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p-Laplacian problems involving critical Hardy–Sobolev exponents

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Abstract. We prove existence, multiplicity, and bifurcation results for p-Laplacian problems involving critical Hardy–Sobolev exponents. Our results are mainly for the case $\lambda \geq \lambda_1$ and extend results in the literature for $0 < \lambda < \lambda_1$. In the absence of a direct sum decomposition, we use critical point theorems based on a cohomological index and a related pseudo-index.

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1. Introduction

Consider the critical p-Laplacian problem

$$\begin{cases} -\Delta_p\, u = \lambda\, |u|^{p-2}\, u + \frac{|u|^{p^*(s)-2}}{|x|^s}\, u & \text{in } \Omega\\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ containing the origin, $1 < p < N$, $\lambda > 0$ is a parameter, $0 < s < p$, and $p^*(s) = (N - s)\, p/(N - p)$ is the critical Hardy–Sobolev exponent. Ghoussoub and Yuan [6] showed, among other things, that this problem has a positive solution when $N \geq p^2$ and $0 < \lambda < \lambda_1$, where $\lambda_1 > 0$ is the first eigenvalue of the eigenvalue problem

$$\begin{cases} -\Delta_p\, u = \lambda\, |u|^{p-2}\, u & \text{in } \Omega\\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

In the present paper we mainly consider the case $\lambda \geq \lambda_1$. Our existence results are the following.

Theorem 1.1. *If $N \geq p^2$ and $0 < \lambda < \lambda_1$, then problem (1.1) has a positive ground state solution.*

Theorem 1.2. *If $N \geq p^2$ and $\lambda > \lambda_1$ is not an eigenvalue of problem (1.2), then problem (1.1) has a nontrivial solution.*

Theorem 1.3. *If*

$$(N - p^2)(N - s) > (p - s)\, p \tag{1.3}$$

and $\lambda \geq \lambda_1$, then problem (1.1) has a nontrivial solution.

Remark 1.4. We note that (1.3) implies $N > p^2$.

Remark 1.5. In the nonsingular case $s = 0$, related results can be found in Degiovanni and Lancelotti [4] for the p-Laplacian and in Mosconi et al. [7] for the fractional p-Laplacian.

Weak solutions of problem (1.1) coincide with critical points of the C^1-functional

$$I_\lambda(u) = \int_\Omega \left[\frac{1}{p}\left(|\nabla u|^p - \lambda\, |u|^p\right) - \frac{1}{p^*(s)}\, \frac{|u|^{p^*(s)}}{|x|^s}\right] dx, \quad u \in W_0^{1,p}(\Omega).$$

Recall that I_λ satisfies the Palais-Smale compactness condition at the level $c \in \mathbb{R}$, or the $(\mathrm{PS})_c$ condition for short, if every sequence $(u_j) \subset W_0^{1,p}(\Omega)$ such that $I_\lambda(u_j) \to c$ and $I'_\lambda(u_j) \to 0$ has a convergent subsequence. Let

$$\mu_s = \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^p\, dx}{\left(\displaystyle\int_\Omega \frac{|u|^{p^*(s)}}{|x|^s}\, dx\right)^{p/p^*(s)}} \tag{1.4}$$

be the best constant in the Hardy–Sobolev inequality, which is independent of Ω (see [6, Theorem 3.1.(1)]). It was shown in [6, Theorem 4.1.(2)] that I_λ satisfies the $(\mathrm{PS})_c$ condition for all

$$c < \frac{p - s}{(N - s)\, p}\, \mu_s^{(N-s)/(p-s)}$$

for any $\lambda > 0$. We will prove Theorems 1.1 – 1.3 by constructing suitable minimax levels below this threshold for compactness. When $0 < \lambda < \lambda_1$, we will show that the infimum of I_λ on the Nehari manifold is below this level. When $\lambda \geq \lambda_1$, I_λ no longer has the mountain pass geometry and a linking type argument is needed. However, the classical linking theorem cannot be used here since the nonlinear operator $-\Delta_p$ does not have linear eigenspaces. We will use a nonstandard linking construction based on sublevel sets as in Perera and Szulkin [11] (see also Perera et al. [9, Proposition 3.23]). Moreover, the standard sequence of eigenvalues of $-\Delta_p$ based on the genus does not give enough information about the structure of the sublevel sets to carry out this construction. Therefore, we will use a different sequence of eigenvalues introduced in Perera [8] that is based on a cohomological index.

For $1 < p < \infty$, eigenvalues of problem (1.2) coincide with critical values of the functional

$$\Psi(u) = \frac{1}{\displaystyle\int_\Omega |u|^p\, dx}, \quad u \in \mathcal{M} = \left\{u \in W_0^{1,p}(\Omega) : \int_\Omega |\nabla u|^p\, dx = 1\right\}.$$

Let $\mathcal{F}$ denote the class of symmetric subsets of $\mathcal{M}$, let $i(M)$ denote the $\mathbb{Z}_2$-cohomological index of $M \in \mathcal{F}$ (see Sect. 2.1), and set

$$\lambda_k := \inf_{M \in \mathcal{F},\, i(M) \geq k}\ \sup_{u \in M}\ \Psi(u), \quad k \in \mathbb{N}.$$

Then $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to \infty$ is a sequence of eigenvalues of (1.2) and

$$\lambda_k < \lambda_{k+1} \implies i(\Psi^{\lambda_k}) = i(\mathcal{M} \setminus \Psi_{\lambda_{k+1}}) = k, \tag{1.5}$$

where $\Psi^a = \{u \in \mathcal{M} : \Psi(u) \leq a\}$ and $\Psi_a = \{u \in \mathcal{M} : \Psi(u) \geq a\}$ for $a \in \mathbb{R}$ (see Perera et al. [9, Propositions 3.52 and 3.53]). We also prove the following bifurcation and multiplicity results for problem (1.1) that do not require $N \geq p^2$. Set

$$V_s(\Omega) = \int_\Omega |x|^{(N-p)\, s/(p-s)}\, dx,$$

and note that

$$\int_\Omega |u|^p\, dx \leq V_s(\Omega)^{(p-s)/(N-s)} \left(\int_\Omega \frac{|u|^{p^*(s)}}{|x|^s}\, dx\right)^{p/p^*(s)} \quad \forall u \in W_0^{1,p}(\Omega) \tag{1.6}$$

by the Hölder inequality.

Theorem 1.6. *If*

$$\lambda_1 - \frac{\mu_s}{V_s(\Omega)^{(p-s)/(N-s)}} < \lambda < \lambda_1,$$

then problem (1.1) *has a pair of nontrivial solutions* $\pm\, u^\lambda$ *such that*

$$\int_\Omega |\nabla u^\lambda|^p\, dx \leq \lambda_1\, (\lambda_1 - \lambda)^{(N-p)/(p-s)}\, V_s(\Omega).$$

Theorem 1.7. *If* $\lambda_k \leq \lambda < \lambda_{k+1} = \cdots = \lambda_{k+m} < \lambda_{k+m+1}$ *for some* $k, m \in \mathbb{N}$ *and*

$$\lambda > \lambda_{k+1} - \frac{\mu_s}{V_s(\Omega)^{(p-s)/(N-s)}}, \tag{1.7}$$

then problem (1.1) *has* m *distinct pairs of nontrivial solutions* $\pm\, u_j^\lambda$, $j = 1, \ldots, m$ *such that*

$$\int_\Omega |\nabla u_j^\lambda|^p\, dx \leq \lambda_{k+1}\, (\lambda_{k+1} - \lambda)^{(N-p)/(p-s)}\, V_s(\Omega). \tag{1.8}$$

In particular, we have the following existence result that is new when $N < p^2$.

Corollary 1.8. *If*

$$\lambda_k - \frac{\mu_s}{V_s(\Omega)^{(p-s)/(N-s)}} < \lambda < \lambda_k$$

for some $k \in \mathbb{N}$, *then problem* (1.1) *has a nontrivial solution.*

Remark 1.9. We note that $\lambda_1 \geq \mu_s / V_s(\Omega)^{(p-s)/(N-s)}$. Indeed, let $\varphi_1 > 0$ be an eigenfunction associated with λ_1. Then

$$\lambda_1 = \frac{\displaystyle\int_\Omega |\nabla \varphi_1|^p \, dx}{\displaystyle\int_\Omega \varphi_1^p \, dx} \geq \frac{\mu_s \left(\displaystyle\int_\Omega \frac{\varphi_1^{p^*(s)}}{|x|^s} \, dx\right)^{p/p^*(s)}}{\displaystyle\int_\Omega \varphi_1^p \, dx} \geq \frac{\mu_s}{V_s(\Omega)^{(p-s)/(N-s)}}$$

by (1.4) and (1.6).

Remark 1.10. Since $V_0(\Omega)$ is the volume of Ω, in the nonsingular case $s = 0$, Theorems 1.6 & 1.7 and Corollary 1.8 reduce to Perera et al. [10, Theorem 1.1 and Corollary 1.2], respectively.

2. Preliminaries

2.1. Cohomological index

The $\mathbb{Z}_2$-cohomological index of Fadell and Rabinowitz [5] is defined as follows. Let W be a Banach space and let $\mathcal{A}$ denote the class of symmetric subsets of $W \setminus \{0\}$. For $A \in \mathcal{A}$, let $\overline{A} = A/\mathbb{Z}_2$ be the quotient space of A with each u and $-u$ identified, let $f : \overline{A} \to \mathbb{R}\mathrm{P}^\infty$ be the classifying map of $\overline{A}$, and let $f^* : H^*(\mathbb{R}\mathrm{P}^\infty) \to H^*(\overline{A})$ be the induced homomorphism of the Alexander-Spanier cohomology rings. The cohomological index of A is defined by

$$i(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ \sup \left\{m \geq 1 : f^*(\omega^{m-1}) \neq 0\right\} & \text{if } A \neq \emptyset, \end{cases}$$

where $\omega \in H^1(\mathbb{R}\mathrm{P}^\infty)$ is the generator of the polynomial ring $H^*(\mathbb{R}\mathrm{P}^\infty) = \mathbb{Z}_2[\omega]$.

Example 2.1. The classifying map of the unit sphere S^{m-1} in $\mathbb{R}^m$, $m \geq 1$ is the inclusion $\mathbb{R}\mathrm{P}^{m-1} \subset \mathbb{R}\mathrm{P}^\infty$, which induces isomorphisms on the cohomology groups H^q for $q \leq m-1$, so $i(S^{m-1}) = m$.

The following proposition summarizes the basic properties of this index.

Proposition 2.2. (Fadell–Rabinowitz [5]) *The index $i : \mathcal{A} \to \mathbb{N} \cup \{0, \infty\}$ has the following properties:*

- (i_1) *Definiteness: $i(A) = 0$ if and only if $A = \emptyset$.*
- (i_2) *Monotonicity: If there is an odd continuous map from A to B (in particular, if $A \subset B$), then $i(A) \leq i(B)$. Thus, equality holds when the map is an odd homeomorphism.*
- (i_3) *Dimension: $i(A) \leq \dim W$.*
- (i_4) *Continuity: If A is closed, then there is a closed neighborhood $N \in \mathcal{A}$ of A such that $i(N) = i(A)$. When A is compact, N may be chosen to be a δ-neighborhood $N_\delta(A) = \{u \in W : dist(u, A) \leq \delta\}$.*
- (i_5) *Subadditivity: If A and B are closed, then $i(A \cup B) \leq i(A) + i(B)$.*

(i_6) *Stability: If SA is the suspension of $A \neq \emptyset$, obtained as the quotient space of $A \times [-1, 1]$ with $A \times \{1\}$ and $A \times \{-1\}$ collapsed to different points, then $i(SA) = i(A) + 1$.*

(i_7) *Piercing property: If A, A_0 and A_1 are closed, and $\varphi : A \times [0, 1] \to A_0 \cup A_1$ is a continuous map such that $\varphi(-u, t) = -\varphi(u, t)$ for all $(u, t) \in A \times [0, 1]$, $\varphi(A \times [0, 1])$ is closed, $\varphi(A \times \{0\}) \subset A_0$ and $\varphi(A \times \{1\}) \subset A_1$, then $i(\varphi(A \times [0, 1]) \cap A_0 \cap A_1) \geq i(A)$.*

(i_8) *Neighborhood of zero: If U is a bounded closed symmetric neighborhood of the origin, then $i(\partial U) = \dim W$.*

2.2. Abstract critical point theorems

We will prove Theorems 1.2 and 1.3 using the following abstract critical point theorem proved in Yang and Perera [13], which generalizes the well-known linking theorem of Rabinowitz [12].

Theorem 2.3. *Let I be a C^1-functional defined on a Banach space W, and let A_0 and B_0 be disjoint nonempty closed symmetric subsets of the unit sphere $S = \{u \in W : \|u\| = 1\}$ such that*

$$i(A_0) = i(S \setminus B_0) < \infty.$$

Assume that there exist $R > r > 0$ and $v \in S \setminus A_0$ such that

$$\sup I(A) \leq \inf I(B), \qquad \sup I(X) < \infty,$$

where

$$\begin{aligned} A &= \{tu : u \in A_0,\ 0 \leq t \leq R\} \cup \{R\,\pi((1-t)\,u + tv) : u \in A_0,\ 0 \leq t \leq 1\}, \\ B &= \{ru : u \in B_0\}, \\ X &= \{tu : u \in A,\ \|u\| = R,\ 0 \leq t \leq 1\}, \end{aligned}$$

and $\pi : W \setminus \{0\} \to S$, $u \mapsto u/\|u\|$ is the radial projection onto S. Let $\Gamma = \{\gamma \in C(X, W) : \gamma(X)$ is closed and $\gamma|_A = id_A\}$, and set

$$c := \inf_{\gamma \in \Gamma} \sup_{u \in \gamma(X)} I(u).$$

Then

$$\inf I(B) \leq c \leq \sup I(X), \tag{2.1}$$

in particular, c is finite. If, in addition, I satisfies the $(\mathrm{PS})_c$ condition, then c is a critical value of I.

Remark 2.4. The linking construction used in the proof of Theorem 2.3 in [13] has also been used in Perera and Szulkin [11] to obtain nontrivial solutions of p-Laplacian problems with nonlinearities that cross an eigenvalue. A similar construction based on the notion of cohomological linking was given in Degiovanni and Lancelotti [3]. See also Perera et al. [9, Proposition 3.23].

Now let I be an even C^1-functional defined on a Banach space W, and let $\mathcal{A}^*$ denote the class of symmetric subsets of W. Let $r > 0$, let $S_r = \{u \in W : \|u\| = r\}$, let $0 < b \leq +\infty$, and let Γ denote the group of odd homeomorphisms of W that are the identity outside $I^{-1}(0, b)$. The pseudo-index of $M \in \mathcal{A}^*$ related to i, S_r, and Γ is defined by

$$i^*(M) = \min_{\gamma \in \Gamma} i(\gamma(M) \cap S_r)$$

(see Benci [2]). We will prove Theorems 1.6 and 1.7 using the following critical point theorem proved in Yang and Perera [13], which generalizes Bartolo et al. [1, Theorem 2.4].

Theorem 2.5. *Let A_0 and B_0 be symmetric subsets of S such that A_0 is compact, B_0 is closed, and*

$$i(A_0) \geq k + m, \qquad i(S \setminus B_0) \leq k$$

for some integers $k \geq 0$ and $m \geq 1$. Assume that there exists $R > r$ such that

$$\sup I(A) \leq 0 < \inf I(B), \qquad \sup I(X) < b,$$

where $A = \{Ru : u \in A_0\}$, $B = \{ru : u \in B_0\}$, and $X = \{tu : u \in A,\ 0 \leq t \leq 1\}$. For $j = k+1, \dots, k+m$, let

$$\mathcal{A}_j^* = \{M \in \mathcal{A}^* : M \text{ is compact and } i^*(M) \geq j\},$$

and set

$$c_j^* := \inf_{M \in \mathcal{A}_j^*} \max_{u \in M} I(u).$$

Then

$$\inf I(B) \leq c_{k+1}^* \leq \cdots \leq c_{k+m}^* \leq \sup I(X),$$

in particular, $0 < c_j^* < b$. If, in addition, I satisfies the $(\mathrm{PS})_c$ condition for all $c \in (0, b)$, then each c_j^* is a critical value of I and there are m distinct pairs of associated critical points.

Remark 2.6. Constructions similar to the one used in the proof of Theorem 2.5 in [13] have also been used in Fadell and Rabinowitz [5] to prove bifurcation results for Hamiltonian systems and in Perera and Szulkin [11] to prove multiplicity results for p-Laplacian problems. See also Perera et al. [9, Proposition 3.44].

2.3. Some estimates

It was shown in [6, Theorem 3.1.(2)] that the infimum in (1.4) is attained by the family of functions

$$u_\varepsilon(x) = \frac{C_{N,p,s}\, \varepsilon^{(N-p)/(p-s)\,p}}{\left[\varepsilon + |x|^{(p-s)/(p-1)}\right]^{(N-p)/(p-s)}}, \quad \varepsilon > 0$$

when $\Omega = \mathbb{R}^N$, where $C_{N,p,s} > 0$ is chosen so that

$$\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^p\, dx = \int_{\mathbb{R}^N} \frac{u_\varepsilon^{p^*(s)}}{|x|^s}\, dx = \mu_s^{(N-s)/(p-s)}.$$

Take a smooth function $\eta : [0, \infty) \to [0, 1]$ such that $\eta(s) = 1$ for $s \le 1/4$ and $\eta(s) = 0$ for $s \ge 1/2$, and set

$$u_{\varepsilon,\delta}(x) = \eta\left(\frac{|x|}{\delta}\right) u_\varepsilon(x), \quad v_{\varepsilon,\delta}(x) = \frac{u_{\varepsilon,\delta}(x)}{\left(\displaystyle\int_{\mathbb{R}^N} \frac{u_{\varepsilon,\delta}^{p^*(s)}}{|x|^s}\, dx\right)^{1/p^*(s)}}, \quad \varepsilon, \delta > 0,$$

so that

$$\int_{\mathbb{R}^N} \frac{v_{\varepsilon,\delta}^{p^*(s)}}{|x|^s}\, dx = 1. \tag{2.2}$$

The following estimates were obtained in [6, Lemma 11.1.(1),(3),(4)]:

$$\int_{\mathbb{R}^N} |\nabla v_{\varepsilon,\delta}|^p\, dx \le \mu_s + C\varepsilon^{(N-p)/(p-s)}, \tag{2.3}$$

$$\int_{\mathbb{R}^N} v_{\varepsilon,\delta}^p\, dx \ge \begin{cases} \dfrac{1}{C}\, \varepsilon^{(p-1)\,p/(p-s)} & \text{if } N > p^2 \\ \dfrac{1}{C}\, \varepsilon^{(p-1)\,p/(p-s)}\, |\log \varepsilon| & \text{if } N = p^2, \end{cases} \tag{2.4}$$

where $C = C(N, p, s, \delta) > 0$ is a constant. While these estimates are sufficient for the proof of Theorem 1.2, we will need the following finer estimates in order to prove Theorem 1.3.

Lemma 2.7. *There exists a constant $C = C(N, p, s) > 0$ such that*

$$\int_{\mathbb{R}^N} |\nabla v_{\varepsilon,\delta}|^p\, dx \le \mu_s + C\Theta_{\varepsilon,\delta}^{(N-p)/(p-s)}, \tag{2.5}$$

$$\int_{\mathbb{R}^N} v_{\varepsilon,\delta}^p\, dx \ge \begin{cases} \dfrac{1}{C}\, \varepsilon^{(p-1)\,p/(p-s)} & \text{if } N > p^2 \\ \dfrac{1}{C}\, \varepsilon^{(p-1)\,p/(p-s)}\, |\log \Theta_{\varepsilon,\delta}| & \text{if } N = p^2, \end{cases} \tag{2.6}$$

where $\Theta_{\varepsilon,\delta} = \varepsilon\, \delta^{-(p-s)/(p-1)}$.

Proof. We have

$$u_{\varepsilon,\delta}(\delta x) = \delta^{-(N-p)/p}\, u_{\Theta_{\varepsilon,\delta},1}(x)$$

and

$$\int_{\mathbb{R}^N} \frac{u_{\varepsilon,\delta}^{p^*(s)}}{|x|^s}\, dx = \int_{\mathbb{R}^N} \frac{u_{\Theta_{\varepsilon,\delta},1}^{p^*(s)}}{|x|^s}\, dx.$$

So

$$v_{\varepsilon,\delta}(\delta x) = \delta^{-(N-p)/p}\, v_{\Theta_{\varepsilon,\delta},1}(x)$$

and hence

$$\nabla v_{\varepsilon,\delta}(\delta x) = \delta^{-N/p}\, \nabla v_{\Theta_{\varepsilon,\delta},1}(x).$$

Then

$$\int_{\mathbb{R}^N} |\nabla v_{\varepsilon,\delta}(x)|^p\, dx = \delta^N \int_{\mathbb{R}^N} |\nabla v_{\varepsilon,\delta}(\delta x)|^p\, dx = \int_{\mathbb{R}^N} |\nabla v_{\Theta_{\varepsilon,\delta},1}(x)|^p\, dx$$

and

$$\int_{\mathbb{R}^N} v^p_{\varepsilon,\delta}(x)\,dx = \delta^N \int_{\mathbb{R}^N} v^p_{\varepsilon,\delta}(\delta x)\,dx = \delta^p \int_{\mathbb{R}^N} v^p_{\Theta_{\varepsilon,\delta},1}(x)\,dx,$$

so (2.5) and (2.6) follow from (2.3) and (2.4), respectively. □

Let i, $\mathcal{M}$, Ψ, and λ_k be as in the introduction, and suppose that $\lambda_k < \lambda_{k+1}$. Then the sublevel set Ψ^{λ_k} has a compact symmetric subset E of index k that is bounded in $L^\infty(\Omega) \cap C^{1,\alpha}_{\text{loc}}(\Omega)$ (see Degiovanni and Lancelotti [4, Theorem 2.3]). Let $\delta_0 = \operatorname{dist}(0, \partial\Omega)$, take a smooth function $\theta : [0,\infty) \to [0,1]$ such that $\theta(s) = 0$ for $s \le 3/4$ and $\theta(s) = 1$ for $s \ge 1$, and set

$$v_\delta(x) = \theta\left(\frac{|x|}{\delta}\right) v(x), \quad v \in E,\ 0 < \delta \le \frac{\delta_0}{2}.$$

Since $E \subset \Psi^{\lambda_k}$ is bounded in $C^1(B_{\delta_0/2}(0))$,

$$\int_\Omega |\nabla v_\delta|^p\,dx \le \int_{\Omega\setminus B_\delta(0)} |\nabla v|^p\,dx + C\int_{B_\delta(0)} \left(|\nabla v|^p + \frac{|v|^p}{\delta^p}\right) dx \le 1 + C\delta^{N-p} \tag{2.7}$$

and

$$\int_\Omega |v_\delta|^p\,dx \ge \int_{\Omega\setminus B_\delta(0)} |v|^p\,dx = \int_\Omega |v|^p\,dx - \int_{B_\delta(0)} |v|^p\,dx \ge \frac{1}{\lambda_k} - C\delta^N, \tag{2.8}$$

where $C = C(N,p,s,\Omega,k) > 0$ is a constant. By (1.6) and (2.8),

$$\int_\Omega \frac{|v_\delta|^{p^*(s)}}{|x|^s}\,dx \ge \frac{1}{C} \tag{2.9}$$

if $\delta > 0$ is sufficiently small.

Now let $\pi : W^{1,p}_0(\Omega)\setminus\{0\} \to \mathcal{M},\ u \mapsto u/\|u\|$ be the radial projection onto $\mathcal{M}$, and set

$$w = \pi(v_\delta), \quad v \in E.$$

If $\delta > 0$ is sufficiently small,

$$\Psi(w) = \frac{\displaystyle\int_\Omega |\nabla v_\delta|^p\,dx}{\displaystyle\int_\Omega |v_\delta|^p\,dx} \le \lambda_k + C\delta^{N-p} < \lambda_{k+1} \tag{2.10}$$

by (2.7) and (2.8), and

$$\int_\Omega \frac{|w|^{p^*(s)}}{|x|^s}\,dx = \frac{\displaystyle\int_\Omega \frac{|v_\delta|^{p^*(s)}}{|x|^s}\,dx}{\left(\displaystyle\int_\Omega |\nabla v_\delta|^p\,dx\right)^{p^*(s)/p}} \ge \frac{1}{C} \tag{2.11}$$

by (2.7) and (2.9). Since $\operatorname{supp} w = \operatorname{supp} v_\delta \subset \Omega \setminus B_{3\delta/4}(0)$ and $\operatorname{supp} \pi(v_{\varepsilon,\delta}) = \operatorname{supp} v_{\varepsilon,\delta} \subset \overline{B_{\delta/2}(0)}$,

$$\operatorname{supp} w \cap \operatorname{supp} \pi(v_{\varepsilon,\delta}) = \emptyset. \tag{2.12}$$

Set

$$E_\delta = \{w : v \in E\}.$$

Lemma 2.8. *For all sufficiently small $\delta > 0$,*

(i) $E_\delta \cap \Psi_{\lambda_{k+1}} = \emptyset$,
(ii) $i(E_\delta) = k$,
(iii) $\pi(v_{\varepsilon,\delta}) \notin E_\delta$.

Proof. (i) follows from (2.10). By (i), $E_\delta \subset \mathcal{M} \backslash \Psi_{\lambda_{k+1}}$ and hence

$$i(E_\delta) \le i(\mathcal{M} \backslash \Psi_{\lambda_{k+1}}) = k$$

by the monotonicity of the index and (1.5). On the other hand, since $E \to E_\delta$, $v \mapsto \pi(v_\delta)$ is an odd continuous map,

$$i(E_\delta) \ge i(E) = k.$$

(ii) follows. (iii) is immediate from (2.12). □

3. Proofs

3.1. Proof of Theorem 1.1

All nontrivial critical points of I_λ lie on the Nehari manifold

$$\mathcal{N} = \Big\{u \in W^{1,p}_0(\Omega) \setminus \{0\} : I'_\lambda(u)\, u = 0\Big\}.$$

We will show that I_λ attains the ground state energy

$$c := \inf_{u \in \mathcal{N}} I_\lambda(u)$$

at a positive critical point.

Since $0 < \lambda < \lambda_1$, $\mathcal{N}$ is closed, bounded away from the origin, and for $u \in W^{1,p}_0(\Omega) \setminus \{0\}$ and $t > 0$, $tu \in \mathcal{N}$ if and only if $t = t_u$, where

$$t_u = \left[\frac{\displaystyle\int_\Omega \big(|\nabla u|^p - \lambda\, |u|^p\big)\, dx}{\displaystyle\int_\Omega \frac{|u|^{p^*(s)}}{|x|^s}\, dx}\right]^{(N-p)/(p-s)\, p}.$$

Moreover,

$$I_\lambda(t_u u) = \sup_{t>0} I_\lambda(tu) = \frac{p-s}{(N-s)\, p}\, \psi_\lambda(u)^{(N-s)/(p-s)},$$

where

$$\psi_\lambda(u) = \frac{\displaystyle\int_\Omega \big(|\nabla u|^p - \lambda\, |u|^p\big)\, dx}{\left(\displaystyle\int_\Omega \frac{|u|^{p^*(s)}}{|x|^s}\, dx\right)^{p/p^*(s)}}.$$

By (2.2)–(2.4),

$$\psi_\lambda(v_{\varepsilon,\delta}) \le \begin{cases} \mu_s - \dfrac{\varepsilon^{(p-1)\,p/(p-s)}}{C} + C\varepsilon^{(N-p)/(p-s)} & \text{if } N > p^2 \\ \mu_s - \dfrac{\varepsilon^{(p-1)\,p/(p-s)}}{C}\,|\log\varepsilon| + C\varepsilon^{(p-1)\,p/(p-s)} & \text{if } N = p^2, \end{cases}$$

and in both cases the last expression is strictly less than μ_s if $\varepsilon > 0$ is sufficiently small, so

$$c \le I_\lambda(t_{v_{\varepsilon,\delta}} v_{\varepsilon,\delta}) < \frac{p-s}{(N-s)\,p}\,\mu_s^{(N-s)/(p-s)}.$$

Then I_λ satisfies the $(\mathrm{PS})_c$ condition by [6, Theorem 4.1.(2)], and hence $I_\lambda|_{\mathcal{N}}$ has a minimizer u_0 by a standard argument. Then $|u_0|$ is also a minimizer, which is positive by the strong maximum principle.

3.2. Proof of Theorem 1.2

We will show that problem (1.1) has a nontrivial solution as long as $\lambda > \lambda_1$ is not an eigenvalue from the sequence (λ_k). Then we have $\lambda_k < \lambda < \lambda_{k+1}$ for some $k \in \mathbb{N}$. Fix $\delta > 0$ so small that the first inequality in (2.10) implies

$$\Psi(w) \le \lambda \quad \forall w \in E_\delta \tag{3.1}$$

and the conclusions of Lemma 2.8 hold. Then let $A_0 = E_\delta$ and $B_0 = \Psi_{\lambda_{k+1}}$, and note that A_0 and B_0 are disjoint nonempty closed symmetric subsets of $\mathcal{M}$ such that

$$i(A_0) = i(\mathcal{M} \setminus B_0) = k \tag{3.2}$$

by Lemma 2.8 (i), (ii) and (1.5). Now let $R > r > 0$, let $v_0 = \pi(v_{\varepsilon,\delta})$, which is in $\mathcal{M} \setminus A_0$ by Lemma 2.8 (iii), and let A, B and X be as in Theorem 2.3.

For $u \in B_0$,

$$I_\lambda(ru) \ge \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right) r^p - \frac{r^{p^*(s)}}{p^*(s)\,\mu_s^{p^*(s)/p}}.$$

Since $\lambda < \lambda_{k+1}$, and $s < p$ implies $p^*(s) > p$, it follows that $\inf I_\lambda(B) > 0$ if r is sufficiently small.

Next we show that $I_\lambda \le 0$ on A if R is sufficiently large. For $w \in A_0$ and $t \ge 0$,

$$I_\lambda(tw) \le \frac{t^p}{p}\left(1 - \frac{\lambda}{\Psi(w)}\right) \le 0$$

by (3.1). Now let $w \in A_0$ and $0 \le t \le 1$, and set $u = \pi((1-t)\,w + tv_0)$. Clearly, $\|(1-t)\,w + tv_0\| \le 1$, and since the supports of w and v_0 are disjoint by (2.12),

$$\int_\Omega \frac{|(1-t)\,w + tv_0|^{p^*(s)}}{|x|^s}\,dx = (1-t)^{p^*(s)} \int_\Omega \frac{|w|^{p^*(s)}}{|x|^s}\,dx + t^{p^*(s)} \int_\Omega \frac{v_0^{p^*(s)}}{|x|^s}\,dx.$$

In view of (2.11), and since

$$\int_\Omega \frac{v_0^{p^*(s)}}{|x|^s}\, dx = \frac{\displaystyle\int_\Omega \frac{v_{\varepsilon,\delta}^{p^*(s)}}{|x|^s}\, dx}{\left(\displaystyle\int_\Omega |\nabla v_{\varepsilon,\delta}|^p\, dx\right)^{p^*(s)/p}} \ge \frac{1}{C}$$

by (2.2) and (2.3) if $\varepsilon > 0$ is sufficiently small, it follows that

$$\int_\Omega \frac{|u|^{p^*(s)}}{|x|^s}\, dx = \frac{\displaystyle\int_\Omega \frac{|(1-t)\, w + tv_0|^{p^*(s)}}{|x|^s}\, dx}{\|(1-t)\, w + tv_0\|^{p^*(s)}} \ge \frac{1}{C}.$$

Then

$$I_\lambda(Ru) \le \frac{R^p}{p} - \frac{R^{p^*(s)}}{p^*(s)} \int_\Omega \frac{|u|^{p^*(s)}}{|x|^s}\, dx \le 0$$

if R is sufficiently large.

Now we show that

$$\sup I_\lambda(X) < \frac{p-s}{(N-s)\, p}\, \mu_s^{(N-s)/(p-s)} \tag{3.3}$$

if $\varepsilon > 0$ is sufficiently small. Noting that

$$X = \{\rho\, \pi((1-t)\, w + tv_0) : w \in E_\delta,\ 0 \le t \le 1,\ 0 \le \rho \le R\},$$

let $w \in E_\delta$ and $0 \le t \le 1$, and set $u = \pi((1-t)\, w + tv_0)$. Then

$$\begin{aligned} \sup_{0 \le \rho \le R} I_\lambda(\rho u) &\le \sup_{\rho \ge 0} \left[\frac{\rho^p}{p} \left(1 - \lambda \int_\Omega |u|^p\, dx\right) - \frac{\rho^{p^*(s)}}{p^*(s)} \int_\Omega \frac{|u|^{p^*(s)}}{|x|^s}\, dx\right] \\ &= \frac{p-s}{(N-s)\, p}\, \psi_\lambda(u)^{(N-s)/(p-s)}, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} \psi_\lambda(u) &= \frac{\left(1 - \lambda \displaystyle\int_\Omega |u|^p\, dx\right)^+}{\left(\displaystyle\int_\Omega \frac{|u|^{p^*(s)}}{|x|^s}\, dx\right)^{p/p^*(s)}} \\ &= \frac{\left(\displaystyle\int_\Omega \left[|(1-t)\, \nabla w + t\, \nabla v_0|^p - \lambda\, |(1-t)\, w + tv_0|^p\right] dx\right)^+}{\left(\displaystyle\int_\Omega \frac{|(1-t)\, w + tv_0|^{p^*(s)}}{|x|^s}\, dx\right)^{p/p^*(s)}} \\ &\le \frac{(1-t)^p \left(1 - \lambda \displaystyle\int_\Omega |w|^p\, dx\right)^+ + t^p \left(1 - \lambda \displaystyle\int_\Omega v_0^p\, dx\right)^+}{\left((1-t)^{p^*(s)} \displaystyle\int_\Omega \frac{|w|^{p^*(s)}}{|x|^s}\, dx + t^{p^*(s)} \int_\Omega \frac{v_0^{p^*(s)}}{|x|^s}\, dx\right)^{p/p^*(s)}} \end{aligned} \tag{3.5}$$

since the supports of w and v_0 are disjoint. Since

$$1 - \lambda \int_\Omega |w|^p \, dx = 1 - \frac{\lambda}{\Psi(w)} \le 0$$

by (3.1),

$$\begin{aligned} \psi_\lambda(u) &\le \psi_\lambda(v_0) \\ &= \frac{\left(\displaystyle\int_\Omega \left[|\nabla v_{\varepsilon,\delta}|^p - \lambda\, v_{\varepsilon,\delta}^p\right] dx\right)^+}{\left(\displaystyle\int_\Omega \frac{v_{\varepsilon,\delta}^{p^*(s)}}{|x|^s}\, dx\right)^{p/p^*(s)}} \\ &\le \begin{cases} \mu_s - \dfrac{\varepsilon^{(p-1)\,p/(p-s)}}{C} + C\varepsilon^{(N-p)/(p-s)} & \text{if } N > p^2 \\ \mu_s - \dfrac{\varepsilon^{(p-1)\,p/(p-s)}}{C}\, |\log \varepsilon| + C\varepsilon^{(p-1)\,p/(p-s)} & \text{if } N = p^2 \end{cases} \end{aligned}$$

by (2.2)–(2.4). In both cases the last expression is strictly less than μ_s if $\varepsilon > 0$ is sufficiently small, so (3.3) follows from (3.4).

The inequalities (2.1) now imply that

$$0 < c < \frac{p-s}{(N-s)\,p}\, \mu_s^{(N-s)/(p-s)}.$$

Then I_λ satisfies the $(\mathrm{PS})_c$ condition by [6, Theorem 4.1.(2)], and hence c is a positive critical value of I_λ by Theorem 2.3.

3.3. Proof of Theorem 1.3

The case where $\lambda > \lambda_1$ is an eigenvalue, but not from the sequence (λ_k), was covered in the proof of Theorem 1.2, so we may assume that $\lambda = \lambda_k < \lambda_{k+1}$ for some $k \in \mathbb{N}$. Take $\delta > 0$ so small that (2.10) and the conclusions of Lemma 2.8 hold, let A_0, B_0 and v_0 be as in the proof of Theorem 1.2, and let A, B and X be as in Theorem 2.3.

As before, $\inf I_\lambda(B) > 0$ if r is sufficiently small, and

$$I_\lambda(R\,\pi((1-t)\,w + tv_0)) \le 0 \quad \forall w \in A_0,\ 0 \le t \le 1$$

if $\Theta_{\varepsilon,\delta}$ is sufficiently small and R is sufficiently large. On the other hand,

$$I_\lambda(tw) \le \frac{t^p}{p}\left(1 - \frac{\lambda_k}{\Psi(w)}\right) \le CR^p \delta^{N-p} \quad \forall w \in A_0,\ 0 \le t \le R$$

by (2.10). It follows that $\sup I_\lambda(A) < \inf I_\lambda(B)$ if δ is also sufficiently small.

It only remains to verify (3.3) for suitable choice of $\delta(\varepsilon)$ and small ε. Maximizing the last expression in (3.5) over $0 \le t \le 1$ gives

$$\psi_\lambda(u) \le \left[\psi_\lambda(v_0)^{(N-s)/(p-s)} + \psi_\lambda(w)^{(N-s)/(p-s)}\right]^{(p-s)/(N-s)}. \tag{3.6}$$

By (2.2), (2.5), and (2.6),

$$\psi_\lambda(v_0) = \frac{\left(\displaystyle\int_\Omega \left[|\nabla v_{\varepsilon,\delta}|^p - \lambda_k\, v_{\varepsilon,\delta}^p\right] dx\right)^+}{\left(\displaystyle\int_\Omega \frac{v_{\varepsilon,\delta}^{p^*(s)}}{|x|^s}\, dx\right)^{p/p^*(s)}} \le \mu_s - \frac{\varepsilon^{(p-1)\,p/(p-s)}}{C} + C\Theta_{\varepsilon,\delta}^{(N-p)/(p-s)}, \tag{3.7}$$

and by (2.10) and (2.11),

$$\psi_\lambda(w) = \frac{\left(1 - \dfrac{\lambda_k}{\Psi(w)}\right)^+}{\left(\displaystyle\int_\Omega \frac{|w|^{p^*(s)}}{|x|^s}\, dx\right)^{p/p^*(s)}} \le C\delta^{N-p}. \tag{3.8}$$

Recalling that $\Theta_{\varepsilon,\delta} = \varepsilon\, \delta^{-(p-s)/(p-1)}$, if there exist $\alpha \in (0, (p-1)/(p-s))$ and a sequence $\varepsilon_j \to 0$ such that, for $\varepsilon = \varepsilon_j$ and $\delta = \varepsilon_j^\alpha$, $\psi_\lambda(v_0) < \mu_s/3$, then $\psi_\lambda(u) \le 2\mu_s/3$ for sufficiently large j by (3.6) and (3.8), which together with (3.4) gives the desired result. So we may assume that for all $\alpha \in (0, (p-1)/(p-s))$, $\psi_\lambda(v_0) \ge \mu_s/3$ for all sufficiently small ε and $\delta = \varepsilon^\alpha$. Since $(p-s)/(N-s) < 1$, then (3.6)–(3.8) with $\delta = \varepsilon^\alpha$ yield

$$\begin{aligned} \psi_\lambda(u) &\le \psi_\lambda(v_0) \left[1 + \left(\frac{\psi_\lambda(w)}{\psi_\lambda(v_0)}\right)^{(N-s)/(p-s)}\right] \\ &\le \psi_\lambda(v_0) + C\, \psi_\lambda(w)^{(N-s)/(p-s)} \le \mu_s - \varepsilon^{(p-1)\,p/(p-s)} \\ &\quad \times \left[\frac{1}{C} - C\varepsilon^{(N-p)(N-s)(\alpha-\alpha_1)/(p-s)} - C\varepsilon^{(N-p)(\alpha_2-\alpha)/(p-1)}\right], \end{aligned}$$

where

$$0 < \alpha_1 := \frac{(p-1)\,p}{(N-p)(N-s)} < \frac{(N-p^2)(p-1)}{(N-p)(p-s)} =: \alpha_2 < \frac{p-1}{p-s}$$

by (1.3). Taking $\alpha \in (\alpha_1, \alpha_2)$ now gives the desired conclusion.

3.4. Proofs of Theorems 1.6 and 1.7

We only give the proof of Theorem 1.7. Proof of Theorem 1.6 is similar and simpler. By [6, Theorem 4.1.(2)], I_λ satisfies the $(\mathrm{PS})_c$ condition for all

$$c < \frac{p-s}{(N-s)\,p}\, \mu_s^{(N-s)/(p-s)},$$

so we apply Theorem 2.5 with b equal to the right-hand side.

By Degiovanni and Lancelotti [4, Theorem 2.3], the sublevel set $\Psi^{\lambda_{k+m}}$ has a compact symmetric subset A_0 with

$$i(A_0) = k + m.$$

We take $B_0 = \Psi_{\lambda_{k+1}}$, so that

$$i(\mathcal{M} \setminus B_0) = k$$

by (1.5). Let $R > r > 0$ and let A, B and X be as in Theorem 2.5. For $u \in \Psi_{\lambda_{k+1}}$,

$$I_\lambda(ru) \geq \frac{r^p}{p} \left(1 - \frac{\lambda}{\lambda_{k+1}}\right) - \frac{r^{p^*(s)}}{p^*(s)\, \mu_s^{p^*(s)/p}}$$

by (1.4). Since $\lambda < \lambda_{k+1}$, and $s < p$ implies $p^*(s) > p$, it follows that $\inf I_\lambda(B) > 0$ if r is sufficiently small. For $u \in A_0 \subset \Psi^{\lambda_{k+1}}$,

$$I_\lambda(Ru) \leq \frac{R^p}{p} \left(1 - \frac{\lambda}{\lambda_{k+1}}\right) - \frac{R^{p^*(s)}}{p^*(s)\, \lambda_{k+1}^{p^*(s)/p}\, V_s(\Omega)^{(p-s)/(N-p)}}$$

by (1.6), so there exists $R > r$ such that $I_\lambda \leq 0$ on A. For $u \in X$,

$$\begin{aligned} I_\lambda(u) &\leq \frac{\lambda_{k+1} - \lambda}{p} \int_\Omega |u|^p\, dx - \frac{1}{p^*(s)\, V_s(\Omega)^{(p-s)/(N-p)}} \left(\int_\Omega |u|^p\, dx\right)^{p^*(s)/p} \\ &\leq \sup_{\rho \geq 0} \left[\frac{(\lambda_{k+1} - \lambda)\, \rho}{p} - \frac{\rho^{p^*(s)/p}}{p^*(s)\, V_s(\Omega)^{(p-s)/(N-p)}}\right] \\ &= \frac{p-s}{(N-s)\, p}\, (\lambda_{k+1} - \lambda)^{(N-s)/(p-s)}\, V_s(\Omega). \end{aligned}$$

So

$$\sup I_\lambda(X) \leq \frac{p-s}{(N-s)\, p}\, (\lambda_{k+1} - \lambda)^{(N-s)/(p-s)}\, V_s(\Omega) < \frac{p-s}{(N-s)\, p}\, \mu_s^{(N-s)/(p-s)}$$

by (1.7). Theorem 2.5 now gives m distinct pairs of (nontrivial) critical points $\pm\, u_j^\lambda$, $j = 1, \ldots, m$ of I_λ such that

$$0 < I_\lambda(u_j^\lambda) \leq \frac{p-s}{(N-s)\, p}\, (\lambda_{k+1} - \lambda)^{(N-s)/(p-s)}\, V_s(\Omega).$$

Since

$$\int_\Omega |\nabla u_j^\lambda|^p\, dx = p\, I_\lambda(u_j^\lambda) + \lambda \int_\Omega |u_j^\lambda|^p\, dx + \frac{p}{p^*(s)} \int_\Omega \frac{|u_j^\lambda|^{p^*(s)}}{|x|^s}\, dx,$$

$$\int_\Omega |u_j^\lambda|^p\, dx \leq V_s(\Omega)^{(p-s)/(N-s)} \left(\int_\Omega \frac{|u_j^\lambda|^{p^*(s)}}{|x|^s}\, dx\right)^{p/p^*(s)}$$

by (1.6), and

$$\int_\Omega \frac{|u_j^\lambda|^{p^*(s)}}{|x|^s}\, dx = \frac{(N-s)\, p}{p-s} \left[I_\lambda(u_j^\lambda) - \frac{1}{p}\, I_\lambda'(u_j^\lambda)\, u_j^\lambda\right] = \frac{(N-s)\, p}{p-s}\, I_\lambda(u_j^\lambda),$$

(1.8) follows. This completes the proof of Theorem 1.7.

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