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Sign-changing solutions for elliptic problems with singular gradient terms and $L^1(\Omega)$ data

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Abstract. In this paper we deal with singular boundary value problems of the type

$$\begin{cases} -\operatorname{div}(a(x,u)\nabla u) + b(x)\dfrac{|\nabla u|^2}{|u|^\theta}\operatorname{sign}(u) = f(x), & \text{in } \Omega, \\ \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where Ω is a open bounded set of $\mathbb{R}^N$ with $N > 2$, $a(x,t)$ is a Carathéodory function with polynomial growth with respect to t, $b(x)$ is bounded and measurable, $\theta \in (0,1)$ and $f(x)$ belongs to $L^1(\Omega)$. The main concern is to consider sign-changing solutions outside the energy space $W_0^{1,2}(\Omega)$.

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Contents

1. Introduction

This paper deals with the study of existence of solutions for a class of quasilinear elliptic problems, with unbounded coefficients and a quadratic-singular lower order term satisfying a sign condition. A simple model problem is

$$\begin{cases} -\operatorname{div}\left(\left[a(x)+|u|^{1-\theta}\right]\nabla u\right)+\dfrac{(1-\theta)}{2}\dfrac{|\nabla u|^2}{|u|^\theta}\operatorname{sign}(u)=f(x), & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded, open set of $\mathbb{R}^N$ $(N>2)$, $\theta\in(0,1)$, a is a measurable function satisfying, for $\alpha,\beta>0$,

$$\alpha\le a(x)\le\beta,$$

and $f(x)$ belongs to $L^1(\Omega)$. The foremost feature of (1.1) is the lower order term that grows quadratically with respect to the gradient and that is singular where u vanishes. These types of nonlinearities have been considered at first in [1] and [2]. The main motivation for the study of this kind of singular equations comes from the Calculus of Variation; indeed, at least formally, (1.1) is the Euler–Lagrange equation of the following functional

$$J(v)=\frac{1}{2}\int_\Omega[a(x)+|v|^{1-\theta}]|\nabla v|^2-\int_\Omega f(x)v,$$

defined on a suitable subset of the energy space $W_0^{1,2}(\Omega)$. It is thus interesting to study the influence of the singular quadratic lower order term in a class of more general, non necessarily variational, problems.

After the already cited [1] and [2], a number of papers has been devoted to the study of positive solutions of problems like

$$\begin{cases} -\operatorname{div}(M(x)\nabla u)+g(u)|\nabla u|^2=f(x), & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where M is a bounded, uniformly elliptic matrix and the function $s\to g(s)$ is singular at the origin, see for instance [4] and [3]. The methods developed in the previous papers require non negative data $f(x)$, in order to conclude that $u>0$ in Ω. In this case the lower order term is well defined inside Ω, even if its singular character is not lost because of the Dirichlet boundary conditions. We refer also to [6] where problems with an unbounded divergence operator and a singular quadratic lower order term are studied, assuming again $f(x)\ge 0$.

The main problem in considering sign-changing data, and in turn possibly sign-changing solutions, is that the region $\{x\in\Omega\ :\ u(x)=0\}$ can be of positive measure and thus the meaning of the lower order term in (1.1) is not clear. This issue is addressed for the first time in [7] and [8], where the authors give a precise meaning to the singular l.o.t. even if the set where u vanishes has non-zero Lebesgue measure.

2. Main assumptions and statement of the existence result

The general problem for which we prove existence of a solution is

$$\begin{cases} -\operatorname{div}\left(a(x,u)\nabla u\right)+b(x)\dfrac{|\nabla u|^2}{|u|^\theta}\operatorname{sign}(u)=f(x), & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $a : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function that satisfies, for $\alpha,\ \beta,\ q > 0$,

$$\alpha(1 + |t|^q) \leq a(x, t) \leq \beta(1 + |t|^q), \tag{2.2}$$

$b : \Omega \to \mathbb{R}$ is a measurable functions such that, for $\zeta,\ \eta > 0$,

$$\zeta \leq b(x) \leq \eta, \tag{2.3}$$

$$0 < \theta < 1, \tag{2.4}$$

$$f \in L^1(\Omega). \tag{2.5}$$

With respect to the existing literature our main references are [6] and [7]. In [6] problem (2.1) is studied in the special case

$$a(x,t) := (1 + |t|^q) \quad \text{and} \quad f(x) \geq 0,\ f \not\equiv 0;$$

existence of positive solutions is proved in the same spirit of [4] and it is shown that the unbounded divergence operator can have a regularizing effect (as in [10]). In [7] the authors consider the case

$$a(x,t) = 1, \quad b(x) = 1, \quad f \in L^m(\Omega) \ \text{ with } \ m = \left(\frac{2^*}{\theta}\right)'; \tag{2.6}$$

they give a meaning to the possibly ill defined lower order term, combining a differentiation lemma for non-Lipschitz compositions with Sobolev functions and Stampacchia's Theorem; thanks to the regularity of the datum, they prove existence of $W_0^{1,2}(\Omega)$, possibly sign-changing, solutions.

In this framework, the contribution of this paper is that we consider sign-changing data with really poor summability, namely $f \in L^1(\Omega)$. This case is not included in [3,4,6], where f has to be positive, nor in the results of [7] and [8] where only solutions in the energy space $W_0^{1,2}(\Omega)$ are considered.
The main difficulties that we have to overcome are, on one side, that our solutions live in the larger space $W_0^{1,\rho}(\Omega)$ with $\rho < \frac{N}{N-1}$, and, on the other one, that the nonconstant bounded coefficient $b(x)$ makes the structure of the equation more difficult to handle. This facts force us to design a special test function that allows us, in some sense, to *desingularize* the problem (see Lemma 3.4 below).

As already said in the Introduction, the first step is to give a proper meaning to the singular lower order term in (2.1). Let us recall here Lemma 2.5 of [7].

Lemma 2.1. *Let $v \in W_0^{1,1}(\Omega)$. If $\frac{|\nabla v|^2}{|v|^\theta}$ is integrable in $\{v \neq 0\}$ then*

$$|v|^{1-\frac{\theta}{2}} \in W_0^{1,2}(\Omega).$$

Moreover

$$\nabla(|v|^{1-\frac{\theta}{2}})(x) = \begin{cases} \left(1 - \frac{\theta}{2}\right) \frac{\nabla|v(x)|}{|v(x)|^{\frac{\theta}{2}}} & a.e. \ in \ \{v \neq 0\} \\ 0 & a.e. \ in \ \{v \neq 0\}. \end{cases}$$

Roughly speaking we can say that, if the singular lower order term is integrable on the set $\{v \neq 0\}$, then the function $h(v) = |v|^{1-\frac{\theta}{2}}$ belongs to $W_0^{1,2}(\Omega)$; moreover its gradient is evaluated as if $s \to h(s)$ were Lipschitz in

$\{v \neq 0\}$ and using Stampacchia's Theorem otherwise. Hence with a slight abuse of notation and following Definition 2.2 of [7], we give the following meaning to the singular lower order term.

Definition 2.2. If the function $v \in W_0^{1,1}(\Omega)$ is such that $|v|^{1-\frac{\theta}{2}} \in W_0^{1,2}(\Omega)$, we define

$$\frac{|\nabla v|^2}{|v|^\theta}(x) := \frac{4}{(2-\theta)^2}\left|\nabla(|v|^{1-\frac{\theta}{2}})(x)\right|^2 = \begin{cases} \frac{|\nabla v(x)|^2}{|v(x)|^\theta} & \text{a.e. in } \{v \neq 0\} \\ 0 & \text{a.e. in } \{v = 0\}. \end{cases}$$

In line with the previous definition, we give the notion of weak solution for the singular problem (2.1).

Definition 2.3. We say that a function u is weak solution of (2.1) if

$$a(x,u)|\nabla u| \in L^\rho(\Omega),\ \forall\, \rho < \frac{N}{N-1}, \qquad |u|^{1-\frac{\theta}{2}} \in W_0^{1,2}(\Omega),$$

and

$$\int_\Omega a(x,u)\nabla u \nabla \varphi + \int_\Omega b(x)\frac{|\nabla u|^2}{|u|^\theta}\text{sign}(u)\varphi = \int_\Omega f(x)\varphi \quad \forall\, \varphi \in C_c^1(\Omega). \quad (2.7)$$

We state now our existence result.

Theorem 2.4. *Suppose that the assumptions* (2.2)*,* (2.3)*,* (2.4) *and* (2.5) *hold true. Then there exists* u *solution of* (2.7)*.*

We prove Theorem 2.4 by means of an approximation procedure. Recalling assumption (2.2), we set for any $n \in \mathbb{N}$

$$a_n(x,t) := \frac{a(x,t)}{1+\frac{1}{n}|t|^q} \quad \text{and} \quad f_n(x) := \frac{f(x)}{1+\frac{1}{n}|f(x)|}.$$

Let us consider the function $u_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, solution of the following approximated problem

$$\int_\Omega a_n(x,u_n)\nabla u_n \nabla \phi + \int_\Omega b(x)\frac{u_n}{(|u_n|+\frac{1}{n})^{\theta+1}}|\nabla u_n|^2\phi = \int_\Omega f_n(x)\phi \qquad (2.8)$$

for every test function ϕ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. For the existence of such a solution we refer to [5] and the references therein. We split the proof of Theorem 2.4 in the following steps:

- preliminary estimates and weak convergence of the sequence $\{u_n\}$;
- strong convergence of the sequence $\{T_k(u_n)\}$ in $W_0^{1,2}(\Omega)$ for every $k > 0$;
- equi-integrability of the lower order term in (2.8);
- passage to the limit.

Remark 2.5. It is well known in the literature that both the lower order term with sign condition and the polynomial growth in the divergence operator can improve the summability of the solution (see [5] and the references therein). Indeed our problem exhibits a superposition of two regularizing effects, in particular the solution given by Theorem 2.4 enjoys the following enhanced regularity properties:

- if $0 < q \le 1-\theta$, u belongs to $W_0^{1,r}(\Omega)$, with $r = \frac{N(2-\theta)}{N-\theta}$;
- if $1-\theta < q \le 1$, u belongs to $W_0^{1,r}(\Omega)$, for every $r < \frac{N(q+1)}{N+q-1}$;
- if $q > 1$, then u belongs to $W_0^{1,2}(\Omega)$.

In the first case, that includes the variational one, the better regularizing effect is due to the presence of the quadratic lower order term. In the remaining two cases, corresponding to higher values of q with respect to $1-\theta$, the enhanced regularity is given by the polynomial growth of the divergence operator. Notice that the interaction between the two regularizing effects is *continuous*, namely when $(1-\theta) = q$ it follows that $\frac{N(2-\theta)}{N-\theta} = \frac{N(q+1)}{N+q-1}$. Since it is possible to deduce these summability results through minor modifications of the proof of Theorem 1.1 in [6], we omit here the proof.

3. Proof of the main result

In the sequel we will use the following auxiliary functions, with $k \ge 0$ and $n \in \mathbb{N}$,

$$T_k(s) = \max\{\min\{k,s\},-k\}, \quad G_k(s) = s - T_k(s),$$
$$\gamma_n(t) := \int_0^t \frac{\tau}{(|\tau|+\frac{1}{n})^{\theta+1}}d\tau, \quad \gamma(t) := \frac{|t|^{1-\theta}}{1-\theta} \quad \text{and} \quad \varphi_\lambda(t) = te^{\lambda t^2}. \tag{3.1}$$

Note that $\lim_{n\to\infty}\gamma_n = \gamma$ and that, for any constants $c, d > 0$, the choice $\lambda = \frac{c^2}{4d^2}$ implies

$$d\varphi_\lambda' - c|\varphi_\lambda| \ge \frac{d}{2}. \tag{3.2}$$

Let us also recall the standard notation for the positive and negative part of a measurable function $w(x)$

$$w(x) = w^+(x) - w^-(x) \quad \text{where} \quad w^+(x) = w\chi_{w\ge 0} \quad w^-(x) = -w\chi_{w<0}.$$

In the next Lemma we give some preliminary estimates on the solution u_n and on the lower order term of problem (2.8).

Lemma 3.1. (Lemma 2.1 of [6]). *Under the same assumptions of Theorem 2.4, for every $n \in \mathbb{N}$ and for every $k \ge 0$, the function u_n, solution of (2.8), satisfies*

$$\int_{\{|u_n|>k\}} b(x)\frac{|u_n||\nabla u_n|^2}{(|u_n|+\frac{1}{n})^{\theta+1}} \le \int_{\{|u_n|>k\}} |f(x)|, \tag{3.3}$$

$$\alpha\int_\Omega (1+|u_n|^q)|\nabla T_k(u_n)|^2 \le k\int_\Omega |f(x)|, \tag{3.4}$$

$$\int_\Omega |u_n|^{q\rho}|\nabla u_n|^\rho + \int_\Omega |\nabla u_n|^\rho < C \quad \textit{for any} \quad \rho < \frac{N}{N-1}, \tag{3.5}$$

where $C = C(\mathcal{S}, f, \alpha, q, N, \Omega)$ is a positive constant that does not depend on u_n.

Proof of Lemma 3.1. Let us prove at first the estimates (3.3) and (3.4). Taking $\phi = \frac{T_j(G_k(u_n))}{j}$, with $j > 0$ and $k \geq 0$, as test function in (2.8) and dropping the energy term we get

$$\int_{\{|u_n|>k+j\}} b(x)\frac{|u_n||\nabla u_n|^2}{(|u_n|+\frac{1}{n})^{\theta+1}} \leq \int_\Omega b(x)\frac{|u_n||\nabla u_n|^2}{(|u_n|+\frac{1}{n})^{\theta+1}}\frac{|T_j(G_k(u_n))|}{j} \leq \int_{\{|u_n|>k\}} |f(x)|.$$

Letting j tend to 0 we deduce (3.3) by Fatou's Lemma. Using $T_k(u_n)$ as test function in (2.8) and dropping the positive lower order term, it follows that

$$\int_\Omega a_n(x,u_n)|\nabla T_k(u_n)|^2 \leq k\int_\Omega |f(x)|.$$

Hence, thanks to assumption (2.2), we deduce (3.4).

To prove (3.5), let us chose $[1-(1+|u_n|)^{1-\sigma}]\text{sgn}(u_n)$, with $\sigma > 1$, as a test function in (2.8). Dropping the positive lower order term and using (2.2), we obtain

$$\alpha(\sigma-1)\int_\Omega \frac{1+|u_n|^q}{(1+|u_n|)^\sigma}|\nabla u_n|^2 \leq \int_\Omega |f(x)|,$$

that is

$$\int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^{\sigma-q}} \leq \frac{1}{\alpha(\sigma-1)}\int_\Omega |f(x)|, \tag{3.6}$$

which holds for every $\sigma > 1$. Let us set $\rho = \frac{N(2+q-\sigma)}{N(q+1)-(\sigma+q)}$ and note that

$$\lim_{\sigma\to 1^+}\rho(\sigma) \nearrow \frac{N}{N-1} > 1.$$

Using Hölder inequality, estimate (3.6) and Sobolev inequality it follows that

$$\begin{aligned}\frac{\mathcal{S}^\rho}{(q+1)^\rho}\Big(\int_\Omega |u_n|^{(q+1)\rho^*}\Big)^{\frac{\rho}{\rho^*}} &\leq \int_\Omega u_n^{q\rho}|\nabla u_n|^\rho \\ &\leq \int_\Omega \frac{|\nabla u_n|^\rho}{(1+|u_n|)^{\frac{\rho(\sigma-q)}{2}}}(1+|u_n|)^{\frac{\rho(\sigma+q)}{2}} \leq \left(\frac{\|f\|_{L^1(\Omega)}}{\alpha(\sigma-1)}\right)^{\frac{\rho}{2}}\left(\int_\Omega (1+|u_n|)^{\frac{\rho(\sigma+q)}{2-\rho}}\right)^{\frac{2-\rho}{2}}.\end{aligned}$$

Noticing that $\frac{\rho}{\rho^*} > \frac{2-\rho}{2}$ and that the previous choice of ρ implies $(q+1)\rho^* = \frac{\rho(\sigma+q)}{2-\rho}$, we deduce, at first, an estimate for the sequence $|u_n|^{(q+1)\rho^*}$ and, secondly, that

$$\int_\Omega |u_n|^{q\rho}|\nabla u_n|^\rho \leq C_1 \quad \text{for any} \quad \rho < \frac{N}{N-1},$$

where $C_1 = C_1(f,\alpha,\mathcal{S},q,N,\Omega)$. As far as the estimate for the sequence $|\nabla u_n|$ is concerned, we have that for $k > 0$

$$\begin{aligned}\int_\Omega |\nabla u_n|^\rho &= \int_\Omega |\nabla T_k(u_n)|^\rho + \int_\Omega |\nabla G_k(u_n)|^\rho \\ &\leq |\Omega|^{\frac{2-\rho}{2}}\|\nabla T_k(u_n)\|^{\frac{\rho}{2}}_{L^2(\Omega)} + \frac{1}{k^{q\rho}}\int_\Omega |u_n|^{q\rho}|\nabla u_n|^\rho \leq C_2 \quad \rho < \frac{N}{N-1},\end{aligned}$$

where the last inequality comes from the previous parts of this proof and $C_2 = C_2(f, \alpha, \mathcal{S}, q, N, \Omega)$. □

Remark 3.2. Thanks to the estimates (3.5), we deduce that there exists $u \in W_0^{1,\rho}(\Omega)$, with $\rho < \frac{N}{N-1}$, such that, up to a not relabeled subsequence, $\{u_n\}$ weakly converges to u in $W_0^{1,\rho}(\Omega)$ and almost everywhere in Ω. Moreover estimate (3.4) implies that, for every $k > 0$, $T_k(u) \in W_0^{1,2}(\Omega)$ and that $\{T_k(u_n)\}$ weakly converges to $T_k(u)$ in $W_0^{1,2}(\Omega)$.

In the next result it is proved that the sequence $\{T_k(u_n)\}$ actually strongly converges to $T_k(u)$ in the energy space $W_0^{1,2}(\Omega)$. As we shall see such a strong convergence is crucial in order to pass to the limit in (2.8).

Remark 3.3. The difference with respect to Proposition 4.7 of [7] is that in that case it is available an estimate in $W_0^{1,2}(\Omega)$ for the sequence u_n and it is possible to take advantage of some cancellation phenomena due to the assumption $b(x) \equiv 1$.

Lemma 3.4. *Under the same assumptions of Theorem 2.4, for any $k > 0$, the sequence $\{T_k(u_n)\}$ strongly converges to $T_k(u)$ in $W_0^{1,2}(\Omega)$ and $\{\nabla u_n\}$ converges, up to a not relabeled subsequence, almost everywhere to ∇u, where u is given by Remark 3.2.*

Proof. We adapt to our case some ideas of [9]. Let us choose as a test function $e^{-\nu\gamma_n(T_s(u_n^-))}\varphi_\lambda(w_n^+)$, where $\varphi_\lambda, \gamma_n$ are defined in (3.1), $\nu = \frac{\eta}{\alpha}$, $s > 0$ and:

$$w_n = T_{2k}[G_l(u_n) + T_k(u_n) - T_k(u)] \qquad 0 < k < l.$$

We get:

$$\begin{aligned}
&\int_\Omega a_n(x, u_n)\nabla u_n \nabla w_n^+ \varphi_\lambda'(w_n^+) e^{-\nu\gamma_n(T_s(u_n^-))} \\
&\quad + \nu \int_\Omega a_n(x, u_n)|\nabla T_s(u_n^-)|^2 \frac{T_s(u_n^-)}{(|T_s(u_n^-)| + \frac{1}{n})^{\theta+1}} e^{-\nu\gamma_n(T_s(u_n^-))}\varphi_\lambda(w_n^+) \\
&\quad + \int_\Omega b(x)\frac{u_n|\nabla u_n|^2}{(|u_n| + \frac{1}{n})^{\theta+1}} e^{-\nu\gamma_n(T_s(u_n^-))}\varphi_\lambda(w_n^+) = \int_\Omega f_n(x) e^{-\nu\gamma_n(T_s(u_n^-))}\varphi_\lambda(w_n^+)
\end{aligned} \tag{3.7}$$

We stress that the second integral on the left hand side above is positive (recall that for us $u_n^- \geq 0$) and our aim is to use it in order to absorb the singular part of the third term. Notice that

$$\begin{aligned}
&\int_\Omega b(x)\frac{u_n|\nabla u_n|^2}{(|u_n| + \frac{1}{n})^{\theta+1}} e^{-\nu\gamma_n(T_s(u_n^-))}\varphi_\lambda(w_n^+) \\
&\quad \geq \int_{\{u_n \leq 0\}} b(x)\frac{u_n|\nabla u_n|^2}{(|u_n| + \frac{1}{n})^{\theta+1}} e^{-\nu\gamma_n(T_s(u_n^-))}\varphi_\lambda(w_n^+) \\
&\quad \geq \int_{\{-k \leq u_n \leq 0\}} b(x)\frac{u_n|\nabla u_n|^2}{(|u_n| + \frac{1}{n})^{\theta+1}} e^{-\nu\gamma_n(T_s(u_n^-))}\varphi_\lambda(w_n^+),
\end{aligned}$$

where the last inequality comes from the fact that $w_n^+ = 0$ where $u_n < -k$. Thus it follows that

$$\begin{aligned}
&\int_{-k\le u_n\le 0} b(x)\frac{u_n|\nabla u_n|^2}{(|u_n|+\frac{1}{n})^{\theta+1}}e^{-\nu\gamma_n(T_s(u_n^-))}\varphi_\lambda(w_n^+)\\
&\quad\ge -\eta\int_{-s\le u_n\le 0}\frac{|u_n|}{(|u_n|+\frac{1}{n})^{\theta+1}}|\nabla\, T_s(u_n)|^2e^{-\nu\gamma_n(T_s(u_n^-))}\varphi_\lambda(w_n^+)\\
&\qquad-\frac{\eta}{s^\theta}\int_\Omega|\nabla T_k(u_n)|^2e^{-\nu\gamma_n(T_s(u_n^-))}\varphi_\lambda(w_n^+). \qquad (3.8)
\end{aligned}$$

Being possible to absorb the second integral in the right hand side of (3.8) with the second integral of the left hand side of (3.7), we get

$$\begin{aligned}
&\int_\Omega a_n(x,u_n)\nabla u_n\nabla w_n^+\varphi_\lambda'(w_n^+)e^{-\nu\gamma_n(T_s(u_n^-))}\\
&\quad\le\int_\Omega f_n(x)\varphi_\lambda(w_n^+)e^{-\nu\gamma_n(T_s(u_n^-))}+\frac{\eta}{s^\theta}\int_\Omega|\nabla T_k(u_n)|^2e^{-\nu\gamma_n(T_s(u_n^-))}\varphi_\lambda(w_n^+). \qquad (3.9)
\end{aligned}$$

In order to rewrite (3.9) in a more convenient way, let us recall that $k < l$ and set $\mathcal{K} := 2k + l$. It is easy to show that

$$\nabla w_n^+\chi_{\{|u_n|\le k\}}=\nabla T_{2k}[T_k(u_n)-T_k(u)]^+\chi_{\{|u_n|\le k\}}=(\nabla T_k(u_n)-\nabla T_k(u))^+\chi_{\{|u_n|\le k\}} \qquad (3.10)$$

and that

$$\nabla T_\mathcal{K}(u_n)\nabla(G_l(u_n)-T_k(u))=\nabla T_\mathcal{K}(u_n)\nabla G_l(u_n)-\nabla T_\mathcal{K}(u_n)\nabla T_k(u)\ge-\nabla T_\mathcal{K}(u_n)\nabla T_k(u). \qquad (3.11)$$

Hence, using (3.10) and thanks to the fact that $\nabla w_n^+ = 0$ if $|u_n| \ge \mathcal{K}$, it results

$$\begin{aligned}
&\int_\Omega a_n(x,u_n)\nabla u_n\nabla w_n^+\varphi_\lambda'(w_n^+)e^{-\nu\gamma_n(T_s(u_n^-))}\\
&\quad=\int_\Omega a_n(x,u_n)\nabla T_k(u_n)\nabla(T_k(u_n)-T_k(u))^+\varphi_\lambda'(w_n^+)e^{-\nu\gamma_n(T_s(u_n^-))}\\
&\qquad+\int_{\{|u_n|>k\}}a_n(x,u_n)\nabla T_\mathcal{K}(u_n)\nabla T_{2k}[G_l(u_n)+k-T_k(u)]^+\varphi_\lambda'(w_n^+)e^{-\nu\gamma_n(T_s(u_n^-))}.
\end{aligned}$$

Moreover using (3.11) it is possible to rewrite the last term in the right hand side above as

$$\begin{aligned}
&\int_{\{|u_n|>k\}\cap\{0\le G_l(u_n)+T_k(u_n)-T_k(u)\le 2k\}}a_n(x,u_n)\nabla T_\mathcal{K}(u_n)\nabla(G_l(u_n)-T_k(u))\varphi_\lambda'(w_n^+)\\
&\quad\ge-\int_{\{|u_n|>k\}\cap\{0\le G_l(u_n)+T_k(u_n)-T_k(u)\le 2k\}}a_n(x,u_n)\nabla T_\mathcal{K}(u_n)\nabla T_k(u)\varphi_\lambda'(w_n^+).
\end{aligned}$$

Thanks to the weak convergence of $\nabla T_k(u_n)$ in $W_0^{1,2}(\Omega)$ for any $k > 0$ (see Remark 3.2), the a.e. convergence of u_n and the fact that $\nabla T_\mathcal{K}(u_n)$ is not zero where $|u_n| \le \mathcal{K}$, it follows that the right hand side above converges to

$$\int_{\{0\le G_l(u)\le 2k\}}a(x,u)\nabla T_\mathcal{K}(u)\nabla T_k(u)\varphi_\lambda'(T_{2k}(G_l(u))^+)e^{-\nu\gamma(T_s(u^-))}\chi_{\{u\ge k\}}=0.$$

Therefore we deduce that

$$
\begin{aligned}
&\int_\Omega a_n(x,u_n)\nabla u_n \nabla w_n^+ \varphi_\lambda'(w_n^+) e^{-\nu\gamma_n(T_s(u_n^-))} \\
&\quad = \int_\Omega a_n(x,u_n)\nabla T_k(u_n)\nabla (T_k(u_n)-T_k(u))^+ \varphi_\lambda'(w_n^+) e^{-\nu\gamma_n(T_s(u_n^-))} + \epsilon_n,
\end{aligned} \tag{3.12}
$$

where ϵ_n converges to zero as $n \to \infty$. Moreover

$$
\begin{aligned}
&\int_\Omega |\nabla T_k(u_n)|^2 e^{-\nu\gamma_n(T_s(u_n^-))} \varphi_\lambda(w_n^+) \\
&\quad \le 2\int_{\{|u_n|\le k\}} |\nabla (T_k(u_n)-T_k(u))^+|^2 e^{-\nu\gamma_n(T_s(u_n^-))}\varphi_\lambda(w_n^+) \\
&\qquad + 2\int_{\{|u_n|\le k\}} |\nabla T_k(u)|^2 e^{-\nu\gamma_n(T_s(u_n^-))}\varphi_\lambda(w_n^+)
\end{aligned} \tag{3.13}
$$

where the last inequality follows from the fact that

$$
\int_{\{|u_n|\le k\}} |\nabla (T_k(u_n)-T_k(u))^-|^2 e^{-\nu\gamma_n(T_s(u_n^-))}\varphi_\lambda(w_n^+) = 0,
$$

because, being $k > l$

$$
\begin{aligned}
\{|u_n|\le k\} \cap \{T_k(u_n)-T_k(u)<0\} \cap \{w_n \ge 0\} \subset \{u_n - T_k(u) < 0\} \\
\cap \{u_n - T_k(u) \ge 0\}
\end{aligned}
$$

Thus, taking into account (3.9), (3.12) and (3.13), we get:

$$
\begin{aligned}
&\int_\Omega a_n(x,u_n)\nabla T_k(u_n)\nabla (T_k(u_n)-T_k(u))^+ \varphi_\lambda'(w_n^+) e^{-\nu\gamma_n(T_s(u_n^-))} \\
&\qquad - 2\frac{\eta}{s^\theta}\int_\Omega |\nabla (T_k(u_n)-T_k(u))^+|^2 e^{-\nu\gamma_n(T_s(u_n^-))}\varphi_\lambda(w_n^+) \\
&\quad \le \int_\Omega f_n(x) e^{-\nu\gamma_n(T_s(u_n^-))}\varphi_\lambda(w_n^+) \\
&\qquad + 2\frac{\eta}{s^\theta}\int_\Omega |\nabla T_k(u)|^2 e^{-\nu\gamma_n(T_s(u_n^-))}\varphi_\lambda(w_n^+) + \epsilon_n.
\end{aligned}
$$

Adding to both sides the term

$$
-\int_\Omega a_n(x,u_n)\nabla T_k(u)\nabla (T_k(u_n)-T_k(u))^+ \varphi_\lambda'(w_n^+) e^{-\nu\gamma_n(T_s(u_n^-))}
$$

and choosing $\lambda = \frac{\eta^2}{\alpha^2 s^{2\theta}}$, in order to apply (3.2), we get:

$$
\begin{aligned}
&\frac{\alpha}{2}\int_\Omega |\nabla (T_k(u_n)-T_k(u))^+|^2 e^{-\nu\gamma_n(T_s(u_n^-))} \le \int_\Omega f_n(x) e^{-\nu\gamma_n(T_s(u_n^-))}\varphi_\lambda(w_n^+) \\
&\quad - \int_\Omega a_n(x,u_n)\nabla T_k(u)\nabla (T_k(u_n)-T_k(u))^+ \varphi_\lambda'(w_n^+) e^{-\nu\gamma_n(T_s(u_n^-))} \\
&\quad + 2\frac{\eta}{s^\theta}\int_\Omega |\nabla T_k(u)|^2 e^{-\nu\gamma_n(T_s(u_n^-))}\varphi_\lambda(w_n^+) + \epsilon_n.
\end{aligned}
$$

To take the limit with respect to n note at first that

$$\lim_{n\to\infty}\int_\Omega a_n(x,u_n)\nabla T_k(u)\nabla(T_k(u_n)-T_k(u))^+\varphi_\lambda'(w_n^+)e^{-\nu\gamma_n(T_s(u_n^-))}=0,$$

because $\nabla T_k(u_n)\rightharpoonup\nabla T_k(u)$ in $(L^2(\Omega))^N$, $u_n\to u$ almost everywhere and $\nabla T_k(u_n)$ is not zero where $|u|\le k$. Moreover, as the sequence $\{w_n^+\}$ converges almost everywhere and in the weak-$*$ topology of $L^\infty(\Omega)$ to $w^+=\big(T_{2k}(G_l(u))\big)^+$, we deduce that

$$\begin{aligned}&\lim_{n\to\infty}\left(\int_\Omega f_n(x)e^{-\nu\gamma_n(T_s(u_n^-))}\varphi_\lambda(w_n^+)+2\frac{\eta}{s^\theta}\int_\Omega|\nabla T_k(u)|^2e^{-\nu\gamma_n(T_s(u_n^-))}\varphi_\lambda(w_n^+)\right)\\&=\int_\Omega f_n(x)e^{-\nu\gamma(T_s(u^-))}\varphi_\lambda(w^+)+2\frac{\eta}{s^\theta}\int_\Omega|\nabla T_k(u)|^2e^{-\nu\gamma(T_s(u^-))}\varphi_\lambda(w^+)\\&\le\varphi_\lambda(2k)\int_{\{u>l\}}|f(x)|.\end{aligned}$$

Hence, being l a free parameter, we let it tend to infinity in order to obtain:

$$\lim_{n\to\infty}\int_\Omega|\nabla(T_k(u_n)-T_k(u))^+|^2=0.$$

Similarly, using $e^{-\nu\gamma_n(T_s(u_n^+))}\varphi_\lambda(w_n^-)$, it is possible to prove that

$$\lim_{n\to\infty}\int_\Omega|\nabla(T_k(u_n)-T_k(u))^-|^2=0.$$

Thus we have proved the strong convergence of the sequence $\{T_k(u_n)\}$ in $W_0^{1,2}(\Omega)$, from which we can infer that, up to a subsequence, $\nabla T_k(u_n)\to\nabla T_k(u)$ almost everywhere in Ω. This in turn implies that, up to a subsequence,

$$\nabla u_n\to\nabla u\quad a.e.\ \text{ in }\Omega.$$

□

Now we focus on the lower order term of (2.8), proving that is uniformly equi-integrable with respect to n.

Lemma 3.5. *Under the assumptions of Theorem 2.4 it follows that the sequence*

$$\left\{b(x)\frac{u_n}{(|u_n|+\frac{1}{n})^{\theta+1}}|\nabla u_n|^2\right\}$$

is uniformly equi-integrable in Ω.

Proof. Following Proposition 4.7 of [7], fix $\nu=\frac{\eta}{\alpha}$ and define for every $\delta>0$

$$v_n(t):=\begin{cases}1-e^{-\nu\gamma_n(\delta)} & \text{if } t>\delta\\ [1-e^{-\nu\gamma_n(t)}]\text{sgn}(t) & \text{if } |t|<\delta\\ e^{-\nu\gamma_n(\delta)}-1 & \text{if } t<-\delta.\end{cases}$$

Note that by construction $v_n \le \omega(\delta)$ where $\omega(\delta) \to 0$ as $\delta \to 0$. Thus using $v_n(u_n)$ as a test function in (2.8) it follows,

$$
\begin{aligned}
&\int_{\{|u_n|\le\delta\}} b(x)\frac{|u_n|}{(|u_n|+\frac{1}{n})^{\theta+1}}|\nabla u_n|^2 \\
&\quad\le \nu\int_{\{|u_n|\le\delta\}} a_n(x,u_n)|\nabla u_n|^2\frac{|u_n|}{(|u_n|+\frac{1}{n})^{\theta+1}}e^{-\nu\gamma_n(u_n)} \\
&\qquad+\int_{\{|u_n|\le\delta\}} b(x)\frac{|u_n|}{(|u_n|+\frac{1}{n})^{\theta+1}}|\nabla u_n|^2(1-e^{-\nu\gamma_n(u_n)}) \\
&\quad\le \int_\Omega |f(x)||v_n(u_n)|.
\end{aligned}
$$

Thus it holds true that

$$
\lim_{\delta\to 0}\sup_n \int_{\{|u_n|\le\delta\}} b(x)\frac{|u_n|}{(|u_n|+\frac{1}{n})^{\theta+1}}|\nabla u_n|^2 = 0 \tag{3.14}
$$

Then for every $\epsilon > 0$ thanks to (3.3), (3.14) and Lemma 3.4 there exist $\delta, k, \mu > 0$ such that, for every $E \subset \Omega$ with $|E| < \mu$, it holds true that

$$
\begin{aligned}
&\int_E b(x)\frac{|u_n|}{(|u_n|+\frac{1}{n})^{\theta+1}}|\nabla u_n|^2 = \int_{\{|u_n|<\delta\}} b(x)\frac{|u_n|}{(|u_n|+\frac{1}{n})^{\theta+1}}|\nabla u_n|^2 \\
&\quad+\int_{\{|u_n|>k\}} b(x)\frac{|u_n|}{(|u_n|+\frac{1}{n})^{\theta+1}}|\nabla u_n|^2 + \int_{\{\delta\le|u_n|\le k\}\cap E} b(x)\frac{|u_n|}{(|u_n|+\frac{1}{n})^{\theta+1}}|\nabla u_n|^2 \\
&\le \frac{\epsilon}{2}+\frac{\eta}{\delta^\theta}\int_E |\nabla T_k(u_n)|^2 \le \epsilon.
\end{aligned}
$$

□

Now we are in position to pass to the limit in equation (2.8) and prove our existence result.

Proof of Theorem 2.4. As already said, thanks to Lemma 3.1, there exists a function $u \in W_0^{1,\rho}(\Omega)$, with $\rho < \frac{N}{N-1}$, such that $u_n \rightharpoonup u$ weakly in $W_0^{1,\rho}(\Omega)$. Thanks to the the almost everywhere convergence of both u_n and ∇u_n, we can conclude also that

$$
\begin{aligned}
a(x,u)|\nabla u| &\in L^\rho(\Omega) \\
a_n(x,u_n)\nabla u_n \rightharpoonup a(x,u)\nabla u \quad &\text{in} \quad (L^\rho(\Omega))^N
\end{aligned}
\quad \text{with} \quad \rho < \frac{N}{N-1}. \tag{3.15}
$$

Thus, for every $\varphi \in C_c^1(\Omega)$, we can pass to the limit with respect to n in the first integral in the left hand side of (2.8). Moreover estimate (3.3) with $k = 0$, assumption (2.3), the almost everywhere convergence of both u_n and ∇u_n and Fatou Lemma imply that

$$
\int_{\{u\neq 0\}} \frac{|\nabla u(x)|^2}{|u(x)|^\theta} \le \frac{1}{\zeta}\int_\Omega |f(x)|.
$$

The latter information allows us to apply Lemma 2.1 and infer that

$$
|u|^{1-\frac{\theta}{2}} \in W_0^{1,2}(\Omega).
$$

Hence, in order to conclude the proof, we have to pass to the limit in the lower order term of (2.8). At first let us prove that for every $\delta > 0$

$$b(x)\frac{u_n(x)|\nabla u_n(x)|^2}{(|u_n(x)|+\frac{1}{n})^{\theta+1}}\chi_{\{|u_n(x)|>\delta\}} \to b(x)\frac{|\nabla u(x)|^2}{|u(x)|^\theta}\chi_{\{|u(x)|>\delta\}} \quad a.e. \text{ in } \Omega. \tag{3.16}$$

In order to show it, note that

$$\frac{u_n|\nabla u_n|^2}{(|u_n|+\frac{1}{n})^{\theta+1}}\chi_{\{|u_n|>\delta\}} = \frac{u_n|\nabla u_n|^2}{(|u_n|+\frac{1}{n})^{\theta+1}}\chi_{\{|u_n|>\delta\}\cap\{|u|>\delta\}}$$
$$+\frac{u_n|\nabla u_n|^2}{(|u_n|+\frac{1}{n})^{\theta+1}}\chi_{\{|u_n|>\delta\}\cap\{|u|<\delta\}} + \frac{u_n|\nabla u_n|^2}{(|u_n|+\frac{1}{n})^{\theta+1}}\chi_{\{|u_n|>\delta\}\cap\{|u|=\delta\}}.$$

The first term converges almost everywhere to $\frac{|\nabla u|^2}{|u|^\theta}\text{sign}(u)\chi_{\{|u|>\delta\}}$, the second converges to zero almost everywhere as n diverges and for the third we have that

$$\frac{u_n|\nabla u_n|^2}{(|u_n|+\frac{1}{n})^{\theta+1}}\chi_{\{|u_n|>\delta\}\cap\{|u|=\delta\}} \le \frac{|u_n||\nabla u_n|^2}{(|u_n|+\frac{1}{n})^{\theta+1}}\chi_{\{|u|=\delta\}} \le \frac{|\nabla u_n|^2}{|u_n|^\theta}\chi_{\{|u|=\delta\}},$$

that converges as well to zero almost everywhere thanks to Stampacchia's Theorem. Thus (3.16) is proved. Now, following the approach of [7], take $\phi \in C_0^1(\Omega)$ and $\epsilon > 0$. Thanks to Lemma 3.5 we can chose δ such that

$$\int_{\{|u_n|\le\delta\}} b(x)\frac{|u_n||\nabla u_n|^2}{(|u_n|+\frac{1}{n})^{\theta+1}} \le \frac{\epsilon}{2\|\phi\|_{L^\infty(\Omega)}}.$$

By means of Fatou's Lemma and recalling Definition 2.2, this also implies

$$\int_{\{|u|\le\delta\}} b(x)\frac{|\nabla u|^2}{|u|^\theta} = \int_{\{|u|\le\delta\}\cap\{u\neq 0\}} b(x)\frac{|\nabla u|^2}{|u|^\theta} \le \frac{\epsilon}{2\|\phi\|_{L^\infty(\Omega)}}.$$

Hence we have

$$\left|\int_\Omega \left(b(x)\frac{u_n|\nabla u_n|^2}{(|u_n|+\frac{1}{n})^{\theta+1}}\phi - b(x)\frac{|\nabla u|^2}{|u|^\theta}\text{sign}(u)\phi\right)\right|$$
$$\le \left|\int_{\{|u_n|>\delta\}} b(x)\frac{u_n|\nabla u_n|^2}{(|u_n|+\frac{1}{n})^{\theta+1}}\phi - \int_{\{|u|>\delta\}} b(x)\frac{|\nabla u|^2}{|u|^\theta}\text{sign}(u)\phi\right| + \epsilon. \tag{3.17}$$

Recalling the equi-integrability property of Lemma 3.5 and (3.16), we deduce that the term in the absolute value in the right hand side above goes to zero as n goes to infinity. Thus for every $\phi \in C_0^1(\Omega)$

$$\limsup_{n\to\infty}\left|\int_\Omega \left(b(x)\frac{u_n|\nabla u_n|^2}{(|u_n|+\frac{1}{n})^{\theta+1}}\phi - b(x)\frac{|\nabla u|^2}{|u|^\theta}\text{sign}(u)\phi\right)\right| \le \epsilon.$$

for any arbitrary $\epsilon > 0$. Hence we can pass to the limit with respect to n in (2.8) and the Theorem is proved. $\square$

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