estimates for $\lambda_0(\mathscr{C}_n(R))$.

4.3 Proof of Theorem 4.1

We begin by comparing $\mathcal{A}_k(s, 3L)$ to $\mathcal{A}_0(3L)$ when $s \geq k$.

Lemma 4.3 *Let $\{\mathcal{A}_k\}_k$ be given by (4.1). Then*

$$\mathcal{A}_k(s, 3L) \leq \mathcal{A}_0(3L) \;\; \forall\, s \geq k,\;\; L > 0 \text{ and } k = 0, 1, 2, \ldots$$

Proof Consider the derivative of $\mathcal{A}_k(s, L)$ with respect to s, which yields

$$\frac{\partial \mathcal{A}_k}{\partial s}(s, 3L) = N_k(s, L)\left[\int_s^{s+3L} f_k(r)\mathrm{d}r\right]^{-2},$$

where

$$N_k(s, L) = \Big(f_k(s+L) - f_k(s+2L)\Big)\int_s^{s+3L} f_k(r)\mathrm{d}r \\ - \Big(f_k(s) - f_k(s+3L)\Big)\int_{s+L}^{s+2L} f_k(r)\mathrm{d}r.$$

By identity (4.2), it is clear that $\lim_{s\to\infty} \mathcal{A}_k(s, 3L) = \mathcal{A}_0(3L)$ for all $L > 0$. Thus, it suffices to show that $N_k(s, L) \geq 0$ for all $s \geq k$ and $L > 0$ for $k = 1, 2, 3, \ldots$ Introduce the function

$$\Phi_k(r, s, L) := \Big[f_k(r+s) f_k(s+L) - f_k(r+s+L) f_k(s)\Big] \\ + \Big[f_k(r+s+L) f_k(s+L) - f_k(r+s) f_k(s+2L)\Big]. \tag{4.5}$$

Then we may express $N_k(s, L)$ as a single integral over $[0, L]$ such that

$$N_k(s, L) = \int_0^L \Big(\Phi_k(r, s, L) - \Phi_k(r, s+L, L)\Big)\mathrm{d}r.$$

Hence, the function $N_k(s, L)$ is *positive* for all $s \geq k$ if the derivative of $\Phi(r, s, L)$ with respect to s is *negative*. Consider each of the square bracket terms $[\ldots]$ in definition (4.5) separately, that is

$$\Psi_k(r, s, L, y) := f_k(r+s+y) f_k(s+L) - f_k(r+s+L-y) f_k(s+2y) \text{ for } y \in \{0, L\},$$

so that $\Phi_k(r, s, L) = \Psi_k(r, s, L, 0) + \Psi_k(r, s, L, L)$.

In order to easily evaluate the derivative of Ψ_k, notice first that the arguments of $f_k(\cdot)$ in each term of Ψ_k sum to a *fixed* value, namely

(i) $2a := 2s + r + L + y$.

Further, introduce the corrections to each argument

(ii) $\epsilon_1 := 2^{-1}(L - r - y)$ and $\epsilon_2 := 2^{-1}(L + r - 3y)$

such that Ψ_k becomes

$$\Psi_k(\ldots) = f_k(a - \epsilon_1) f_k(a + \epsilon_1) - f_k(a - \epsilon_2) f_k(a + \epsilon_2) \\ = \frac{e^{-2a}}{(k!)^2}\left[\left(a^2 - \epsilon_1^2\right)^k - \left(a^2 - \epsilon_2^2\right)^k\right].$$

Since $a = a(s)$ is the only quantity in the above expression that depends s, we obtain

$$\frac{\partial \Psi_k}{\partial s}(\dots) = \frac{2e^{-2a}}{(k!)^2}\left[\left(a^2-\epsilon_1^2\right)^{k-1}\left(ka+\epsilon_1^2-a^2\right)-\left(a^2-\epsilon_2^2\right)^{k-1}\left(ka+\epsilon_2^2-a^2\right)\right]. \tag{4.6}$$

By (i)–(ii) and since $r \le L$, we always have the ordering

$$|\epsilon_1(y)| \le |\epsilon_2(y)| \le a(s, y) \ \ \forall\, s \ge k \text{ and } y \in \{0, L\},$$

which means (4.6) is negative whenever the factor $(-a^2 + \epsilon_1^2 + ka)$ is negative. The latest claim is easily verified as $|\epsilon_1| \le a - k$, and therefore $\epsilon_1^2 - (a-k)^2 = (ka - \epsilon_1^2 - a^2) + k(a-k) \le 0$. Hence, for any $y \in \{0, L\}$, $0 \le r \le L$ and $s \ge k$, we conclude that

$$\frac{\partial \Psi_k}{\partial s}(\dots) \le 0 \implies \frac{\partial \Phi_k}{\partial s}(\dots) \le 0 \implies N_k(s, L) \ge 0.$$

□

By the latest lemma, any shifted n-iterate Cantor set $C_n(\pi R^2) + s$ with $s \ge k$ satisfies

$$\int_{C_{n+1}(\pi R^2)+s} f_k(r)\mathrm{d}r \le \mathcal{A}_0(\pi R^2/3^n) \int_{C_n(\pi R^2)+s} f_k(r)\mathrm{d}r \quad \text{for } n = 0, 1, 2, \dots,$$

which combined with (2.8) and then identity (4.4), yields

$$\begin{aligned}\int_{C_{n+1}(\pi R^2)+s} f_k(r)\mathrm{d}r &\le \int_{C_0(\pi R^2)+s} f_k(r)\mathrm{d}r \prod_{j=0}^{n} \mathcal{A}_0(\pi R^2/3^j)\\ &\le \lambda_0(\mathscr{C}_0(R)) \prod_{j=0}^{n} \mathcal{A}_0(\pi R^2/3^j) = \lambda_0(\mathscr{C}_{n+1}(R)).\end{aligned} \tag{4.7}$$

Next, we relate the integrals of f_k over the shifted n-iterates to the non-shifted n-iterates.

Lemma 4.4 *Let $L > 0$. Then for every fixed $k, n = 0, 1, 2, \dots$, we have*

(A) $\int_{C_n(L)\cap[k,\infty[} f_k(r)\mathrm{d}r \le \int_{C_n(L)+k} f_k(r)\mathrm{d}r$ *and*
(B) $\int_{C_n(L)\cap[0,k]} f_k(r)\mathrm{d}r \le \int_{C_n(L)+k} f_k(r)\mathrm{d}r$.

Proof For case (A), since $f_k(r)$ is monotonically decreasing for $r > k$, it suffices to verify

$$|C_n(L) \cap [k, r]| \le |(C_n(L) + k) \cap [k, r]| \ \ \forall\, r \ge k.$$

In terms of the Cantor function $\mathcal{G}_{L,n}$ in (2.9), the above claim reads

$$\mathcal{G}_{L,n}(r) - \mathcal{G}_{L,n}(k) \leq \mathcal{G}_{L,n}(r-k) \ \ \forall\, r \geq k,$$

which is the same subadditivity property as in (2.10).

For case (B), consider the reflection of elements $C_n(L) \cap [0, k]$ about the point k, that is, consider the subset

$$\mathcal{R}_{n,k} := \{r \geq k \mid 2k - r \in C_n(L) \cap [0, k]\}, \tag{4.8}$$

By (2.7), we have that

$$\int_{C_n(L)\cap[0,k]} f_k(r)\mathrm{d}r \leq \int_{\mathcal{R}_{n,k}} f_k(r)\mathrm{d}r.$$

Similarly to (A), in order to prove (B), it suffices to show that

$$|\mathcal{R}_{n,k} \cap [k, r]| \leq |(C_n(L) + k) \cap [k, r]| = L \cdot \mathcal{G}_{L,n}(r-k) \ \ \forall\, r \geq k. \tag{4.9}$$

By definition (4.8), the set $\mathcal{R}_{n,k}$ satisfies

$$|\mathcal{R}_{n,k} \cap [k, r]| = |C_n(L) \cap [2k - r, k]| = L\big(\mathcal{G}_{L,n}(k) - \mathcal{G}_{L,n}(2k - r)\big).$$

We now apply the subadditivity of $\mathcal{G}_{L,n}$ to $\mathcal{G}_{L,n}(k) = \mathcal{G}_{L,n}((r-k) + (2k - r))$, from which claim (4.9) follows. □

Now, combine inequality (4.7) with Lemma 4.4, to conclude

$$\lambda_k(\mathscr{C}_n(R)) \leq 2\lambda_0(\mathscr{C}_n(R)) \ \ \forall\ \ k, n \geq 0,$$

which is a restatement of Theorem 4.1.

4.4 Proof of Theorem 4.2

We formulate a precise asymptotic estimate for the first eigenvalue.

Proposition 4.1 *There exist positive, finite constants $a_1 \leq a_2$ such that for each $n = 0, 1, 2, \ldots$*

$$a_1 \leq \frac{\left(2\pi R^2 + 1\right)^{\frac{\ln 2}{\ln 3}}}{2^n\left(1 - e^{-\pi R^2/3^n}\right)} \cdot \lambda_0(\mathscr{C}_n(R)) \leq a_2 \ \ \forall\, \pi R^2 \in [0, 3^n/2].$$

Proof Combine the two identities (4.3), (4.4) to obtain

$$\lambda_0(\mathscr{C}_n(R)) = \left(1 - e^{-\pi R^2/3^n}\right) \prod_{j=1}^{n} \left(1 + e^{-2\pi R^2/3^j}\right) \text{ for } n = 0, 1, 2, \dots$$

By the above identity, it is sufficient to show that

$$a_1 \le \left(2\pi R^2 + 1\right)^{\frac{\ln 2}{\ln 3}} \prod_{j=1}^{n} \frac{1}{2} \left(1 + e^{-2\pi R^2/3^j}\right) \le a_2 \;\; \forall\, \pi R^2 \in [0, 3^n/2].$$

Exchange the product for a sum, and the above inequality is equivalent to

$$\ln a_1 \le \sum_{j=1}^{n} \ln\left(1 + e^{-x/3^j}\right) - \left(n - \frac{\ln(x+1)}{\ln 3}\right) \ln 2 \le \ln a_2 \; \forall\, x \in [0, 3^n].$$

The above inequality can now be proven by means of the following two claims

(i) *there exists a finite, positive constant* β *such that*

$$0 \le \sum_{j=1}^{\infty} \left[\ln\left(1 + y^{1/3^j}\right) - y^{1/3^j} \ln 2\right] \le \beta \textit{ for } y \in [0, 1], \textit{ and}$$

(ii) *there exist finite constants* $\gamma_1 \le \gamma_2$ *such that*

$$\gamma_1 \le \sum_{j=1}^{n} e^{-x/3^j} - \left(n - \frac{\ln(x+1)}{\ln 3}\right) \le \gamma_2 \textit{ for } x \in [0, 3^n].$$

For claim (i), consider the non-negative function $g(y) := \ln(1 + y) - y \ln 2$ for $y \in [0, 1]$. Since $|g'(y)| \le g'(0) = 1 - \ln 2$ for all $y \in [0, 1]$, $g(y)$ can be bounded from above by the linear spline

$$h(y) := g'(0) \cdot \begin{cases} y, & y \in [0, 1/2] \\ (1 - y), & y \in [1/2, 1]. \end{cases}$$

Thus, the sum in claim (i) is bounded by $\sum_{j=1}^{\infty} h(y^{1/3^j})$ for $y \in [0, 1]$. Since $g(0) = h(0) = 0$, we may always assume that $y > 0$. Further, observe that for any $0 < y \le 1$, we have that $y^{1/3^j} \nearrow 1$ as $j \to \infty$. In particular, for any fixed $0 < y \le 1$, there exists a *smallest* $j_0 \in \mathbb{N}$ such that $y^{1/3^j} \ge 1/2$ for all $j \ge j_0$. Based on our choice j_0, we split the sum

$$\sum_{j=1}^{\infty} h\left(y^{1/3^j}\right) = \sum_{j=1}^{j_0-1} h\left(y^{1/3^j}\right) + \sum_{j=j_0}^{\infty} h\left(y^{1/3^j}\right) = g'(0) \left[\sum_{j=0}^{j_0-1} y^{1/3^j} + \sum_{j=j_0}^{\infty} \left(1 - y^{1/3^j}\right)\right],$$

and consider each sum separately. While the first sum is possibly empty, in the non-empty case, introduce the variable $z_1 := y^{1/3^{j_0-1}} \in [0, 1/2]$ such that

$$\sum_{j=1}^{j_0-1} y^{1/3^j} = \sum_{j=0}^{j_0-2} z_1^{3^j} \leq \sum_{j=0}^{\infty} z_1^{3^j} \leq \sum_{j=0}^{\infty} 2^{-3^j} =: \mathcal{S}_1. \tag{4.10}$$

Similarly for the second sum, introduce the variable $z_2 := y^{1/3^{j_0}} \in [1/2, 1]$ such that

$$\sum_{j=j_0}^{\infty} \left(1 - y^{1/3^j}\right) = \sum_{j=0}^{\infty} \left(1 - z_2^{1/3^j}\right) \leq \sum_{j=0}^{\infty} \left(1 - 2^{-1/3^j}\right) =: \mathcal{S}_2. \tag{4.11}$$

By direct comparison with the geometric series, that is, $2^{-3^j} \leq 2^{-j}$ and $1 - 2^{-1/3^j} \leq 3^{-j} \ln(2)$, both series $\mathcal{S}_1$ and $\mathcal{S}_2$ are convergent. Hence, claim (i) follows with $\beta = g'(0)(\mathcal{S}_1 + \mathcal{S}_2)$.

Claim (ii) is proven by similar means as (i), where we split the sum $\sum_{j=1}^{n} e^{-x/3^j}$. In particular, for a fixed $x \in [0, 3^n]$, we split the sum at the point $j_1 := \max\{\lfloor \frac{\ln(x)}{\ln(3)} \rfloor, 0\}$ such that $\sum_{j=1}^{n} = \sum_{j=1}^{j_1} + \sum_{j=j_1+1}^{n}$. If the first sum is non-empty, set $z_3 := e^{-x/3^{j_1}} \in [0, e^{-1}]$ such that

$$\sum_{j=1}^{j_1} e^{-x/3^j} = \sum_{j=1}^{j_1} z_3^{3^j} \leq \sum_{j=1}^{\infty} z_3^{3^j} \leq \sum_{j=1}^{\infty} e^{-3^j} =: \mathcal{S}_3, \tag{4.12}$$

which is a convergent series. For the second sum, we utilize for $y \geq 0$ the inequalities $1 - y \leq e^{-y} \leq 1$ to obtain lower and upper estimates. By comparison with the geometric series and since $x/3^{j_1+1} \leq 1$, we conclude that

$$-\frac{3}{2} \leq \sum_{j=j_1+1}^{n} e^{-x/3^j} - (n - j_1) \leq 0. \tag{4.13}$$

Finally, by combining estimates (4.12)–(4.13) with the bounds $\frac{\ln(x+1)}{\ln(3)} - 1 \leq j_1 \leq \frac{\ln(x+1)}{\ln(3)}$, claim (ii) follows with constants $\gamma_1 = -3/2$ and $\gamma_2 = 1 + \mathcal{S}_3$. □

Now, by applying the estimates of Proposition 4.1 with constants $a_1 \leq a_2$ to Theorem 4.1, we obtain Theorem 4.2 with constants $c_1 = a_1 \leq c_2 = 2a_2$.

Remark (Numerical estimates) From the proof of Proposition 4.1, we are also able to retrieve some numerical estimates for the constants $a_1 \leq a_2$. It should, however, be noted that the method chosen in the proof is *not* meant to produce *optimal* constants. Nevertheless, with $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ defined as in (4.10)–(4.12), we obtain the estimates

$$\ln a_1 = -\frac{3}{2}\ln 2 \approx -1.0397 \text{ and}$$
$$\ln a_2 = (1-\ln 2)(\mathcal{S}_1+\mathcal{S}_2) + (1+\mathcal{S}_3)\ln 2 \approx 1.1713.$$

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Appendix A. Omitted Proofs in Sect. 2.3

We shall prove the following two properties for the integrands $\{f_k(r) := \frac{r^k}{k!}e^{-r}\}_{k=0}^{\infty}$:

$$f_k(k-r) \le f_k(k+r) \ \ \forall\, r \in [0,k] \quad \text{for } k = 1,2,3,\dots \tag{A.1}$$

and

$$\int_E f_k(r)\mathrm{d}r \le \int_0^{|E|} f_0(r)\mathrm{d}r = 1 - e^{-|E|} \quad \text{for } k = 0,1,2,\dots, \tag{A.2}$$

where E is some measurable subset of $\mathbb{R}_+$.

Proof (Property (A.1)) It is sufficient to show that the fraction $\delta_k(r) := \frac{f_k(k-r)}{f_k(k+r)} \le 1$ for all $r \in [0,k]$. By differentiation, $\delta_k'(r) \le 0$ for all $r \in [0,k]$ and since $\delta_k(0) = 1$, we are done. □

Proof (Property (A.2)) Since every f_k is normalized, i.e., $\|f_k\|_1 = 1$, and $f_k(r)$ is monotonically increasing for $0 < r < k$ and decreasing for $r > k$, we may assume E to be an interval of finite measure. Define the function

$$g_k(L,s) := \int_0^L f_0(r)\mathrm{d}r - \int_s^{s+L} f_k(r)\mathrm{d}r,$$

and note that it suffices to show that $g_k(L,s) \ge 0$ for all $L, s \ge 0$ and every k. Differentiating g_k with respect to L,

$$\frac{\partial g_k}{\partial L}(s,L) = f_0(L) - f_k(s+L) = e^{-L}\Big(1 - \frac{e^{-s}}{k!}(s+L)^k\Big),$$

reveals a critical point at $L = L_0 := \sqrt[k]{e^s k!} - s$. By the second derivative test, it follows that $L = L_0$ represents a maximum for $g_k(L, s)$ with $s > 0$ fixed. Since the other possible extrema occur when $L = 0$ or $L \to \infty$, which both can easily be verified to yield a non-negative $g_k(L, s)$, we conclude that $g_k(L, s)$ is *always* non-negative. □

References

1. Abreu, L. D., Speckbacher, MAbreu, L. D., Speckbacher, M.: Donoho–Logan large sieve principles for modulation and polyanalytic Fock spaces. arXiv:1808.02258 (2018)
2. Abreu, L.D., Gröchenig, K., Romero, J.L.: Harmonic analysis in phase space and finite Weyl–Heisenberg ensembles. J. Stat. Phys. **174**, 1104–1136 (2019)
3. Bourgain, J., Dyatlov, S.: Spectral gaps without the pressure condition. Ann. Math. **187**(3), 825–867 (2018)
4. Daubechies, I.: Time–frequency localization operators: a geometric phase space approach. IEEE Trans. Inf. Theory **34**(4), 605–612 (1988)
5. Doboš, J.: The standard Cantor function is subadditive. Proc. Am. Math. Soc. **124**(11), 3425–3426 (1996)
6. Dyatlov, S.: An introduction to fractal uncertainty principle. J. Math. Phys. **60**, 081505 (2019)
7. Dyatlov, S., Jin, L.: Dolgopyat's method and the fractal uncertainty principle. Anal. PDE **11**(6), 1457–1485 (2018)
8. Dyatlov, S., Zahl, J.: Spectral gaps, additive energy, and a fractal uncertainty principle. Geom. Funct. Anal. **26**, 1011–1094 (2016)
9. Folland, Gerald B.: Harmonic Analysis in Phase Space. vol. 122. Annals of Mathematics Studies. Princeton University Press, Princeton, N.J. (1989)
10. Folland, G., Sitaram, A.: The uncertainty principle: a mathematical survey. J. Fourier Anal. Appl. **3**(3), 207–238 (1997)
11. Gröchenig, K.: Foundations of Time–Frequency Analysis. Applied and Numerical Harmonic Analysis. Birkhäuser, Boston (2001)
12. Reznikov, A.: Sharp constants in the Paneyah–Logvinenko–Sereda theorem. Comptes rendus - Mathématique **348**(3), 141–144 (2010)
13. Slepian, D., Pollak, H.O.: Prolate spheroidal wave functions, Fourier analysis and uncertainty - I. Bell Syst. Tech. J. **40**, 43–63 (1961)