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Harmonic Functions, Conjugate Harmonic Functions and the Hardy Space H^1 in the Rational Dunkl Setting

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Abstract
In this work we extend the theory of the classical Hardy space H^1 to the rational Dunkl setting. Specifically, let Δ be the Dunkl Laplacian on a Euclidean space $\mathbb{R}^N$. On the half-space $\mathbb{R}_+ \times \mathbb{R}^N$, we consider systems of conjugate $(\partial_t^2 + \Delta_{\mathbf{x}})$-harmonic functions satisfying an appropriate uniform L^1 condition. We prove that the boundary values of such harmonic functions, which constitute the real Hardy space H^1_Δ, can be characterized in several different ways, namely by means of atoms, Riesz transforms, maximal functions or Littlewood–Paley square functions.

Keywords Rational Dunkl theory · Hardy spaces · Cauchy–Riemann equations · Riesz transforms · Maximal operators

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In memory of Elias M. Stein (1931–2018).

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1 Introduction

Real Hardy spaces on $\mathbb{R}^N$ have their origin in the study of holomorphic functions of one variable in the upper half-plane $\mathbb{R}^2_+ = \{z = x + iy \in \mathbb{C} : y > 0\}$. The theorem of Burkholder et al. [5] asserts that a real-valued harmonic function u on $\mathbb{R}^2_+$ is the real part of a holomorphic function $F(z) = u(z) + iv(z)$ satisfying the L^p condition

$$\sup_{y>0} \int_{\mathbb{R}} |F(x + iy)|^p \, dx < \infty, \quad 0 < p < \infty,$$

if and only if the nontangential maximal function $u^*(x) = \sup_{|x-x'|<y} |u(x' + iy)|$ belongs to $L^p(\mathbb{R})$. Here $0 < p < \infty$. The N-dimensional theory was then developed in Stein and Weiss [36] and Fefferman and Stein [19], where the notion of holomorphy was replaced by conjugate harmonic functions. To be more precise, a system of C^2 functions

$$\begin{aligned} &\boldsymbol{u}(x_0, x_1, \dots, x_N) \\ &\quad = (u_0(x_0, x_1, \dots, x_N), u_1(x_0, x_1, \dots, x_N), \dots, u_N(x_0, x_1, \dots, x_N)), \end{aligned}$$

where $x_0 > 0$, satisfies the generalized Cauchy–Riemann equations if

$$\frac{\partial u_j}{\partial x_i} = \frac{\partial u_i}{\partial x_j} \quad \forall\, 0 \le i \ne j \le N \quad \text{and} \quad \sum_{j=0}^{N} \frac{\partial u_j}{\partial x_j} = 0. \tag{1.1}$$

One says that $\boldsymbol{u}$ has the L^p property if

$$\sup_{x_0>0} \int_{\mathbb{R}^N} |\boldsymbol{u}(x_0, x_1, \dots, x_N)|^p \, dx_1 \dots dx_N < \infty. \tag{1.2}$$

As in the case $N = 1$, if $1 - \frac{1}{N} < p < \infty$ and $u_0(x_0, x_1, \dots, x_N)$ is a harmonic function, there is a system $\boldsymbol{u} = (u_0, u_1, \dots, u_N)$ of C^2 functions satisfying (1.1) and (1.2) if and only if

$$u_0^*(\mathbf{x}) = \sup_{\|\mathbf{x}-\mathbf{x}'\|<x_0} |u_0(x_0, \mathbf{x}')|$$

belongs to $L^p(\mathbb{R}^N)$. Here $\mathbf{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$ and similarly $\mathbf{x}' = (x'_1, \dots, x'_N)$. Then u_0 has a limit f_0 in the sense of distributions, as $x_0 \searrow 0$, and u_0 is the Poisson integral of f_0. It turns out that the set of all distributions obtained in this way, which forms the so-called real Hardy space $H^p(\mathbb{R}^N)$, can be equivalently characterized in terms of real analysis (see [19]), namely by means of various maximal functions, square functions or Riesz transforms. Another important contribution to this theory lies in the atomic decomposition introduced by Coifman [7] and extended to spaces of homogeneous type by Coifman and Weiss [8].

The goal of this paper is to study harmonic functions, conjugate harmonic functions, and related Hardy space H^1 for the Dunkl Laplacian Δ (see Sect. 2). We shall prove that these objects have properties analogous to the classical ones. In particular, the related real Hardy space H^1_Δ, which can be defined as the set of boundary values of $(\partial_t^2 + \Delta_{\mathbf{x}})$-harmonic functions satisfying a relevant L^1 property, can be characterized by appropriate maximal functions, square functions, Riesz transforms or atomic decompositions. Apart from the square function characterization, this was achieved previously in [3] and [13] in the one-dimensional case, as well as in the product case.

Hardy spaces associated with semigroups of linear operators have a long history. Let us present a small and selected part of it. Muckenhoupt and Stein [26] introduced a notion of conjugacy for the one–dimensional Bessel operator, which initiated a study of Hardy spaces in the Bessel setting, continued subsequently in [4]. In [20] and [6], the authors developed a theory of real Hardy spaces H^p on homogeneous nilpotent Lie groups, associated either with a sublaplacian (if the group is stratified) or with a Rockland operator (if the group is graded). Another important contribution is the theory of local Hardy spaces in [22], which has several applications, e.g., in the study of Hardy spaces associated with the twisted laplacian [25] or with Schrödinger operators with certain (large) potentials [17]. Hardy spaces associated with semigroups whose kernels satisfy Gaussian bounds were studied in [24]. There, the theory of tent spaces [9,33] was used to produce specific atomic decompositions for Hardy spaces defined by square functions. This theory was further enhanced in [11,37] via characterizations by means of maximal functions.

In the one-dimensional case and in the product case considered in [3,13], the Dunkl kernel can be expressed explicitly in terms of classical special functions (Bessel functions or the confluent hypergeometric function). Thus its behavior is fully understood. In the general case considered in the present paper, no such information is available. Therefore an essential part of our work consists in estimating the Dunkl kernel, the heat kernel, the Poisson kernel, and their derivatives (see Sects. 3–5). As observed in [3], the heat kernel satisfies no Gaussian bound in the Dunkl setting. However, as shown in Sect. 4, some Gaussian-type bounds hold provided the Euclidean distance is replaced by the orbit distance (3.3). Similarly for the Poisson kernel, whose estimates in terms of the orbit distance resemble the analysis on spaces of homogeneous type (see Sect. 5). These crucial observations allow us to adapt the techniques of [11,24,37] in order to obtain atomic, maximal function, and square function characterizations of the Hardy space H^1_Δ. As far as the Riesz transform characterization of H^1_Δ is concerned, we use the maximum principle for Dunkl–Laplace subharmonic functions, together with estimates for the Dunkl and Poisson kernels.

Let us finally mention some further works in the continuation of the present paper. In [14] another atomic decomposition for the Hardy H^1_Δ space is obtained. The article [23] provides characterizations of the Hardy space associated with the Dunkl harmonic oscillator, while [15] is devoted to non-radial multipliers associated with the Dunkl transform.

1.1 Notation

- As usual, $\mathbb{N}=\{0,1,2,\dots\}$ denotes the set of nonnegative integers.
- The Euclidean space $\mathbb{R}^N$ is equipped with the standard inner product

$$\langle \mathbf{x},\mathbf{y}\rangle=\sum\nolimits_{j=1}^{N} x_j\,y_j$$

and the corresponding norm $\|\mathbf{x}\|=\big(\sum_{j=1}^{N}|x_j|^2\big)^{1/2}$. Throughout the paper,

$$B(\mathbf{x},r)=\{\,\mathbf{y}\in\mathbb{R}^N\mid \|\mathbf{x}-\mathbf{y}\|<r\,\}$$

stands for the ball with center $\mathbf{x}\in\mathbb{R}^N$ and radius $r>0$. Finally, $\mathbb{R}_+^{N+1}$ denotes the half-space $(0,\infty)\times\mathbb{R}^N$ in $\mathbb{R}^{N+1}$.
- In $\mathbb{R}^N$, the directional derivative along ξ is denoted by ∂_ξ. As usual, for every multi-index $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_N)\in\mathbb{N}^N$, we set $|\alpha|=\sum_{j=1}^{N}\alpha_j$ and

$$\partial^{\alpha}=\partial_{e_1}^{\alpha_1}\circ\partial_{e_2}^{\alpha_2}\circ\cdots\circ\partial_{e_N}^{\alpha_N}\,,$$

where $\{e_1,e_2,\dots,e_N\}$ is the canonical basis of $\mathbb{R}^N$. The additional subscript $\mathbf{x}$ in $\partial_{\mathbf{x}}^{\alpha}$ means that the partial derivative ∂^{α} is taken with respect to the variable $\mathbf{x}\in\mathbb{R}^N$.
- The symbol $\sim$ between two positive expressions f,g means that their ratio $\frac{f}{g}$ is bounded from above and below by positive constants.
- The symbol $\lesssim$ (respectively $\gtrsim$) between two nonnegative expressions f,g means that there exists a constant $C>0$ such that $f\leq Cg$ (respectively $f\geq Cg$).
- We denote by $C_0(\mathbb{R}^N)$ the space of all continuous functions on $\mathbb{R}^N$ vanishing at infinity, by $C_c^{\infty}(\mathbb{R}^N)$ the space of all smooth functions on $\mathbb{R}^N$ with compact support, and by $\mathcal{S}(\mathbb{R}^N)$ the Schwartz class on $\mathbb{R}^N$. If $m\in\mathbb{N}$ and Ω is an open subset of $\mathbb{R}^N$, then f is a C^m function on Ω if f and all partial derivatives $\partial^{\alpha}f$, $|\alpha|\leq m$, are continuous functions on Ω.
- If J is a measurable subset of $\mathbb{R}^N$, then χ_J denotes the characteristic function of J, that is, $\chi_J(\mathbf{x})=1$ if $\mathbf{x}\in J$ and $\chi_J(\mathbf{x})=0$ otherwise.
- Throughout the paper, C,C',c, etc. stand for positive constants, whose values may vary from occurrence to occurrence.

Further notation is defined in the next two sections.

2 Statement of the Results

In this section we first collect basic facts concerning Dunkl operators, the Dunkl Laplacian, and the corresponding heat and Poisson semigroups. For details we refer the reader to [12,30,32]. Next we state our main results.

In the Euclidean space $\mathbb{R}^N$ the reflection σ_α with respect to the hyperplane $\alpha^\perp$ orthogonal to a nonzero vector $\alpha \in \mathbb{R}^N$ is given by

$$\sigma_\alpha(\mathbf{x}) = \mathbf{x} - 2\frac{\langle \mathbf{x}, \alpha\rangle}{\|\alpha\|^2}\alpha.$$

A finite set $R \subset \mathbb{R}^N \setminus \{0\}$ is called a *root system* if $\sigma_\alpha(R) = R$ for every $\alpha \in R$. We shall consider normalized reduced root systems, that is, $\|\alpha\|^2 = 2$ for every $\alpha \in R$. The finite group G generated by the reflections σ_α is called the *Weyl group* (*reflection group*) of the root system. We shall denote by $\mathcal{O}(\mathbf{x})$, resp. $\mathcal{O}(B)$ the G-orbit of a point $\mathbf{x} \in \mathbb{R}^N$, resp. a subset $B \subset \mathbb{R}^N$. A *multiplicity function* is a G-invariant function $k : R \to \mathbb{C}$, which will be fixed and ≥ 0 throughout this paper.

Given a root system R and a multiplicity function k, the *Dunkl operators* T_ξ are the following deformations of directional derivatives ∂_ξ by difference operators:

$$\begin{aligned} T_\xi f(\mathbf{x}) &= \partial_\xi f(\mathbf{x}) + \sum_{\alpha\in R} \frac{k(\alpha)}{2}\langle\alpha, \xi\rangle \frac{f(\mathbf{x}) - f(\sigma_\alpha(\mathbf{x}))}{\langle\alpha, \mathbf{x}\rangle} \\ &= \partial_\xi f(\mathbf{x}) + \sum_{\alpha\in R^+} k(\alpha)\langle\alpha, \xi\rangle \frac{f(\mathbf{x}) - f(\sigma_\alpha(\mathbf{x}))}{\langle\alpha, \mathbf{x}\rangle}. \end{aligned}$$

Here R^+ is any fixed positive subsystem of R. The Dunkl operators T_ξ, which were introduced in [12], commute pairwise and are skew-symmetric with respect to the G-invariant measure $dw(\mathbf{x}) = w(\mathbf{x})\, d\mathbf{x}$, where

$$w(\mathbf{x}) = \prod_{\alpha\in R} |\langle\alpha, \mathbf{x}\rangle|^{k(\alpha)} = \prod_{\alpha\in R^+} |\langle\alpha, \mathbf{x}\rangle|^{2k(\alpha)}.$$

Set $T_j = T_{e_j}$, where $\{e_1, \dots, e_N\}$ is the canonical basis of $\mathbb{R}^N$. The *Dunkl Laplacian* associated with R and k is the differential-difference operator $\Delta = \sum_{j=1}^n T_j^2$, which acts on C^2 functions by

$$\Delta f(\mathbf{x}) = \Delta_{\mathrm{eucl}} f(\mathbf{x}) + \sum_{\alpha\in R} k(\alpha)\delta_\alpha f(\mathbf{x}) = \Delta_{\mathrm{eucl}} f(\mathbf{x}) + 2\sum_{\alpha\in R^+} k(\alpha)\delta_\alpha f(\mathbf{x}),$$

where

$$\delta_\alpha f(\mathbf{x}) = \frac{\partial_\alpha f(\mathbf{x})}{\langle\alpha, \mathbf{x}\rangle} - \frac{f(\mathbf{x}) - f(\sigma_\alpha(\mathbf{x}))}{\langle\alpha, \mathbf{x}\rangle^2}.$$

The operator Δ is essentially self-adjoint on $L^2(dw)$ (see for instance [2, Theorem 3.1]) and generates the heat semigroup

$$H_t f(\mathbf{x}) = e^{t\Delta} f(\mathbf{x}) = \int_{\mathbb{R}^N} h_t(\mathbf{x}, \mathbf{y}) f(\mathbf{y})\, dw(\mathbf{y}). \tag{2.1}$$

Here the heat kernel $h_t(\mathbf{x},\mathbf{y})$ is a C^∞ function in all variables $t>0, \mathbf{x}\in\mathbb{R}^N, \mathbf{y}\in\mathbb{R}^N$, which satisfies

$$h_t(\mathbf{x},\mathbf{y}) = h_t(\mathbf{y},\mathbf{x})> 0 \quad \text{and} \quad \int_{\mathbb{R}^N} h_t(\mathbf{x},\mathbf{y})\, dw(\mathbf{y}) = 1.$$

Notice that (2.1) defines a strongly continuous semigroup of linear contractions on $L^p(dw)$, for every $1\le p<\infty$.

The Poisson semigroup $P_t = e^{-t\sqrt{-\Delta}}$ is given by the subordination formula

$$P_t f(\mathbf{x}) = \pi^{-1/2}\int_0^\infty e^{-u}\exp\Big(\frac{t^2}{4u}\Delta\Big) f(\mathbf{x})\frac{du}{\sqrt{u}} \tag{2.2}$$

and solves the boundary value problem

$$\begin{cases} (\partial_t^2+\Delta_{\mathbf{x}})\, u(t,\mathbf{x}) = 0\\ u(0,\mathbf{x}) = f(\mathbf{x})\end{cases}$$

in the half-space $\mathbb{R}^{1+N}_+ = (0,\infty)\times\mathbb{R}^N\subset\mathbb{R}^{1+N}$ (see [31, Sect. 5]). Let $e_0 = (1,0,\dots,0)$, $e_1 = (0,1,\dots,0)$,..., $e_N = (0,0,\dots,1)$ be the canonical basis in $\mathbb{R}^{1+N}$. In order to unify our notation we shall denote the variable t by x_0 and set $T_0 = \partial_{e_0}$.

Our goal is to study real harmonic functions of the operator

$$\mathcal{L} = T_0^2+\Delta = \sum_{j=0}^N T_j^2. \tag{2.3}$$

The operator $\mathcal{L}$ is the Dunkl Laplacian associated with the root system R, considered as a subset of $\mathbb{R}^{1+N}$ under the embedding $R\subset\mathbb{R}^N\hookrightarrow\mathbb{R}\times\mathbb{R}^N$.

We say that a system

$$\mathbf{u} = (u_0,u_1,\dots,u_N), \quad \text{where } u_j = u_j(x_0,\underbrace{x_1,\dots,x_N}_{\mathbf{x}}) \ \ \forall\, 0\le j\le N,$$

of C^1 real functions on $\mathbb{R}^{1+N}_+$ satisfies the generalized Cauchy–Riemann equations if

$$\begin{cases} T_iu_j = T_ju_i \quad \forall\, 0\le i\ne j\le N,\\ \sum_{j=0}^N T_ju_j = 0.\end{cases} \tag{2.4}$$

In this case each component u_j is $\mathcal{L}$-harmonic, i.e., $\mathcal{L}u_j=0$.

We say that a system $\mathbf{u}$ of C^2 real $\mathcal{L}$-harmonic functions on $\mathbb{R}^{1+N}_+$ belongs to the Hardy space $\mathcal{H}^1$ if it satisfies both (2.4) and the L^1 condition

$$\|\mathbf{u}\|_{\mathcal{H}^1} = \sup_{x_0>0}\big\||\mathbf{u}(x_0,\cdot)|\big\|_{L^1(dw)} = \sup_{x_0>0}\int_{\mathbb{R}^N}|\mathbf{u}(x_0,\mathbf{x})|\,dw(\mathbf{x})<\infty,$$

where $|\mathbf{u}(x_0,\mathbf{x})| = \Big(\sum_{j=0}^N|u_j(x_0,\mathbf{x})|^2\Big)^{1/2}$.

We are now ready to state our first main result.

Theorem 2.1 *Let u_0 be a $\mathcal{L}$-harmonic function in the upper half-space $\mathbb{R}^{1+N}_+$. Then there are $\mathcal{L}$-harmonic functions u_j $(j = 1, \dots, N)$ such that $\mathbf{u} = (u_0, u_1, \dots, u_N)$ belongs to $\mathcal{H}^1$ if and only if the nontangential maximal function*

$$u_0^*(\mathbf{x}) = \sup_{\|\mathbf{x}'-\mathbf{x}\|<x_0} |u_0(x_0, \mathbf{x}')| \tag{2.5}$$

belongs to $L^1(dw)$. In this case, the norms $\|u_0^*\|_{L^1(dw)}$ and $\|\mathbf{u}\|_{\mathcal{H}^1}$ are moreover equivalent.

If $\mathbf{u} \in \mathcal{H}^1$, we shall prove that the limit $f(\mathbf{x}) = \lim_{x_0 \to 0} u_0(x_0, \mathbf{x})$ exists in $L^1(dw)$ and $u_0(x_0, \mathbf{x}) = P_{x_0} f(\mathbf{x})$. This leads to consider the so-called real Hardy space

$$H^1_\Delta = \{f(\mathbf{x}) = \lim_{x_0 \to 0} u_0(x_0, \mathbf{x}) \mid (u_0, u_1, \dots, u_N) \in \mathcal{H}^1\},$$

equipped with the norm

$$\|f\|_{H^1_\Delta} = \|(u_0, u_1, \dots, u_N)\|_{\mathcal{H}^1}.$$

Let us denote by

$$\mathcal{M}_P f(\mathbf{x}) = \sup_{\|\mathbf{x}-\mathbf{x}'\|<t} \left|P_t f(\mathbf{x}')\right| \tag{2.6}$$

the nontangential maximal function associated with the Poisson semigroup $P_t = e^{-t\sqrt{-\Delta}}$. According to Theorem 2.1, H^1_Δ coincides with the following subspace of $L^1(dw)$:

$$H^1_{\max,P} = \{f \in L^1(dw) \mid \|f\|_{H^1_{\max,P}} := \|\mathcal{M}_P f\|_{L^1(dw)} < \infty\}. \tag{2.7}$$

Moreover, the norms $\|f\|_{H^1_\Delta}$ and $\|f\|_{H^1_{\max,P}}$ are equivalent.

Our task is to prove other characterizations of H^1_Δ by means of real analysis.

A Characterization by the Heat Maximal Function

Let

$$\mathcal{M}_H f(\mathbf{x}) = \sup_{\|\mathbf{x}-\mathbf{x}'\|^2<t} |H_t f(\mathbf{x}')|$$

be the nontangential maximal function associated with the heat semigroup $H_t = e^{t\Delta}$ and set

$$H^1_{\max,H} = \{f \in L^1(dw) \mid \|f\|_{H^1_{\max,H}} := \|\mathcal{M}_H f\|_{L^1(dw)} < \infty\}. \tag{2.8}$$

Theorem 2.2 *The spaces H^1_Δ and $H^1_{\max,H}$ coincide and the corresponding norms $\|f\|_{H^1_\Delta}$ and $\|f\|_{H^1_{\max,H}}$ are equivalent.*

B Characterization by Square Functions

For every $1 \leq p \leq \infty$, the operators $Q_t = t\sqrt{-\Delta}e^{-t\sqrt{-\Delta}}$ are uniformly bounded on $L^p(dw)$ (this is a consequence of the estimates (4.3), (5.7) and (5.4)). Consider the square function

$$Sf(\mathbf{x}) = \left(\iint_{\|\mathbf{x}-\mathbf{y}\|<t} |Q_t f(\mathbf{y})|^2 \frac{dt\, dw(\mathbf{y})}{t\, w(B(\mathbf{x}, t))}\right)^{1/2} \tag{2.9}$$

and the space

$$H^1_{\text{square}} = \{f \in L^1(dw) \mid \|Sf\|_{L^1(dw)} < \infty\}.$$

Theorem 2.3 *The spaces H^1_Δ and H^1_{square} coincide and the corresponding norms $\|f\|_{H^1_\Delta}$ and $\|Sf\|_{L^1(dw)}$ are equivalent.*

Remark 2.4 The square function characterization of H^1_Δ is also valid for $Q_t = t^2\Delta\, e^{t^2\Delta}$.

C Characterization by Riesz Transforms

The Riesz transforms, which are defined in the Dunkl setting by

$$R_j f = T_j(-\Delta)^{-1/2} f$$

(see Sect. 8), are bounded operators on $L^p(dw)$, for every $1 < p < \infty$ (cf. [1]). In the limit case $p = 1$, they turn out to be bounded operators from H^1_Δ into $H^1_\Delta \subset L^1(dw)$. This leads to consider the space

$$H^1_{\text{Riesz}} = \{f \in L^1(dw) \mid \|R_j f\|_{L^1(w)} < \infty,\ \forall\, 1 \leq j \leq N\}.$$

Theorem 2.5 *The spaces H^1_Δ and H^1_{Riesz} coincide and the corresponding norms $\|f\|_{H^1_\Delta}$ and*

$$\|f\|_{H^1_{\text{Riesz}}} := \|f\|_{L^1(dw)} + \sum\nolimits_{j=1}^N \|R_j f\|_{L^1(dw)}.$$

are equivalent.

D Characterization by Atomic Decompositions

Let us define atoms in the spirit of [24]. Given a Euclidean ball B in $\mathbb{R}^N$, we shall denote its radius by r_B and its G-orbit by $\mathcal{O}(B)$. For any positive integer M, let $\mathcal{D}(\Delta^M)$ be the domain of Δ^M as an (unbounded) operator on $L^2(dw)$.

Definition 2.6 Let $1 < q \leq \infty$ and let M be a positive integer. A function $a \in L^2(dw)$ is said to be a $(1, q, M)$*–atom* if there exist $b \in \mathcal{D}(\Delta^M)$ and a ball B such that

- $a = \Delta^M b$,
- $\operatorname{supp}(\Delta^\ell b) \subset \mathcal{O}(B) \ \forall\, 0 \leq \ell \leq M$,
- $\|(r_B^2 \Delta)^\ell b\|_{L^q(dw)} \leq r_B^{2M} w(B)^{\frac{1}{q}-1} \ \forall\, 0 \leq \ell \leq M$.

Let us remark that $\|a\|_{L^1(dw)} \leq |G|^{1-\frac{1}{q}}$, where $|G|$ denotes the number of elements of G. This follows easily from the above conditions (with $\ell = M$) by using Hölder's inequality.

Definition 2.7 An $L^1(dw)$–function f belongs to $H^1_{(1,q,M)}$ if there exist $(1, q, M)$–atoms a_j and $\lambda_j \in \mathbb{C}$ such that $\sum_j |\lambda_j| < \infty$ and

$$f = \sum\nolimits_j \lambda_j a_j \,. \tag{2.10}$$

Moreover,

$$\|f\|_{H^1_{(1,q,M)}} = \inf \left\{ \sum\nolimits_j |\lambda_j| \right\},$$

where the infimum is taken over all representations (2.10).

Notice that the series (2.10) converges (absolutely) in $L^1(dw)$ and almost everywhere. Moreover, the results in our paper ensure that the convergence of the series (2.10) holds in the Hardy space H^1 as well.

Theorem 2.8 *The spaces H^1_Δ and $H^1_{(1,q,M)}$ coincide and the corresponding norms are equivalent.*

Let us briefly describe the organization of the proofs of the results. Clearly, $H^1_{(1,q_1,M)} \subset H^1_{(1,q_2,M)}$ for $1 < q_2 \leq q_1 \leq \infty$. The proof $(u_0, u_1, \dots, u_N) \in \mathcal{H}^1$ implies $u_0^* \in L^1(dw)$, which is actually the inclusion $H^1_\Delta \subset H^1_{\max,P}$, is presented in Sect. 7, see Proposition 7.6. The proof is based on $\mathcal{L}$-subharmonicity of certain function constructed from $\mathbf{u}$ (see Sect. 6). The converse to Proposition 7.6 is proved at the very end of Sect. 11. Inclusions: $H^1_\Delta \subset H^1_{\text{Riesz}} \subset H^1_\Delta$ are shown in Sect. 8. Further, $H^1_{(1,q,M)} \subset H^1_{\text{Riesz}}$ for M large is proved in Sect. 9. Section 10 is devoted to proving $H^1_{\max,H} = H^1_{\max,\text{P}}$. The proofs of $H^1_{\max,H} \subset H^1_{(1,\infty,M)}$ for every $M \geq 1$ are presented in Sect. 11. Inclusion: $H^1_{(1,q,M)} \subset H^1_{\max,H}$ for every $M \geq 1$ is proved in Sect. 12. Finally, $H^1_{(1,2,M)} \subset H^1_{\text{square}} \subset H^1_{(1,2,M)}$ are established in Sect. 13.

3 Dunkl Kernel, Dunkl Transform and Dunkl Translations

The purpose of this section is to collect some facts about the Dunkl kernel, the Dunkl transform and Dunkl translations. General references are [10,12,30,32]. At the end of this section we shall derive estimates for the Dunkl translations of radial functions. These estimates will be used later to obtain bounds for the heat kernel and for the Poisson kernel, as well as for their derivatives, and furthermore upper and lower bounds for the Dunkl kernel.

We begin with some notation. Given a root system R in $\mathbb{R}^N$ and a multiplicity function $k \geq 0$, let

$$\gamma = \sum\nolimits_{\alpha\in R^+} k(\alpha) \quad \text{and} \quad \mathbf{N} = N + 2\gamma. \tag{3.1}$$

The number $\mathbf{N}$ is called the homogeneous dimension, because of the scaling property

$$w(B(t\mathbf{x}, tr)) = t^{\mathbf{N}} w(B(\mathbf{x}, r)).$$

Observe that

$$w(B(\mathbf{x}, r)) \sim r^N \prod_{\alpha\in R} (\,|\langle \alpha, \mathbf{x}\rangle| + r\,)^{k(\alpha)}.$$

Thus the measure w is doubling, that is, there is a constant $C > 0$ such that

$$w(B(\mathbf{x}, 2r)) \leq C\, w(B(\mathbf{x}, r)).$$

Moreover, there exists a constant $C \geq 1$ such that, for every $\mathbf{x} \in \mathbb{R}^N$ and for every $r_2 \geq r_1 > 0$,

$$C^{-1}\Big(\frac{r_2}{r_1}\Big)^N \leq \frac{w(B(\mathbf{x}, r_2))}{w(B(\mathbf{x}, r_1))} \leq C\Big(\frac{r_2}{r_1}\Big)^{\mathbf{N}}. \tag{3.2}$$

Set

$$V(\mathbf{x}, \mathbf{y}, t) = \max\big\{w(B(\mathbf{x}, t)),\, w(B(\mathbf{y}, t))\big\}.$$

Finally, let

$$d(\mathbf{x}, \mathbf{y}) = \min_{\sigma\in G} \|\mathbf{x} - \sigma(\mathbf{y})\| \tag{3.3}$$

denote the distance between two G-orbits $\mathcal{O}(\mathbf{x})$ and $\mathcal{O}(\mathbf{y})$. Obviously, $\mathcal{O}(B(\mathbf{x}, r)) = \{\mathbf{y}\in\mathbb{R}^N \mid d(\mathbf{y}, \mathbf{x}) < r\}$ and

$$w(B(\mathbf{x}, r)) \leq w(\mathcal{O}(B(\mathbf{x}, r))) \leq |G|\, w(B(\mathbf{x}, r)).$$

3.1 Dunkl Kernel

For fixed $\mathbf{x} \in \mathbb{R}^N$, the *Dunkl kernel* $\mathbf{y} \longmapsto E(\mathbf{x}, \mathbf{y})$ is the unique solution to the system

$$\begin{cases} T_\xi f = \langle \xi, \mathbf{x}\rangle\, f \quad \forall\, \xi \in \mathbb{R}^N, \\ f(0) = 1. \end{cases}$$

The following integral formula was obtained by Rösler [28] :

$$E(\mathbf{x},\mathbf{y})=\int_{\mathbb{R}^N} e^{\langle\eta,\mathbf{y}\rangle}\,d\mu_{\mathbf{x}}(\eta), \tag{3.4}$$

where $\mu_{\mathbf{x}}$ is a probability measure supported in the convex hull conv $\mathcal{O}(\mathbf{x})$ of the G-orbit of $\mathbf{x}$. The function $E(\mathbf{x},\mathbf{y})$, which generalizes the exponential function $e^{\langle\mathbf{x},\mathbf{y}\rangle}$, extends holomorphically to $\mathbb{C}^N\times\mathbb{C}^N$ and satisfies the following properties :

- $E(0,\mathbf{y})=1\quad\forall\,\mathbf{y}\in\mathbb{C}^N$,
- $E(\mathbf{x},\mathbf{y})=E(\mathbf{y},\mathbf{x})\quad\forall\,\mathbf{x},\mathbf{y}\in\mathbb{C}^N$,
- $E(\lambda\mathbf{x},\mathbf{y})=E(\mathbf{x},\lambda\mathbf{y})\quad\forall\,\lambda\in\mathbb{C},\ \forall\,\mathbf{x},\mathbf{y}\in\mathbb{C}^N$,
- $E(\sigma(\mathbf{x}),\sigma(\mathbf{y}))=E(\mathbf{x},\mathbf{y})\quad\forall\,\sigma\in G,\ \forall\,\mathbf{x},\mathbf{y}\in\mathbb{C}^N$,
- $\overline{E(\mathbf{x},\mathbf{y})}=E(\bar{\mathbf{x}},\bar{\mathbf{y}})\quad\forall\,\mathbf{x},\mathbf{y}\in\mathbb{C}^N$,
- $E(\mathbf{x},\mathbf{y})>0\quad\forall\,\mathbf{x},\mathbf{y}\in\mathbb{R}^N$,
- $|E(i\mathbf{x},\mathbf{y})|\le 1\quad\forall\,\mathbf{x},\mathbf{y}\in\mathbb{R}^N$,
- $|\partial_{\mathbf{y}}^{\alpha}E(\mathbf{x},\mathbf{y})|\le\|\mathbf{x}\|^{|\alpha|}\max_{\sigma\in G}e^{\operatorname{Re}\langle\sigma(\mathbf{x}),\mathbf{y}\rangle}\quad\forall\,\alpha\in\mathbb{N}^N,\ \forall\,\mathbf{x}\in\mathbb{R}^N,\ \forall\,\mathbf{y}\in\mathbb{C}^N$.

3.2 Dunkl Transform

The *Dunkl transform* is defined on $L^1(dw)$ by

$$\mathcal{F}f(\xi)=c_k^{-1}\int_{\mathbb{R}^N} f(\mathbf{x})E(\mathbf{x},-i\xi)\,dw(\mathbf{x}),$$

where

$$c_k=\int_{\mathbb{R}^N} e^{-\frac{\|\mathbf{x}\|^2}{2}}\,dw(\mathbf{x})>0\,.$$

In the limit case $k\equiv 0$, the Dunkl transform boils down to the classical Fourier transform

$$\hat{f}(\xi)=(2\pi)^{-N/2}\int_{\mathbb{R}^N} f(\mathbf{x})\,e^{-i\,\langle\xi,\mathbf{x}\rangle}\,d\mathbf{x}\,.$$

The following properties hold for the Dunkl transform (see [10,32]):

- The Dunkl transform is a topological automorphisms of the Schwartz space $\mathcal{S}(\mathbb{R}^N)$.
- (*Inversion formula*) For every $f\in\mathcal{S}(\mathbb{R}^N)$ and actually for every $f\in L^1(dw)$ such that $\mathcal{F}f\in L^1(dw)$, we have

$$f(\mathbf{x})=(\mathcal{F})^2 f(-\mathbf{x})\qquad\forall\,\mathbf{x}\in\mathbb{R}^N.$$

- (*Plancherel Theorem*) The Dunkl transform extends to an isometric automorphism of $L^2(dw)$.
- The Dunkl transform of a radial function is again a radial function.

- (*Scaling*) For $\lambda \in \mathbb{R}^*$, we have

$$\mathcal{F}(f_\lambda)(\xi) = \mathcal{F}f(\lambda\xi), \quad \text{where } f_\lambda(\mathbf{x}) = |\lambda|^{-\mathbf{N}} f(\lambda^{-1}\mathbf{x}).$$

- Via the Dunkl transform, the Dunkl operator T_η corresponds to the multiplication by $\pm i\,\langle \eta, \cdot\,\rangle$. Specifically,

$$\begin{cases} \mathcal{F}(T_\eta f) = i\,\langle \eta, \cdot\,\rangle\, \mathcal{F}f, \\ T_\eta(\mathcal{F}f) = -i\,\mathcal{F}(\langle \eta, \cdot\,\rangle f). \end{cases}$$

  In particular, $\mathcal{F}(\Delta f)(\xi) = -\|\xi\|^2 \mathcal{F}f(\xi)$.

3.3 Dunkl Translations and Dunkl Convolution

The *Dunkl translation* $\tau_{\mathbf{x}} f$ of a function $f \in \mathcal{S}(\mathbb{R}^N)$ by $\mathbf{x} \in \mathbb{R}^N$ is defined by

$$\tau_{\mathbf{x}} f(\mathbf{y}) = c_k^{-1} \int_{\mathbb{R}^N} E(i\xi, \mathbf{x})\, E(i\xi, \mathbf{y})\, \mathcal{F}f(\xi)\, dw(\xi). \tag{3.5}$$

Notice the following properties of Dunkl translations:

- each translation $\tau_{\mathbf{x}}$ is a continuous linear map of $\mathcal{S}(\mathbb{R}^N)$ into itself, which extends to a contraction on $L^2(dw)$,
- (*Identity*) $\tau_0 = I$,
- (*Symmetry*) $\tau_{\mathbf{x}} f(\mathbf{y}) = \tau_{\mathbf{y}} f(\mathbf{x}) \quad \forall\, \mathbf{x}, \mathbf{y} \in \mathbb{R}^N,\ \forall\, f \in \mathcal{S}(\mathbb{R}^N)$,
- (*Scaling*) $\tau_{\mathbf{x}}(f_\lambda) = (\tau_{\lambda^{-1}\mathbf{x}} f)_\lambda \quad \forall\, \lambda > 0,\ \forall\, \mathbf{x} \in \mathbb{R}^N,\ \forall\, f \in \mathcal{S}(\mathbb{R}^N)$,
- (*Commutativity*) the Dunkl translations $\tau_{\mathbf{x}}$ and the Dunkl operators T_ξ all commute,
- (*Skew–symmetry*)

$$\int_{\mathbb{R}^N} \tau_{\mathbf{x}} f(\mathbf{y})\, g(\mathbf{y})\, dw(\mathbf{y}) = \int_{\mathbb{R}^N} f(\mathbf{y})\, \tau_{-\mathbf{x}} g(\mathbf{y})\, dw(\mathbf{y}) \forall \mathbf{x} \in \mathbb{R}^N, \forall f, g \in \mathcal{S}(\mathbb{R}^N).$$

The latter formula allows us to define the Dunkl translations $\tau_{\mathbf{x}} f$ in the distributional sense for $f \in L^p(dw)$ with $1 \le p \le \infty$. In particular,

$$\int_{\mathbb{R}^N} \tau_{\mathbf{x}} f(\mathbf{y})\, dw(\mathbf{y}) = \int_{\mathbb{R}^N} f(\mathbf{y})\, dw(\mathbf{y}) \quad \forall\, \mathbf{x} \in \mathbb{R}^N,\ \forall\, f \in \mathcal{S}(\mathbb{R}^N).$$

Finally, notice that $\tau_{\mathbf{x}} f$ is given by (3.5), if $f \in L^1(dw)$ and $\mathcal{F}f \in L^1(dw)$.

The *Dunkl convolution* of two reasonable functions (for instance Schwartz functions) is defined by

$$\begin{aligned} (f * g)(\mathbf{x}) &= c_k\, \mathcal{F}^{-1}[(\mathcal{F}f)(\mathcal{F}g)](\mathbf{x}) \\ &= \int_{\mathbb{R}^N} (\mathcal{F}f)(\xi)\, (\mathcal{F}g)(\xi)\, E(\mathbf{x}, i\xi)\, dw(\xi) \quad \forall\, \mathbf{x} \in \mathbb{R}^N \end{aligned}$$

or, equivalently, by

$$(f*g)(\mathbf{x}) = \int_{\mathbb{R}^N} f(\mathbf{y})\,\tau_{\mathbf{x}}g(-\mathbf{y})\,dw(\mathbf{y}) \qquad \forall\,\mathbf{x}\in\mathbb{R}^N.$$

3.4 Dunkl Translations of Radial Functions

The following specific formula was obtained by Rösler [29] for the Dunkl translations of (reasonable) radial functions $f(\mathbf{x}) = \tilde{f}(\|\mathbf{x}\|)$:

$$\tau_{\mathbf{x}} f(-\mathbf{y}) = \int_{\mathbb{R}^N} (\tilde{f}\circ A)(\mathbf{x},\mathbf{y},\eta)\,d\mu_{\mathbf{x}}(\eta) \qquad \forall\,\mathbf{x},\mathbf{y}\in\mathbb{R}^N. \tag{3.6}$$

Here

$$A(\mathbf{x},\mathbf{y},\eta) = \sqrt{\|\mathbf{x}\|^2+\|\mathbf{y}\|^2-2\langle\mathbf{y},\eta\rangle} = \sqrt{\|\mathbf{x}\|^2-\|\eta\|^2+\|\mathbf{y}-\eta\|^2}$$

and $\mu_{\mathbf{x}}$ is the probability measure occurring in (3.4), which is supported in conv $\mathcal{O}(\mathbf{x})$.

In the remaining part of this section, we shall derive estimates for the Dunkl translations of certain radial functions. Recall that $d(\mathbf{x},\mathbf{y})$ denotes the distance of the orbits $\mathcal{O}(\mathbf{x})$ and $\mathcal{O}(\mathbf{y})$ (see (3.3)). Let us begin with the following elementary estimates (see, e.g., [1]), which hold for $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$ and $\eta\in$ conv $\mathcal{O}(\mathbf{x})$:

$$A(\mathbf{x},\mathbf{y},\eta)\ge d(\mathbf{x},\mathbf{y}) \tag{3.7}$$

and

$$\begin{cases} \|\nabla_{\mathbf{y}}\{A(\mathbf{x},\mathbf{y},\eta)^2\}\| \le 2\,A(\mathbf{x},\mathbf{y},\eta), & \\ |\partial_{\mathbf{y}}^{\beta}\{A(\mathbf{x},\mathbf{y},\eta)^2\}| \le 2 & \text{if } |\beta|=2, \\ \partial_{\mathbf{y}}^{\beta}\{A(\mathbf{x},\mathbf{y},\eta)^2\} = 0 & \text{if } |\beta|>2. \end{cases} \tag{3.8}$$

Hence

$$\|\nabla_{\mathbf{y}}A(\mathbf{x},\mathbf{y},\eta)\|\le 1 \tag{3.9}$$

and, more generally,

$$|\partial_{\mathbf{y}}^{\beta}(\theta\circ A)(\mathbf{x},\mathbf{y},\eta)| \le C_\beta\,A(\mathbf{x},\mathbf{y},\eta)^{m-|\beta|} \qquad \forall\,\beta\in\mathbb{N}^N,$$

if $\theta\in C^\infty(\mathbb{R}\smallsetminus\{0\})$ is a homogeneous symbol of order $m\in\mathbb{R}$, i.e.,

$$\left|\left(\tfrac{d}{dx}\right)^{\beta}\theta(x)\right| \le C_\beta\,|x|^{m-\beta} \qquad \forall\,x\in\mathbb{R}\smallsetminus\{0\},\ \forall\,\beta\in\mathbb{N}.$$

Similarly,

$$|\partial_{\mathbf{y}}^{\beta}(\tilde{\theta}\circ A)(\mathbf{x},\mathbf{y},\eta)| \le C_\beta\,\big\{1+A(\mathbf{x},\mathbf{y},\eta)\big\}^{m-|\beta|} \qquad \forall\,\beta\in\mathbb{N}^N,$$

if $\tilde{\theta} \in C^{\infty}(\mathbb{R})$ is an even inhomogeneous symbol of order $m \in \mathbb{R}$, i.e.,

$$\left|\left(\tfrac{d}{dx}\right)^{\beta}\tilde{\theta}(x)\right| \leq C_{\beta}\,(1+|x|)^{m-\beta} \qquad \forall\, x \in \mathbb{R}\,,\ \forall\, \beta \in \mathbb{N}\,.$$

Consider the radial function

$$q(\mathbf{x}) = c_M\,(1+\|\mathbf{x}\|^2)^{-M/2}$$

on $\mathbb{R}^N$, where $M > \mathbf{N}$ and $c_M > 0$ is a normalizing constant such that $\int_{\mathbb{R}^N} q(\mathbf{x})\,dw(\mathbf{x}) = 1$. Notice that $\tilde{q}(x) = c_M\,(1+x^2)^{-M/2}$ is an even inhomogeneous symbol of order $-M$. The following estimate holds for the translates $q_t(\mathbf{x},\mathbf{y}) = \tau_{\mathbf{x}} q_t(-\mathbf{y})$ of $q_t(\mathbf{x}) = t^{-\mathbf{N}} q(t^{-1}\mathbf{x})$.

Proposition 3.1 *There exists a constant $C > 0$ (depending on M) such that*

$$0 \leq q_t(\mathbf{x},\mathbf{y}) \leq C\,V(\mathbf{x},\mathbf{y},t)^{-1} \qquad \forall\, t>0,\ \forall\,\mathbf{x},\mathbf{y} \in \mathbb{R}^N.$$

Proof By scaling we can reduce to $t = 1$. Fix $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$. We shall prove that

$$\begin{aligned}\int_{\mathbb{R}^N} (1 + A(\mathbf{x},\mathbf{y},\eta))^{-M} d\mu_{\mathbf{x}}(\eta) &\sim \int_{\mathbb{R}^N} (1 + A(\mathbf{x},\mathbf{y},\eta)^2)^{-M/2} d\mu_{\mathbf{x}}(\eta)\\ &= q_1(\mathbf{x},\mathbf{y}) \leq C V(\mathbf{x},\mathbf{y},1)^{-1}. \end{aligned} \tag{3.10}$$

Set $\bar{B} = \{\mathbf{y}' \in \mathbb{R}^N \mid \|\mathbf{y}'-\mathbf{y}\| \leq 1\}$. By continuity, the function $\bar{B} \ni \mathbf{y}' \longmapsto q_1(\mathbf{x},\mathbf{y}')$ reaches a maximum $K = q_1(\mathbf{x},\mathbf{y}_0) \geq 0$ on the ball $\bar{B}$ at some point $\mathbf{y}_0 \in \bar{B}$. For every $\mathbf{y}' \in \bar{B}$, we have

$$\begin{aligned} 0 \leq q_1(\mathbf{x},\mathbf{y}_0) - q_1(\mathbf{x},\mathbf{y}') &= \int_{\mathbb{R}^N} \left\{(\tilde{q}\circ A)(\mathbf{x},\mathbf{y}_0,\eta) - (\tilde{q}\circ A)(\mathbf{x},\mathbf{y}',\eta)\right\} d\mu_{\mathbf{x}}(\eta)\\ &= \int_{\mathbb{R}^N}\int_0^1 \frac{\partial}{\partial s}\,(\tilde{q}\circ A)(\mathbf{x},\underbrace{\mathbf{y}'+s(\mathbf{y}_0-\mathbf{y}')}_{\mathbf{y}_s},\eta)\,ds\,d\mu_{\mathbf{x}}(\eta)\\ &\leq \|\mathbf{y}_0-\mathbf{y}'\| \int_{\mathbb{R}^N}\int_0^1 |(\tilde{q}'\circ A)(\mathbf{x},\mathbf{y}_s,\eta)|\,ds\,d\mu_{\mathbf{x}}(\eta)\\ &\leq M\,\|\mathbf{y}_0-\mathbf{y}'\| \int_{\mathbb{R}^N}\int_0^1 (\tilde{q}\circ A)(\mathbf{x},\mathbf{y}_s,\eta)\,ds\,d\mu_{\mathbf{x}}(\eta)\\ &= M\,\|\mathbf{y}_0-\mathbf{y}'\| \int_0^1 q_1(\mathbf{x},\mathbf{y}_s)\,ds\\ &\leq M\,\|\mathbf{y}_0-\mathbf{y}'\|\,K\,. \end{aligned}$$

Here we have used (3.9) and the elementary estimate

$$|\tilde{q}'(x)| \leq M\,\tilde{q}(x) \qquad \forall\, x \in \mathbb{R}\,.$$

Hence

$$q_1(\mathbf{x},\mathbf{y}') \geq q_1(\mathbf{x},\mathbf{y}_0) - |q_1(\mathbf{x},\mathbf{y}_0) - q_1(\mathbf{x},\mathbf{y}')| \geq K - \frac{K}{2} = \frac{K}{2}\,,$$

if $\mathbf{y}' \in \bar{B} \cap B(\mathbf{y}_0, r)$ with $r = \frac{1}{2M}$. Moreover, as $w(\bar{B} \cap B(\mathbf{y}_0, r)) \sim w(\bar{B})$, we have

$$\begin{aligned} 1 = \int_{\mathbb{R}^N} q_1(\mathbf{x},\mathbf{y}')dw(\mathbf{y}') &\geq \int_{\bar{B}\cap B(\mathbf{y}_0,r)} q_1(\mathbf{x},\mathbf{y}')\,dw(\mathbf{y}') \\ &\geq \frac{K}{2} w(\bar{B} \cap B(\mathbf{y}_0, r)) \geq \frac{K}{C}\, w(\bar{B})\,. \end{aligned}$$

Therefore

$$0 \leq q_1(\mathbf{x},\mathbf{y}) \leq K \leq C\, w(B(\mathbf{y},1))^{-1}.$$

We deduce (3.10) by using the symmetry $q_1(\mathbf{x},\mathbf{y}) = q_1(\mathbf{y},\mathbf{x})$. □

Consider next a radial function f satisfying

$$|f(\mathbf{x})| \lesssim (1+\|\mathbf{x}\|)^{-M-\kappa} \qquad \forall\, \mathbf{x} \in \mathbb{R}^N$$

with $M > \mathbf{N}$ and $\kappa \geq 0$. Then the following estimate holds for the translates $f_t(\mathbf{x},\mathbf{y}) = \tau_{\mathbf{x}} f_t(-\mathbf{y})$ of $f_t(\mathbf{x}) = t^{-\mathbf{N}} f(t^{-1}\mathbf{x})$.

Corollary 3.2 *There exists a constant $C > 0$ such that*

$$|f_t(\mathbf{x},\mathbf{y})| \leq C\, V(\mathbf{x},\mathbf{y},t)^{-1}\Big(1+\frac{d(\mathbf{x},\mathbf{y})}{t}\Big)^{-\kappa} \qquad \forall\, t>0,\ \forall\, \mathbf{x},\mathbf{y} \in \mathbb{R}^N.$$

Proof By scaling we can reduce to $t = 1$. By using (3.6), (3.7), and (3.10) we get

$$\begin{aligned} |f_1(\mathbf{x},\mathbf{y})| &\lesssim \int_{\mathbb{R}^N} \big(1+A(\mathbf{x},\mathbf{y},\eta)\big)^{-M}\big(1+A(\mathbf{x},\mathbf{y},\eta)\big)^{-\kappa} d\mu_{\mathbf{x}}(\eta) \\ &\leq C\, V(\mathbf{x},\mathbf{y},1)^{-1}\big(1+d(\mathbf{x},\mathbf{y})\big)^{-\kappa}. \end{aligned}$$

□

Notice that the space of radial Schwartz functions f on $\mathbb{R}^N$ identifies with the space of even Schwartz functions $\tilde{f}$ on $\mathbb{R}$, which is equipped with the norms

$$\|\tilde{f}\|_{\mathcal{S}_m} = \max_{0\leq j\leq m} \sup_{x\in\mathbb{R}} (1+|x|)^m \left|\Big(\frac{d}{dx}\Big)^j \tilde{f}(x)\right| \qquad \forall\, m \in \mathbb{N}\,. \tag{3.11}$$

Proposition 3.3 *For every $\kappa \geq 0$, there exist $C \geq 0$ and $m \in \mathbb{N}$ such that, for all even Schwartz functions $\tilde{\psi}^{\{1\}}, \tilde{\psi}^{\{2\}}$ and for all even nonnegative integers ℓ_1, ℓ_2, the convolution kernel*

$$\Psi_{s,t}(\mathbf{x},\mathbf{y}) = c_k^{-1} \int_{\mathbb{R}^N} (s\|\xi\|)^{\ell_1} \psi^{\{1\}}(s\|\xi\|)\,(t\|\xi\|)^{\ell_2} \psi^{\{2\}}(t\|\xi\|)\, E(\mathbf{x}, i\xi)\, E(-\mathbf{y}, i\xi)\, dw(\xi)$$

satisfies

$$|\Psi_{s,t}(\mathbf{x},\mathbf{y})| \leq C\, \|\psi^{\{1\}}\|_{\mathcal{S}_{m+\ell_1+\ell_2}} \|\psi^{\{2\}}\|_{\mathcal{S}_{m+\ell_1+\ell_2}} \times \min\left\{\left(\tfrac{s}{t}\right)^{\ell_1}, \left(\tfrac{t}{s}\right)^{\ell_2}\right\} V(\mathbf{x},\mathbf{y}, s+t)^{-1}\left(1 + \frac{d(\mathbf{x},\mathbf{y})}{s+t}\right)^{-\kappa},$$

for every $s, t > 0$ and for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$.

Proof By continuity of the inverse Dunkl transform in the Schwartz setting, there exists a positive even integer m and a constant $C > 0$ such that

$$\sup_{\mathbf{z}\in\mathbb{R}^N} (1+\|\mathbf{z}\|)^{M+\kappa}\, |\mathcal{F}^{-1} f(\mathbf{z})| \leq C\, \|\tilde{f}\|_{\mathcal{S}_m},$$

for every even function $\tilde{f} \in C^m(\mathbb{R})$ with $\|\tilde{f}\|_{\mathcal{S}_m} < \infty$. Consider first the case $0 < s \leq t = 1$. Then

$$\|(s\xi)^{\ell_1}\, \tilde{\psi}^{\{1\}}(s\xi)\, \xi^{\ell_2}\, \tilde{\psi}^{\{2\}}(\xi)\|_{\mathcal{S}_m} \leq C\, \|\psi^{\{1\}}\|_{\mathcal{S}_m} \|\psi^{\{2\}}\|_{\mathcal{S}_{m+\ell_1+\ell_2}}\, s^{\ell_1}.$$

According to Corollary 3.2, we deduce that

$$\begin{aligned} |\Psi_{s,1}(\mathbf{x},\mathbf{y})| &\leq C\, \|\psi^{\{1\}}\|_{\mathcal{S}_{m+\ell_1+\ell_2}} \|\psi^{\{2\}}\|_{\mathcal{S}_{m+\ell_1+\ell_2}}\, s^{\ell_1}\, V(\mathbf{x},\mathbf{y},1)^{-1} \big(1+d(\mathbf{x},\mathbf{y})\big)^{-\kappa} \\ &\leq C\, \|\psi^{\{1\}}\|_{\mathcal{S}_{m+\ell_1+\ell_2}} \|\psi^{\{2\}}\|_{\mathcal{S}_{m+\ell_1+\ell_2}}\, s^{\ell_1}\, V(\mathbf{x},\mathbf{y},s+1)^{-1} \left(1+\frac{d(\mathbf{x},\mathbf{y})}{s+1}\right)^{-\kappa}. \end{aligned}$$

In the case $s = 1 \geq t > 0$, we have similarly

$$|\Psi_{1,t}(\mathbf{x},\mathbf{y})| \leq C\, \|\psi^{\{1\}}\|_{\mathcal{S}_{m+\ell_1+\ell_2}} \|\psi^{\{2\}}\|_{\mathcal{S}_{m+\ell_1+\ell_2}}\, t^{\ell_2}\, V(\mathbf{x},\mathbf{y},1+t)^{-1} \left(1+\frac{d(\mathbf{x},\mathbf{y})}{1+t}\right)^{-\kappa}.$$

The general case is obtained by scaling. □

4 Heat Kernel and Dunkl Kernel

Via the Dunkl transform, the heat semigroup $H_t = e^{t\Delta}$ is given by

$$H_t f(\mathbf{x}) = \mathcal{F}^{-1}\big(e^{-t\|\xi\|^2} \mathcal{F} f(\xi)\big)(\mathbf{x}).$$

Alternately (see, e.g., [32])

$$H_t f(\mathbf{x}) = f * h_t(\mathbf{x}) = \int_{\mathbb{R}^N} h_t(\mathbf{x}, \mathbf{y})\, f(\mathbf{y})\, dw(\mathbf{y}),$$

where the heat kernel $h_t(\mathbf{x}, \mathbf{y})$ is a smooth positive radial convolution kernel. Specifically, for every $t > 0$ and for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$,

$$h_t(\mathbf{x}, \mathbf{y}) = c_k^{-1}\,(2t)^{-\mathbf{N}/2}\, e^{-\frac{\|\mathbf{x}\|^2+\|\mathbf{y}\|^2}{4t}}\, E\Big(\frac{\mathbf{x}}{\sqrt{2t}}, \frac{\mathbf{y}}{\sqrt{2t}}\Big) = \tau_{\mathbf{x}} h_t(-\mathbf{y}), \tag{4.1}$$

where

$$h_t(\mathbf{x}) = \tilde{h}_t(\|\mathbf{x}\|) = c_k^{-1}\,(2t)^{-\mathbf{N}/2}\, e^{-\frac{\|\mathbf{x}\|^2}{4t}}.$$

In particular,

$$\begin{aligned} &h_t(\mathbf{x}, \mathbf{y}) = h_t(\mathbf{y}, \mathbf{x}) > 0,\\ &\int_{\mathbb{R}^N} h_t(\mathbf{x}, \mathbf{y})\, dw(\mathbf{y}) = 1,\\ &h_t(\mathbf{x}, \mathbf{y}) \le c_k^{-1}\,(2t)^{-\mathbf{N}/2}\, e^{-\frac{d(\mathbf{x},\mathbf{y})^2}{4t}}. \end{aligned} \tag{4.2}$$

4.1 Upper Heat Kernel Estimates

We prove now Gaussian bounds for the heat kernel and its derivatives, in the spirit of spaces of homogeneous type, except that the metric $\|\mathbf{x} - \mathbf{y}\|$ is replaced by the orbit distance $d(\mathbf{x}, \mathbf{y})$ (see (3.3)). In comparison with (4.2), the main difference lies in the factor $t^{\mathbf{N}/2}$, which is replaced by the volume of appropriate balls.

Theorem 4.1 (a) Time derivatives*: for any nonnegative integer m, there are constants* $C, c > 0$ *such that*

$$\big|\partial_t^m\, h_t(\mathbf{x}, \mathbf{y})\big| \le C\, t^{-m}\, V(\mathbf{x}, \mathbf{y}, \sqrt{t}\,)^{-1}\, e^{-c\, d(\mathbf{x},\mathbf{y})^2/t}, \tag{4.3}$$

for every $t > 0$ *and for every* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$.

(b) Hölder bounds*: for any nonnegative integer m, there are constants* $C, c > 0$ *such that*

$$\big|\partial_t^m h_t(\mathbf{x}, \mathbf{y}) - \partial_t^m h_t(\mathbf{x}, \mathbf{y}')\big| \le C\, t^{-m}\, \Big(\frac{\|\mathbf{y}-\mathbf{y}'\|}{\sqrt{t}}\Big)\, V(\mathbf{x}, \mathbf{y}, \sqrt{t}\,)^{-1}\, e^{-c\, d(\mathbf{x},\mathbf{y})^2/t}, \tag{4.4}$$

for every $t > 0$ *and for every* $\mathbf{x}, \mathbf{y}, \mathbf{y}' \in \mathbb{R}^N$ *such that* $\|\mathbf{y}-\mathbf{y}'\| < \sqrt{t}$.

(c) Dunkl derivative*: for any* $\xi \in \mathbb{R}^N$ *and for any nonnegative integer m, there are constants* $C, c > 0$ *such that*

$$\Big|\, T_{\xi,\mathbf{x}}\, \partial_t^m\, h_t(\mathbf{x}, \mathbf{y})\Big| \le C\, t^{-m-1/2}\, V(\mathbf{x}, \mathbf{y}, \sqrt{t}\,)^{-1}\, e^{-c\, d(\mathbf{x},\mathbf{y})^2/t}, \tag{4.5}$$

for all $t>0$ *and* $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$.

(d) Mixed derivatives*: for any nonnegative integer m and for any multi-indices* α,β*, there are constants* $C,c>0$ *such that, for every* $t>0$ *and for every* $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$,

$$\left|\partial_t^m\partial_{\mathbf{x}}^\alpha\partial_{\mathbf{y}}^\beta h_t(\mathbf{x},\mathbf{y})\right|\le C\,t^{-m-\frac{|\alpha|}{2}-\frac{|\beta|}{2}}\,V(\mathbf{x},\mathbf{y},\sqrt{t}\,)^{-1}\,e^{-c\,d(\mathbf{x},\mathbf{y})^2/t},\tag{4.6}$$

for every $t>0$ *and for every* $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$.

Proof The proof relies on the expression

$$h_t(\mathbf{x},\mathbf{y})=\int_{\mathbb{R}^N}\tilde{h}_t\big(A(\mathbf{x},\mathbf{y},\eta)\big)\,d\mu_{\mathbf{x}}(\eta)\tag{4.7}$$

and on the properties of $A(\mathbf{x},\mathbf{y},\eta)$.

(a) Consider first the case $m=0$. By scaling we can reduce to $t=1$. On the one hand, we use (3.7) to estimate

$$\begin{aligned}c_k\,2^{\mathbf{N}/2}\,h_1(\mathbf{x},\mathbf{y})&=\int_{\mathbb{R}^N}e^{-A(\mathbf{x},\mathbf{y},\eta)^2/8}\,e^{-A(\mathbf{x},\mathbf{y},\eta)^2/8}\,d\mu_{\mathbf{x}}(\eta)\\&\le e^{-d(\mathbf{x},\mathbf{y})^2/8}\int_{\mathbb{R}^N}e^{-A(\mathbf{x},\mathbf{y},\eta)^2/8}\,d\mu_{\mathbf{x}}(\eta)\,.\end{aligned}$$

On the other hand, it follows from Proposition 3.1 and Corollary 3.2 that

$$\int_{\mathbb{R}^N}e^{-\,c\,A(\mathbf{x},\mathbf{y},\eta)^2}d\mu_{\mathbf{x}}(\eta)\lesssim V(\mathbf{x},\mathbf{y},1)^{-1}\,,$$

for any fixed $c>0$. Hence

$$h_1(\mathbf{x},\mathbf{y})\lesssim V(\mathbf{x},\mathbf{y},1)^{-1}\,e^{-\,d(\mathbf{x},\mathbf{y})^2/8}\,.$$

Consider next the case $m>0$. Observe that $\partial_t^m\tilde{h}_t(x)$ is equal to $t^{-m}\tilde{h}_t(x)$ times a polynomial in $\frac{x^2}{t}$. Therefore

$$\left|\partial_t^m\,\tilde{h}_t(x)\right|\le C_m\,t^{-m}\,\tilde{h}_{2t}(x)\,.\tag{4.8}$$

By differentiating (4.7) and by using (4.8), we deduce that

$$\left|\partial_t^m\,h_t(\mathbf{x},\mathbf{y})\right|\le C_m\,t^{-m}\,h_{2t}(\mathbf{x},\mathbf{y})\,.$$

We conclude by using the case $m=0$.

(b) Observe now that $\tilde{\mathfrak{h}}_t(x)=\partial_x\,\partial_t^m\,\tilde{h}_t(x)$ is equal to $\frac{x}{t^{m+1}}\tilde{h}_t(x)$ times a polynomial in $\frac{x^2}{t}$, hence

$$\left|\tilde{\mathfrak{h}}_t(x)\right|\le C_m\,t^{-m-1/2}\,\tilde{h}_{2t}(x)\,.\tag{4.9}$$

By differentiating (4.7) and by using (3.9) and (4.3), we estimate

$$\begin{aligned}|\partial_t^m h_t(\mathbf{x},\mathbf{y})-\partial_t^m h_t(\mathbf{x},\mathbf{y}')| &= \Big|\int_{\mathbb{R}^N}\big\{\partial_t^m \tilde{h}_t(A(\mathbf{x},\mathbf{y},\eta))-\partial_t^m \tilde{h}_t(A(\mathbf{x},\mathbf{y}',\eta))\big\}d\mu_{\mathbf{x}}(\eta)\Big|\\
&= \Big|\int_{\mathbb{R}^N}\int_0^1 \frac{\partial}{\partial s}\partial_t^m \tilde{h}_t(A(\mathbf{x},\underbrace{\mathbf{y}'+s(\mathbf{y}-\mathbf{y}')}_{\mathbf{y}_s},\eta))\,ds\,d\mu_{\mathbf{x}}(\eta)\Big|\\
&\le \|\mathbf{y}-\mathbf{y}'\|\int_0^1\int_{\mathbb{R}^N}\big|\tilde{\mathfrak{h}}_t(A(\mathbf{x},\mathbf{y}_s,\eta))\big|\,d\mu_{\mathbf{x}}(\eta)\,ds\\
&\le C_m\,t^{-m}\,\frac{\|\mathbf{y}-\mathbf{y}'\|}{\sqrt{t}}\int_0^1 h_{2t}(\mathbf{x},\mathbf{y}_s)\,ds\\
&\le C_m'\,t^{-m}\,\frac{\|\mathbf{y}-\mathbf{y}'\|}{\sqrt{t}}\int_0^1 V(\mathbf{x},\mathbf{y}_s,\sqrt{2t}\,)\,e^{-c\,\frac{d(\mathbf{x},\mathbf{y}_s)^2}{2t}}\,ds\,.\end{aligned}$$

In order to conclude, notice that

$$V(\mathbf{x},\mathbf{y}_s,\sqrt{2t}\,)\sim V(\mathbf{x},\mathbf{y},\sqrt{t}\,) \tag{4.10}$$

under the assumption $\|\mathbf{y}-\mathbf{y}'\|<\sqrt{t}$ and let us show that, for every $c>0$, there exists $C\ge 1$ such that

$$C^{-1}\,e^{-\frac{3}{2}c\,\frac{d(\mathbf{x},\mathbf{y})^2}{t}}\le e^{-c\,\frac{d(\mathbf{x},\mathbf{y}_s)^2}{t}}\le C\,e^{-\frac{1}{2}c\,\frac{d(\mathbf{x},\mathbf{y})^2}{t}}. \tag{4.11}$$

As long as $d(\mathbf{x},\mathbf{y})\le C\sqrt{t}$, all expressions in (4.11) are indeed comparable to 1. On the other hand, if $d(\mathbf{x},\mathbf{y})\ge\sqrt{32\,t}$, then

$$\begin{aligned}|d(\mathbf{x},\mathbf{y})^2-d(\mathbf{x},\mathbf{y}_s)^2| &= |d(\mathbf{x},\mathbf{y})-d(\mathbf{x},\mathbf{y}_s)|\,\{d(\mathbf{x},\mathbf{y})+d(\mathbf{x},\mathbf{y}_s)\}\\
&\le \|\mathbf{y}-\mathbf{y}_s\|\,\{2\,d(\mathbf{x},\mathbf{y})+\|\mathbf{y}-\mathbf{y}_s\|\}\le\sqrt{2\,t}\,\{2\,d(\mathbf{x},\mathbf{y})+\sqrt{2\,t}\,\}\\
&\le \sqrt{8\,t}\,d(\mathbf{x},\mathbf{y})+2\,t\le\frac{1}{2}\,d(\mathbf{x},\mathbf{y})^2+2\,t\,.\end{aligned}$$

Hence

$$\frac{1}{2}\,d(\mathbf{x},\mathbf{y})^2/t-2\le d(\mathbf{x},\mathbf{y}_s)^2/t\le\frac{3}{2}\,d(\mathbf{x},\mathbf{y})^2/t+2\,.$$

(c) By symmetry, we can replace $T_{\xi,\mathbf{x}}$ by $T_{\xi,\mathbf{y}}$. Consider first the contribution of the directional derivative in $T_{\xi,\mathbf{y}}$. By differentiating (4.7) and by using (4.9) and (4.3), we estimate as above

$$\begin{aligned}|\partial_{\xi,\mathbf{y}}\partial_t^m h_t(\mathbf{x},\mathbf{y})| &\le \|\xi\|\int_{\mathbb{R}^N}|\tilde{\mathfrak{h}}_t(A(\mathbf{x},\mathbf{y},\eta))|\,d\mu_{\mathbf{x}}(\eta)\\
&\le C\,t^{-m-1/2}\,h_{2t}(\mathbf{x},\mathbf{y})\\
&\le C\,t^{-m-1/2}\,V(\mathbf{x},\mathbf{y},\sqrt{t}\,)^{-1}\,e^{-c\,d(\mathbf{x},\mathbf{y})^2/t}.\end{aligned}$$

Consider next the contributions

$$\frac{\partial_t^m h_t(\mathbf{x},\mathbf{y}) - \partial_t^m h_t(\mathbf{x},\sigma_\alpha(\mathbf{y}))}{\langle\alpha,\mathbf{y}\rangle} \tag{4.12}$$

of the difference operators in $T_{\xi,\mathbf{y}}$. If $|\langle\alpha,\mathbf{y}\rangle| > \sqrt{t/2}$, we use (4.3) and estimate separately each term in (4.12). If $|\langle\alpha,\mathbf{y}\rangle| \le \sqrt{t/2}$, we estimate again

$$\begin{aligned}\Big|\frac{\partial_t^m h_t(\mathbf{x},\mathbf{y}) - \partial_t^m h_t(\mathbf{x},\sigma_\alpha(\mathbf{y}))}{\langle\alpha,\mathbf{y}\rangle}\Big| &\le \sqrt{2}\int_{\mathbb{R}^N}\int_0^1 |\tilde{\mathfrak{h}}_t(A(\mathbf{x},\mathbf{y}_s,\eta))|ds\,d\mu_{\mathbf{x}}(\eta)\\ &\le C\,t^{-m-1/2}\int_0^1 h_{2t}(\mathbf{x},\mathbf{y}_s)\,ds\\ &\le C\,t^{-m-1/2}\int_0^1 V(\mathbf{x},\mathbf{y}_s,\sqrt{2t})^{-1}\,e^{-c\,\frac{d(\mathbf{x},\mathbf{y}_s)^2}{2t}}\,ds\\ &\le C\,t^{-m-1/2}\,V(\mathbf{x},\mathbf{y},\sqrt{t})^{-1}\,e^{-c\,\frac{d(\mathbf{x},\mathbf{y})^2}{t}}.\end{aligned}$$

In the last step we have used (4.10) and (4.11), which hold as $\|\mathbf{y}_s-\mathbf{y}\|\le\sqrt{t}$.

(d) This time, we use (3.8) to estimate

$$\big|\,\partial_{\mathbf{y}}^{\beta}\partial_t^m\tilde{h}_t\big(A(\mathbf{x},\mathbf{y},\eta)\big)\big| \le C_{m,\beta}\,t^{-m-\frac{|\beta|}{2}}\,\tilde{h}_{2t}\big(A(\mathbf{x},\mathbf{y},\eta)\big)\,. \tag{4.13}$$

Firstly, by differentiating (4.7) and by using (4.13), we obtain

$$\big|\,\partial_t^m\partial_{\mathbf{y}}^{\beta}h_t(\mathbf{x},\mathbf{y})\big| \le C_{m,\beta}\,t^{-m-\frac{|\beta|}{2}}\,h_{2t}(\mathbf{x},\mathbf{y})\,. \tag{4.14}$$

Secondly, by differentiating

$$h_t(\mathbf{x},\mathbf{y}) = \int_{\mathbb{R}^N} h_{t/2}(\mathbf{x},\mathbf{z})\,h_{t/2}(\mathbf{z},\mathbf{y})\,dw(\mathbf{z})\,,$$

by using (4.14) and by symmetry, we get

$$\big|\,\partial_t^m\partial_{\mathbf{x}}^{\alpha}\partial_{\mathbf{y}}^{\beta}h_t(\mathbf{x},\mathbf{y})\big| \le C_{m,\alpha,\beta}\,t^{-m-\frac{|\alpha|}{2}-\frac{|\beta|}{2}}\,h_{2t}(\mathbf{x},\mathbf{y})\,.$$

We conclude by using (4.3). □

4.2 Lower Heat Kernel Estimates

We begin with an auxiliary result.

Lemma 4.2 *Let $\tilde{f}$ be a smooth bump function on $\mathbb{R}$ such that $0\le\tilde{f}\le 1$, $\tilde{f}(x)=1$ if $|x|\le\frac{1}{2}$ and $\tilde{f}(x)=0$ if $|x|\ge 1$. Set as usual*

$$f(\mathbf{x}) = \tilde{f}(\|\mathbf{x}\|) \quad \textit{and} \quad f(\mathbf{x},\mathbf{y}) = \tau_{\mathbf{x}}f(-\mathbf{y}).$$

Then $0 \leq f(\mathbf{x},\mathbf{y}) \leq 1$ *and* $f(\mathbf{x},\mathbf{y}) = 0$ *if* $d(\mathbf{x},\mathbf{y}) \geq 1$*. Moreover, there exists a positive constant* c_1 *such that*

$$\sup_{\mathbf{y}\in\mathcal{O}(B(\mathbf{x},1))} f(\mathbf{x},\mathbf{y}) \geq \frac{c_1}{w(B(\mathbf{x},1))}, \tag{4.15}$$

for every $\mathbf{x} \in \mathbb{R}^N$.

Proof All claims follow from (3.6) and (3.7). Let us prove the last one. On the one hand, by translation invariance,

$$\int_{\mathbb{R}^N} f(\mathbf{x},\mathbf{y})\, dw(\mathbf{y}) = \int_{\mathbb{R}^N} f(\mathbf{y})\, dw(\mathbf{y}) \geq w(B(0,1/2)).$$

On the other hand,

$$\int_{\mathbb{R}^N} f(\mathbf{x},\mathbf{y})\, dw(\mathbf{y}) = \int_{\mathcal{O}(B(\mathbf{x},1))} f(\mathbf{x},\mathbf{y})\, dw(\mathbf{y}) \leq |G|\, w(B(\mathbf{x},1)) \sup_{\mathbf{y}\in\mathcal{O}(B(\mathbf{x},1))} f(\mathbf{x},\mathbf{y}).$$

This proves (4.15) with $c_1 = \frac{w(B(0,1/2))}{|G|}$. □

Proposition 4.3 *There exist positive constants* c_2 *and* ε *such that*

$$h_t(\mathbf{x},\mathbf{y}) \geq \frac{c_2}{w(B(\mathbf{x},\sqrt{t}))},$$

for every $t > 0$ *and* $\mathbf{x},\mathbf{y} \in \mathbb{R}^N$ *satisfying* $\|\mathbf{x}-\mathbf{y}\| \leq \varepsilon\sqrt{t}$.

Proof By scaling it suffices to prove the proposition for $t = 2$. According to Lemma 4.2, applied to $\tilde{h}_1 \gtrsim \tilde{f}$, there exists $c_3 > 0$ and, for every $\mathbf{x} \in \mathbb{R}^N$, there exists $\mathbf{y}(\mathbf{x}) \in \mathcal{O}(B(\mathbf{x},1))$ such that

$$h_1(\mathbf{x},\mathbf{y}(\mathbf{x})) \geq c_3\, w(B(\mathbf{x},1))^{-1}.$$

This estimate holds true around $\mathbf{y}(\mathbf{x})$, according to (4.4), Specifically, there exists $0 < \varepsilon < 1$ (independent of $\mathbf{x}$) such that

$$h_1(\mathbf{x},\mathbf{y}) \geq \tfrac{c_3}{2}\, w(B(\mathbf{x},1))^{-1} \qquad \forall\, \mathbf{y} \in B(\mathbf{y}(\mathbf{x}),\varepsilon).$$

By using the semigroup property and the symmetry of the heat kernel, we deduce that

$$\begin{aligned} h_2(\mathbf{x},\mathbf{x}) &= \int h_1(\mathbf{x},\mathbf{y})\, h_1(\mathbf{y},\mathbf{x})\, dw(\mathbf{y}) \\ &\geq \int_{B(\mathbf{y}(\mathbf{x}),\varepsilon)} h_1(\mathbf{x},\mathbf{y})^2\, dw(\mathbf{y}) \\ &\geq w(B(\mathbf{y}(\mathbf{x}),\varepsilon)\, (\tfrac{c_3}{2})^2\, w(B(\mathbf{x},1))^{-2}. \end{aligned}$$

By using the fact that the balls $B(\mathbf{y}(\mathbf{x}), \varepsilon)$, $B(\mathbf{x}, 1)$, $B(\mathbf{x}, \sqrt{2})$ have comparable volumes and by using again (4.4), we conclude that

$$h_2(\mathbf{x}, \mathbf{y}) \geq c_4\, w(B(\mathbf{x}, \sqrt{2}))^{-1},$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ sufficiently close. □

A standard argument, which we include for the reader's convenience, allows us to deduce from such a near on diagonal estimate the following global lower Gaussian bound.

Theorem 4.4 *There exist positive constants C and c such that*

$$h_t(\mathbf{x}, \mathbf{y}) \geq \frac{C}{\min\{w(B(\mathbf{x}, \sqrt{t})), w(B(\mathbf{y}, \sqrt{t}))\}}\, e^{-c\,\|\mathbf{x}-\mathbf{y}\|^2/t}, \tag{4.16}$$

for every $t > 0$ and for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$.

Proof We resume the notation of Proposition 4.3. For $s \in \mathbb{R}$, we define $\lceil s \rceil$ to be the smallest integer larger than or equal to s. Assume that $\|\mathbf{x} - \mathbf{y}\|^2/t \geq 1$ and set $n = \lceil 4\|\mathbf{x} - \mathbf{y}\|^2/(\varepsilon^2 t) \rceil \geq 4$. Let $\mathbf{x}_i = \mathbf{x} + i(\mathbf{y} - \mathbf{x})/n$ $(i = 0, \dots, n)$, so that $\mathbf{x}_0 = \mathbf{x}$, $\mathbf{x}_n = \mathbf{y}$, and $\|\mathbf{x}_{i+1} - \mathbf{x}_i\| = \|\mathbf{x} - \mathbf{y}\|/n$. Consider the balls $B_i = B(\mathbf{x}_i, \frac{\varepsilon}{4}\sqrt{t/n})$ and observe that

$$\begin{aligned}\|\mathbf{y}_{i+1} - \mathbf{y}_i\| &\leq \|\mathbf{y}_i - \mathbf{x}_i\| + \|\mathbf{x}_i - \mathbf{x}_{i+1}\| + \|\mathbf{x}_{i+1} - \mathbf{y}_{i+1}\| < \frac{\varepsilon}{4}\sqrt{\frac{t}{n}} + \frac{\varepsilon}{2}\sqrt{\frac{t}{n}} + \frac{\varepsilon}{4}\sqrt{\frac{t}{n}} \\ &= \varepsilon\sqrt{\frac{t}{n}}\end{aligned}$$

if $\mathbf{y}_i \in B_i$ and $\mathbf{y}_{i+1} \in B_{i+1}$. By using the semigroup property, Proposition 4.3 and the behavior of the ball volume, we estimate

$$\begin{aligned}h_t(\mathbf{x}, \mathbf{y}) &= \int_{\mathbb{R}^N} \dots \int_{\mathbb{R}^N} h_{t/n}(\mathbf{x}, \mathbf{y}_1) h_{t/n}(\mathbf{y}_1, \mathbf{y}_2) \dots h_{t/n}(\mathbf{y}_{n-1}, \mathbf{y})\, dw(\mathbf{y}_1) \dots dw(\mathbf{y}_{n-1}) \\ &\geq c_2^{n-1} \int_{B_1} \dots \int_{B_{n-1}} w(B(\mathbf{x}, \sqrt{t/n}))^{-1} \dots w(B(\mathbf{y}_{n-1}, \sqrt{t/n}))^{-1} \\ &\quad \times dw(\mathbf{y}_1) \dots dw(\mathbf{y}_{n-1}) \\ &\geq c_3^{n-1} w(B(\mathbf{x}, \sqrt{t/n}))^{-1} \frac{w(B_1) \dots w(B_{n-1})}{w(B(\mathbf{x}_1, \sqrt{t/n})) \dots w(B(\mathbf{x}_{n-1}, \sqrt{t/n}))} \\ &\geq c_5^{n-1} w(B(\mathbf{x}, \sqrt{t}))^{-1} = c_5^{-1} w(B(\mathbf{x}, \sqrt{t}))^{-1} e^{-n \ln c_5^{-1}} \\ &\geq C\, w(B(\mathbf{x}, \sqrt{t}))^{-1} e^{-c\frac{\|\mathbf{x}-\mathbf{y}\|^2}{t}}.\end{aligned}$$

We conclude by symmetry. □

By combining (4.3) and (4.16), we obtain in particular the following near on diagonal estimates. Notice that the ball volumes $w(B(\mathbf{x},\sqrt{t}\,))$ and $w(B(\mathbf{y},\sqrt{t}\,))$ are comparable under the assumptions below.

Corollary 4.5 *For every $c>0$, there exists $C>0$ such that*

$$\frac{C^{-1}}{w(B(\mathbf{x},\sqrt{t}\,))} \le h_t(\mathbf{x},\mathbf{y}) \le \frac{C}{w(B(\mathbf{x},\sqrt{t}\,))},$$

for every $t>0$ and $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$ such that $\|\mathbf{x}-\mathbf{y}\|\le c\sqrt{t}$.

4.3 Estimates of the Dunkl Kernel

According to (4.1), the heat kernel estimates (4.3) and (4.16) imply the following results, which partially improve upon known estimates for the Dunkl kernel. Notice that $\mathbf{x}$ can be replaced by $\mathbf{y}$ in the ball volumes below.

Corollary 4.6 *There are constants $c\ge 1$ and $C\ge 1$ such that*

$$\frac{C^{-1}}{w(B(\mathbf{x},1))}\, e^{\frac{\|\mathbf{x}\|^2+\|\mathbf{y}\|^2}{2}}\, e^{-c\,\|\mathbf{x}-\mathbf{y}\|^2} \le E(\mathbf{x},\mathbf{y}) \le \frac{C}{w(B(\mathbf{x},1))}\, e^{\frac{\|\mathbf{x}\|^2+\|\mathbf{y}\|^2}{2}}\, e^{-c^{-1}d(\mathbf{x},\mathbf{y})^2},$$

for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$. In particular,

- *for every $\varepsilon>0$, there exists $C\ge 1$ such that*

$$\frac{C^{-1}}{w(B(\mathbf{x},1))}\, e^{\frac{\|\mathbf{x}\|^2+\|\mathbf{y}\|^2}{2}} \le E(\mathbf{x},\mathbf{y}) \le \frac{C}{w(B(\mathbf{x},1))}\, e^{\frac{\|\mathbf{x}\|^2+\|\mathbf{y}\|^2}{2}},$$

  for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$ satisfying $\|\mathbf{x}-\mathbf{y}\|<\varepsilon$;
- *there exist $c>0$ and $C>0$ such that*

$$E(\lambda\mathbf{x},\mathbf{y}) \ge \frac{C}{w(B(\sqrt{\lambda}\,\mathbf{x},1))}\, e^{\lambda(1-c\,\|\mathbf{x}-\mathbf{y}\|^2)},$$

  for all $\lambda\ge 1$ and for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$ with $\|\mathbf{x}\|=\|\mathbf{y}\|=1$.

5 Poisson Kernel in the Dunkl Setting

The Poisson semigroup $P_t=e^{-t\sqrt{-\Delta}}$ is subordinated to the heat semigroup $H_t=e^{t\Delta}$ by (2.2) and correspondingly for their integral kernels

$$p_t(\mathbf{x},\mathbf{y}) = \pi^{-1/2}\int_0^\infty e^{-u}\, h_{\frac{t^2}{4u}}(\mathbf{x},\mathbf{y})\,\frac{du}{\sqrt{u}}\,. \tag{5.1}$$

This subordination formula enables us to transfer properties of the heat kernel $h_t(\mathbf{x},\mathbf{y})$ to the Poisson kernel $p_t(\mathbf{x},\mathbf{y})$. For instance,

$$\begin{aligned} &p_t(\mathbf{x},\mathbf{y}) = p_t(\mathbf{y},\mathbf{x}) > 0,\\ &\int_{\mathbb{R}^N} p_t(\mathbf{x},\mathbf{y})\,dw(\mathbf{y}) = 1,\\ &p_t(\mathbf{x},\mathbf{y}) = \tau_{\mathbf{x}} p_t(-\mathbf{y}), \end{aligned} \tag{5.2}$$

where

$$p_t(\mathbf{x}) = \tilde{p}_t(\|\mathbf{x}\|) = c_k'\, t\,\big(t^2 + \|\mathbf{x}\|^2\big)^{-\frac{\mathbf{N}+1}{2}} \tag{5.3}$$

and

$$c_k' = \frac{2^{\mathbf{N}/2}\,\Gamma(\frac{\mathbf{N}+1}{2})}{\sqrt{\pi}\,c_k} > 0\,.$$

The following global bounds hold for the Poisson kernel and its derivatives.

Proposition 5.1 (a) Upper and lower bounds*: there is a constant* $C \geq 1$ *such that*

$$\frac{C^{-1}}{V(\mathbf{x},\mathbf{y},t+\|\mathbf{x}-\mathbf{y}\|)}\,\frac{t}{t+\|\mathbf{x}-\mathbf{y}\|} \leq p_t(\mathbf{x},\mathbf{y}) \leq \frac{C}{V(\mathbf{x},\mathbf{y},t+d(\mathbf{x},\mathbf{y}))}\,\frac{t}{t+d(\mathbf{x},\mathbf{y})} \tag{5.4}$$

for every $t>0$ *and for every* $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$.

(b) Dunkl gradient*: for every* $\xi\in\mathbb{R}^N$*, there is a constant* $C>0$ *such that*

$$\big|T_{\xi,\mathbf{y}}\, p_t(\mathbf{x},\mathbf{y})\big| \leq \frac{C}{V(\mathbf{x},\mathbf{y},t+d(\mathbf{x},\mathbf{y}))}\,\frac{1}{t+d(\mathbf{x},\mathbf{y})} \tag{5.5}$$

for all $t>0$ *and* $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$.

(c) Mixed derivatives: for any nonnegative integer m and for any multi-index β, there is a constant $C\geq 0$ *such that, for every* $t>0$ *and for every* $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$,

$$\big|\partial_t^m \partial_{\mathbf{y}}^{\beta}\, p_t(\mathbf{x},\mathbf{y})\big| \leq C\, p_t(\mathbf{x},\mathbf{y})\big(t+d(\mathbf{x},\mathbf{y})\big)^{-m-|\beta|} \times \begin{cases} 1 & \text{if } m=0,\\ 1+\frac{d(\mathbf{x},\mathbf{y})}{t} & \text{if } m>0. \end{cases} \tag{5.6}$$

Moreover, for any nonnegative integer m and for any multi-indices β, β′, there is a constant $C\geq 0$ *such that, for every* $t>0$ *and for every* $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$,

$$\big|\partial_t^m \partial_{\mathbf{x}}^{\beta} \partial_{\mathbf{y}}^{\beta'}\, p_t(\mathbf{x},\mathbf{y})\big| \leq C\, t^{-m-|\beta|-|\beta'|}\, p_t(\mathbf{x},\mathbf{y})\,. \tag{5.7}$$

Notice that, by symmetry, (5.5) holds also with $T_{\xi,\mathbf{x}}$ instead of $T_{\xi,\mathbf{y}}$.

Proof (a) The Poisson kernel bounds (5.4) are obtained by inserting the heat kernel bounds (4.3) and (4.16) in the subordination formula (5.1). For a detailed proof we refer the reader to [16, Proposition 6].

(b) The Dunkl gradient estimate (5.5) is deduced similarly from (4.5).

(c) The estimate (5.6) is proved directly. As $(t,x)\longmapsto(t^2+x^2)^{-(\mathbf{N}+1)/2}$ is a homogeneous symbol of order $-\mathbf{N}-1$ on $\mathbb{R}^2$, we have

$$\begin{cases} |\partial_x^\beta\,\tilde{p}_t(x)|\le C_\beta\,(t+|x|)^{-\beta}\,\tilde{p}_t(x)\\ |\partial_t^m\,\partial_x^\beta\,\tilde{p}_t(x)|\le C_{m,\beta}\,t^{-1}(t+|x|)^{1-m-\beta}\,\tilde{p}_t(x)\end{cases}\qquad \forall\, t>0,\ \forall\, x\in\mathbb{R}, \tag{5.8}$$

for every positive integer m and for every nonnegative integer β. By using (3.6), (3.7), (5.2), (5.3) and (5.8), we estimate

$$\begin{aligned}\big|\partial_{\mathbf{y}}^\beta\,p_t(\mathbf{x},\mathbf{y})\big|&\le\int_{\mathbb{R}^N}\big|\,\partial_{\mathbf{y}}^\beta\,\tilde{p}_t(A(\mathbf{x},\mathbf{y},\eta))\big|\,d\mu_{\mathbf{x}}(\eta)\\ &\le C_\beta\int_{\mathbb{R}^N}\big(t+A(\mathbf{x},\mathbf{y},\eta)\big)^{-|\beta|}\,\tilde{p}_t(A(\mathbf{x},\mathbf{y},\eta))\,d\mu_{\mathbf{x}}(\eta)\\ &\le C_\beta\,\big(t+d(\mathbf{x},\mathbf{y})\big)^{-|\beta|}\,p_t(\mathbf{x},\mathbf{y})\end{aligned}$$

and, similarly,

$$\big|\partial_t^m\partial_{\mathbf{y}}^\beta\,p_t(\mathbf{x},\mathbf{y})\big|\le C_{m,\beta}\,t^{-1}\big(t+d(\mathbf{x},\mathbf{y})\big)^{1-m-|\beta|}\,p_t(\mathbf{x},\mathbf{y}),$$

for every positive integer m. Finally, (5.7) is deduced from (5.6) by using the semigroup property. More precisely, by differentiating

$$p_t(\mathbf{x},\mathbf{y})\;=\int_{\mathbb{R}^N}p_{t/2}(\mathbf{x},\mathbf{z})\,p_{t/2}(\mathbf{z},\mathbf{y})\,dw(\mathbf{z})\,,$$

by using (5.6) and by symmetry, we obtain

$$\begin{aligned}\big|\partial_t^m\partial_{\mathbf{x}}^\beta\partial_{\mathbf{y}}^{\beta'}\,p_t(\mathbf{x},\mathbf{y})\big|&\lesssim t^{-m-|\beta|-|\beta'|}\int_{\mathbb{R}^N}p_{t/2}(\mathbf{x},\mathbf{z})\,p_{t/2}(\mathbf{z},\mathbf{y})\,dw(\mathbf{z})\\ &=t^{-m-|\beta|-|\beta'|}\,p_t(\mathbf{x},\mathbf{y})\,.\end{aligned}$$

□

Notice the following straightforward consequence of the upper bound in (5.4):

$$\mathcal{M}_P\,f(\mathbf{x})\lesssim\sum_{\sigma\in G}\mathcal{M}_{HL}\,f(\sigma(\mathbf{x}))\,, \tag{5.9}$$

where $\mathcal{M}_{HL}$ denotes the Hardy–Littlewood maximal function on the space of homogeneous type $(\mathbb{R}^N,\,\|\mathbf{x}-\mathbf{y}\|,\,dw)$. Likewise, (4.3) yields

$$\mathcal{M}_H\,f(\mathbf{x})\lesssim\sum_{\sigma\in G}\mathcal{M}_{HL}\,f(\sigma(\mathbf{x}))\,.$$

Observe that the Poisson kernel is an approximation of the identity in the following sense.

Proposition 5.2 *Given any compact subset $K \subset \mathbb{R}^N$, any $r > 0$ and any $\varepsilon > 0$, there exists $t_0 = t_0(K, r, \varepsilon) > 0$ such that, for every $0 < t < t_0$ and for every $\mathbf{x} \in K$,*

$$\int_{\|\mathbf{x}-\mathbf{y}\|>r} p_t(\mathbf{x}, \mathbf{y})\, dw(\mathbf{y}) < \varepsilon\,.$$

Proof Let K be a compact subset of $\mathbb{R}^N$ and let $r, \varepsilon > 0$. Fix $\mathbf{x}_0 \in K$ and consider $f \in C_c^\infty(\mathbb{R}^N)$ such that $0 \le f \le 1$, $f = 1$ on $B(\mathbf{x}_0, r/4)$ and $\operatorname{supp} f \subset B(\mathbf{x}_0, r/2)$. By the inversion formula,

$$f(\mathbf{x}) - P_t f(\mathbf{x}) = c_k^{-1} \int_{\mathbb{R}^N} (1 - e^{-t\,\|\xi\|})\, E(i\xi, \mathbf{x})\, \mathcal{F}f(\xi)\, dw(\xi)\,,$$

hence

$$|f(\mathbf{x}) - P_t f(\mathbf{x})| \le c_k^{-1} \int_{\mathbb{R}^N} \big(1 - e^{-t\,\|\xi\|}\big)\, |\mathcal{F}f(\xi)|\, dw(\xi)\,. \tag{5.10}$$

As $\mathcal{F}f \in \mathcal{S}(\mathbb{R}^N)$, (5.10) implies that there is $t_0 = t_0(\mathbf{x}_0, r, \varepsilon) > 0$ such that

$$\sup_{\mathbf{x}\in\mathbb{R}^N} |f(\mathbf{x}) - P_t f(\mathbf{x})| < \varepsilon \quad \forall\, 0 < t < t_0\,.$$

In particular, for every $0 < t < t_0$ and for every $\mathbf{x} \in B(\mathbf{x}_0, r/4)$, we have

$$\begin{aligned} 0 \le \int_{\|\mathbf{x}-\mathbf{y}\|>r} p_t(\mathbf{x}, \mathbf{y})\, dw(\mathbf{y}) &= 1 - \int_{\|\mathbf{x}-\mathbf{y}\|\le r} p_t(\mathbf{x}, \mathbf{y})\, dw(\mathbf{y}) \\ &\le f(\mathbf{x}) - \int_{\|\mathbf{x}-\mathbf{y}\|\le r} p_t(\mathbf{x}, \mathbf{y}) f(\mathbf{y})\, dw(\mathbf{y}) \le |f(\mathbf{x}) - P_t f(\mathbf{x})| < \varepsilon\,. \end{aligned}$$

We easily conclude the proof by compactness. □

The following results follow from (5.4), (5.9), and Proposition 5.2.

Corollary 5.3 *Let f be a bounded continuous function on $\mathbb{R}^N$. Then its Poisson integral $u(t, \mathbf{x}) = P_t f(\mathbf{x})$ is also bounded and continuous on $[0, \infty) \times \mathbb{R}^N$.*

Corollary 5.4 *Let $f \in L^p(dw)$ with $1 \le p \le \infty$. Then for almost every $\mathbf{x} \in \mathbb{R}^N$,*

$$\lim_{t\to 0} \sup_{\|\mathbf{y}-\mathbf{x}\|<t} \big|P_t f(\mathbf{y}) - f(\mathbf{x})\big| = 0.$$

Moreover, for $f \in L^p(dw)$, $1 \le p < \infty$, we have $\lim_{t\to 0} \|P_t f - f\|_{L^p(dw)} = 0$.

Remark 5.5 The assertion of Proposition 5.2 remains valid with the same proof if $p_t(\mathbf{x}, \mathbf{y})$ is replaced by $\Phi_t(\mathbf{x}, \mathbf{y}) = \tau_{\mathbf{x}} \Phi_t(-\mathbf{y})$, where $\Phi \in \mathcal{S}(\mathbb{R}^N)$ is radial, nonnegative, and $\int \Phi(\mathbf{x})\, dw(\mathbf{x}) = 1$.

6 Conjugate Harmonic Functions—Subharmonicity

For $\sigma \in G$, let $f^\sigma(\mathbf{x}) = f(\sigma(\mathbf{x}))$. It is easy to check that

$$\begin{aligned} T_\xi f^\sigma(\mathbf{x}) &= (T_{\sigma\xi} f)^\sigma(\mathbf{x}), \quad \sigma \in G,\ \mathbf{x}, \xi \in \mathbb{R}^N, \\ (\Delta f^\sigma)(\mathbf{x}) &= (\Delta f)^\sigma(\mathbf{x}). \end{aligned} \tag{6.1}$$

Let $\{\sigma_{ij}\}_{i,j=1}^N$ denote the matrix of $\sigma \in G$ written in the canonical basis $e_1, \dots, e_N$ of $\mathbb{R}^N$. Clearly, $\{\sigma_{ij}\}_{i,j=1}^N$ belongs to the group $\mathbf{O}(N, \mathbb{R})$ of the orthogonal $N \times N$ matrices.

Lemma 6.1 *Assume that* $\mathbf{u}(x_0, \mathbf{x}) = (u_0(x_0, \mathbf{x}), u_1(x_0, \mathbf{x}), \dots, u_N(x_0, \mathbf{x}))$ *satisfies the Cauchy–Riemann equations* (2.4). *For* $\sigma \in G$*, set*

$$u_{\sigma,0}(x_0, \mathbf{x}) = u_0(x_0, \sigma(\mathbf{x})), \quad u_{\sigma,j}(x_0, \mathbf{x}) = \sum_{i=1}^N \sigma_{ij} u_i(x_0, \sigma(\mathbf{x})),\ \ j = 1, 2, \dots, N. \tag{6.2}$$

Then $\mathbf{u}_\sigma(x_0, \mathbf{x}) = (u_{\sigma,0}(x_0, \mathbf{x}), u_{\sigma,1}(x_0, \mathbf{x}), \dots, u_{\sigma,N}(x_0, \mathbf{x}))$ *satisfies the Cauchy–Riemann equations. Moreover,*

$$|\mathbf{u}_\sigma(x_0, \mathbf{x})| = |\mathbf{u}(x_0, \sigma(\mathbf{x}))|. \tag{6.3}$$

Proof Let $1 \le k, j \le N$. Then

$$T_k u_{\sigma,j}(x_0, \mathbf{x}) = \sum_{i=1}^N \sigma_{ij} T_k(u_i(x_0, \sigma\cdot))(\mathbf{x}) = \sum_{i=1}^N \sigma_{ij} \sum_{\ell=1}^N \sigma_{\ell k}(T_\ell u_i)(x_0, \sigma(\mathbf{x})), \tag{6.4}$$

and, similarly,

$$T_j u_{\sigma,k}(x_0, \mathbf{x}) = \sum_{i=1}^N \sigma_{ik} \sum_{\ell=1}^N \sigma_{\ell j}(T_\ell u_i)(x_0, \sigma(\mathbf{x})). \tag{6.5}$$

Recall that $T_\ell u_i = T_i u_\ell$. Hence, (6.5) becomes

$$T_j u_{\sigma,k}(x_0, \mathbf{x}) = \sum_{i=1}^N \sigma_{ik} \sum_{\ell=1}^N \sigma_{\ell j}(T_i u_\ell)(x_0, \sigma(\mathbf{x})). \tag{6.6}$$

Now we see that (6.4) and (6.6) are equal. The proof that $T_k u_{\sigma,0} = T_0 u_{\sigma,k}$ is straightforward. The second equality of (2.4) follows directly from (6.6) and the fact that $\sigma^{-1} = \sigma^*$.

Since $\{\sigma_{ij}\} \in \mathbf{O}(N, \mathbb{R})$,

$$|u_{\sigma,0}(x_0, \mathbf{x})|^2 + \sum_{j=1}^{N} |u_{\sigma,j}(x_0, \mathbf{x})|^2 = |u_0(x_0, \sigma(\mathbf{x}))|^2 + \sum_{j=1}^{N} \left| \sum_{i=1}^{N} \sigma_{ij} u_i(x_0, \sigma(\mathbf{x})) \right|^2$$
$$= |u_0(x_0, \sigma(\mathbf{x}))|^2 + \sum_{i=1}^{N} |u_i(x_0, \sigma(\mathbf{x}))|^2, \quad (6.7)$$

which proves (6.3). □

Let

$$F(t, \mathbf{x}) = \{\mathbf{u}_\sigma(t, \mathbf{x})\}_{\sigma \in G}. \quad (6.8)$$

We shall always assume that $\mathbf{u}$ and $\mathbf{u}_\sigma$ are related by (6.2). Then, by (6.3),

$$|F(x_0, \mathbf{x})|^2 = \sum_{\sigma \in G} \sum_{\ell=0}^{N} |u_{\sigma,\ell}(x_0, \mathbf{x})|^2 = \sum_{\sigma \in G} |\mathbf{u}_\sigma(x_0, \mathbf{x})|^2 = \sum_{\sigma \in G} |\mathbf{u}(x_0, \sigma(\mathbf{x}))|^2.$$

Observe that $|F(x_0, \mathbf{x})| = |F(x_0, \sigma(\mathbf{x}))|$ for every $\sigma \in G$.

Consequently, for every $\alpha \in R$,

$$\sum_{\sigma \in G} \sum_{\ell=0}^{N} \Big(u_{\sigma,\ell}(x_0, \mathbf{x}) - u_{\sigma,\ell}(x_0, \sigma_\alpha(\mathbf{x})) \Big) \cdot u_{\sigma,\ell}(x_0, \mathbf{x})$$
$$= \frac{1}{2} \sum_{\sigma \in G} \sum_{\ell=0}^{N} \Big| u_{\sigma,\ell}(x_0, \mathbf{x}) - u_{\sigma,\ell}(x_0, \sigma_\alpha(\mathbf{x})) \Big|^2. \quad (6.9)$$

We shall need the following auxiliary lemma.

Lemma 6.2 *For every $\varepsilon > 0$ there is $\delta > 0$ such that for every matrix $A = \{a_{ij}\}_{i,j=0}^{N}$ with real entries a_{ij} one has*

$$\|A\|^2 \leq \varepsilon \Big((\,trA)^2 + \sum_{i<j} (a_{ij} - a_{ji})^2 \Big) + (1 - \delta) \|A\|_{\mathrm{HS}}^2,$$

where $\|A\|_{\mathrm{HS}}$ denotes the Hilbert–Schmidt norm of A.

Proof The lemma was proved in [13]. For the convenience of the reader we present a short proof. The inequality is known for trace zero symmetric A (see Stein and Weiss [36, Lemma 2.2]). By homogeneity we may assume that $\|A\|_{\mathrm{HS}} = 1$. Assume that the inequality does not hold. Then there is $\varepsilon > 0$ such that for every $n > 0$ there is $A_n = \{a_{ij}^{\{n\}}\}_{i,j=0}^{N}$, $\|A_n\|_{\mathrm{HS}} = 1$ such that

$$\|A_n\|^2 > \varepsilon \Big((\mathrm{tr} A_n)^2 + \sum_{i<j} (a_{ij}^{\{n\}} - a_{ji}^{\{n\}})^2 \Big) + \left(1 - \frac{1}{n}\right) \|A_n\|_{\mathrm{HS}}^2.$$

Thus there is a subsequence n_s such that $A_{n_s} \to A$, $\|A\|_{\mathrm{HS}} = 1$ and

$$\|A\|^2 \geq \varepsilon \left((\mathrm{tr} A)^2 + \sum_{i<j} (a_{ij} - a_{ji})^2 \right) + \|A\|^2_{\mathrm{HS}}.$$

But then $A = A^*$ and $\mathrm{tr} A = 0$, and so, $\|A\|^2 \geq \|A\|^2_{\mathrm{HS}}$. This contradicts the already known inequality. □

We now state and prove the main theorem of Sect. 6, which is the analog in the Dunkl setting of a Euclidean subharmonicity property (see [34, Chapter VII, Sect. 3.1]) and which was proved in the product case in [13, Proposition 4.1]. Recall (2.3) that $\mathcal{L} = T_0^2 + \Delta$.

Theorem 6.3 *There is an exponent $0 < q < 1$ which depends on k such that if $\mathbf{u} = (u_0, u_1, \ldots, u_N) \in C^2$ satisfies the Cauchy–Riemann equations (2.4), then the function $|F|^q$ is $\mathcal{L}$-subharmonic, that is, $\mathcal{L}(|F|^q)(t, \mathbf{x}) \geq 0$ on the set where $|F| > 0$.*

Proof Observe that $|F|^q$ is C^2 on the set where $|F| > 0$. Let $\cdot$ denote the inner product in $\mathbb{R}^{(N+1)\cdot|G|}$. For $j = 0, 1, \ldots, N$, we have

$$\partial_{e_j} |F|^q = q|F|^{q-2}\Big((\partial_{e_j} F) \cdot F\Big)$$
$$\partial^2_{e_j} |F|^q = q(q-2)|F|^{q-4}\Big((\partial_{e_j} F) \cdot F\Big)^2 + q|F|^{q-2}\Big((\partial^2_{e_j} F) \cdot F + |\partial_{e_j} F|^2\Big).$$

Recall that $|F(x_0, \mathbf{x})| = |F(x_0, \sigma(\mathbf{x}))|$. Hence,

$$\begin{aligned} \mathcal{L}|F|^q &= q(q-2)|F|^{q-4} \left\{ \left(\sum_{j=0}^{N} \Big((\partial_{e_j} F) \cdot F\Big)^2 \right\} \right. \\ &\quad + q|F|^{q-2} \left\{ \left(\sum_{j=0}^{N} \partial^2_{e_j} F + 2 \sum_{\alpha \in R^+} \frac{k(\alpha)}{\langle \alpha, \mathbf{x} \rangle} \partial_\alpha F \right) \cdot F + \sum_{j=0}^{N} |\partial_{e_j} F|^2 \right\}. \end{aligned} \tag{6.10}$$

Since $T_j T_\ell = T_\ell T_j$, we conclude from (2.4) applied to $\mathbf{u}_\sigma$ that for $\ell = 0, 1, \ldots, N$, we have

$$\begin{aligned} &\sum_{j=0}^{N} \partial^2_{e_j} u_{\sigma,\ell}(x_0, \mathbf{x}) + 2 \sum_{\alpha \in R^+} \frac{k(\alpha)}{\langle \alpha, \mathbf{x} \rangle} \partial_\alpha u_{\sigma,\ell}(x_0, \mathbf{x}) \\ &\quad = \sum_{\alpha \in R^+} k(\alpha) \|\alpha\|^2 \frac{u_{\sigma,\ell}(x_0, \mathbf{x}) - u_{\sigma,\ell}(x_0, \sigma_\alpha(\mathbf{x}))}{\langle \alpha, \mathbf{x} \rangle^2}. \end{aligned}$$

Thus,

$$\begin{aligned}
&\left(\sum_{j=0}^{N} \partial_{e_j}^2 F + 2\sum_{\alpha\in R^+} \frac{k(\alpha)}{\langle \alpha, \mathbf{x}\rangle}\partial_\alpha F\right)\cdot F \\
&= \sum_{\sigma\in G}\sum_{\ell=0}^{N}\left(\sum_{j=0}^{N} \partial_{e_j}^2 u_{\sigma,\ell}(x_0,\mathbf{x}) + 2\sum_{\alpha\in R^+}\frac{k(\alpha)}{\langle \alpha, \mathbf{x}\rangle}\partial_\alpha u_{\sigma,\ell}(x_0,\mathbf{x})\right)u_{\sigma,\ell}(x_0,\mathbf{x}) \\
&= \sum_{\sigma\in G}\sum_{\ell=0}^{N}\sum_{\alpha\in R^+} k(\alpha)\|\alpha\|^2\frac{u_{\sigma,\ell}(x_0,\mathbf{x}) - u_{\sigma,\ell}(x_0,\sigma_\alpha(\mathbf{x}))}{\langle \alpha, \mathbf{x}\rangle^2}u_{\sigma,\ell}(x_0,\mathbf{x}) \\
&= \sum_{\alpha\in R^+}\frac{k(\alpha)\|\alpha\|^2}{\langle \alpha, \mathbf{x}\rangle^2}\sum_{\sigma\in G}\sum_{\ell=0}^{N}\Big(u_{\sigma,\ell}(x_0,\mathbf{x}) - u_{\sigma,\ell}(x_0,\sigma_\alpha(\mathbf{x}))\Big)u_{\sigma,\ell}(x_0,\mathbf{x}) \\
&= \frac{1}{2}\sum_{\alpha\in R^+}\frac{k(\alpha)\|\alpha\|^2}{\langle \alpha, \mathbf{x}\rangle^2}\sum_{\sigma\in G}\sum_{\ell=0}^{N}\Big(u_{\sigma,\ell}(x_0,\mathbf{x}) - u_{\sigma,\ell}(x_0,\sigma_\alpha(\mathbf{x}))\Big)^2
\end{aligned} \tag{6.11}$$

Thanks to (6.10) and (6.11), it suffices to prove that there is $0 < q < 1$ such that

$$\begin{aligned}
&(2-q)\sum_{j=0}^{N}\Big((\partial_{e_j}F(x_0,\mathbf{x}))\cdot F(x_0,\mathbf{x})\Big)^2 \\
&\le \frac{1}{2}|F(x_0,\mathbf{x})|^2\sum_{\sigma\in G}\sum_{\ell=0}^{N}\sum_{\alpha\in R^+}\frac{k(\alpha)\|\alpha\|^2}{\langle \alpha, \mathbf{x}\rangle^2}\Big(u_{\sigma,\ell}(x_0,\mathbf{x}) - u_{\sigma,\ell}(x_0,\sigma_\alpha(\mathbf{x}))\Big)^2 \\
&\quad + |F(x_0,\mathbf{x})|^2\left(\sum_{j=0}^{N}|\partial_{e_j}F(x_0,\mathbf{x})|^2\right).
\end{aligned} \tag{6.12}$$

Set

$$B_\sigma = \begin{bmatrix} \partial_{e_0}u_{\sigma,0} & \partial_{e_0}u_{\sigma,1} & \dots & \partial_{e_0}u_{\sigma,N} \\ \partial_{e_1}u_{\sigma,0} & \partial_{e_1}u_{\sigma,1} & \dots & \partial_{e_1}u_{\sigma,N} \\ & & \dots & \\ \partial_{e_N}u_{\sigma,0} & \partial_{e_N}u_{\sigma,1} & \dots & \partial_{e_N}u_{\sigma,N} \end{bmatrix}.$$

Let $\mathbf{B} = \{B_\sigma\}_{\sigma\in G}$ be matrix with $N+1$ rows and $(N+1)\cdot|G|$ columns. It represents a linear operator (denoted by $\mathbf{B}$) from $\mathbb{R}^{(N+1)\cdot|G|}$ into $\mathbb{R}^{1+N}$. Let $\|\mathbf{B}\|$ be its norm.

Observe that for $0 < q < 1$, we have

$$(2-q)\sum_{j=0}^{N}\Big((\partial_{e_j}F)\cdot F\Big)^2 \leq (2-q)|F|^2\|\mathbf{B}\|^2,$$
$$|F|^2\sum_{j=0}^{N}|\partial_{e_j}F|^2 = |F|^2\|\mathbf{B}\|_{\mathrm{HS}}^2.$$

Clearly,

$$\|\mathbf{B}\|^2 \leq \sum_{\sigma\in G}\|B_\sigma\|^2, \quad \|\mathbf{B}\|_{\mathrm{HS}}^2 = \sum_{\sigma\in G}\|B_\sigma\|_{\mathrm{HS}}^2.$$

Therefore the inequality (6.12) will be proven if we show that

$$(2-q)\sum_{\sigma\in G}\|B_\sigma\|^2 \leq \sum_{\sigma\in G}\|B_\sigma\|_{\mathrm{HS}}^2 + \frac{1}{2}\sum_{\sigma\in G}\sum_{\ell=0}^{N}\sum_{\alpha\in R^+}\frac{k(\alpha)\|\alpha\|^2}{\langle\alpha,\mathbf{x}\rangle^2}\Big(u_{\sigma,\ell}(x_0,\mathbf{x})-u_{\sigma,\ell}(x_0,\sigma_\alpha(\mathbf{x}))\Big)^2. \tag{6.13}$$

Recall that

$$\gamma = \sum_{j=1}^{N}\sum_{\alpha\in R^+}\frac{k(\alpha)\langle\alpha,e_j\rangle^2}{\|\alpha\|^2} = \sum_{j=0}^{N}\sum_{\alpha\in R^+}\frac{k(\alpha)\langle\alpha,e_j\rangle^2}{\|\alpha\|^2}$$

(see (3.1)). By applying first the Cauchy–Riemann equations (2.4) and next the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}(\operatorname{tr}B_\sigma)^2 &= \left(-\sum_{j=1}^{N}\sum_{\alpha\in R^+}k(\alpha)\langle\alpha,e_j\rangle\frac{u_{\sigma,j}(x_0,\mathbf{x})-u_{\sigma,j}(x_0,\sigma_\alpha(\mathbf{x}))}{\langle\alpha,\mathbf{x}\rangle}\right)^2\\
&\leq \left(\sum_{j=1}^{N}\sum_{\alpha\in R^+}\frac{k(\alpha)\langle\alpha,e_j\rangle^2}{\|\alpha\|^2}\right)\\
&\quad\times\left(\sum_{j=1}^{N}\sum_{\alpha\in R^+}\|\alpha\|^2k(\alpha)\frac{\big(u_{\sigma,j}(x_0,\mathbf{x})-u_{\sigma,j}(x_0,\sigma_\alpha(\mathbf{x}))\big)^2}{\langle\alpha,\mathbf{x}\rangle^2}\right)\\
&\leq \gamma\sum_{j=0}^{N}\sum_{\alpha\in R^+}\|\alpha\|^2k(\alpha)\frac{\big(u_{\sigma,j}(x_0,\mathbf{x})-u_{\sigma,j}(x_0,\sigma_\alpha(\mathbf{x}))\big)^2}{\langle\alpha,\mathbf{x}\rangle^2}.\end{aligned} \tag{6.14}$$

Utilizing again the Cauchy–Riemann equations (2.4), we get

$$
\begin{aligned}
&\sum_{0\le i<j\le N} \big(\partial_{e_i} u_{\sigma,j}(x_0,\mathbf{x}) - \partial_{e_j} u_{\sigma,i}(x_0,\mathbf{x})\big)^2 \\
&= \sum_{j=1}^{N} \left(\sum_{\alpha\in R^+} k(\alpha)\langle\alpha, e_j\rangle \frac{u_{\sigma,0}(x_0,\mathbf{x}) - u_{\sigma,0}(x_0,\sigma_\alpha(\mathbf{x}))}{\langle\alpha,\mathbf{x}\rangle} \right)^2 \\
&\quad + \sum_{1\le i<j\le N} \left(\sum_{\alpha\in R^+} -k(\alpha)\langle\alpha, e_i\rangle \frac{u_{\sigma,j}(x_0,\mathbf{x}) - u_{\sigma,j}(x_0,\sigma_\alpha(\mathbf{x}))}{\langle\alpha,\mathbf{x}\rangle} \right. \\
&\quad \left. + k(\alpha)\langle\alpha, e_j\rangle \frac{u_{\sigma,i}(x_0,\mathbf{x}) - u_{\sigma,i}(x_0,\sigma_\alpha(\mathbf{x}))}{\langle\alpha,\mathbf{x}\rangle} \right)^2 \\
&\le 2 \left(\sum_{j=0}^{N} \sum_{\alpha\in R^+} \frac{k(\alpha)\langle\alpha, e_j\rangle^2}{\|\alpha\|^2} \right) \\
&\quad \times \left(\sum_{j=0}^{N} \sum_{\alpha\in R^+} \|\alpha\|^2 k(\alpha) \frac{\big(u_{\sigma,j}(x_0,\mathbf{x}) - u_{\sigma,j}(x_0,\sigma_\alpha(\mathbf{x}))\big)^2}{\langle\alpha,\mathbf{x}\rangle^2} \right).
\end{aligned}
\tag{6.15}
$$

Using the auxiliary Lemma 6.2 together with (6.14) and (6.15) we have that for every $\varepsilon > 0$ there is $0 < \delta < 1$ such that

$$
\begin{aligned}
\sum_{\sigma\in G} \|B_\sigma\|^2 &\le (1-\delta) \sum_{\sigma\in G} \|B_\sigma\|_{\mathrm{HS}}^2 \\
&\quad + 3\varepsilon\gamma \sum_{\sigma\in G} \sum_{j=0}^{N} \sum_{\alpha\in R^+} \|\alpha\|^2 k(\alpha) \frac{\big(u_{\sigma,j}(x_0,\mathbf{x}) - u_{\sigma,j}(x_0,\sigma_\alpha(\mathbf{x}))\big)^2}{\langle\alpha,\mathbf{x}\rangle^2}.
\end{aligned}
\tag{6.16}
$$

Taking $\varepsilon > 0$ such that $3\varepsilon\gamma \le \frac{1}{4}$ and utilizing (6.16) we deduce that (6.13) holds for q such that $(1-\delta) \le (2-q)^{-1}$. □

7 Harmonic Functions in the Dunkl Setting

In this section we characterize certain $\mathcal{L}$-harmonic functions in the half-space $\mathbb{R}^{1+N}_+$ by adapting the classical proofs (see, e.g., [19,34,36]). Let us first construct an auxiliary barrier function.

7.1 Barrier Function

For fixed $\delta > 0$, let $v_1, \ldots, v_s \in \mathbb{R}^N$ be a set of vectors of the unit sphere in $S^{N-1} = \{\mathbf{x} \in \mathbb{R}^N : \|\mathbf{x}\| = 1\}$ which forms a δ–net on S^{N-1}. Let $M, \varepsilon > 0$. Define

$$\mathcal{V}_m(x_0, \mathbf{x}) = 2M\varepsilon x_0 + \varepsilon E\Big(\frac{\varepsilon\pi}{4}\mathbf{x}, v_m\Big)\cos\Big(\frac{\varepsilon\pi}{4}x_0\Big), \quad m = 1, \ldots, s, \tag{7.1}$$

(cf. [34, Chapter VII, Sect. 1.2] in the classical setting). The function $\mathcal{V}_m$ is $\mathcal{L}$-harmonic and strictly positive on $[0, \varepsilon^{-1}] \times \mathbb{R}^N$. Set

$$\mathcal{V}(x_0, \mathbf{x}) = \sum_{m=1}^{s} \mathcal{V}_m(x_0, \mathbf{x}).$$

By Corollary 4.6,

$$\lim_{\|\mathbf{x}\|\to\infty} \mathcal{V}(x_0, \mathbf{x}) = \infty \quad \text{uniformly in } x_0 \in [0, \varepsilon^{-1}]. \tag{7.2}$$

7.2 Maximum Principle and the Mean Value Property

As we have already remarked in Sect. 2, the operator $\mathcal{L}$ is the Dunkl–Laplace operator associated with the root system R as a subset of $\mathbb{R}^{1+N} = \mathbb{R} \times \mathbb{R}^N$. We shall denote the element of $\mathbb{R}^{1+N}$ by $\boldsymbol{x} = (x_0, \mathbf{x})$. The associated measure will be denoted by $\boldsymbol{w}$. Clearly, $d\boldsymbol{w}(\boldsymbol{x}) = w(\mathbf{x})\, d\mathbf{x}\, dx_0$. Moreover, $E(\boldsymbol{x}, \boldsymbol{y}) = e^{x_0 y_0} E(\mathbf{x}, \mathbf{y})$. We shall slightly abuse notation and use the same letter σ for the action of the group G in $\mathbb{R}^{1+N}$, so $\sigma(\boldsymbol{x}) = \sigma(x_0, \mathbf{x}) = (x_0, \sigma(\mathbf{x}))$.

The following weak maximum principle for $\mathcal{L}$-subharmonic functions was actually proved in Theorem 4.2 of Rösler [27].

Theorem 7.1 *Let $\Omega \subset \mathbb{R}^{1+N}$ be open, bounded, and $\overline{\Omega} \subset (0, \infty) \times \mathbb{R}^N$. Assume that Ω is G-invariant, that is, $(x_0, \sigma(\mathbf{x})) \in \Omega$ for $(x_0, \mathbf{x}) \in \Omega$ and all $\sigma \in G$. Let $f \in C^2(\Omega) \cap C(\overline{\Omega})$ be real-valued and $\mathcal{L}$-subharmonic. Then*

$$\max_{\overline{\Omega}} f = \max_{\partial\Omega} f.$$

Let $f^{\{r\}}(\boldsymbol{x}) = \chi_{B(0,r)}(\boldsymbol{x})$ be the characteristic function of the ball in $\mathbb{R}^{1+N}$. Set

$$f(r, \boldsymbol{x}, \boldsymbol{y}) = \tau_{\boldsymbol{x}} f^{\{r\}}(-\boldsymbol{y}).$$

Clearly, $0 \le f(r, \boldsymbol{x}, \boldsymbol{y}) \le 1$. The following mean value theorem was proved in [21, Theorem 3.2].

Theorem 7.2 *Let $\Omega \subset \mathbb{R}^{1+N}$ be an open and G-invariant set and let u be a C^2 function in Ω. Then u is $\mathcal{L}$-harmonic if and only if u has the following mean value property:*

for all $\boldsymbol{x} \in \Omega$ *and* $\rho > 0$ *such that* $B(\boldsymbol{x}, \rho) \subset \Omega$*, we have*

$$u(\boldsymbol{x}) = \frac{1}{\boldsymbol{w}(B(0,r))} \int_{\Omega} f(r, \boldsymbol{x}, \boldsymbol{y}) u(\boldsymbol{y}) d\boldsymbol{w}(\boldsymbol{y}) \quad \text{for } 0 < r < \rho/3.$$

7.3 Characterizations of $\mathcal{L}$-Harmonic Functions in the Upper Half-Space

Theorem 7.3 *Suppose that u is a* C^2 *function on* $\mathbb{R}_+^{1+N}$*. Then u is a Poisson integral of a bounded function on* $\mathbb{R}^N$ *if and only if u is* $\mathcal{L}$*-harmonic and bounded.*

Proof The proof is identical to that of Stein [34]. Clearly, the Poisson integral of a bounded function is bounded and $\mathcal{L}$-harmonic. To prove the converse assume that u is $\mathcal{L}$-harmonic and bounded, so $|u| \leq M$. Set $f_n(\mathbf{x}) = u(\frac{1}{n}, \mathbf{x})$ and $u_n(x_0, \mathbf{x}) = P_{x_0} f_n(\mathbf{x})$. Then $U_n(x_0, \mathbf{x}) = u(x_0 + \frac{1}{n}, \mathbf{x}) - u_n(x_0, \mathbf{x})$ is $\mathcal{L}$-harmonic, $|U_n| \leq 2M$, continuous on $[0, \infty) \times \mathbb{R}^N$, and $U_n(0, \mathbf{x}) = 0$. We shall prove that $U_n \equiv 0$. Fix $(y_0, \mathbf{y}) \in \mathbb{R}_+^{1+N}$. Set

$$U(x_0, \mathbf{x}) = U_n(x_0, \mathbf{x}) + \mathcal{V}(x_0, \mathbf{x})$$

and consider the function U on the closure of the set $\Omega = (0, \varepsilon^{-1}) \times B(0, R)$, with $\varepsilon > 0$ small and R large enough. Then U is $\mathcal{L}$-harmonic in Ω, continuous on $\bar{\Omega}$, and positive on the boundary of the $\partial\Omega$. Thus, by the maximum principle, U is positive in $\bar{\Omega}$, so

$$U_n(y_0, \mathbf{y}) > -2M\varepsilon y_0 - \sum_{m=1}^{s} \varepsilon E\Big(\frac{\varepsilon\pi}{4}\mathbf{y}, v_m\Big) \cos\Big(\frac{\varepsilon\pi}{4} y_0\Big).$$

Letting $\varepsilon \to 0$ we obtain $U_n(y_0, \mathbf{y}) \geq 0$. The same argument applied to $-u$ gives $-U_n(y_0, \mathbf{y}) \geq 0$, so $U_n \equiv 0$, which can be written as

$$u\Big(x_0 + \frac{1}{n}, \mathbf{x}\Big) = P_{x_0} f_n(\mathbf{x}) = \int p_{x_0}(\mathbf{x}, \mathbf{y}) f_n(\mathbf{y})\, dw(\mathbf{y}). \tag{7.3}$$

Clearly $|f_n| \leq M$, so by the *-weak compactness, there is a subsequence n_j and $f \in L^\infty(\mathbb{R}^N)$ such that for $\varphi \in L^1(dw)$, we have

$$\lim_{j\to\infty} \int \varphi(\mathbf{y}) f_{n_j}(\mathbf{y})\, dw(\mathbf{y}) = \int \varphi(\mathbf{y}) f(\mathbf{y})\, dw(\mathbf{y}).$$

So,

$$\begin{aligned} u(x_0, \mathbf{x}) &= \lim_{j\to\infty} u\Big(x_0 + \frac{1}{n_j}, \mathbf{x}\Big) = \lim_{j\to\infty} \int p_{x_0}(\mathbf{x}, \mathbf{y}) f_{n_j}(\mathbf{y})\, dw(\mathbf{y}) \\ &= \int p_{x_0}(\mathbf{x}, \mathbf{y}) f(\mathbf{y})\, dw(\mathbf{y}). \end{aligned}$$

□

Corollary 7.4 *If u is $\mathcal{L}$-harmonic and bounded in $\mathbb{R}^{1+N}_+$ then u has a nontangential limit at almost every point of the boundary.*

Theorem 7.5 *Suppose that u is a C^2-function on $\mathbb{R}^{1+N}_+$. If $1 < p < \infty$ then u is a Poisson integral of an $L^p(dw)$ function if and only if u is $\mathcal{L}$-harmonic and*

$$\sup_{x_0>0} \|u(x_0,\cdot)\|_{L^p(dw)} < \infty. \tag{7.4}$$

If $p = 1$ then u is a Poisson integral of a bounded measure ω if and only if u is $\mathcal{L}$-harmonic and

$$\sup_{x_0>0} \|u(x_0,\cdot)\|_{L^1(dw)} < \infty. \tag{7.5}$$

Moreover, if $u^* \in L^1(dw)$ (see (2.5)), then $d\omega(\mathbf{x}) = f(\mathbf{x})dw(\mathbf{x})$, where $f \in L^1(dw)$.

Proof Assume that either (7.4) or (7.5) holds. Then, by Theorem 7.2, for every $\varepsilon > 0$,

$$\sup_{x_0>0} \sup_{\mathbf{x}\in\mathbb{R}^N} |u(x_0+\varepsilon,\mathbf{x})| \le C_\varepsilon < \infty. \tag{7.6}$$

Set $f_n(\mathbf{x}) = u(\frac{1}{n},\mathbf{x})$. From Theorem 7.3 we conclude that $u(\frac{1}{n}+x_0,\mathbf{x}) = P_{x_0} f_n(\mathbf{x})$. Moreover, there is a subsequence n_j such that f_{n_j} converges weakly-* to $f \in L^p(dw)$ (if $1 < p < \infty$) or to a measure ω (if $p = 1$). In both cases u is the Poisson integral either of f or ω. If additionally $u^* \in L^1(dw)$, then the measure ω is absolutely continuous with respect to dw. □

7.4 Proof of a Part of Theorem 2.1

We are now in a position to prove a part of Theorem 2.1, which is stated in the following proposition. The converse is proven at the very end of Sect. 11 (see Proposition 11.5).

Proposition 7.6 *Assume that $\mathbf{u} \in \mathcal{H}^1$. Then*

$$\|\mathbf{u}^*\|_{L^1(dw)} \le C\|\mathbf{u}\|_{\mathcal{H}^1}. \tag{7.7}$$

Proof Fix $\varepsilon > 0$. Set $u_{j,\varepsilon}(x_0,\mathbf{x}) = u_j(\varepsilon+x_0,\mathbf{x})$, $f_{j,\varepsilon}(\mathbf{x}) = u_j(\varepsilon,\mathbf{x})$. Then, by Theorem 7.2, the $\mathcal{L}$-harmonic function $u_{j,\varepsilon}(x_0,\mathbf{x})$ is bounded and continuous on the closed set $[0,\infty)\times\mathbb{R}^N$. In particular, $f_{j,\varepsilon} \in L^\infty \cap L^1(dw) \cap C^2$. By Theorem 7.3,

$$u_{j,\varepsilon}(x_0,\mathbf{x}) = P_{x_0} f_{j,\varepsilon}(\mathbf{x}).$$

It is not difficult to conclude using (5.7) (with $m = 0$) that $\lim_{\|(x_0,\mathbf{x})\|\to\infty} |u_{j,\varepsilon}(x_0,\mathbf{x})| = 0$. Thus also $\lim_{\|\mathbf{x}\|\to\infty} f_{j,\varepsilon}(\mathbf{x}) = 0$. Set $\mathbf{u}_\varepsilon = (u_{0,\varepsilon}, u_{1,\varepsilon}, \ldots, u_{N,\varepsilon})$. Clearly, $\mathbf{u}_\varepsilon \in \mathcal{H}^1$. Let $F_\varepsilon(x_0,\mathbf{x}) = F(\varepsilon+x_0,\mathbf{x})$, where $F(x_0,\mathbf{x})$ is defined by (6.8). Set $\boldsymbol{f}_\varepsilon(\mathbf{x}) = |F(\varepsilon,\mathbf{x})|$. Let $0 < q < 1$ be as in Theorem 6.3 and $p = q^{-1} > 1$. Observe that the function $|F_\varepsilon(x_0,\mathbf{x})|^q - P_{x_0}(\boldsymbol{f}_\varepsilon^q)(\mathbf{x})$ vanishes for $x_0 = 0$ and

$$\lim_{\|(x_0,\mathbf{x})\|\to\infty} \Big(|F_\varepsilon(x_0,\mathbf{x})|^q - P_{x_0}(\boldsymbol{f}_\varepsilon^q)(\mathbf{x})\Big) = 0.$$

So, by Theorem 6.3 and the maximum principle (see Theorem 7.1),

$$|\mathbf{u}(\varepsilon + x_0, \mathbf{x})|^q \leq |F_\varepsilon(x_0, \mathbf{x})|^q \leq P_{x_0}(f_\varepsilon^q)(\mathbf{x}). \tag{7.8}$$

Set $\mathbf{u}_\varepsilon^*(\mathbf{x}) = \sup_{\|\mathbf{x}-\mathbf{y}\|<x_0} |\mathbf{u}(\varepsilon + x_0, \mathbf{y})|$. Then, by (7.8) and (5.9),

$$\|\mathbf{u}_\varepsilon^*\|_{L^1(dw)} \leq C_p \|f_\varepsilon^q\|_{L^p(dw)}^p = C_p \|f_\varepsilon\|_{L^1(dw)} \leq C_p \|\mathbf{u}\|_{\mathcal{H}^1}.$$

Since $\mathbf{u}_\varepsilon^*(\mathbf{x}) \to \mathbf{u}^*(\mathbf{x})$ as $\varepsilon \to 0$ and the convergence is monotone, we use the Lebesgue monotone convergence theorem and get (7.7). □

From Theorem 7.5 and Proposition 7.6 we obtain the following corollary.

Corollary 7.7 *If* $\mathbf{u} \in \mathcal{H}^1$, *then there are* $f_j \in L^1(dw)$, $j = 0, 1, \ldots, N$, *such that* $|f_j(\mathbf{x})| \leq \mathbf{u}^*(\mathbf{x})$ *and* $u_j(x_0, \mathbf{x}) = P_{x_0} f_j(\mathbf{x})$. *Moreover, the limit* $\lim_{x_0 \to 0} u_j(x_0, \mathbf{x}) = f_j(\mathbf{x})$ *exists in* $L^1(dw)$.

8 Riesz Transform Characterization of H^1_Δ

8.1 Riesz Transforms

The Riesz transforms in the Dunkl setting are defined by

$$\mathcal{F}(R_j f)(\xi) = -i \frac{\xi_j}{\|\xi\|} (\mathcal{F} f)(\xi), \quad j = 1, 2, \ldots, N.$$

They are bounded operators on $L^2(dw)$. Clearly,

$$R_j f = -T_{e_j}(-\Delta)^{-1/2} f = - \lim_{\varepsilon \to 0^+,\, M \to \infty} c \int_\varepsilon^M T_{e_j} e^{t\Delta} f \frac{dt}{\sqrt{t}},$$

and the convergence is in $L^2(dw)$ for $f \in L^2(dw)$. It follows from [1] that R_j are bounded operators on $L^p(dw)$ for $1 < p < \infty$.

Our task is to define $R_j f$ for $f \in L^1(dw)$. To this end we set

$$\mathcal{T} = \{\varphi \in L^2(dw) : (\mathcal{F}\varphi)(\xi)(1 + \|\xi\|)^n \in L^2(dw),\ n = 0, 1, 2, \ldots\}.$$

It is not difficult to check that if $\varphi \in \mathcal{T}$, then $\varphi \in C_0(\mathbb{R}^N)$ and $R_j \varphi \in C_0(\mathbb{R}^N) \cap L^2(dw)$. Moreover, for fixed $\mathbf{y} \in \mathbb{R}^N$ the function $p_t(\mathbf{x}, \mathbf{y})$ belongs to $\mathcal{T}$. Now $R_j f$, for $f \in L^1(dw)$, is defined in a weak sense as a functional on $\mathcal{T}$, by

$$\langle R_j f, \varphi \rangle = - \int_{\mathbb{R}^N} f(\mathbf{x}) R_j \varphi(\mathbf{x})\, dw(\mathbf{x}).$$

8.2 Proof of Theorem 2.5

Assume that $f \in L^1(dw)$ is such that $R_j f$ belong to $L^1(dw)$ for $j = 1, 2, \dots, N$. Set $f_0(\mathbf{x}) = f(\mathbf{x})$, $f_j(\mathbf{x}) = R_j f(\mathbf{x})$, $u_0(x_0, \mathbf{x}) = P_{x_0} f(\mathbf{x})$, $u_j(x_0, \mathbf{x}) = P_{x_0} f_j(\mathbf{x})$. Then $\mathbf{u} = (u_0, u_1, \dots, u_n)$ satisfies (2.4). Moreover,

$$\sup_{x_0>0} \int_{\mathbb{R}^N} |u_j(x_0, \mathbf{x})|\, dw(\mathbf{x}) \le \|f_j\|_{L^1(dw)} \quad \text{for } j = 0, 1, \dots, N.$$

Thus $\mathbf{u} \in \mathcal{H}^1$ and

$$\|f\|_{H^1_\Delta} = \|\mathbf{u}\|_{\mathcal{H}^1} \le \|f\|_{L^1(dw)} + \sum_{j=1}^{N} \|R_j f\|_{L^1(dw)}.$$

We turn to prove the converse. Assume that $f_0 \in H^1_\Delta$. By the definition of H^1_Δ there is a system $\mathbf{u} = (u_0, u_1, \dots, u_N) \in \mathcal{H}^1$ such that $f_0(\mathbf{x}) = \lim_{x_0 \to 0} u_0(x_0, \mathbf{x})$ (convergence in $L^1(dw)$). Set $f_j(\mathbf{x}) = \lim_{x_0 \to 0} u_j(x_0, \mathbf{x})$, where limits exist in $L^1(dw)$ (see Corollary 7.7). We have $u_j(x_0, \mathbf{x}) = P_{x_0} f_j(\mathbf{x})$. It suffices to prove that $R_j f_0 = f_j$. To this end, for $\varepsilon > 0$, let $f_{j,\varepsilon}(\mathbf{x}) = u_j(\varepsilon, \mathbf{x})$, $u_{j,\varepsilon}(x_0, \mathbf{x}) = u_j(x_0 + \varepsilon, \mathbf{x})$. Then $f_{j,\varepsilon} \in L^1(dw) \cap C_0(\mathbb{R}^N)$. In particular, $f_{j,\varepsilon} \in L^2(dw)$. Set $g_j = R_j f_{0,\varepsilon}$, $v_j(x_0, \mathbf{x}) = P_{x_0} g_j(\mathbf{x})$. Then $\mathbf{v} = (u_{0,\varepsilon}, v_1, \dots, v_N)$ satisfies the Cauchy–Riemann equations (2.4). Therefore, $T_j u_{0,\varepsilon}(x_0, \mathbf{x}) = T_0 u_{j,\varepsilon}(x_0, \mathbf{x}) = T_0 v_j(x_0, \mathbf{x})$. Hence, $u_{j,\varepsilon}(x_0, \mathbf{x}) - v_j(x_0, \mathbf{x}) = c_j(\mathbf{x})$. But $\lim_{x_0 \to \infty} u_{j,\varepsilon}(x_0, \mathbf{x}) = 0 = \lim_{x_0 \to \infty} v_j(x_0, \mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}^N$. Consequently, $u_{j,\varepsilon}(x_0, \mathbf{x}) = v_j(x_0, \mathbf{x})$. Thus, $f_{j,\varepsilon} = R_j f_{0,\varepsilon}$. Since $\lim_{\varepsilon \to 0} f_{j,\varepsilon} = f_j$ in $L^1(dw)$ and $R_j f_{0,\varepsilon} \to R f_0$ in the sense of distributions, we have $f_j = R_j f_0$.

9 Inclusion $H^1_{(1,q,M)} \subset H^1_\Delta$

In this section we show that the atomic space $H^1_{(1,q,M)}$ with $M > \mathbf{N}$ is contained in the Hardy space H^1_Δ and there exists $C = C_{k,q,M}$ such that

$$\|f\|_{H^1_\Delta} \le C \|f\|_{H^1_{(1,q,M)}}. \tag{9.1}$$

Let $f \in H^1_{(1,q,M)}$. According to Theorem 2.5, it is enough to show that $R_j f \in L^1(dw)$ and $\|R_j f\|_{L^1(dw)} \le C \|f\|_{H^1_{(1,q,M)}}$. By the definition of the atomic space, there is a sequence a_j of $(1, q, M)$ atoms and $\lambda_i \in \mathbb{C}$, $(\lambda_i) \in \ell^1$, such that $f = \sum_i \lambda_i a_i$ and $\sum_i |\lambda_i| \le 2\|f\|_{H^1_{(1,q,M)}}$. Observe that the series converges in $L^1(dw)$, hence $R_j f = \sum_i \lambda_j R_j a_j$ in the sense of distributions. Therefore it suffices to prove that there is a constant $C > 0$ such $\|R_j a\|_{L^1(dw)} \le C$ for every a being a $(1, q, M)$-atom. Our proof follows ideas of [24]. Let $b \in \mathcal{D}(\Delta^M)$ and $B(\mathbf{y_0}, r)$ be as in the definition of

$(1, q, M)$ atom. Since R_j is bounded on $L^q(dw)$, by the Hölder inequality, we have

$$\|R_j a\|_{L^1(\mathcal{O}(B(\mathbf{y_0}, 4r)))} \leq C.$$

In order to estimate $R_j a$ on the set $\mathcal{O}(B(\mathbf{y_0}, 4r))^c$ we write

$$\begin{aligned} R_j a &= c_k'' \int_0^\infty T_{j,\mathbf{x}} e^{t\Delta} a \frac{dt}{\sqrt{t}} \\ &= c_k'' \int_0^{r^2} T_{j,\mathbf{x}} e^{t\Delta} a \frac{dt}{\sqrt{t}} + c_k'' \int_{r^2}^\infty T_{j,\mathbf{x}} e^{t\Delta} (\Delta)^M b \frac{dt}{\sqrt{t}} \\ &= c_k'' \int_0^{r^2} T_{j,\mathbf{x}} e^{t\Delta} a \frac{dt}{\sqrt{t}} + c_k'' \int_{r^2}^\infty T_{j,\mathbf{x}} \partial_t^M e^{t\Delta} b \frac{dt}{\sqrt{t}} \\ &= R_{j,0} a + R_{j,\infty} a. \end{aligned}$$

Further, using (4.5) with $m = 0$ together with (3.2), we get

$$\begin{aligned} |R_{j,0} a(\mathbf{x})| &\leq C \int_0^{r^2} \int_{\mathbb{R}^N} t^{-1} w(B(\mathbf{y}, \sqrt{t}))^{-1} e^{-cd(\mathbf{x},\mathbf{y})^2/t} |a(\mathbf{y})|\, dw(\mathbf{y}) dt \\ &\leq C \frac{r^{\mathbf{N}+1}}{d(\mathbf{x}, \mathbf{y})^{\mathbf{N}+1} w(B(\mathbf{y}_0, r))}. \end{aligned} \tag{9.2}$$

To estimate $R_{j,\infty} a$ we recall that $\|b\|_{L^1(dw)} \leq r^{2M}$. Using (4.5) with $m = M$, we obtain

$$\begin{aligned} |R_{j,\infty} a(\mathbf{x})| &\leq C \int_{r^2}^\infty \int_{\mathbb{R}^N} t^{-M-1} w(B(\mathbf{y}, \sqrt{t}))^{-1} e^{-cd(\mathbf{x},\mathbf{y})^2/t} |b(\mathbf{y})|\, dw(\mathbf{y}) dt \\ &\leq C \frac{r^{2M}}{d(\mathbf{x}, \mathbf{y})^{2M} w(B(\mathbf{y}_0, r))}. \end{aligned} \tag{9.3}$$

Obviously, (9.2) and (9.3) combined with (3.2) imply $\|R_j a\|_{L^1(\mathcal{O}(B(\mathbf{y_0}, 4r))^c)} \leq C$.

10 Maximal Functions

Let $\Phi(\mathbf{x})$ be a radial continuous function such that $|\Phi(\mathbf{x})| \leq C(1 + \|\mathbf{x}\|)^{-\kappa-\beta}$ with $\kappa > \mathbf{N}$. Set $\Phi_t(\mathbf{x}) = t^{-N} \Phi(t^{-1}\mathbf{x})$ and $\Phi_t(\mathbf{x}, \mathbf{y}) = \tau_{\mathbf{x}} \Phi_t(-\mathbf{y})$. Then, by Corollary 3.2,

$$|\Phi_t(\mathbf{x}, \mathbf{y})| \leq C\, V(\mathbf{x}, \mathbf{y}, t)^{-1} \left(1 + \frac{d(\mathbf{x}, \mathbf{y})}{t}\right)^{-\beta}.$$

Set $\mathcal{M}_{\Phi,a} f(\mathbf{x}) = \sup_{\|\mathbf{x}-\mathbf{y}\| < at} |\Phi_t f(\mathbf{y})|$, where

$$\Phi_t f(\mathbf{x}) = \Phi_t * f(\mathbf{x}) = \int_{\mathbb{R}^N} \Phi_t(\mathbf{x}, \mathbf{y})\, f(\mathbf{y})\, dw(\mathbf{y}).$$

If $a = 1$, we write $\mathcal{M}_\Phi$ instead of $\mathcal{M}_{\Phi,1}$. By definition a function $f \in L^1(dw)$ belongs to $H^1_{\max,\Phi}$ if $\mathcal{M}_\Phi f \in L^1(dw)$. In this case $\|f\|_{H^1_{\max,\Phi}} := \|\mathcal{M}_\Phi f\|_{L^1(dw)}$. Recall that if $\Phi(\mathbf{x}) = p_1(\mathbf{x})$ (see (5.3)), then we write $\mathcal{M}_P$, $H^1_{\max,P}$ and $\|\cdot\|_{H^1_{\max,P}}$ for the corresponding maximal function, space and norm respectively (see (2.6) and (2.7)).

10.1 The Space $\mathcal{N}$

The space $H^1_{\max,\Phi}$ is related to the following tent space $\mathcal{N}$.

Definition 10.1 For $a > 0$, $\lambda > \mathbf{N}$ and suitable functions $u(t, \mathbf{x})$, set

$$u^*_a(\mathbf{x}) = \sup_{\|\mathbf{x}-\mathbf{y}\|<at} |u(t, \mathbf{y})| \quad \text{and} \quad u^{**}_\lambda(\mathbf{x}) = \sup_{\mathbf{y}\in\mathbb{R}^N,\, t>0} |u(t, \mathbf{y})| \left(\frac{t}{\|\mathbf{y}-\mathbf{x}\|+t}\right)^{\lambda}.$$

The *tent space* $\mathcal{N}_a$ is then defined by

$$\mathcal{N}_a = \{u(t, \mathbf{x}) : \|u\|_{\mathcal{N}_a} = \|u^*_a\|_{L^1(dw)} < \infty\}.$$

If $a = 1$, we use the simplified notation $\mathcal{N}$, $\|u\|_{\mathcal{N}}$ and u^* (cf. (2.5)).

Obviously, if $u(t, \mathbf{x}) = \Phi_t f(\mathbf{x})$, then $\|f\|_{H^1_{\max,\Phi}} = \|u\|_{\mathcal{N}}$.

Lemma 10.2 *There are constants $C, c_\lambda, C_\lambda > 0$ such that*

$$\|u\|_{\mathcal{N}_a} \le C\left(\frac{a+b}{b}\right)^{\mathbf{N}} \|u\|_{\mathcal{N}_b}, \tag{10.1}$$

$$c_\lambda\|u\|_{\mathcal{N}} \le \|u^{**}_\lambda\|_{L^1(dw)} \le C_\lambda\|u\|_{\mathcal{N}}. \tag{10.2}$$

Proof Similar to the proofs in [35, Chapter II] and [20, p. 114]. □

If $\Omega \subset \mathbb{R}^N$ is an open set, then the tent over Ω is given by

$$\widehat{\Omega} = \left((0,\infty)\times\mathbb{R}^N\right) \setminus \bigcup_{\mathbf{x}\in\Omega^c} \Gamma(\mathbf{x}), \quad \text{where } \Gamma(\mathbf{x}) = \{(t, \mathbf{y}) : \|\mathbf{x}-\mathbf{y}\| < 4t\}.$$

The space $\mathcal{N}$ admits the following atomic decomposition (see [35]).

Definition 10.3 A function $A(t, \mathbf{x})$ is an atom for $\mathcal{N}$ if there is a ball B such that

- supp $A \subset \hat{B}$,
- $\|A\|_{L^\infty} \le w(B)^{-1}$.

Clearly, $\|A\|_{\mathcal{N}} \le 1$ for every atom A for $\mathcal{N}$. Moreover, every $u \in \mathcal{N}$ can be written as $u = \sum_j \lambda_j A_j$, where the functions A_j are atoms for $\mathcal{N}$, $\lambda_j \in \mathbb{C}$, and $\sum_j |\lambda_j| \le C\,\|u\|_{\mathcal{N}}$.

Proposition 10.4 *Let* $u(t,\mathbf{x}) = P_t f(\mathbf{x})$ *and* $v(t,\mathbf{x}) = t^n \frac{d^n}{dt^n} P_t f(\mathbf{x})$. *Then for* $f \in L^1(dw)$ *we have*

$$\|v\|_{\mathcal{N}} \leq C_n \|u\|_{\mathcal{N}}.$$

Proof Assume that $\|u\|_{\mathcal{N}} < \infty$. Clearly, $v(t,\mathbf{x}) = 2^n Q_{t/2} P_{t/2} f(\mathbf{x})$, where $Q_t = t^n \frac{d^n}{dt^n} P_t$. Set $u^{\{1\}}(t,\mathbf{x}) = u(\frac{t}{2},\mathbf{x})$. Then

$$\|u^{\{1\}}\|_{\mathcal{N}} \leq C\|u\|_{\mathcal{N}}.$$

According to the atomic decomposition, $u^{\{1\}} = \sum_j c_j A_j$, where the functions A_j are atoms for $\mathcal{N}$, $c_j \in \mathbb{C}$, and $\sum |c_j| \lesssim \|u\|_{\mathcal{N}}$, (see Definition 10.3). Thus, by Lemma 10.2, we have

$$v(t,\mathbf{x}) = 2^n \sum_j c_j Q_{t/2} A_j(t,\mathbf{x}),$$

$$Q_{t/2} A_j(t,\mathbf{x}) = \int_{\mathbb{R}^N} Q_{t/2}(\mathbf{x},\mathbf{y}) A_j(t,\mathbf{y}) dw(\mathbf{y}).$$

From Proposition 5.1 and Definition 10.3 we conclude that $\|Q_{t/2}A_j(t,\mathbf{x})\|_{\mathcal{N}} \leq C$. □

10.2 Calderón Reproducing Formula

Fix $m \in \mathbb{N}$ sufficiently large. Let $\tilde{\Theta} \in C^m(\mathbb{R})$ be an even function such that $\|\tilde{\Theta}\|_{\mathcal{S}^m} < \infty$ (see (3.11)). Set $\Theta(\mathbf{x}) = \tilde{\Theta}(\|\mathbf{x}\|)$ and assume that $\int_{\mathbb{R}^N} \Theta(\mathbf{x})\, dw(\mathbf{x}) = 0$. Write

$$L^2\Big(\mathbb{R}_+^{1+N}, \frac{dt}{t}\, dw(\mathbf{x})\Big) = L^2(dw\, dt/t).$$

The Plancherel theorem for the Dunkl transform implies that

$$\|\Theta_t * f(\mathbf{x})\|_{L^2(dw\, dt/t)} \leq C\, \|f\|_{L^2(dw)}. \tag{10.3}$$

Thus, $f \mapsto \Theta_t * f(\mathbf{x})$ is a bounded linear operator from $L^2(dw)$ into $L^2(dw\, dt/t)$. By duality, for $F(t,\mathbf{x}) \in L^2(dw\, dt/t)$, the limit

$$\lim_{\varepsilon\to 0^+} \int_\varepsilon^{\varepsilon^{-1}} (\Theta_t * F(t,\cdot))(\mathbf{x}) \frac{dt}{t}$$

exists in $L^2(\mathbb{R}^N, dw)$ and defines a bounded linear operator π_Θ from $L^2(dw\, dt/t)$ into $L^2(\mathbb{R}^N, dw)$, that is,

$$\|\pi_\Theta F\|_{L^2(dw)} \leq C\, \|F\|_{L^2(dw\, dt/t)}. \tag{10.4}$$

With the customary abuse of notation, we write

$$\pi_\Theta F(\mathbf{x}) = \int_0^\infty (\Theta_t * F(t,\cdot))(\mathbf{x})\frac{dt}{t} = \int_0^\infty \int_{\mathbb{R}^N} \Theta_t(\mathbf{x},\mathbf{y})F(t,\mathbf{y})\,dw(\mathbf{y})\frac{dt}{t}.$$

Let Φ be a radial C^∞ function on $\mathbb{R}^N$ such that

$$0 \le \Phi \le 1\,, \quad \operatorname{supp}\Phi \subset B(0,1/4)\,, \quad \Phi = 1 \text{ on } B(0,1/8)\,,$$

and let κ be an integer $> \mathbf{N}/2$. Then

$$\Psi = \Delta^{2\kappa}(\Phi * \Phi) = (\Delta^\kappa \Phi) * (\Delta^\kappa \Phi).$$

is radial, real-valued and

$$\operatorname{supp}\Psi \subset B(0,1/2)\,, \quad \int_{\mathbb{R}^N} \Psi(\mathbf{x})\,dw(\mathbf{x}) = 0\,,$$
$$\mathcal{F}\Psi(\xi) = c_k\,\|\xi\|^{4\kappa}\,(\mathcal{F}\Phi)^2(\xi) = c_k\,\|\xi\|^{4\kappa}\,|\mathcal{F}\Phi(\xi)|^2.$$

Moreover, it follows from (3.6) and (3.7) that

$$\Phi_t(\mathbf{x},\mathbf{y}) = 0 \quad \text{if} \quad d(\mathbf{x},\mathbf{y}) > t/4 \quad \text{and} \quad \Psi_t(\mathbf{x},\mathbf{y}) = 0 \quad \text{if} \quad d(\mathbf{x},\mathbf{y}) > t/2. \tag{10.5}$$

Furthermore,

$$\int_{\mathbb{R}^N} \Psi_t(\mathbf{x},\mathbf{y})\,dw(\mathbf{y}) = \int_{\mathbb{R}^N} \Psi_t(\mathbf{x},\mathbf{y})\,dw(\mathbf{x}) = 0 \qquad \forall\, t > 0$$

and the following Calderón reproducing formulae hold in $L^2(dw)$, for $n = 1, 2, \dots$:

$$f = \lim_{\varepsilon\to 0^+} c'_n \int_\varepsilon^{\varepsilon^{-1}} \Psi_t(t\sqrt{-\Delta})^n\, e^{-t\sqrt{-\Delta}} f\,\frac{dt}{t} = \lim_{\varepsilon\to 0^+} c' \int_\varepsilon^{\varepsilon^{-1}} \Psi_t(-t^2\Delta)\, e^{t^2\Delta} f\,\frac{dt}{t}. \tag{10.6}$$

We may think about the operator in the second equality in (10.6) as the identity operator obtained by the composition of the bounded linear operator

$$L^2(dw) \ni f \mapsto -t^2\Delta e^{t^2\Delta} f(\mathbf{x}) \in L^2(dw\,dt/t)$$

with the linear operator $c'\pi_\Psi$, which is bounded from $L^2(dw\,dt/t)$ into $L^2(dw)$ (see (10.3) and (10.4)).

Recall that m is a large positive integer. Let $\Phi^{\{j\}}(\mathbf{x}) = \tilde{\Phi}^{\{j\}}(\|\mathbf{x}\|)$, $j = 1, 2$, where $\tilde{\Phi}^{\{j\}}$ are even C^m-functions such that

$$\|\tilde{\Phi}^{\{j\}}\|_{\mathcal{S}^m} < \infty \tag{10.7}$$

and

$$\int_{\mathbb{R}^N} \Phi^{\{j\}}(\mathbf{x})\, dw(\mathbf{x}) = 1 \qquad (j = 1, 2). \tag{10.8}$$

By rescaling possibly $\Phi^{\{j\}}$, we may assume that

$$f = \lim_{\varepsilon \to 0^+} c_j'' \int_\varepsilon^{\varepsilon^{-1}} \Psi_t \Phi_t^{\{j\}} f \frac{dt}{t} \qquad \forall\, f \in L^2(dw), \tag{10.9}$$

where the limit takes place in $L^2(dw)$. Moreover, by Lemma 10.2, there is a constant $C_s > 0$ such that if $u^{\{j\}}(t, \mathbf{x}) = \Phi_t^{\{j\}} f(\mathbf{x})$ and $v^{\{j\}}(t, \mathbf{x}) = \Phi_{ts}^{\{j\}} f(\mathbf{x}) = u^{\{j\}}(st, \mathbf{x})$, then

$$C_s^{-1} \|v^{\{j\}}\|_{\mathcal{N}} \le \|u^{\{j\}}\|_{\mathcal{N}} \le C_s \|v^{\{j\}}\|_{\mathcal{N}}.$$

We are in a position to state the main results of this section.

Proposition 10.5 *For $\Phi^{\{1\}}$ and $\Phi^{\{2\}}$ as above and every $f \in L^2(dw)$, we have*

$$\begin{aligned} \|\Phi_t^{\{1\}} f\|_{\mathcal{N}_\alpha} = \|\mathcal{M}_{\Phi^{\{1\}},\alpha} f\|_{L^1(dw)} &\le C_{\Phi^{\{1\}},\Phi^{\{2\}},\alpha,\alpha'} \|\mathcal{M}_{\Phi^{\{2\}},\alpha'} f\|_{L^1(dw)} \\ &= C_{\Phi^{\{1\}},\Phi^{\{2\}},\alpha,\alpha'} \|\Phi_t^{\{2\}} f\|_{\mathcal{N}_{\alpha'}}. \end{aligned}$$

Proof Let $\Psi^{\{1\}} = \Phi^{\{1\}} - \Phi^{\{2\}}$. Then $\Psi^{\{1\}}$ is radial and thanks to (10.8), we have $|\mathcal{F}\Psi^{\{1\}}(\xi)| \le C\|\xi\|^2$ for $\|\xi\| < 1$. It suffices to prove that

$$\|\Psi_t^{\{1\}} f\|_{\mathcal{N}} \le C \|\Phi_t^{\{2\}} f\|_{\mathcal{N}}.$$

Using (10.9), we obtain

$$\Psi_t^{\{1\}} f = \lim_{\varepsilon \to 0^+} c_2'' \int_\varepsilon^{\varepsilon^{-1}} \Psi_t^{\{1\}} \Psi_s \Phi_s^{\{2\}} f \frac{ds}{s},$$

where the limit takes place both in $L^2(dw)$ and pointwise, because $\Psi_t^{\{1\}}(\mathbf{y}, \mathbf{w}) \in L^2(dw(\mathbf{w}))$. According to Proposition 3.3, for any $\eta, \ell > 0$ such that $\ell \le 4\kappa$, the integral kernel $K_{t,s}(\mathbf{y}, \mathbf{z})$ of the operator $\Psi_t^{\{1\}} \Psi_s$ satisfies

$$|K_{t,s}(\mathbf{y}, \mathbf{z})| \le C_{\eta,\ell} \min\Big(\Big(\frac{t}{s}\Big)^2, \Big(\frac{s}{t}\Big)^\ell\Big) \frac{1}{V(\mathbf{y}, \mathbf{z}, s+t)} \Big(1 + \frac{d(\mathbf{y}, \mathbf{z})}{s+t}\Big)^{-\mathbf{N}-\eta}.$$

We take $\mathbf{N} < \lambda < \eta < \ell$. Then for $\|\mathbf{x} - \mathbf{y}\| < t$, we have

$$\int_{\mathbb{R}^N} |K_{t,s}(\mathbf{y}, \mathbf{z})| \Big(1 + \frac{d(\mathbf{x}, \mathbf{z})}{s}\Big)^\lambda dw(\mathbf{z}) \le C' \min\Big(\Big(\frac{s}{t}\Big)^{\ell-\lambda}, \Big(\frac{t}{s}\Big)^2\Big). \tag{10.10}$$

Therefore, using (10.10), we obtain

$$\begin{aligned}\sup_{\|\mathbf{x}-\mathbf{y}\|<t} |\Psi_t^{\{1\}} f(\mathbf{y})| &= |c_2''| \sup_{\|\mathbf{x}-\mathbf{y}\|<t} \left| \lim_{\varepsilon\to 0^+} \int_\varepsilon^{\varepsilon^{-1}} \int_{\mathbb{R}^N} K_{t,s}(\mathbf{y},\mathbf{z}) \Phi_s^{\{2\}} f(\mathbf{z})\, dw(\mathbf{z}) \frac{ds}{s} \right| \\ &\leq |c_2''| \sup_{\mathbf{z},s} |\Phi_s^{\{2\}} f(\mathbf{z})| \Big(1+\frac{d(\mathbf{x},\mathbf{z})}{s}\Big)^{-\lambda} \\ &\quad \times \sup_{\|\mathbf{x}-\mathbf{y}\|<t} \int_0^\infty \int_{\mathbb{R}^N} |K_{t,s}(\mathbf{y},\mathbf{z})| \Big(1+\frac{d(\mathbf{x},\mathbf{z})}{s}\Big)^{\lambda} dw(\mathbf{z}) \frac{ds}{s} \\ &\leq C \sup_{\mathbf{z},s} |\Phi_s^{\{2\}} f(\mathbf{z})| \Big(1+\frac{d(\mathbf{x},\mathbf{z})}{s}\Big)^{-\lambda}.\end{aligned}$$

The proof is complete, by applying (10.2). □

Remark 10.6 It follows from the proof of Proposition 10.5 that if $\Theta \in \mathcal{S}(\mathbb{R}^N)$ is radial and $\int_{\mathbb{R}^N} \Theta(\mathbf{x})\, dw(\mathbf{x}) = 0$, and $\Phi^{\{2\}}$ is as above, then for $f \in L^2(dw)$, we have

$$\|\Theta_t f\|_{\mathcal{N}} \leq C \|\Phi_t^{\{2\}} f\|_{\mathcal{N}}.$$

Proposition 10.7 *For a function $\Phi^{\{1\}}$ as above and $\alpha > 0$ there is a constant $C_{\Phi^{\{1\}},\alpha} > 0$ such that*

$$\|\mathcal{M}_{\Phi^{\{1\}},\alpha} f\|_{L^1(dw)} \leq C_{\Phi^{\{1\}},\alpha} \|\mathcal{M}_P f\|_{L^1(dw)}, \quad \textit{for } f \in L^1(dw) \cap L^2(dw).$$

Proof For a positive integer n (large), set $\phi(\xi) = e^{-\|\xi\|}\Big(\sum_{j=0}^{n+1} \frac{\|\xi\|^j}{j!}\Big)$. Then

$$|\phi(\xi) - 1| \leq C\, \|\xi\|^{n+1} \quad \text{for} \quad \|\xi\| < 1.$$

So ϕ is a C^n function such that $|\partial^\beta \phi(\xi)| \leq C_\beta \exp(-\|\xi\|/2)$, $|\beta| \leq n$. Put $\Phi^{\{2\}} = c_k^{-1} \mathcal{F}^{-1}\phi$. Applying Proposition 10.5, we have

$$\|\Phi_t^{\{1\}} f\|_{\mathcal{N}} \lesssim \|\Phi_t^{\{2\}} f\|_{\mathcal{N}}.$$

Notice that $\frac{d^j}{dt^j} P_t f(\mathbf{x}) = \mathcal{F}^{-1}(\|\xi\|^j e^{-t\|\xi\|} \mathcal{F} f(\xi))(\mathbf{x})$. Hence, from Proposition 10.4 we conclude,

$$\|\Phi_t^{\{2\}} f\|_{\mathcal{N}} \leq C \sum_{j=0}^{n+1} \Big\| t^j \frac{d^j}{dt^j} P_t f \Big\|_{\mathcal{N}} \leq C' \|P_t f\|_{\mathcal{N}}.$$

□

Lemma 10.8 *$H^1_{\max,H} \subset H^1_{\max,P}$ and there is a constant $C > 0$ such that*

$$\|\mathcal{M}_P f\|_{L^1(dw)} \leq C \|\mathcal{M}_H f\|_{L^1(dw)} \quad \textit{for } f \in L^1(dw). \tag{10.11}$$

Proof The proof is standard. Let $f \in L^1(dw)$. Set $u(t, \mathbf{x}) = e^{t^2\Delta} f(\mathbf{x})$. By the subordination formula (2.2) for fixed $t > 0$, we have

$$\begin{aligned}
\sup_{\|\mathbf{x}'-\mathbf{x}\|<t} |P_t f(\mathbf{x}')| &\leq \frac{1}{2\sqrt{\pi}} \int_0^\infty \sup_{\|\mathbf{x}'-\mathbf{x}\|<t} |u(ts, \mathbf{x}')| e^{-\frac{1}{4s^2}} \frac{ds}{s^2} \\
&= \frac{1}{2\sqrt{\pi}} \int_0^\infty \sup_{\|\mathbf{x}'-\mathbf{x}\|<t} |u(ts, \mathbf{x}')| \left(\frac{ts}{\|\mathbf{x}-\mathbf{x}'\| + ts} \right)^\lambda \\
&\quad \times \left(\frac{\|\mathbf{x}-\mathbf{x}'\| + ts}{ts} \right)^\lambda e^{-\frac{1}{4s^2}} \frac{ds}{s^2} \\
&\leq \frac{1}{2\sqrt{\pi}} \int_0^\infty u_\lambda^{**}(\mathbf{x}) \Big(\frac{1+s}{s} \Big)^\lambda e^{-\frac{1}{4s^2}} \frac{ds}{s^2} \\
&\leq C u_\lambda^{**}(\mathbf{x}).
\end{aligned}$$

Now the lemma follows from (10.2). □

Note that Propositions 10.5 and 10.7 together with Lemma 10.8 imply that

$$H^1_{\max,\Phi\{1\}} \cap L^2(dw) = H^1_{\max,H} \cap L^2(dw) = H^1_{\max,P} \cap L^2(dw)$$

and for $f \in L^2(dw)$, we have

$$\|\mathcal{M}_{\Phi\{1\}} f\|_{L^1(dw)} \sim \|\mathcal{M}_H f\|_{L^1(dw)} \sim \|\mathcal{M}_P f\|_{L^1(dw)}. \tag{10.12}$$

Our task is to remove the assumption $f \in L^2(dw)$ from (10.12).

Lemma 10.9 *Assume that $f \in H^1_{\max,P}$. Then $P_t f \in L^2(dw)$ for every $t > 0$ and*

$$\lim_{t\to 0} \|P_t f - f\|_{H^1_{\max,P}} = 0. \tag{10.13}$$

Proof Proposition 5.1 implies that $P_t f \in L^2(dw)$. To prove (10.13) we follow, e.g., [18, proof of (6.5)].

First observe that there is a constant $C > 0$ such that for every $A > 0$ and $t > 0$, we have

$$\Big\| \sup_{s>At,\ \|\mathbf{x}-\mathbf{y}\|<s} |P_{t+s} f(\mathbf{y}) - P_s f(\mathbf{y})| \Big\|_{L^1(dw(\mathbf{x}))} \leq C A^{-1} \|f\|_{L^1(dw)}. \tag{10.14}$$

To see (10.14) fix $\mathbf{z} \in \mathbb{R}^N$. For $s > At$, thanks to (5.4), we have

$$\begin{aligned} |p_{s+t}(\mathbf{y},\mathbf{z}) - p_s(\mathbf{y},\mathbf{z})| &= \left| \int_0^t \partial_u p_{s+u}(\mathbf{y},\mathbf{z})\,du \right| \\ &\leq C \int_0^t \frac{1}{u+s+d(\mathbf{y},\mathbf{z})} w(B(\mathbf{z}, s+u+d(\mathbf{y},\mathbf{z})))^{-1}\,du \\ &\leq C \int_0^t \frac{1}{s+d(\mathbf{y},\mathbf{z})} w(B(\mathbf{z}, s+d(\mathbf{y},\mathbf{z})))^{-1}\,du \\ &\leq \frac{C}{A} \frac{s}{s+d(\mathbf{y},\mathbf{z})} w(B(\mathbf{z}, s+d(\mathbf{y},\mathbf{z})))^{-1}. \end{aligned}$$

Since $s + d(\mathbf{x},\mathbf{z}) \leq s + d(\mathbf{x},\mathbf{y}) + d(\mathbf{y},\mathbf{z}) \leq s + \|\mathbf{x}-\mathbf{y}\| + d(\mathbf{y},\mathbf{z}) \leq 2(s + d(\mathbf{y},\mathbf{z}))$, we obtain

$$\sup_{\|\mathbf{x}-\mathbf{y}\|<s} |p_{s+t}(\mathbf{y},\mathbf{z}) - p_s(\mathbf{y},\mathbf{z})| \leq \frac{C}{A} \frac{s}{s+d(\mathbf{x},\mathbf{z})} w(B(\mathbf{z}, s+d(\mathbf{x},\mathbf{z})))^{-1}, \quad (10.15)$$

which implies (10.14).

In order to finish the proof of (10.13) assume that $f \in H^1_{\max,P}$. Using (10.14), we get

$$\begin{aligned} \|P_t f - f\|_{H^1_{\max,P}} &\leq \Big\| \sup_{s>At,\ \|\mathbf{x}-\mathbf{y}\|<s} |P_{t+s} f(\mathbf{y}) - P_s f(\mathbf{y})| \Big\|_{L^1(dw(\mathbf{x}))} \\ &\quad + \Big\| \sup_{s\leq At,\ \|\mathbf{x}-\mathbf{y}\|<s} |P_{t+s} f(\mathbf{y}) - P_s f(\mathbf{y})| \Big\|_{L^1(dw(\mathbf{x}))} \\ &\leq CA^{-1}\|f\|_{L^1(dw)} + \Big\| \sup_{s\leq At,\ \|\mathbf{x}-\mathbf{y}\|<s} |P_{s+t} f(\mathbf{y}) - f(\mathbf{x})| \Big\|_{L^1(dw(\mathbf{x}))} \\ &\quad + \Big\| \sup_{s\leq At,\ \|\mathbf{x}-\mathbf{y}\|<s} |P_s f(\mathbf{y}) - f(\mathbf{x})| \Big\|_{L^1(dw(\mathbf{x}))} \\ &\leq CA^{-1}\|f\|_{L^1(dw)} \\ &\quad + 2\Big\| \sup_{s\leq(A+1)t,\ \|\mathbf{x}-\mathbf{y}\|<s} |P_s f(\mathbf{y}) - f(\mathbf{x})| \Big\|_{L^1(dw(\mathbf{x}))}. \end{aligned}$$

Fix $\varepsilon > 0$ and take $A = C\varepsilon^{-1}$. Corollary 5.4 implies

$$\lim_{t\to 0} \sup_{s\leq(A+1)t,\ \|\mathbf{x}-\mathbf{y}\|<s} |P_s f(\mathbf{y}) - f(\mathbf{x})| = 0 \quad \text{for almost every } \mathbf{x} \in \mathbb{R}^N.$$

Since $\sup_{s\leq(A+1)t,\ \|\mathbf{x}-\mathbf{y}\|<s} |P_s f(\mathbf{y}) - f(\mathbf{x})| \leq 2\mathcal{M}_P f(\mathbf{x}) \in L^1(dw(\mathbf{x}))$, the proof is complete by applying the Lebesgue dominated convergence theorem. □

Lemma 10.10 *Let $\varphi \in \mathcal{S}(\mathbb{R}^N)$ be a radial function. There is a constant $C > 0$ such that for all $\varepsilon > 0$ and $u(t,\mathbf{x}) \in \mathcal{N}$ if $u_\varepsilon(t,\mathbf{x}) = u(t,\cdot) * \varphi_\varepsilon(\mathbf{x})$, then*

$$\|u_\varepsilon\|_{\mathcal{N}} \leq C\|u\|_{\mathcal{N}}.$$

Proof Let $\lambda > \mathbf{N}$ and $M > 0$ be large enough. For fixed $\mathbf{x} \in \mathbb{R}^N$ we have

$$\begin{aligned} u_\varepsilon^*(\mathbf{x}) &\le C \sup_{t\ge\varepsilon,\, \|\mathbf{x}-\mathbf{y}\|<t} \int_{\mathbb{R}^N} |u(t,\mathbf{z})|\Big(1+\frac{d(\mathbf{z},\mathbf{x})}{t}\Big)^{-\lambda}\Big(1+\frac{d(\mathbf{z},\mathbf{y})}{t}\Big)^{\lambda} |\varphi_\varepsilon(\mathbf{y},\mathbf{z})|\,dw(\mathbf{z}) \\ &\quad + C \sup_{0<t<\varepsilon}\ \sup_{d(\mathbf{x},\mathbf{y})<t} \int_{\mathbb{R}^N} |u(t,\mathbf{z})|\Big(1+\frac{d(\mathbf{x},\mathbf{y})}{\varepsilon}\Big)^{M} V(\mathbf{y},\mathbf{z},\varepsilon)^{-1} \\ &\quad \times \Big(1+\frac{d(\mathbf{x},\mathbf{z})}{\varepsilon}\Big)^{-M} dw(\mathbf{z}) \\ &\le C'_\lambda \sum_{\sigma\in G} u_\lambda^{**}(\sigma(\mathbf{x})) + C'_M \int_{\mathbb{R}^N} u^*(\mathbf{z}) w(B(\mathbf{z},\varepsilon))^{-1}\Big(1+\frac{d(\mathbf{x},\mathbf{z})}{\varepsilon}\Big)^{-M} dw(\mathbf{z}). \end{aligned}$$

Integrating the inequality with respect to $dw(\mathbf{x})$ and applying (10.2) we obtain the lemma. □

Theorem 10.11 *Let* $\Phi^{\{1\}}$ *satisfies* (10.7) *and* (10.8). *Then the spaces* $H^1_{\max,\Phi^{\{1\}}}$, $H^1_{\max,H}$, *and* $H^1_{\max,P}$ *coincide and the corresponding norms are equivalent* (*cf.* (10.12)).

Proof Assume that $f \in H^1_{\max,P}$. Using Lemma 10.9 we take a sequence $t_n \to 0$, $n = 0,1,\ldots$, such that $\|P_{t_0} f\|_{H^1_{\max,P}} \le 2\|f\|_{H^1_{\max,P}}$, $\|P_{t_{n+1}} f - P_{t_n} f\|_{H^1_{\max,P}} \le 2^{-n}\|f\|_{H^1_{\max,P}}$. Then $f = P_{t_0} f + \sum_{n=1}^\infty (P_{t_n} f - P_{t_{n-1}} f) =: g_0 + \sum_{n=1}^\infty g_n$, with the convergence in $L^1(dw)$. The functions $g_n \in L^2(dw) \cap H^1_{\max,P}$, so, by (10.12),

$$\|\mathcal{M}_{\Phi^{\{1\}}} f\|_{L^1(dw)} \le \sum_{j=0}^\infty \|\mathcal{M}_{\Phi^{\{1\}}} g_j\|_{L^1(dw)} \le C \sum_{j=0}^\infty \|\mathcal{M}_P g_j\|_{L^1(dw)} \le 3C\|f\|_{H^1_{\max,P}}.$$

We now turn to prove the converse. Suppose that $f \in H^1_{\max,\Phi^{\{1\}}}$. Then using Lemma 10.10 and the fact that $\|f * h_\varepsilon\|_{L^2(dw)} \le \|f\|_{L^1(dw)}\|h_\varepsilon\|_{L^2(dw)}$ we conclude that $f_\varepsilon = f * h_\varepsilon \in H^1_{\max,\Phi^{\{1\}}} \cap L^2(dw)$ and $\sup_{\varepsilon>0}\|f_\varepsilon\|_{H^1_{\max,\Phi^{\{1\}}}} \le C\|f\|_{H^1_{\max,\Phi^{\{1\}}}}$. Applying (10.12) we get $\sup_{\varepsilon>0}\|f_\varepsilon\|_{H^1_{\max,H}} \le C'\|f\|_{H^1_{\max,\Phi^{\{1\}}}}$. Observe that $\lim_{\varepsilon\to 0}\mathcal{M}_H f_\varepsilon(\mathbf{x}) = \mathcal{M}_H f(\mathbf{x})$ for almost all $\mathbf{x} \in \mathbb{R}^N$ and the convergence is monotone. Hence, by the Lebesgue monotone convergence theorem, we get $\|f\|_{H^1_{\max,H}} \le C'\|f\|_{H^1_{\max,\Phi^{\{1\}}}}$. Finally, the inequality $\|f\|_{H^1_{\max,P}} \le C\|f\|_{H^1_{\max,\Phi^{\{1\}}}}$ is obtained from Lemma 10.8. □

11 Atomic Decompositions; Inclusion $H^1_{\max,H} \subset H^1_{(1,\infty,M)}$

In next theorem we show that all elements in $H^1_{\max,H} \cap L^2(dw) = H^1_{\max,P} \cap L^2(dw)$ admit atomic decompositions into $(1,\infty,M)$-atoms. The $L^2(dw)$ condition is removed afterwards in Theorem 11.4.

Theorem 11.1 *For every positive integer M, there is a constant $C_M > 0$ such that every element $f \in H^1_{\max,H} \cap L^2(dw) = H^1_{\max,P} \cap L^2(dw)$ can be written as*

$$f = \sum \lambda_j a_j,$$

where a_j are $(1,\infty,M)$-atoms, $\sum |\lambda_j| \leq C_M \|\mathcal{M}_P f\|_{L^1(dw)}$. Moreover, the convergence takes place in $L^2(dw)$.

Proof This result is known for Hardy spaces associated with semigroups satisfying Gaussian bounds on spaces of homogeneous type (see [11] and [37]). The proof presented here is a straightforward adaptation of [37] with the difference that tents are now constructed with respect to the orbit distance $d(\mathbf{x},\mathbf{y})$. We include details for the convenience of readers unfamiliar with [11] and [37]. More experienced readers may skip the proof and jump to Theorem 11.4.

Without loss of generality, we may assume that M is an even integer $> 2\mathbf{N}$.

Step 1. Reproducing formulae. Let Φ, Ψ be as in the Calderón reproducing formula with $\kappa = M/2$ (see Sect. 10). Set

$$\varphi(\xi) = \mathcal{F}(\Phi)(\xi) = \tilde{\varphi}(\|\xi\|),$$
$$\psi(\xi) = \mathcal{F}(\Psi)(\xi) = c_k\|\xi\|^{2M}|\varphi(\xi)|^2 = \tilde{\psi}(\|\xi\|) = c_k\|\xi\|^{2M}|\tilde{\varphi}(\|\xi\|)|^2.$$

Then there is a constant c such that

$$f = \lim_{\varepsilon\to 0^+} c\int_\varepsilon^{\varepsilon^{-1}} \Psi_t(-t^2\Delta)e^{t^2\Delta} f\frac{dt}{t} = c\,\pi_\Psi(-t^2\Delta e^{t^2\Delta} f)$$

with convergence in $L^2(dw)$ (see Sect. 10.2). We have

$$\mathcal{F}f(\xi) = \lim_{\varepsilon\to 0^+} c_k c\int_\varepsilon^{\varepsilon^{-1}} t^2\|\xi\|^2\tilde{\psi}(t\|\xi\|)e^{-t^2\|\xi\|^2}\mathcal{F}f(\xi)\frac{dt}{t}.$$

For $\xi \neq 0$, set

$$\eta(\xi) = c_k c\int_1^\infty t^2\|\xi\|^2\tilde{\psi}(t\|\xi\|)e^{-t^2\|\xi\|^2}\frac{dt}{t} = c_k c\int_{\|\xi\|}^\infty t^2\tilde{\psi}(t)e^{-t^2}\frac{dt}{t}.$$

Put $\eta(0) = 1$. Then η is a Schwartz class radial real-valued function. Set $\Xi(\mathbf{x}) = c_k^{-1}\mathcal{F}^{-1}\eta(\mathbf{x})$. Then $\Xi \in \mathcal{S}(\mathbb{R}^N)$, $\int \Xi(\mathbf{x})\,dw(\mathbf{x}) = 1$, and

$$c\int_a^b \Psi_t t^2\Delta e^{t^2\Delta} f\frac{dt}{t} = \Xi_a f - \Xi_b f. \tag{11.1}$$

Step 2. Space of orbits. Let $X = \mathbb{R}^N/G$ be the space of orbits equipped with the metric $d(\mathcal{O}(\mathbf{x}), \mathcal{O}(\mathbf{y})) = d(\mathbf{x}, \mathbf{y})$ and the measure $\boldsymbol{m}(A) = w\Big(\bigcup_{\mathcal{O}(\mathbf{x})\in A} \mathcal{O}(\mathbf{x})\Big)$. So $(X, d, \boldsymbol{m})$ is the space of homogeneous type in the sense of Coifman–Weiss. The space X can be identified with a positive Weyl chamber. Any open set in X of finite measure admits the following easily proved Whitney type covering lemma.

Lemma 11.2 *Suppose that $\boldsymbol{\Omega} \subset X$ is an open set with finite measure. Then there is a sequence of balls $B_X(\mathcal{O}(\mathbf{x}_{\{n\}}), r_{\{n\}})$ such that $r_{\{n\}} = d(\mathcal{O}(\mathbf{x}_{\{n\}}), \boldsymbol{\Omega}^c)$,*

$$\bigcup_{n\in\mathbb{N}} B_X(\mathcal{O}(\mathbf{x}_{\{n\}}), r_{\{n\}}/2) = \boldsymbol{\Omega},$$

the balls $B_X(\mathcal{O}(\mathbf{x}_{\{n\}}), r_{\{n\}}/10)$ are disjoint.

Step 3. Decomposition of $\mathbb{R}^{N+1}_+$. Assume that $f \in H^1_{\max,H} \cap L^2(dw)$. Let

$$\begin{aligned} F(t,\mathbf{x}) &= \Big(|t^2\Delta e^{t^2\Delta} f(\mathbf{x})| + |\Xi_t f(\mathbf{x})|\Big), \\ \boldsymbol{F}(t,\mathbf{x}) &= \sup_{\sigma\in G} F(t,\sigma(\mathbf{x})), \end{aligned}$$

and

$$\mathcal{M}f(\mathbf{x}) = \sup_{d(\mathbf{x},\mathbf{y})<5t} F(t,\mathbf{y}) = \sup_{\|\mathbf{x}-\mathbf{y}\|<5t} \boldsymbol{F}(t,\mathbf{y}).$$

Then, by Proposition 10.5 and Remark 10.6, we have $\|\mathcal{M}f\|_{L^1(dw)} \leq C\|f\|_{H^1_{\max,H}}$. Observe that $\mathcal{M}f(\sigma(\mathbf{x})) = \mathcal{M}f(\mathbf{x})$. Therefore $\mathcal{M}f(\mathbf{x})$ can be identified with the function $\boldsymbol{\mathcal{M}}f(\mathcal{O}(\mathbf{x}))$ on X. Moreover, $\|\mathcal{M}f(\mathbf{x})\|_{L^1(dw)} = \|\boldsymbol{\mathcal{M}}f(\mathcal{O}(\mathbf{x}))\|_{L^1(\boldsymbol{m})}$. For an open set $\boldsymbol{\Omega} \subset X$, let

$$\hat{\boldsymbol{\Omega}} = \{(t,\mathcal{O}(\mathbf{x})) : B_X(\mathcal{O}(\mathbf{x}), 4t) \subset \boldsymbol{\Omega}\}$$

be the tent over $\boldsymbol{\Omega}$. For $j \in \mathbb{Z}$ define

$$\boldsymbol{\Omega}_j = \{\mathcal{O}(\mathbf{x}) \in X : \boldsymbol{\mathcal{M}}f(\mathcal{O}(\mathbf{x})) > 2^j\}, \quad \Omega_j = \{\mathbf{x} \in \mathbb{R}^N : \mathcal{M}f(\mathbf{x}) > 2^j\}.$$

Then $\boldsymbol{\Omega}_j$ is open in X, $\Omega_j = \bigcup_{\mathcal{O}(\mathbf{x})\in\boldsymbol{\Omega}_j} \mathcal{O}(\mathbf{x})$, $\boldsymbol{m}(\boldsymbol{\Omega}_j) = w(\Omega_j)$,

$$\sum_j 2^j w(\Omega_j) \sim \|\mathcal{M}f\|_{L^1(dw)} \sim \|f\|_{H^1_{\max,H}}.$$

Clearly, $\widehat{\Omega}_j = \{(t,\mathbf{x}) \in \mathbb{R}^{N+1}_+ : (t,\mathcal{O}(\mathbf{x})) \in \widehat{\boldsymbol{\Omega}}_j\}$. Set $\mathbf{T}_j = \hat{\Omega}_j \setminus \hat{\Omega}_{j+1}$. Then,

$$\operatorname{supp} F(t,\mathbf{x}) \subset \bigcup_{j\in\mathbb{Z}} \hat{\Omega}_j = \bigcup_{j\in\mathbb{Z}} (\hat{\Omega}_j \setminus \hat{\Omega}_{j+1}) = \bigcup_{j\in\mathbb{Z}} \mathbf{T}_j \tag{11.2}$$

Let $B_X(\mathcal{O}(\mathbf{x}_{\{n,j\}}), r_{\{n,j\}}/2))$, $\mathbf{x}_{\{n,j\}} \in \mathbb{R}^N$, $n = 1, 2, \ldots$, be a Whitney covering of $\mathbf{\Omega}_j$. Set

$$Q_{\{n,j\}} = \{\mathbf{x} \in \mathbb{R}^N : \mathcal{O}(\mathbf{x}) \in B_X(\mathcal{O}(\mathbf{x}_{\{n,j\}}), r_{\{n,j\}}/2))\} = \mathcal{O}(B(\mathbf{x}_{\{n,j\}}, r_{\{n,j\}}/2)).$$

Obviously, $w(B(\mathbf{x}_{\{n,j\}}, r_{\{n,j\}}/2)) \leq w(Q_{\{n,j\}}) \leq |G| w(B(\mathbf{x}_{\{n,j\}}, r_{\{n,j\}}/2))$. We define a cone over a G-invariant set E as

$$\mathcal{R}(E) = \{(t, \mathbf{y}) : d(\mathbf{y}, E) < 2t\}.$$

For $n = 1, 2, \ldots$, let

$$\mathbf{T}_{\{n,j\}} = \mathbf{T}_j \cap \left(\mathcal{R}(Q_{\{n,j\}}) \setminus \bigcup_{i=0}^{n-1} \mathcal{R}(Q_{\{i,j\}}) \right), \quad \mathcal{R}(Q_{\{0,j\}}) = \emptyset.$$

Clearly, $\hat{\Omega}_j \subset \bigcup_{n\in\mathbb{N}} \mathcal{R}(Q_{\{n,j\}})$, $\mathbf{T}_{\{n,j\}} \cap \mathbf{T}_{\{n',j'\}} = \emptyset$ if $(j,n) \neq (j',n')$. Thus we have

$$\operatorname{supp} F(t, \mathbf{x}) \subset \bigcup_{j\in\mathbb{Z}} \bigcup_{n\in\mathbb{N}} \mathbf{T}_{\{n,j\}}. \tag{11.3}$$

Step 4. Decomposition of f and $L^2(dw)$-convergence. Set $G(t, \mathbf{x}) = -t^2 \Delta e^{t^2\Delta} f(\mathbf{x})$, $G_{\{n,j\}}(t, \mathbf{x}) = \chi_{\mathbf{T}_{\{n,j\}}}(t, \mathbf{x}) G(t, \mathbf{x})$. By our assumption $f \in L^2(dw) \cap H^1_{\max,H}$, hence $G \in L^2(dw\, dt/t)$ and $G(t, \mathbf{x}) = \sum_{n\in\mathbb{N},\, j\in\mathbb{Z}} G_{\{n,j\}}(t, \mathbf{x})$, where the series converges unconditionally in $L^2(dw\, dt/t)$, because the sets $\mathbf{T}_{\{n,j\}}$ are pairwise disjoint. Since π_Ψ is a bounded linear operator from $L^2(dw\, dt/t)$ into $L^2(dw)$, we get

$$f = \sum_{j\in\mathbb{Z},\, n\in\in\mathbb{N}} c\, \pi_\Psi\big(G_{\{n,j\}}\big) =: \sum_{j\in\mathbb{Z},\, n\in\mathbb{N}} f_{\{n,j\}}, \tag{11.4}$$

where the series converges unconditionally in $L^2(dw)$.

Step 5. What remains to be done. Let $\lambda_{\{n,j\}} = 2^j w(Q_{\{n,j\}})$. Then

$$\sum_{j\in\mathbb{Z},\, n\in\mathbb{N}} |\lambda_{\{n,j\}}| = \sum_{j\in\mathbb{Z},\, n\in\mathbb{N}} 2^j w(Q_{\{n,j\}}) \lesssim \sum_{j\in\mathbb{Z}} 2^j w(\Omega_j) \sim \|f\|_{H^1_{\max,H}}.$$

Our task is to prove (in Steps 6–9) that, thanks to the choice of the sets $\mathbf{T}_{\{n,j\}}$, the functions $a_{\{n,j\}} = \lambda^{-1}_{\{n,j\}} f_{\{n,j\}}$ are proportional to $(1, \infty, M)$-atoms. Once this is done, the series in (11.4) converges (absolutely) in $L^1(dw)$ as well. Moreover,

$$f = \sum_{j\in\mathbb{Z},\, n\in\mathbb{N}} f_{\{n,j\}} = \sum_{j\in\mathbb{Z},\, n\in\mathbb{N}} \lambda_{\{n,j\}} a_{\{n,j\}}$$

will be the desired atomic decomposition.

Step 6. Functions $b_{\{n,j\}}$. Support of $\Delta^m b_{\{n,j\}}$ for $m = 0, 1, \dots, M$. Observe that

$$\begin{aligned} a_{\{n,\,j\}} &= (\lambda_{\{n,\,j\}})^{-1} c \lim_{\varepsilon\to 0^+} \int_\varepsilon^{r_{\{n,\,j\}}} \Psi_t\Big(\chi_{\mathbf{T}_{\{n,\,j\}}} t^2(-\Delta)e^{t^2\Delta} f\Big)\frac{dt}{t} \\ &= (\lambda_{\{n,\,j\}})^{-1} c \lim_{\varepsilon\to 0^+} \int_\varepsilon^{r_{\{n,\,j\}}} t^{2M}(-\Delta)^M \Phi_t\Phi_t\Big(\chi_{\mathbf{T}_{\{n,\,j\}}} t^2(-\Delta)e^{t^2\Delta} f\Big)\frac{dt}{t}. \end{aligned} \tag{11.5}$$

Indeed, if $t > r_{\{n,\,j\}}$ and $(t, \mathbf{y}) \in \mathcal{R}(Q_{\{n,\,j\}})$ then

$$d(\mathbf{y}, (\Omega_j)^c) \le d(\mathbf{y}, Q_{\{n,\,j\}}) + \frac{1}{2} r_{\{n,\,j\}} + d(\mathbf{x}_{\{n,\,j\}}, (\Omega_j)^c) \le 2t + \frac{1}{2}t + t = \frac{7}{2}t. \tag{11.6}$$

Hence $(t, \mathbf{y}) \notin \mathbf{T}_{\{n,\,j\}}$, which gives (11.5).

As a consequence of (10.5) and (11.5), we have

$$\operatorname{supp} a_{\{n,\,j\}} \subset \Big\{\mathbf{x} \in \mathbb{R}^N : d(\mathbf{x}, \mathbf{x}_{\{n,\,j\}}) \le \frac{7}{2} r_{\{n,\,j\}}\Big\} = \mathcal{O}\Big(B\Big(\mathbf{x}_{\{n,\,j\}}, \frac{7}{2} r_{\{n,\,j\}}\Big)\Big). \tag{11.7}$$

Let

$$b_{\{n,\,j\}} = (\lambda_{\{n,\,j\}})^{-1} c \lim_{\varepsilon\to 0^+} \int_\varepsilon^{r_{\{n,\,j\}}} t^{2M} \Phi_t\Phi_t\Big(\chi_{\mathbf{T}_{\{n,\,j\}}} t^2(-\Delta)e^{t^2\Delta} f\Big)\frac{dt}{t}. \tag{11.8}$$

One can prove using the Dunkl transform that $b_{\{n,\,j\}} \in \mathcal{D}(\Delta^M)$ and

$$(-\Delta)^m b_{\{n,\,j\}} = (\lambda_{\{n,\,j\}})^{-1} c \lim_{\varepsilon\to 0^+} \int_\varepsilon^{r_{\{n,\,j\}}} t^{2M}(-\Delta)^m \Phi_t\Phi_t\Big(\chi_{\mathbf{T}_{\{n,\,j\}}} t^2(-\Delta)e^{t^2\Delta} f\Big)\frac{dt}{t} \tag{11.9}$$

for $m = 1, 2, \dots M$, because Δ^m is closed on $L^2(dw)$. Taking a sequence $\varepsilon_\ell \to 0^+$ instead of $\varepsilon \to 0^+$ if necessary, we may assume that the convergence in (11.5), (11.8), and (11.9) holds in $L^2(dw)$ and almost everywhere. By the same arguments,

$$\operatorname{supp} \Delta^m b_{\{n,\,j\}} \subset \mathcal{O}\Big(B\Big(\mathbf{x}_{\{n,\,j\}}, \frac{7}{2} r_{\{n,\,j\}}\Big)\Big). \tag{11.10}$$

Note also that $\Delta^m b_{\{n,\,j\}}(\mathbf{x}) \neq 0$ implies that there is $(t, \mathbf{y}) \in \hat{\Omega}_j$ such that $d(\mathbf{x}, \mathbf{y}) < t$. Then $\mathcal{O}(\mathbf{x}) \in B_X(\mathcal{O}(\mathbf{y}), t) \subset B_X(\mathcal{O}(\mathbf{y}), 4t) \subset \boldsymbol{\Omega}_j$. Hence,

$$\operatorname{supp} \Delta^m b_{\{n,\,j\}} \subset \Omega_j. \tag{11.11}$$

Clearly, $a_{\{n,j\}} = (-\Delta)^M b_{\{n,j\}} = \Delta^M b_{\{n,j\}}$, because M is an even integer.

Step 7. Size of $\Delta^m b_{\{n,j\}}$ for $m = 0, 1, \ldots, M-1$. Suppose that $(t, \mathbf{y})$ is such that $\chi_{\mathbf{T}_{\{n,j\}}}(t, \mathbf{y}) = 1$. Then $(t, \mathbf{y}) \in (\hat{\Omega}_{j+1})^c$, so $|t^2\Delta e^{t^2\Delta} f(\mathbf{y})| \leq 2^{j+1}$. Consequently,

$$\begin{aligned}
&|\Delta^m b_{\{n,j\}}(\mathbf{x})| \\
&= \frac{c}{\lambda_{\{n,j\}}} \left| \lim_{\ell\to\infty} \int_{\varepsilon_\ell}^{r_{\{n,j\}}} t^{2M-2m} (t^2(-\Delta))^m \Phi_t \Phi_t (\chi_{\mathbf{T}_{\{n,j\}}} t^2(-\Delta) e^{t^2\Delta} f)(\mathbf{x}) \frac{dt}{t} \right| \\
&= \frac{c}{\lambda_{\{n,j\}}} \left| \lim_{\ell\to\infty} \int_{\varepsilon_\ell}^{r_{\{n,j\}}} \int_{\mathbb{R}^N} t^{2M-2m} K_t^m(\mathbf{x}, \mathbf{y}) (\chi_{\mathbf{T}_{\{n,j\}}}(t, \mathbf{y}) t^2(-\Delta) e^{t^2\Delta} f(\mathbf{y})) dw(\mathbf{y}) \frac{dt}{t} \right|,
\end{aligned}$$

where $K_t^m(\mathbf{x}, \mathbf{y})$ is the integral kernel of the operator $(-t^2\Delta)^m \Phi_t \Phi_t$. Recall that

$$|K_t^m(\mathbf{x}, \mathbf{y})| \leq C w(B(\mathbf{x}, t))^{-1}$$

and

$$K_t^m(\mathbf{x}, \mathbf{y}) = 0 \quad \text{for } d(\mathbf{x}, \mathbf{y}) > t/2$$

(see (10.5) and Corollary 3.2). Thus,

$$\begin{aligned}
|\Delta^m b_{\{n,j\}}(\mathbf{x})| &\leq C(\lambda_{\{n,j\}})^{-1} 2^{j+1} \int_0^{r_{\{n,j\}}} \int_{\mathbb{R}^N} t^{2M-2m} |K_t^m(\mathbf{x}, \mathbf{y})| dw(\mathbf{y}) \frac{dt}{t} \\
&\leq C(\lambda_{\{n,j\}})^{-1} 2^{j+1} \int_0^{r_{\{n,j\}}} t^{2M-2m} \frac{dt}{t} \\
&= C(\lambda_{\{n,j\}})^{-1} 2^j (r_{\{n,j\}})^{2M-2m} \\
&= C w(Q_{\{n,j\}})^{-1} (r_{\{n,j\}})^{2M-2m}. \qquad (11.12)
\end{aligned}$$

Step 8. Key lemma. It remains to estimate

$$a_{\{n,j\}}(\mathbf{x}) = \frac{c}{\lambda_{\{n,j\}}} \lim_{\ell\to\infty} \int_{\varepsilon_\ell}^{\varepsilon_\ell^{-1}} \int_{\mathbb{R}^N} \Psi_t(\mathbf{x}, \mathbf{y}) \chi_{\mathbf{T}_{\{n,j\}}}(t, \mathbf{y}) (t^2(-\Delta) e^{t^2\Delta} f)(\mathbf{y}) \, dw(\mathbf{y}) \frac{dt}{t}.$$

Let $E_{\{n,j\}} = \bigcup_{i=1}^n Q_{\{i,j\}}$. Then

$$\begin{aligned}
\chi_{\mathbf{T}_{\{n,j\}}}(t, \mathbf{y}) &= \chi_{\hat{\Omega}_j}(t, \mathbf{y}) \chi_{(\hat{\Omega}_{j+1})^c}(t, \mathbf{y}) \chi_{\mathcal{R}(E_{\{n,j\}})}(t, \mathbf{y}) \chi_{(\mathcal{R}(E_{\{n-1,j\}}))^c}(t, \mathbf{y}) \\
&= \chi_1(t, \mathbf{y}) \chi_2(t, \mathbf{y}) \chi_3(t, \mathbf{y}) \chi_4(t, \mathbf{y}). \qquad (11.13)
\end{aligned}$$

The following lemma (see [37, Lemma 4.2]) plays a crucial role in the remaining part of the proof of Theorem 11.1.

Lemma 11.3 *For every $\mathbf{x} \in \Omega_j$ and every function χ_s, $s = 1, 2, 3, 4$, there are numbers $0 < \delta_s \leq \omega_s$ such that $\omega_s \leq 3\delta_s$ and either $\Psi_t(\mathbf{x}, \mathbf{y})\chi_s(t, \mathbf{y}) = 0$ for every $0 < t < \delta_s$ or $\Psi_t(\mathbf{x}, \mathbf{y})\chi_s(t, \mathbf{y}) = \Psi_t(\mathbf{x}, \mathbf{y})$ for every $0 < t < \delta_s$ and either $\Psi_t(\mathbf{x}, \mathbf{y})\chi_s(t, \mathbf{y}) = 0$ for every $t > \omega_s$ or $\Psi_t(\mathbf{x}, \mathbf{y})\chi_s(t, \mathbf{y}) = \Psi_t(\mathbf{x}, \mathbf{y})$ for every $t > \omega_s$.*

Proof For the reader's convenience, we include a short proof along the lines of [37]. Fix $t > 0$ and define $\chi_1'(\mathbf{y}) = \chi_{[4t,\infty)}(d(\mathbf{y}, \Omega_j^c))$, $\chi_2'(\mathbf{y}) = \chi_{(-\infty,4t)}(d(\mathbf{y}, \Omega_{j+1}^c))$, $\chi_3'(\mathbf{y}) = \chi_{(-\infty,2t)}(d(\mathbf{y}, E_{\{n,j\}}))$, $\chi_4'(\mathbf{y}) = \chi_{[2t,\infty)}(d(\mathbf{y}, E_{\{n-1,j\}}))$. Clearly, $\chi_s'(\mathbf{y}) = \chi_s(t,\mathbf{y})$ for $s = 1, 2, 3, 4$. If $d(\mathbf{x},\mathbf{y}) \geq t$, then $\Psi_t(\mathbf{x},\mathbf{y}) = \Psi_t(\mathbf{x},\mathbf{y})\chi_s(t,\mathbf{y}) = 0$. Therefore, to finish the proof, we assume that $d(\mathbf{x},\mathbf{y}) < t$. Then

$$-t + d(A,\mathbf{x}) < d(A,\mathbf{y}) < t + d(A,\mathbf{x}) \text{ for } A = \Omega_j^c, \Omega_{j+1}^c, E_{\{n,j\}}, E_{\{n-1,j\}}.$$

We are in a position to define consecutively δ_s and ω_s.

(1) If $d(\mathbf{x}, \Omega_j^c) < 3t$ or $d(\mathbf{x}, \Omega_j^c) > 5t$, then $\chi_1'(\mathbf{y}) = 0$ and $\chi_1'(\mathbf{y}) = 1$ respectively, so we put $\delta_1 = \frac{1}{5}d(\mathbf{x}, \Omega_j^c)$ and $\omega_1 = \frac{1}{3}d(\mathbf{x}, \Omega_j^c)$.
(2) If $d(\mathbf{x}, \Omega_{j+1}^c) < 3t$ or $d(\mathbf{x}, \Omega_{j+1}^c) > 5t$, then $\chi_2'(\mathbf{y}) = 1$ and $\chi_2'(\mathbf{y}) = 0$ respectively. Hence we set $\delta_2 = \frac{1}{5}d(\mathbf{x}, \Omega_{j+1}^c)$ and $\omega_2 = \frac{1}{3}d(\mathbf{x}, \Omega_{j+1}^c)$ if $d(\mathbf{x}, \Omega_{j+1}^c) \neq 0$, $\delta_2 = \omega_2 = \delta_1$ otherwise.
(3) If $d(\mathbf{x}, E_{\{n,j\}}) < t$ or $d(\mathbf{x}, E_{\{n,j\}}) > 3t$, then $\chi_3'(\mathbf{y}) = 1$ and $\chi_3'(\mathbf{y}) = 0$ respectively. Thus we put $\delta_3 = \frac{1}{3}d(\mathbf{x}, E_{\{n,j\}})$ and $\omega_3 = d(\mathbf{x}, E_{\{n,j\}})$ if $d(\mathbf{x}, E_{\{n,j\}}) \neq 0$, $\delta_3 = \omega_3 = \delta_1$ otherwise.
(4) If $d(\mathbf{x}, E_{\{n-1,j\}}) < t$ or $d(\mathbf{x}, E_{\{n-1,j\}}) > 3t$, then $\chi_4'(\mathbf{y}) = 0$ and $\chi_4'(\mathbf{y}) = 1$ respectively, so we put $\delta_4 = \frac{1}{3}d(\mathbf{x}, E_{\{n-1,j\}})$ and $\omega_4 = d(\mathbf{x}, E_{\{n-1,j\}})$ if $d(\mathbf{x}, E_{\{n-1,j\}}) \neq 0$, $\delta_4 = \omega_4 = \delta_1$ otherwise. □

We finish Step 8 by the remark (see Case 1 of the proof of the lemma) that if $t > \omega_1 > 0$ then

$$\Psi_t(\mathbf{x},\mathbf{y})\chi_{\mathbf{T}_{\{n,j\}}}(t,\mathbf{y}) = 0.$$

Step 9. Estimates for $a_{\{n,j\}}$. We shall prove that

$$|a_{\{n,j\}}(\mathbf{x})| \leq Cw(Q_{\{n,j\}})^{-1}. \tag{11.14}$$

Fix $\mathbf{x} \in \Omega_j$. Recall that $\operatorname{supp} a_{\{n,j\}} \subset \Omega_j$. Let $J = \bigcup_{s=1}^4 [\delta_s, \omega_s]$, $I = (0,\infty) \setminus J$, where δ_s, ω_s are from Lemma 11.3. Obviously, $I = (a_1, b_1) \cup \ldots \cup (a_m, b_m)$, where $m \leq 5$, $a_1 = 0$, $b_m = \infty$, and (a_l, b_l) are connected disjoint components of I. Clearly,

$$\begin{aligned} &\left|a_{\{n,j\}}(\mathbf{x})\right| \\ &\leq \sum_{s=1}^4 (\lambda_{\{n,j\}})^{-1} c \int_{\delta_s}^{\omega_s} \int_{\mathbb{R}^N} \left|\Psi_t(\mathbf{x},\mathbf{y})\chi_{\mathbf{T}_{\{n,j\}}}(t,\mathbf{y})(t^2(-\Delta)e^{t^2\Delta} f)(\mathbf{y})\right| dw(\mathbf{y})\frac{dt}{t} \\ &\quad + \sum_{s=1}^m (\lambda_{\{n,j\}})^{-1} c \left|\int_{a_s}^{b_s} \int_{\mathbb{R}^N} \Psi_t(\mathbf{x},\mathbf{y})\chi_{\mathbf{T}_{\{n,j\}}}(t,\mathbf{y})(t^2(-\Delta)e^{t^2\Delta} f)(\mathbf{y})\, dw(\mathbf{y})\frac{dt}{t}\right|, \end{aligned}$$

where some of the integrals are understood as improper ones (see Steps 6 and 7).

Consider the integral over $[\delta_s, \omega_s]$. Take $t \in [\delta_s, \omega_s]$ and $\mathbf{y}$ such that the integrant $\Big|\Psi_t(\mathbf{x},\mathbf{y})\chi_{\mathbf{T}_{\{n,j\}}}(t,\mathbf{y})(t^2(-\Delta)e^{t^2\Delta}f)(\mathbf{y})\Big| \neq 0$. Then $(t,\mathbf{y}) \notin \hat{\Omega}_{j+1}$. Thus, there is $\mathbf{x}'$ such that $d(\mathbf{y},\mathbf{x}') < 4t$ and $\mathbf{x}' \notin \Omega_{j+1}$, which means that $\mathcal{M}f(\mathbf{x}') \leq 2^{j+1}$. Consequently, $|t^2(-\Delta)e^{t^2\Delta}f(\mathbf{y})| \leq 2^{j+1}$. Hence,

$$
\begin{aligned}
&(\lambda_{\{n,j\}})^{-1}c\int_{\delta_s}^{\omega_s}\int_{\mathbb{R}^N}\Big|\Psi_t(\mathbf{x},\mathbf{y})\chi_{\mathbf{T}_{\{n,j\}}}(t,\mathbf{y})(t^2(-\Delta)e^{t^2\Delta}f)(\mathbf{y})\Big|\,dw(\mathbf{y})\frac{dt}{t}\\
&\quad\leq (\lambda_{\{n,j\}})^{-1}2^{j+1}c\int_{\delta_s}^{\omega_s}\int_{\mathbb{R}^N}\Big|\Psi_t(\mathbf{x},\mathbf{y})\Big|\,dw(\mathbf{y})\frac{dt}{t}\\
&\quad\leq C'(\lambda_{\{n,j\}})^{-1}2^{j+1}c\int_{\delta_s}^{\omega_s}\frac{dt}{t}\\
&\quad\leq Cw(Q_{\{n,j\}})^{-1}, \qquad (11.15)
\end{aligned}
$$

because $0 < \omega_s \leq 3\delta_s$.

We turn to estimate the integrals over $[a_s, b_s]$. Assume that

$$
(\lambda_{\{n,j\}})^{-1}c\left|\int_{a_s}^{b_s}\int_{\mathbb{R}^N}\Psi_t(\mathbf{x},\mathbf{y})\chi_{\mathbf{T}_{\{n,j\}}}(t,\mathbf{y})(t^2(-\Delta)e^{t^2\Delta}f)(\mathbf{y})\,dw(\mathbf{y})\frac{dt}{t}\right| > 0.
$$

By Lemma 11.3 for fixed $\mathbf{x} \in \Omega_j$ and $s \in \{1, 2, \ldots, m\}$, either $\chi_{\mathbf{T}_{\{n,j\}}}(t,\mathbf{y}) \equiv 0$ for all $t \in [a_s, b_s]$ and $d(\mathbf{x},\mathbf{y}) < t$ or $\chi_{\mathbf{T}_{\{n,j\}}}(t,\mathbf{y}) \equiv 1$ for all $t \in [a_s, b_s]$ and $d(\mathbf{x},\mathbf{y}) < t$. So the letter holds. This gives that for every $t \in [a_s, b_s]$ and $\mathbf{y}$ such that $d(\mathbf{x},\mathbf{y}) < t$, we have $(t,\mathbf{y}) \notin \hat{\Omega}_{j+1}$. So there is $\mathbf{x}'$ (which depends on $(t,\mathbf{y})$) such that $d(\mathbf{y},\mathbf{x}') < 4t$ and $\mathcal{M}f(\mathbf{x}') < 2^{j+1}$. Note that $d(\mathbf{x},\mathbf{x}') < d(\mathbf{x},\mathbf{y}) + d(\mathbf{y},\mathbf{x}') < 5t$. Consequently, for every $t \in [a_s, b_s]$, we have

$$
2^{j+1} \geq \mathcal{M}f(\mathbf{x}') \geq \sup_{d(\mathbf{x}',\mathbf{z})<5t}|\Xi_t f(\mathbf{z})| \geq |\Xi_t f(\mathbf{x})|.
$$

Finally, in our case

$$
\begin{aligned}
&(\lambda_{\{n,j\}})^{-1}c\left|\int_{a_s}^{b_s}\int_{\mathbb{R}^N}\Psi_t(\mathbf{x},\mathbf{y})\chi_{\mathbf{T}_{\{n,j\}}}(t,\mathbf{y})(t^2(-\Delta)e^{t^2\Delta}f)(\mathbf{y})\,dw(\mathbf{y})\frac{dt}{t}\right|\\
&\quad= (\lambda_{\{n,j\}})^{-1}c\left|\int_{a_s}^{b_s}\int_{\mathbb{R}^N}\Psi_t(\mathbf{x},\mathbf{y})(t^2(-\Delta)e^{t^2\Delta}f)(\mathbf{y})\,dw(\mathbf{y})\frac{dt}{t}\right|\\
&\quad= (\lambda_{\{n,j\}})^{-1}\left|\Xi_{a_s}f(\mathbf{x}) - \Xi_{b_s}f(\mathbf{x})\right|\\
&\quad\leq Cw(Q_{\{n,j\}})^{-1}, \qquad (11.16)
\end{aligned}
$$

where in the last equality we have used (11.1). The estimates (11.15) and (11.16) give (11.14). Recall that $w(Q_{\{n,j\}}) \sim w(B(\mathbf{x}_{\{n,j\}}, 7r_{\{n,j\}}/2))$. Hence, from (11.14), (11.12), (11.7), and (11.10) we deduce Step 5. The proof of Theorem 11.1 is complete. □

Having Lemma 10.9 together with Theorems 11.1 and 10.11 we are in a position to complete the proof of the atomic decomposition of $H^1_{\max,H}$ functions. This is stated in the theorem below.

Theorem 11.4 *There is a constant $C > 0$ such that every function $f \in H^1_{\max,H}$ can be written as*

$$f = \sum \lambda_j a_j,$$

where a_j are $(1, \infty, M)$-atoms, $\sum |\lambda_j| \le C\|\mathcal{M}_H f\|_{L^1(dw)}$.

Proof Recall that $\|f\|_{H^1_{\max,H}} \sim \|f\|_{H^1_{\max,P}}$ (see Theorem 10.11). Take a sequence g_n $(n = 0, 1, \ldots)$ as in the proof of Theorem 10.11. Then $g_n \in H^1_{\max,P} \cap L^2(dw)$, $f = \sum_{n=0}^{\infty} g_n$, and $\sum_{n=0}^{\infty} \|g_n\|_{H^1_{\max,P}} \le 3\|f\|_{H^1_{\max,P}}$. By Theorem 11.1 the functions g_n admit atomic decompositions into $(1, \infty, M)$-atoms, that is, $g_n = \sum_m \lambda_{n,m} a_{n,m}$, where the functions $a_{n,m}$ are $(1, \infty, M)$-atoms, $\lambda_{n,m} \in \mathbb{C}$, and $\sum_m |\lambda_{n,m}| \le C\|g_n\|_{\max,P}$. Finally,

$$f = \sum_{n,m} \lambda_{n,m} a_{n,m}$$

is the desired atomic decomposition. □

We are in a position to complete the proof of Theorem 2.1, by proving the following proposition, which is the converse to Proposition 7.6.

Proposition 11.5 *Assume that u_0 is $\mathcal{L}$-harmonic and satisfies $u_0^* \in L^1(dw)$. Then there is a system $\mathbf{u} = (u_0, u_1, \ldots, u_N) \in \mathcal{H}^1$ such that $\|\mathbf{u}\|_{\mathcal{H}^1} \le C\|u_0^*\|_{L^1(dw)}$.*

Proof By Theorem 7.5 we have $u_0(t, \mathbf{x}) = P_t f_0(\mathbf{x})$, where $f_0 \in L^1(dw)$. So $f_0 \in H^1_{\max,\, P}$ and $\|f_0\|_{H^1_{\max,\, P}} = \|u_0^*\|_{L^1(dw)}$. Using Theorem 11.4 and then (9.1) we obtain that $f_0 \in H^1_\Delta$ and $\|f_0\|_{H^1_\Delta} \le C\|u_0^*\|_{L^1(dw)}$. □

12 Inclusion $H^1_{(1,q,M)} \subset H^1_{\max,H}$

In this section we shall prove that, for every integer $M \ge 1$ and for every $1 < q \le \infty$, we have $H^1_{(1,q,M)} \subset H^1_{\max,H}$ and

$$\|f\|_{H^1_{\max,H}} \le C_{M,q}\|f\|_{H^1_{(1,q,M)}}.$$

It suffices to establish that there is a constant $C_{M,q} > 0$ such that

$$\|a\|_{H^1_{\max,H}} \le C_{M,q},$$

for every $(1, q, M)$-atom a. Since every $(1, q, M)$-atom is a $(1, q, 1)$-atom, we may reduce to $M = 1$.

Assume that a is a $(1, q, 1)$-atom associated with a set $\mathcal{B} = \bigcup_{\sigma \in G} B(\sigma(\mathbf{y}_0), r)$. Then there is a function $b \in \mathcal{D}(\Delta)$ such that $a = \Delta b$, $\operatorname{supp} \Delta^j b \subset \mathcal{B}$, $\|\Delta^j b\|_{L^q(dw)} \leq r^{2-2j} w(\mathcal{B})^{\frac{1}{q}-1}$, $j = 0, 1$. Set $u(t, \mathbf{x}) = e^{t^2\Delta} a(\mathbf{x})$. Observe that

$$\|u^*\|_{L^q(dw)} \leq C_q \|a\|_{L^q(dw)} \leq w(\mathcal{B})^{\frac{1}{q}-1}$$

(see (2.5) for the definition of u^*). Thus, by the doubling property of the measure $dw(\mathbf{x})\, d\mathbf{x}$ and the Hölder inequality,

$$\int_{d(\mathbf{x}, \mathbf{y}_0) \leq 8r} u^*(\mathbf{x})\, dw(\mathbf{x}) \leq C'_q.$$

We turn to estimate $u^*(\mathbf{x})$ on $d(\mathbf{x}, \mathbf{y}_0) > 8r$. Clearly,

$$\begin{aligned} u^*(\mathbf{x}) &\leq \sup_{0 < t < d(\mathbf{x}, \mathbf{y}_0)/4,\, d(\mathbf{x}', \mathbf{x}) < t} |e^{t^2\Delta} \Delta b(\mathbf{x}')| + \sup_{t > d(\mathbf{x}, \mathbf{y}_0)/4,\, d(\mathbf{x}', \mathbf{x}) < t} |e^{t^2\Delta} \Delta b(\mathbf{x}')| \\ &= J_1(\mathbf{x}) + J_2(\mathbf{x}). \end{aligned} \tag{12.1}$$

Recall that $\|b\|_{L^1(dw)} \leq r^2$ and note that

$$e^{t^2\Delta} \Delta = \Delta e^{t^2\Delta} = \frac{d}{ds} e^{s\Delta}\Big|_{s=t^2}.$$

To deal with J_1 we note that if $d(\mathbf{x}', \mathbf{x}) < t \leq d(\mathbf{x}, \mathbf{x}_0)/4$, $d(\mathbf{x}, \mathbf{y}_0) > 4r$, and $d(\mathbf{y}, \mathbf{y}_0) < r$, then $d(\mathbf{x}', \mathbf{y}) \sim d(\mathbf{x}, \mathbf{y}_0)$. So, using (4.3), we have

$$\left| \frac{d}{ds} h_s(\mathbf{x}', \mathbf{y}) \right|\Big|_{s=t^2} \leq \frac{C}{t^2 w(B(\mathbf{y}_0, d(\mathbf{y}_0, \mathbf{x})))} e^{-c' d(\mathbf{y}_0, \mathbf{x})^2/t^2}.$$

Hence,

$$J_1(\mathbf{x}) \lesssim w(B(\mathbf{y}_0, d(\mathbf{x}, \mathbf{y}_0)))^{-1} \frac{r^2}{d(\mathbf{x}, \mathbf{y}_0)^2}.$$

In order to estimate J_2, we observe from (4.3) that for $t > d(\mathbf{x}, \mathbf{y})$ and $d(\mathbf{y}, \mathbf{y}_0) < r < t$, we have

$$\left| \frac{d}{ds} h_s(\mathbf{x}', \mathbf{y}) \right|\Big|_{s=t^2} \leq \frac{C}{t^2 w(B(\mathbf{y}_0, d(\mathbf{y}_0, \mathbf{x})))}.$$

Consequently,

$$J_2(\mathbf{x}) \lesssim w(B(\mathbf{y}_0, d(\mathbf{x}, \mathbf{y}_0)))^{-1} \frac{r^2}{d(\mathbf{x}, \mathbf{y}_0)^2}.$$

Now

$$\int_{d(\mathbf{x},\mathbf{y}_0)>8r} u^*(\mathbf{x})\,dw(\mathbf{x}) \lesssim \sum_{j=3}^{\infty} \int_{2^j r<d(\mathbf{x},\mathbf{y}_0)\le 2^{j+1}r} \frac{r^2}{w(B(\mathbf{y}_0, d(\mathbf{x},\mathbf{y}_0)))d(\mathbf{x},\mathbf{y}_0)^2}\,dw(\mathbf{x})$$
$$\lesssim \sum_{j=3}^{\infty} 2^{-2j} = C.$$

13 Square Function Characterization

In this section we prove Theorem 2.3. More precisely we show that the atomic Hardy space $H^1_{(1,2,M)}$ coincides with the Hardy space defined by the square function (2.9) with $Q_t = t\sqrt{-\Delta}\,e^{-t\sqrt{-\Delta}}$. This is achieved by mimicking arguments in [24]. The proof for $Q_t = t^2(-\Delta)\,e^{t^2\Delta}$ is similar.

13.1 Tent Spaces T_2^p on Spaces of Homogeneous Type

The square function characterization of the Hardy space $H^1_{(1,2,M)}$ can be related with the so called tent space T_2^1. The tent spaces on Euclidean spaces were introduced in [9] and then extended on spaces of homogeneous type (see, e.g. [33]). For more details we refer the reader to [35].

For a measurable function $F(t,\mathbf{x})$ on $(0,\infty)\times\mathbb{R}^N$, let

$$\mathcal{A}F(\mathbf{x}) := \left(\int_0^\infty \int_{\|\mathbf{y}-\mathbf{x}\|<t} |F(t,\mathbf{y})|^2 \frac{dw(\mathbf{y})}{w(B(\mathbf{x},t))}\frac{dt}{t}\right)^{1/2}.$$

Definition 13.1 For $1 \le p < \infty$ the tent space T_2^p is defined to be

$$T_2^p = \{F : \|F\|_{T_2^p} := \|\mathcal{A}F\|_{L^p(dw)} < \infty\}.$$

Clearly, by the doubling property,

$$\|F\|^2_{T_2^2} = \|\mathcal{A}F\|^2_{L^2(dw)} \sim \int_0^\infty \int_{\mathbb{R}^N} |F(t,\mathbf{y})|^2 \frac{dw(\mathbf{y})dt}{t}. \tag{13.1}$$

Remark 13.2 Let Ψ be as in (10.6). Recall that, by (10.4) and (13.1), the linear operator π_Ψ is bounded from $L^2(dw\,dt/t)$ into $L^2(dw)$. Furthermore, by using the Dunkl transform, one can easily prove that, if $F(t,\mathbf{x}) = Q_t f(\mathbf{x})$ with $f \in L^2(dw)$, then

$$\|F\|_{T_2^2} = \|Sf\|_{L^2(dw)} \sim \|f\|_{L^2(dw)}$$

and $f = c_1\pi_\Psi(F)$.

The tent space T^1_2 on the space of homogeneous type admits the following atomic decomposition (see, e.g., [33]).

Definition 13.3 A measurable function $A(t,\mathbf{x})$ is a T^1_2-*atom* if there is a ball $B \subset \mathbb{R}^N$ such that

- $\operatorname{supp} A \subset \widehat{B}$
- $\iint_{(0,\infty)\times\mathbb{R}^N} |A(t,\mathbf{x})|^2\,dw(\mathbf{x})\,\frac{dt}{t} \le w(B)^{-1}$.

A function F belongs to T^1_2 if and only if there are sequences A_j of T^1_2-atoms and $\lambda_j \in \mathbb{C}$ such that

$$\sum_j \lambda_j A_j = F, \quad \sum_j |\lambda_j| \sim \|F\|_{T^1_2},$$

where the convergence is in T^1_2 norm and almost everywhere.

The Hölder inequality immediately gives that there is a constant $C > 0$ such that for every function $A(t,\mathbf{x})$ being a T^1_2-atom one has

$$\|A\|_{T^1_2} \le C.$$

Observe that for $f \in L^1(dw)$, the function $F(t,\mathbf{x}) = Q_t f(\mathbf{x})$ is well defined. Moreover, $\mathcal{A}F(\mathbf{x}) = Sf(\mathbf{x})$ and $\|Sf\|_{L^1(dw)} = \|F\|_{T^1_2}$.

Remark 13.4 According to the proof of atomic decomposition of T^1_2 presented in [33], the function $\lambda_j A_j$ can be taken of the form $\lambda_j A_j(t,\mathbf{x}) = \chi_{S_j}(t,\mathbf{x})F(t,\mathbf{x})$, where S_j are disjoint, $\mathbb{R}^{N+1}_+ = \bigcup S_j$, and S_j is contained in a tent $\widehat{B}_j$.

So, if $F \in T^1_2 \cap T^2_2$, then F can be decomposed into atoms such that $F(t,\mathbf{x}) = \sum_j \lambda_j A_j(t,\mathbf{x})$ and the convergence is in T^1_2, T^2_2, and pointwise.

Lemma 13.5 *The map* $(P_s F)(t,\mathbf{x}) = \int p_s(\mathbf{x},\mathbf{y})F(t,\mathbf{y})\,dw(\mathbf{y})$ *is bounded on* T^1_2. *Moreover, there is a constant* $C > 0$ *independent of* $s > 0$ *such that* $\|P_s F\|_{T^1_2} \le C\|F\|_{T^1_2}$.

Proof Let $F(t,\mathbf{x}) = \sum_j \lambda_j A_j(t,\mathbf{x})$ be an atomic decomposition of $F \in T^1_2$ as described above. Since $p_s(\mathbf{x},\mathbf{y}) \ge 0$, it suffices to prove that there is a constant $C > 0$ such that

$$\Big\| P_s|A| \Big\|_{T^1_2} \le C$$

for every atom A of T^1_2. To this end let $B = B(\mathbf{x}_0, r)$ be a ball associated with A. Obviously, $P_s|A|(t,\mathbf{x}') = 0$ for $t > r$.

Case 1 $s > r$. Then, by (5.4) and the Hölder inequality,

$$P_s|A|(t,\mathbf{x}') \le \frac{Cs}{s + d(\mathbf{x}_0,\mathbf{x}')}\frac{w(B(\mathbf{x}_0,r))^{1/2}}{w(B(\mathbf{x}_0, s + d(\mathbf{x}_0,\mathbf{x}')))}\left(\int |A(t,\mathbf{y})|^2 dw(\mathbf{y})\right)^{1/2}.$$

If $\|\mathbf{x} - \mathbf{x}'\| < t \le r$, then $s + d(\mathbf{x}_0, \mathbf{x}') \sim s + d(\mathbf{x}_0, \mathbf{x})$, because, by our assumption, $s > r$. Hence,

$$\begin{aligned} \Big\| P_s|A| \Big\|_{T_2^1} &\le C \int \frac{s}{s + d(\mathbf{x}_0, \mathbf{x})} \frac{w(B(\mathbf{x}_0, r))^{1/2}}{w(B(\mathbf{x}_0, s + d(\mathbf{x}_0, \mathbf{x})))} \\ &\quad \times \left(\int_0^r \int_{\|\mathbf{x}-\mathbf{x}'\|<t} \int |A(t, \mathbf{y})|^2 dw(\mathbf{y}) \frac{dw(\mathbf{x}')dt}{w(B(\mathbf{x}, t))t} \right)^{1/2} dw(\mathbf{x}) \\ &\le C \int \frac{s}{s + d(\mathbf{x}_0, \mathbf{x})} \frac{dw(\mathbf{x})}{w(B(\mathbf{x}_0, s + d(\mathbf{x}_0, \mathbf{x})))} \le C, \end{aligned}$$

where to get the second to last inequality we first integrated with respect to $dw(\mathbf{x}')$ and then used the definition of T_2^1-atom.

Case 2 $s \le r$. Recall that P_s is a contraction on $L^2(dw)$. Hence,

$$\begin{aligned} \|\mathcal{A}P_s|A|\|_{L^1(\mathcal{O}(B(\mathbf{x}_0,4r)),\, dw)} &\le Cw(B(\mathbf{x}_0, r))^{1/2} \|\mathcal{A}P_s|A|\|_{L^2(dw)} \\ &\le Cw(B(\mathbf{x}_0, r))^{1/2} \|P_s|A|\|_{T_2^2} \\ &\le Cw(B(\mathbf{x}_0, r))^{1/2} \||A|\|_{T_2^2} \le C. \end{aligned} \tag{13.2}$$

If $d(\mathbf{x}, \mathbf{x}_0) > 4r$, $\|\mathbf{x}' - \mathbf{x}\| < t < r$, and $\|\mathbf{x}_0 - \mathbf{y}\| < r$, then $s + d(\mathbf{x}', \mathbf{y}) \sim s + d(\mathbf{x}, \mathbf{x}_0)$. Now we proceed as in Case 1 to get the required bound on $\mathcal{O}(B(\mathbf{x}_0, 4r))^c$. □

Lemma 13.6 *The family P_s forms an approximation of the identity in T_2^1, that is,*

$$\lim_{s \to 0} \|P_s F - F\|_{T_2^1} = 0.$$

Proof According to Lemma 13.5, it suffices to establish that for every A being a T_2^1-atom, we have

$$\lim_{s \to 0} \|P_s A - A\|_{T_2^1} = \lim_{s \to 0} \|\mathcal{A}(P_s A - A)\|_{L^1(dw)} = 0. \tag{13.3}$$

Let A be such an atom and let $B = B(\mathbf{x}_0, r)$ be its associated ball. To prove (13.3) it suffices to consider $0 < s < r$.

If $d(\mathbf{x}, \mathbf{x}_0) > 4r$, $\|\mathbf{y} - \mathbf{x}_0\| < r$, and $\|\mathbf{x} - \mathbf{x}'\| < t < r$, then $s + d(\mathbf{x}', \mathbf{y}) \sim d(\mathbf{x}, \mathbf{x}_0)$, so

$$|P_s A(t, \mathbf{x}')| \le \frac{Cs}{s + d(\mathbf{x}_0, \mathbf{x})} \frac{w(B(\mathbf{x}_0, r))^{1/2}}{w(B(\mathbf{x}_0, s + d(\mathbf{x}_0, \mathbf{x})))} \left(\int |A(t, \mathbf{y})|^2 dw(\mathbf{y}) \right)^{1/2}.$$

Since $\operatorname{supp} A \cap \{(t, \mathbf{x}') : \|\mathbf{x}' - \mathbf{x}\| < t < r\} = \emptyset$, we have

$$|\mathcal{A}(P_s A - A)(\mathbf{x})| = |\mathcal{A}(P_s A)(\mathbf{x})| \le \frac{Cs}{s + d(\mathbf{x}_0, \mathbf{x})} \frac{1}{w(B(\mathbf{x}_0, s + d(\mathbf{x}_0, \mathbf{x})))}.$$

Hence,

$$\lim_{s\to 0}\int_{d(\mathbf{x},\mathbf{x}_0)>4r} |\mathcal{A}(P_s A - A)(\mathbf{x})|\, dw(\mathbf{x}) = 0.$$

We now turn to estimate $\|\mathcal{A}(P_s A - A)\|_{L^1(\mathcal{O}(B(\mathbf{x}_0,4r)),dw)}$. Observe that

$$\begin{aligned}|(P_s A - A)(t,\mathbf{x}')| \leq 2\mathcal{M}_P A(t,\mathbf{x}') \quad \text{and } \|\mathcal{M}_P A(t,\mathbf{x}')\|_{L^2(dw(\mathbf{x}'))}\\ \leq C\|A(t,\mathbf{x}')\|_{L^2(dw(\mathbf{x}'))}.\end{aligned}$$

Moreover, $\lim_{s\to 0}\|P_s A(t,\mathbf{x}') - A(t,\mathbf{x}')\|_{L^2(dw(\mathbf{x}'))} = 0$ for almost every $t > 0$. Therefore, applying the Hölder inequality and (13.1), we have

$$\begin{aligned}&\limsup_{s\to 0}\|\mathcal{A}(P_s A - A)\|_{L^1(\mathcal{O}(B(\mathbf{x}_0,4r)))}\\ &\quad\leq \limsup_{s\to 0} C w(B)^{1/2}\|\mathcal{A}(P_s A - A)\|_{L^2(\mathcal{O}(B(\mathbf{x}_0,4r)))}\\ &\quad\leq \limsup_{s\to 0} C w(B)^{1/2}\left(\int_0^r\int |P_s A(t,\mathbf{x}) - A(t,\mathbf{x})|^2\frac{dw(\mathbf{x})\,dt}{t}\right)^{1/2} = 0,\end{aligned}$$

where in the last equality we have used the Lebesgue dominated convergence theorem. □

13.2 Proof of Theorem 2.3

The inclusion $H^1_{(1,2,M)} \subset H^1_{\text{square}}$ will be established once we prove the following lemma.

Lemma 13.7 *For every positive integer M, there exists a constant $C_M > 0$ such that, for every $(1, 2, M)$-atom a, we have*

$$\|F(t,\mathbf{x})\|_{T^1_2} \leq C_M, \ \textit{where } F(t,\mathbf{x}) = Q_t a(\mathbf{x}).$$

Proof Let a be a $(1, 2, M)$–atom, $M \geq 1$, associated with a ball $B = B(\mathbf{x}_0, r)$. By definition $a = \Delta^M b$ with $\Delta^\ell b$ (for $\ell = 0, 1, \ldots, M$) satisfying relevant support and size conditions (see Definition 2.6). By the Hölder inequality,

$$\|Sa\|_{L^1(\mathcal{O}(8B))} \lesssim \|Sa\|_{L^2(\mathcal{O}(8B))} w(\mathcal{O}(8B))^{1/2} \lesssim 1.$$

If $d(\mathbf{x},\mathbf{x}_0) > 8r$ then choose $n \geq 3$ such that $2^n r \leq d(\mathbf{x},\mathbf{x}_0) < 2^{n+1} r$ and split the integral as below

$$\begin{aligned}Sa(\mathbf{x})^2 &= \int\int_{t>\|\mathbf{x}-\mathbf{y}\|} |Q_t a(\mathbf{y})|^2 w(B(\mathbf{y},t))^{-1}\, dw(\mathbf{y})\,\frac{dt}{t}\\ &= \int_0^{2^n r/4}\int_{t>\|\mathbf{x}-\mathbf{y}\|} + \int_{2^n r/4}^{\infty}\int_{t>\|\mathbf{x}-\mathbf{y}\|} = I_1 + I_2.\end{aligned}$$

Define $a_1 = \Delta^{M-1}b$. Then by the definition of the atom $\|a_1\|_{L^1(w)} \leq r^2$. Note that

$$Q_t(a) = Q_t(\Delta a_1) = (\Delta Q_t)(a_1) = t(\partial_t Q_t)^3(a_1).$$

Estimation for I_1. If $z \in \mathcal{O}(B)$ and $\|\mathbf{x}-\mathbf{y}\| < t \leq 2^n r/4$, then $2^n r \lesssim d(\mathbf{z},\mathbf{y})$. Therefore, thanks to (5.4) and (5.7) with $m = 3$, we have

$$\begin{aligned}|Q_t a(\mathbf{y})|^2 &= \left|\int t(\partial_t^3)(p_t(\mathbf{y},\mathbf{z}))a_1(\mathbf{z})\,dw(\mathbf{z})\right|^2\\ &\lesssim \left(\int d(\mathbf{z},\mathbf{y})^{-2}\frac{t}{t+d(\mathbf{z},\mathbf{y})}V(\mathbf{z},\mathbf{y},t+d(\mathbf{z},\mathbf{y}))^{-1}|a_1(\mathbf{z})|\,dw(\mathbf{z})\right)^2\\ &\lesssim (2^n r)^{-4}\frac{t^2}{(2^n r)^2}w(B(\mathbf{x}_0,2^n r))^{-2}\|a_1\|^2_{L^1(dw)}.\end{aligned}$$

Consequently,

$$\begin{aligned}I_1 &\lesssim \left(\int_0^{2^n r} t\,dt\right)w(B(\mathbf{x}_0,2^n r))^{-2}\|a_1\|^2_{L^1(dw)}(2^n r)^{-4}(2^n r)^{-2}\\ &\lesssim 2^{-4n}w(B(\mathbf{x}_0,2^n r))^{-2}.\end{aligned}$$

Estimation for I_2. In this case $t \geq 2^n r/4$, so thanks to (5.7) with $m = 3$ we have

$$\begin{aligned}|Q_t a(\mathbf{y})|^2 &= \left(\int t(\partial_t^3)(p_t(\mathbf{y},\mathbf{z}))a_1(\mathbf{z})\,dw(\mathbf{z})\right)^2\\ &\lesssim \left(\int t^{-2}\frac{t}{t+d(\mathbf{z},\mathbf{y})}V(\mathbf{z},\mathbf{y},t+d(\mathbf{z},\mathbf{y}))^{-1}|a_1(\mathbf{z})|\,dw(\mathbf{z})\right)^2\\ &\lesssim t^{-4}w(B(\mathbf{x}_0,2^n r))^{-2}\|a_1\|^2_{L^1(dw)}.\end{aligned}$$

Consequently,

$$I_2 \lesssim \left(\int_{2^n r/4}^{\infty} t^{-5}\,dt\right)w(B(\mathbf{x}_0,2^n r))^{-2}\|a_1\|^2_{L^1(dw)} \lesssim 2^{-4n}w(B(\mathbf{x}_0,2^n r))^{-2}.$$

Finally,

$$\|Sa\|_{L^1(\mathcal{O}(8B)^c)} \lesssim \sum_{n\geq 3}\int_{2^n r<d(\mathbf{x},\mathbf{x}_0)\leq 2^{n+1}r} 2^{-2n}w(B(\mathbf{x}_0,2^n r))^{-1}dw(\mathbf{x}) \lesssim 1.$$

□

The opposite inclusion $H^1_{\text{square}} \subset H^1_{(1,2,M)}$ is contained in the following proposition.

Proposition 13.8 *Let M be a positive integer. Assume that for $f \in L^1(dw)$ the function $F(t,\mathbf{x}) = Q_t f(\mathbf{x})$ belongs to T_2^1. Then there are $\lambda_j \in \mathbb{C}$ and a_j being $(1,2,M)$-atoms such that*

$$f = \sum_j \lambda_j a_j \quad \textit{and} \quad \sum_j |\lambda_j| \leq C\|F\|_{T_2^1}.$$

The constant C depends on M but it is independent of f.

Proof We start our proof under the additional assumption $f \in L^2(dw)$. Then $F(t,\mathbf{x}) = Q_t f(\mathbf{x}) \in T_2^1 \cap T_2^2$. The proof in this case is the same as that of [24, Theorem 4.1]. The only difference is to control support of functions $\Delta^s b_j$. For the convenience of the reader we provide its sketch.

Let $F = \sum_j \lambda_j A_j$ be a T_2^1 atomic decomposition of the function $Q_t f(\mathbf{x})$ as it is described in Remark 13.4. In particular, $\sum_j |\lambda_j| \leq C\|Sf\|_{L^1(dw)}$. Let $\Psi^{\{1\}}$ be a radial C^∞ function supported by $B(0,1/4)$ such that $\int_0^\infty \Psi_t Q_t \frac{dt}{t}$ forms a Calderón reproducing formula, where $\Psi = \Delta^{M+1}\Psi^{\{1\}}$. By (10.4), Remarks 13.2 and 13.4, we have

$$f = \pi_\Psi F = \sum_j \lambda_j \pi_\Psi A_j \tag{13.4}$$

and the series converges unconditionally in $L^2(dw)$. Then $\operatorname{supp} A_j \subset \widehat{B}_j$. Let $B_j = B(\mathbf{y}_j, r_j)$ be a ball associated with A_j. Recall that Δ^m is closed on $L^2(dw)$ for every positive integer m. Set $a_j = \pi_\Psi(A_j)$. We have $a_j = \Delta^M b_j$, where

$$b_j = \lim_{\varepsilon \to 0^+} \int_\varepsilon^{r_j/4} t^{2M}\left(t^2\Delta\Psi_t^{\{1\}} A_j\right)\frac{dt}{t}.$$

Clearly, $\operatorname{supp} b_j \subset \mathcal{O}(B(\mathbf{y}_j, 2r_j))$. The same argument as in the proof of Lemma 4.11. in [24] shows that for every $s = 0, 1, 2, \ldots, M$, the function

$$b_{j,s} = \Delta^s b_j = \lim_{\varepsilon \to 0^+} \int_\varepsilon^{r_j/4} t^{2M}\left(\Delta^s t^2\Delta\Psi_t^{\{1\}} A_j\right)\frac{dt}{t}$$

is supported by $\mathcal{O}(B(\mathbf{y}_j, 2r_j))$ and its $L^2(w)$-norm is bounded by $r^{2M-2s}w(B_j)^{-1/2}$. Thus a_j are proportional to $(1,2,M)$-atoms. In particular, $\|a_j\|_{L^1(dw)} \leq C$ and, consequently, the series (13.4) converges (absolutely) in $L^1(dw)$.

To remove the additional assumption $f \in L^2(dw)$ we recall that the Poisson kernel is an approximation of the identity in $L^1(dw)$ and in H^1_{square} (this is actually Lemma 13.6). Moreover, $P_t f \in L^2(dw)$ for $t > 0$ and $f \in L^1(dw)$. Thus, for $f \in H^1_{\text{square}}$, we can write $f = \sum_n g_n$, where the series converges in $L^1(dw)$, $g_n \in L^2(dw)$, and $\sum_n \|g_n\|_{H^1_{\text{square}}} \leq 3\|f\|_{H^1_{\text{square}}}$. Further we continue as in the proof of Theorem 11.20 to obtain the desired atomic decomposition. □

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