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# Capacity solution to a coupled system of parabolic–elliptic equations in Orlicz–Sobolev spaces

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**Abstract.** The existence of a capacity solution to a coupled nonlinear parabolic–elliptic system is analyzed, the elliptic part in the parabolic equation being of the form $-\operatorname{div} a(x,t,u,\nabla u)$. The growth and the coercivity conditions on the monotone vector field $a$ are prescribed by an $N$-function, $M$, which does not have to satisfy a $\Delta_2$ condition. Therefore we work with Orlicz–Sobolev spaces which are not necessarily reflexive. We use Schauder's fixed point theorem to prove the existence of a weak solution to certain approximate problems. Then we show that some subsequence of approximate solutions converges in a certain sense to a capacity solution.

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## 1. Introduction

In recent years, there has been an increasing interest in the study of various mathematical problems involving the operators satisfying non-polynomial growth conditions instead of having the usual $p$-structure which employ the standard theory of monotone operators relying on the Sobolev space $W^{1,p}(\Omega)$, the origins of which can be traced back to the work of Orlicz in the 1930s. Later on, Polish and Czechoslovak mathematicians investigated the modular function spaces (see, for example, Musielak [19] and Krasnoselskii and Rutickii [18]). Many properties of Sobolev spaces have been extended to Orlicz–Sobolev spaces, mainly by Dankert [7] Donaldson and Trudinger [9] and O'Neil [20] (see also [1] for an excellent account of those works). At present, the operators satisfying non-polynomial growth arouse much interest with the development of elastic mechanics, electro-rheological fluids as an important class of non-Newtonian fluids (sometimes referred to as smart fluids). The electro-rheological fluids are characterized by their ability to highly change in their mechanical properties under the influence of an external electromagnetic field. A mathematical model of electro-rheological fluids was proposed by Rajagopal and Růžička [21,22] we refer for instance to [4] and [6] for different non-standard growth conditions and to [5] and [17] for some recent existence results in the context of non-polynomial growth. According to Diening [8] we are strongly convinced that these more general spaces will become increasingly important in modeling modern materials.

This paper deals with the existence of a capacity solution to a coupled system of parabolic–elliptic equations, whose unknowns are the temperature inside a semiconductor material, $u$, and the electric potential, $\varphi$, namely

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div} a(x,t,u,\nabla u) = \rho(u)|\nabla\varphi|^2 \;\text{ in } Q_T = \Omega \times (0,T), \\ \operatorname{div}(\rho(u)\nabla\varphi) = 0 \;\text{ in } Q_T, \\ \varphi = \varphi_0 \text{ on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \;\text{ in } \Omega, \\ u = 0 \;\text{ on } \partial\Omega \times (0,T), \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is the space region occupied by the semiconductor, $Au = -\operatorname{div} a(x,t,u,\nabla u)$ is a Leray-Lions operator defined on $W_0^{1,x}L_M(Q_T)$, $M$ is an appropriate $N$-function, and the functions $\varphi_0$ and $u_0$ are given.

The functional spaces to deal with these problems are Orlicz-Sobolev spaces. In general, Orlicz–Sobolev spaces are neither reflexive nor separable.

This problem may be regarded as a generalization of the so-called thermistor problem arising in electromagnetism [3,13,14].

Since we are dealing with a nonuniformly elliptic problem (see assumption (3.6) on $\rho(s)$ below), one readily realizes that the search of weak solutions to problem (1.1) are not well suited. Indeed, $\rho(s)$ may converge to zero as $|s|$ tends to infinity and as a result, if $u$ is unbounded in $Q_T$, the elliptic equation becomes degenerate at points where $u$ is infinity and, therefore, no a priori estimates for $\nabla\varphi$ will be available and thus, $\varphi$ may not belong to a Sobolev space. Instead of $\varphi$, we may consider the function $\Phi = \rho(u)|\nabla\varphi|^2$ as a whole and then show that belongs to $L^2(Q_T)^d$. This means that a new formulation of the original system is possible and the solution to this new formulation will be called capacity solution.

The concept of capacity solution was first introduced by Xu in [25] in the analysis of a modified version of the thermistor problem. The same author applied this concept to more general settings where weaker assumptions [24] or mixed boundary conditions [26] are considered.

The existence of a capacity solution of (1.1) in the classical Sobolev spaces has been proved by González Montesinos and Ortegón Gallego in [14].

After establishing the continuity of a certain mapping, we use Schauder's fixed point theorem to prove the existence of a weak solution to an approximate problem. Then we show that some subsequence of approximate solutions converges in a certain sense to a capacity solution.

The main goal of this paper is to prove the existence of a capacity solution of (1.1) in the sense of Definition 4.1 (see Sect. 3) for a general $N$-function, $M$, along with the lack of reflexivity in this setting combined with the nonuniformly elliptic character of the elliptic equation.

This work is organized as follows. In Sect. 2 we recall some well-known properties and results on Orlicz-Sobolev spaces. Section 3 is devoted to specify the assumptions on data. In Sect. 4 we give the definition of a capacity solution of (1.1). Finally, in Sect. 5 we present the existence result and its proof.

## 2. Preliminaries

In this section we present some well-known results on Orlicz and Orlicz–Sobolev spaces. Most of them can be found in [1,10–12,15,16] and [18].

Let $M\colon \mathbb{R}^+ \to \mathbb{R}^+$ be an $N$-function, i.e., $M$ is a convex function, with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \to 0$ as $t \to 0$ and $\frac{M(t)}{t} \to \infty$ as $t \to \infty$. Equivalently, $M$ admits the representation: $M(t) = \int_0^t m(s)\,\mathrm{d}s$ where $m\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing and right continuous function, with $m(0) = 0$, $m(t) > 0$ for $t > 0$, and $m(t) \to \infty$ as $t \to \infty$. The $N$-function $\bar{M}$ conjugate to $M$ is defined by $\bar{M}(t) = \int_0^t \bar{m}(s)\,\mathrm{d}s$, where $\bar{m}\ :\ \mathbb{R}^+ \to \mathbb{R}^+$ is given by $\bar{m}(t) = \sup\{s \,/\, m(s) \leq t\}$.

The $N$-function $M$ is said to satisfy the $\Delta_2$-condition if, for some $k > 0$,

$$M(2t) \leq k\,M(t) \quad \text{for all } t \geq 0. \tag{2.1}$$

When this inequality holds only for $t \geq t_0 > 0$, $M$ is said to satisfy the $\Delta_2$-condition near infinity.

Let $P$ and $M$ be two $N$-functions. The notation $P \ll M$ means that $P$ grows essentially less rapidly than $M$, i.e., for each $\varepsilon > 0$,

$$\frac{P(t)}{M(\varepsilon t)} \to 0 \quad \text{as } t \to \infty. \tag{2.2}$$

This is the case if and only if, for each $\varepsilon > 0$,

$$\frac{M^{-1}(t)}{P^{-1}(\varepsilon t)} \to 0 \quad \text{as } t \to \infty. \tag{2.3}$$

We will extend these $N$-functions into even functions on all $\mathbb{R}$. Let $\Omega$ be an open subset of $\mathbb{R}^d$, $d \in \mathbb{N}$. The Orlicz class $\mathcal{L}_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions $u$ on $\Omega$ such that:

$$\int_\Omega M(u(x))\,\mathrm{d}x < +\infty \quad \left(\text{resp. } \int_\Omega M\Big(\frac{u(x)}{\lambda}\Big)\,\mathrm{d}x < +\infty \text{ for some } \lambda > 0\right). \tag{2.4}$$

Notice that $L_M(\Omega)$ is a Banach space under the so-called Luxemburg norm, namely

$$\|u\|_{M,\Omega} = \inf\left\{\lambda > 0 \,/\, \int_\Omega M\Big(\frac{u(x)}{\lambda}\Big)\,\mathrm{d}x \le 1\right\}, \tag{2.5}$$

and $\mathcal{L}_M(\Omega)$ is a convex subset of $L_M(\Omega)$. Indeed, $L_M(\Omega)$ is the linear hull of $\mathcal{L}_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\bar{\Omega}$ is denoted by $E_M(\Omega)$. The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if $M$ satisfies the $\Delta_2$-condition, for all $t$ or for $t$ large according to whether $\Omega$ has infinite measure or not.

The dual of $E_M(\Omega)$ can be identified with $L_{\bar{M}}(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)\,\mathrm{d}x$, and the dual norm on $L_{\bar{M}}(\Omega)$ is equivalent to $\|.\|_{\bar{M},\Omega}$. The space $L_M(\Omega)$ is reflexive if and only if $M$ and $\bar{M}$ satisfy the $\Delta_2$-condition, for all $t$ or for $t$ large, according to whether $\Omega$ has infinite measure or not.

In $L_M(\Omega)$ we define the Orlicz norm $||u||_{(M)}$ by

$$||u||_{(M)} = \sup \int_\Omega u(x)v(x)\,\mathrm{d}x, \tag{2.6}$$

where the supremum is taken over all $v \in E_{\bar{M}(\Omega)}$ such that $||v||_{\bar{M},\Omega} \le 1$. An important inequality in $L_M(\Omega)$ is the following:

$$\int_\Omega M(u(x))\,\mathrm{d}x \le ||u||_{(M)} \text{ for all } u \in L_M(\Omega) \text{ such that } ||u||_{(M)} \le 1, \tag{2.7}$$

wherefrom we readily deduce

$$\int_\Omega M\left(\frac{u(x)}{||u||_{(M)}}\right)\mathrm{d}x \le 1 \text{ for all } u \in L_M(\Omega)\backslash\{0\}. \tag{2.8}$$

It can be shown that the norm $||\cdot||_{(M)}$ is equivalent to the Luxemburg norm $\|\cdot\|_{M,\Omega}$. Indeed,

$$||u||_{M,\Omega} \le ||u||_{(M)} \le 2||u||_{M,\Omega} \text{ for all } u \in L_M(\Omega). \tag{2.9}$$

Also, the Hölder inequality holds

$$\int_\Omega |u(x)v(x)|\,\mathrm{d}x \le ||u||_{M,\Omega}||v||_{(\bar{M})} \text{ for all } u \in L_M(\Omega) \text{ and } v \in L_{\bar{M}}(\Omega),$$

in particular, if $\Omega$ has finite measure, Hölder's inequality yields the continuous inclusion $L_M(\Omega) \subset L^1(\Omega)$.

We now turn to the Orlicz–Sobolev space. $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of all functions $u$ such that $u$ and its distributional derivatives up to order one lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$). This is a Banach space under the norm

$$\|u\|_{1,M,\Omega} = \sum_{|\alpha|\le 1} \|\nabla^\alpha u\|_{M,\Omega}. \tag{2.10}$$

Thus $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $d+1$ copies of $L_M(\Omega)$. Denoting this product by $\Pi L_M$, we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\bar{M}})$ and $\sigma(\Pi L_M, \Pi L_{\bar{M}})$. The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\bar{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$.

Convergence in norm in Orlicz or Orlicz–Sobolev spaces is rather strict when $M$ does not satisfies the $\Delta_2$-condition. To this end, it is very convenient to introduce the concept of modular convergence.

**Definition 2.1.** Let $(u_n) \subset L_M(\Omega)$ and $u \subset L_M(\Omega)$. We say that $u_n$ converges to $u$ for the modular convergence in $L_M(\Omega)$ if for some $\lambda > 0$, $\int_\Omega M\Big(\frac{u_n - u}{\lambda}\Big)$ $\mathrm{d}x \to 0$. Let $(u_n) \subset W^1L_M(\Omega)$ and $u \subset W^1L_M(\Omega)$. We say that $(u_n)$ converges to $u$ for the modular convergence in $W^1L_M(\Omega)$ if $\nabla^\alpha u_n$ converges to $\nabla^\alpha u$ for the modular convergence in $L_M(\Omega)$, for all multiindex $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d$ such that $|\alpha| = \alpha_1 + \cdots + \alpha_d \le 1$ and $\alpha_j \ge 0$ for all $1 \le j \le d$.

If $M$ satisfies the $\Delta_2$-condition on (near infinity only when $\Omega$ has finite measure), then modular convergence coincides with norm convergence. This is not true in the general case. For instance, consider the following 1D example: $\Omega = (0,1)$, $M(s) = e^{|s|} - |s| - 1$, $u_n(x) = \log\Big(1 + \frac{1}{n\sqrt{x}}\Big)$, $n \in \mathbb{N}$, $x \in (0,1)$. Since $M(u_n(x)) \le \frac{1}{n\sqrt{x}}$ we have $\lim_{n\to\infty} \int_0^1 M(u_n) = 0$ and thus $u_n \to 0$ in $L_M(0,1)$ for the modular convergence. On the other hand, for $\varepsilon > 0$ we obtain

$$M\left(\frac{u_n(x)}{\varepsilon}\right) \ge \frac{1}{n^{1/\varepsilon}x^{1/(2\varepsilon)}} - \frac{1}{\varepsilon n\sqrt{x}} - 1,$$

so that

$$\int_0^1 M\left(\frac{u_n}{\varepsilon}\right) = +\infty \text{ for all } 0 < \varepsilon < \frac{1}{2},$$

and, consequently, $(u_n)$ does not converge in the norm of $L_M(0,1)$.

The following result shows that the modular convergence in $L_M$ implies, in particular, the convergence in the weak-$*$ topology $\sigma(L_M, L_{\bar{M}})$.

**Lemma 2.2.** ([11,16]) *Let $(u_n) \subset L_M(\Omega)$, $u \in L_M(\Omega)$ and $v \in L_{\bar{M}}(\Omega)$ such that $u_n \to u$ with respect to the modular convergence. Then,*

1. *$u_n v \to uv$ strongly in $L^1(\Omega)$. In particular, $\int_\Omega u_n v \to \int_\Omega uv$.*
2. *Furthermore, if $(v_n) \subset L_{\bar{M}}(\Omega)$ is such that $v_n \to v$ with respect to the modular convergence, then $u_n v_n \to uv$ strongly in $L^1(\Omega)$.*

*Proof.* Let $\lambda > 0$ and $\mu > 0$ such that $M((u_n - u)/\lambda) \to 0$ strongly in $L^1(\Omega)$ and $\bar{M}(v/\mu) \in L^1(\Omega)$. Take a subsequence $(u_{n_k})_k$ such that $u_{n_k} v \to uv$ a.e. in $\Omega$. Then

$$|u_{n_k} v - uv| \le \lambda\mu \left[M\left(\frac{u_{n_k} - u}{\lambda}\right) + \bar{M}\left(\frac{v}{\mu}\right)\right],$$

and thus, from Lebesgue's dominated convergence theorem it yields $u_{n_k} v \to uv$ strongly in $L^1(\Omega)$. Since the limit $uv$ does not depend on the subsequence $(u_{n_k})_k$, it is the whole sequence $(u_n)$ that converges strongly in $L^1(\Omega)$. In order to show the second assertion, we have

$$u_n v_n - uv = (u_n - u)(v_n - v) + (u_n v - uv) + (v_n u - uv),$$

and thus, if $\mu > 0$ is such that $\bar{M}((v_n - v)/\mu) \to 0$ strongly in $L^1(\Omega)$, it yields

$$|u_n v_n - uv| \le \lambda\mu \left[ M\left(\frac{u_n - u}{\lambda}\right) + \bar{M}\left(\frac{v_n - v}{\mu}\right)\right] + |u_n v - uv| + |v_n u - uv|,$$

and using the first assertion, we deduce the desired result. □

**Lemma 2.3.** *Assume that the open set $\Omega \subset \mathbb{R}^d$ has finite measure. If $P \ll M$ and $u_n \to u$ for the modular convergence in $L_M(\Omega)$, then $u_n \to u$ strongly in $E_P(\Omega)$.*

*Proof.* By Theorem 2.1 in [23] we have $u_n, u \in E_P(\Omega)$. Let $\epsilon > 0$ be arbitrary. There exists $\lambda > 0$ such that

$$\int_\Omega M\Big(\frac{u_n - u}{\lambda}\Big) \to 0, \text{ when } n \to \infty.$$

Therefore, there exists $h \in L^1(\Omega)$ such that

$$M\Big(\frac{u_n - u}{\lambda}\Big) \le h \text{ a.e. in } \Omega,$$

for a subsequence still denoted $u_n$. Now choose $t_0 > 0$ such that

$$\frac{P(\frac{t}{\epsilon})}{M(\frac{t}{\lambda})} \le 1, \text{ when } t \ge t_0.$$

Then,

$$P\Big(\frac{u_n - u}{\epsilon}\Big) \le M\Big(\frac{u_n - u}{\lambda}\Big) + P\Big(\frac{t_0}{\epsilon}\Big) \le h + P\Big(\frac{t_0}{\epsilon}\Big) \text{ a.e. } \; x \in \Omega.$$

Since $h + P(\frac{t_0}{\epsilon}) \in L^1(\Omega)$, we have

$$P\Big(\frac{u_n - u}{\epsilon}\Big) \to 0 \text{ in } L^1(\Omega),$$

by Lebesgue's dominated convergence theorem. As $\epsilon > 0$ is arbitrary, we have $u_n \to u$ in $E_P(\Omega)$. □

Let $W^{-1}L_{\bar{M}}(\Omega)$ (resp. $W^{-1}E_{\bar{M}}(\Omega)$) denote the space of distributions on $\Omega$ which can be written as sums of derivatives of order up to one of functions in $L_{\bar{M}}(\Omega)$ (resp. $E_{\bar{M}}(\Omega)$). It is a Banach space under the usual quotient norm.

If the open set $\Omega$ has the segment property, then the space $\mathcal{D}(\Omega)$ is dense in $W_0^1 L_M(\Omega)$ for the modular convergence and for the topology $\sigma(\Pi L_M, \Pi L_{\bar{M}})$ (see [15]). Consequently, the action of a distribution in $W^{-1}L_{\bar{M}}(\Omega)$ on an element of $W_0^1 L_M(\Omega)$ is well defined. For more details the reader is referred to [1,18].

For $K > 0$, we define the truncation at height $K$, $T_K\colon \mathbb{R} \mapsto \mathbb{R}$ by

$$T_K(s) = \min(K, \max(s, -K)) = \begin{cases} s & \text{if } |s| \le K, \\ Ks/|s| & \text{if } |s| > K, \end{cases} \tag{2.11}$$

The following abstract lemmas will be applied to the truncation operators.

**Lemma 2.4.** ([15]). *Let $F\colon \mathbb{R} \mapsto \mathbb{R}$ be a Lipschitz continuous function such that $F(0) = 0$. Let $M$ be an $N$-function and let $u \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$).*

*Then $F(u) \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Moreover, if the set of discontinuity points $D$ of $F'$ is finite, then*

$$\frac{\partial}{\partial x_i}F(u) = \begin{cases} F'(u)\dfrac{\partial u}{\partial x_i} & a.e.\ in\ \{x \in \Omega \,/\, u(x) \notin D\}, \\ 0 & a.e.\ in\ \{x \in \Omega \,/\, u(x) \in D\}. \end{cases}$$

**Lemma 2.5.** ([15]) *Let $F : \mathbb{R} \mapsto \mathbb{R}$ be a Lipschitz continuous function such that $F(0) = 0$. We assume that the set of discontinuity points of $F'$ is finite. Let $M$ be an $N$-function, then the mapping $F\colon W^1L_M(\Omega) \mapsto W^1L_M(\Omega)$ is sequentially continuous with respect to the weak-$*$ topology $\sigma(\Pi L_M, \Pi E_{\bar{M}})$.*

Let $\Omega$ be a bounded open subset of $\mathbb{R}^d$, $d \in \mathbb{N}$, $T > 0$ and set $Q_T = \Omega \times (0,T)$. Let $M$ be an $N$-function. For each $\alpha = (\alpha_1,\ldots,\alpha_d) \in \mathbb{Z}^d$, with $\alpha_j \geq 0$ for all $j$, $1 \leq j \leq d$, denote by $\nabla_x^\alpha$ the distributional derivative on $Q_T$ of order $\alpha$ with respect to the variable $x \in \Omega$, and $|\alpha| = \alpha_1 + \cdots + \alpha_d$. The inhomogeneous Orlicz–Sobolev spaces are defined as follows,

$$W^{1,x}L_M(Q_T) = \{u \in L_M(Q_T) \,/\, \nabla_x^\alpha u \in L_M(Q_T) \text{ for all } \alpha \text{ with } |\alpha| \leq 1\},$$
$$W^{1,x}E_M(Q_T) = \{u \in E_M(Q_T) \,/\, \nabla_x^\alpha u \in E_M(Q_T) \text{ for all } \alpha \text{ with } |\alpha| \leq 1\}.$$

The last space is a subspace of the first one, and both are Banach spaces under the norm,

$$\|u\| = \sum_{|\alpha|\leq 1} \|\nabla_x^\alpha u\|_{M,Q_T}.$$

We can easily show that they form a complementary system when $\Omega$ satisfies the segment property [15]. These spaces are considered as subspaces of the product space $\Pi L_M(Q_T) = L_M(Q_T)^{d+1}$. We shall also consider the weak-$*$ topologies $\sigma(\Pi L_M, \Pi E_{\bar{M}})$ and $\sigma(\Pi L_M, \Pi L_{\bar{M}})$. If $u \in W^{1,x}L_M(Q_T)$ then the function $t \longmapsto u(t) = u(t,\cdot)$ is defined a.e. in $[0,T]$ with values in $W^1L_M(\Omega)$. If, further, $u \in W^{1,x}E_M(Q_T)$ then the concerned function is a $W^1E_M(\Omega)$-valued and is strongly measurable. Furthermore the following embedding holds: $W^{1,x}E_M(Q_T) \subset L^1(0,T;W^1E_M(\Omega))$. The space $W^{1,x}L_M(Q_T)$ is not in general separable. If $u \in W^{1,x}L_M(Q_T)$, we can not conclude that the function $u(t)$ is measurable on $(0,T)$. However, the scalar function $t \mapsto \|u(t)\|_{M,\Omega}$ is in $L^1(0,T)$. The space $W_0^{1,x}E_M(Q_T)$ is defined as the (norm) closure in $W^{1,x}E_M(Q_T)$ of $\mathcal{D}(Q_T)$. We can easily show as in [16] that when $\Omega$ has the segment property, then each element $u$ of the closure of $\mathcal{D}(Q_T)$ with respect of the weak-$*$ topology $\sigma(\Pi L_M, \Pi E_{\bar{M}})$ is a limit, in $W^{1,x}L_M(Q_T)$, of some sequence $(u_n) \subset \mathcal{D}(Q_T)$ for the modular convergence, i.e., there exists $\lambda > 0$ such that for all $|\alpha| \leq 1$,

$$\int_{Q_T} M\left(\frac{\nabla_x^\alpha u_n - \nabla_x^\alpha u}{\lambda}\right) \mathrm{d}x\,\mathrm{d}t \to 0 \text{ as } n \to \infty.$$

From Lemma 2.2, this implies that $(u_n)$ converges to $u$ in $W^{1,x}L_M(Q_T)$ for the weak-$*$ topology $\sigma(\Pi L_M, \Pi L_{\bar{M}})$. Consequently,

$$\overline{\mathcal{D}(Q_T)}^{\sigma(\Pi L_M, \Pi E_{\bar{M}})} = \overline{\mathcal{D}(Q_T)}^{\sigma(\Pi L_M, \Pi L_{\bar{M}})}.$$

This space will be denoted by $W_0^{1,x}L_M(Q_T)$. Furthermore, $W_0^{1,x}L_M(Q_T) \cap \Pi E_M = W_0^{1,x}E_M(Q_T)$. Poincaré's inequality also holds in $W_0^{1,x}L_M(Q_T)$, i.e., there is a constant $C > 0$ such that for all $u \in W_0^{1,x}L_M(Q_T)$ one has,

$$\sum_{|\alpha|\leq 1} \|\nabla_x^\alpha u\|_{M,Q_T} \leq C \sum_{|\alpha|=1} \|\nabla_x^\alpha u\|_{M,Q_T}.$$

Thus both sides of the last inequality are equivalent norms on $W_0^{1,x}L_M(Q_T)$. We have then the following complementary system

$$\begin{pmatrix} W_0^{1,x}L_M(Q_T) & F \\ W_0^{1,x}E_M(Q_T) & F_0 \end{pmatrix}$$

$F$ being the dual space of $W_0^{1,x}E_M(Q_T)$. It is also, except for an isomorphism, the quotient of $\Pi L_{\bar{M}}$ by the polar set $W_0^{1,x}E_M(Q_T)^{\perp}$, and will be denoted by $F = W^{-1,x}L_{\bar{M}}(Q_T)$ and it can be shown that,

$$W^{-1,x}L_{\bar{M}}(Q_T) = \left\{ f = \sum_{|\alpha|\leq 1} \nabla_x^\alpha f_\alpha \,/\, f_\alpha \in L_{\bar{M}}(Q_T) \right\}.$$

This space will be equipped with the usual quotient norm

$$\|f\| = \inf \sum_{|\alpha|\leq 1} \|f_\alpha\|_{\bar{M},Q_T},$$

where the infimum is taken over all possible decompositions $f = \sum_{|\alpha|\leq 1} \nabla_x^\alpha f_\alpha$, $f_\alpha \in L_{\bar{M}}(Q_T)$. The space $F_0$ is then given by,

$$F_0 = \left\{ f = \sum_{|\alpha|\leq 1} \nabla_x^\alpha f_\alpha \,/\, f_\alpha \in E_{\bar{M}}(Q_T) \right\},$$

and is denoted by $F_0 = W^{-1,x}E_{\bar{M}}(Q_T)$.

**Remark 2.6.** We can easily check, using Lemma 2.4, that each Lipschitz continuous mapping $F$, with $F(0) = 0$, acts in the inhomogeneous Orlicz–Sobolev space of order one $W^{1,x}L_M(Q_T)$ and $W_0^{1,x}L_M(Q_T)$ with values in the same space, respectively.

In the sequel, we will make use of the following results which concern mollification with respect to time and space variables and some trace results. For a function $u \in L^1(Q_T)$ we introduce the function $\tilde{u} \in L^1(\Omega \times \mathbb{R})$ as $\tilde{u}(x,s) = u(x,s)\chi_{(0,T)}$ and define, for all $\mu > 0$, $t \in [0,T]$ and a.e. $x \in \Omega$, the function $u_\mu$ given as follows

$$u_\mu(x,t) = \mu \int_{-\infty}^{t} \tilde{u}(x,s)\exp(\mu(s-t))\,\mathrm{d}s. \tag{2.12}$$

**Lemma 2.7.** ([11])

1. *Let* $u \in L_M(Q_T)$. *Then* $u_\mu \in C([0,T]; L_M(\Omega))$ *and* $u_\mu \to u$ *as* $\mu \to +\infty$ *in* $L_M(Q_T)$ *for the modular convergence.*
2. *Let* $u \in W^{1,x}L_M(Q_T)$. *Then* $u_\mu \in C([0,T]; W^1L_M(\Omega))$ *and* $u_\mu \to u$ *as* $\mu \to +\infty$ *in* $W^{1,x}L_M(Q_T)$ *for the modular convergence.*
3. *Let* $u \in E_M(Q_T)$ *(respectively,* $u \in W^{1,x}E_M(Q_T)$*). Then* $u_\mu \to u$ *as* $\mu \to +\infty$ *strongly in* $E_M(Q_T)$ *(respectively, strongly in* $W^{1,x}E_M(Q_T)$*).*
4. *Let* $u \in W^{1,x}L_M(Q_T)$ *then* $\frac{\partial u_\mu}{\partial t} = \mu(u - u_\mu) \in W^{1,x}L_M(Q_T)$.
5. *Let* $(u_n) \subset W^{1,x}L_M(Q_T)$ *and* $u \in W^{1,x}L_M(Q_T)$ *such that* $u_n \to u$ *strongly in* $W^{1,x}L_M(Q_T)$ *(respectively, for the modular convergence). Then, for all* $\mu > 0$, $(u_n)_\mu \to u_\mu$ *strongly in* $W^{1,x}L_M(Q_T)$ *(respectively, for the modular convergence).*

**Lemma 2.8.** ([11]) *Let* $M$ *be an* $N$*-function. Let* $(u_n) \subset W^{1,x}L_M(Q_T)$ *such that,* $u_n \rightharpoonup u$ *weakly-∗ in* $W^{1,x}L_M(Q_T)$ *for* $\sigma(\Pi L_M, \Pi E_{\bar{M}})$ *and* $\frac{\partial u_n}{\partial t} = h_n + k_n$ *in* $\mathcal{D}'(Q_T)$ *with* $(h_n)$ *bounded in* $W^{-1,x}L_{\bar{M}}(Q_T)$ *and* $(k_n)$ *bounded in the space* $L^1(Q_T)$. *Then,* $u_n \to u$ *strongly in* $L^1_{\text{loc}}(Q_T)$.

*If further,* $u_n \in W_0^{1,x}L_M(Q_T)$ *then* $u_n \to u$ *strongly in* $L^1(Q_T)$.

**Lemma 2.9.** ([12]) *Let* $\Omega$ *be a bounded open subset of* $\mathbb{R}^d$ *with the segment property. Consider the Banach space*

$$W = \left\{ u \in W_0^{1,x}L_M(Q_T) \,/\, \frac{\partial u}{\partial t} \in W^{-1,x}L_{\bar{M}}(Q_T) + L^1(Q_T) \right\}.$$

*Then the embedding* $W \subset C([0,T]; L^1(\Omega))$ *holds true and is continuous.*

**Lemma 2.10.** ([12]) *Let* $M$ *be an* $N$*-function. If* $\mathcal{F}$ *is bounded in* $W_0^{1,x}L_M(Q_T)$ *and* $\left\{ \frac{\partial f}{\partial t} \,/\, f \in \mathcal{F} \right\}$ *is bounded in* $W^{-1,x}L_{\bar{M}}(Q_T)$ *then* $\mathcal{F}$ *is relatively compact in* $L^1(Q_T)$.

**Lemma 2.11.** ([12]) *Let* $Y$ *be a Banach space such that* $L^1(\Omega) \subset Y$ *with continuous embedding. If* $\mathcal{F}$ *is bounded in* $W_0^{1,x}L_M(Q_T)$ *and is relatively compact in* $L^1(0,T;Y)$ *then* $\mathcal{F}$ *is relatively compact in* $E_P(Q_T)$ *for all* $P \ll M$.

## 3. Assumptions and statement of the main results

In the sequel, $\Omega$ is a bounded open set in $\mathbb{R}^d$, $d \geq 2$ an integer, $T > 0$ is given and $Q_T = \Omega \times (0,T)$. We consider the Banach space $\mathbf{W}$ given as follows

$$\mathbf{W} = \left\{ v \in W_0^{1,x}L_M(Q_T) \,/\, \frac{\partial v}{\partial t} \in W^{-1,x}L_{\bar{M}}(Q_T) \right\}$$

provided with its standard norm

$$||v||_{\mathbf{W}} = ||v||_{W^{1,x}L_M(Q_T)} + \left\| \frac{\partial v}{\partial t} \right\|_{W^{-1,x}L_{\bar{M}}(Q_T)}.$$

Throughout this paper $\langle \cdot,\cdot\rangle$ stands for the duality pairing between the spaces $W^{1,x}L_M(Q_T)\cap L^2(Q_T)$ and $W^{-1,x}L_{\bar M}(Q_T)+L^2(Q_T)$ or between $W_0^{1,x}L_M(Q_T)$ and $W^{-1,x}L_{\bar M}(Q_T)$, and we assume the following assumptions:

$$M(t)=\int_0^{|t|} m(s)\,\mathrm{d}s \text{ and } P(t)=\int_0^{|t|} p(s)\,\mathrm{d}s \text{ are two } N\text{-functions such that} \\ P\ll M, \text{ and } t\le p(t) \text{ for all } t\ge 0. \tag{3.1}$$

Consider a second order partial differential operator

$$A\colon D(A)\subset W_0^{1,x}L_M(Q_T)\mapsto W^{-1,x}L_{\bar M}(Q_T)$$

in divergence form $A(u)=-\operatorname{div} a(x,t,u,\nabla u)$, where $a:\Omega\times(0,T)\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}^d$ is a Carathéodory function satisfying, for almost every $(x,t)\in Q_T$ and for all $s,s_1,s_2\in\mathbb{R}$, $\xi,\xi^*\in\mathbb{R}^d$,

$$|a(x,t,s,\xi)|\le\zeta\Big[c(x,t)+\bar M^{-1}(P(ks))+\bar M^{-1}(M(k|\xi|))\Big], \tag{3.2}$$

$$|a(x,t,s_1,\xi)-a(x,t,s_2,\xi)|\le\zeta\Big[e(x,t)+|s_1|+|s_2|+P^{-1}(kM(|\xi|))\Big], \tag{3.3}$$

$$(a(x,t,s,\xi)-a(x,t,s,\xi^*))(\xi-\xi^*)\ge\alpha M(|\xi-\xi^*|), \tag{3.4}$$

$$a(x,t,s,0)=0, \tag{3.5}$$

where $c\in E_{\bar M}(Q_T)$, $e\in E_P(Q_T)$ and $\alpha$, $\zeta$, $k>0$ are given real numbers.

$$\rho\in C(\mathbb{R}) \text{ and there exists } \bar\rho\in\mathbb{R} \text{ such that } 0<\rho(s)\le\bar\rho, \text{ for all } s\in\mathbb{R}, \tag{3.6}$$

$$\varphi_0\in L^2(0,T;H^1(\Omega))\cap L^\infty(Q_T), \tag{3.7}$$

$$u_0\in L^2(\Omega). \tag{3.8}$$

**Lemma 3.1.** *Let $P\colon\mathbb{R}\mapsto\mathbb{R}$ be an $N$-function with the representation $P(t)=\int_0^{|t|} p(s)\,\mathrm{d}s$, such that $s\le p(s)$, for all $s\ge 0$. Then the following continuous inclusions hold true: $L_P(\Omega)\hookrightarrow L^2(\Omega)\hookrightarrow L_{\bar P}(\Omega)$. In particular, $W_0^1L_P(\Omega)\hookrightarrow H_0^1(\Omega)$ and $H^{-1}(\Omega)\hookrightarrow W^{-1}L_{\bar P}(\Omega)$.*

*Furthermore, if $M$ is an $N$-function such that $P\ll M$, then the same continuous inclusions hold true for $M$, that is, $L_M(\Omega)\hookrightarrow L^2(\Omega)\hookrightarrow L_{\bar M}(\Omega)$, $W_0^1L_M(\Omega)\hookrightarrow H_0^1(\Omega)$ and $H^{-1}(\Omega)\hookrightarrow W^{-1}L_{\bar M}(\Omega)$.*

*Proof.* We have $P(t)=\int_0^{|t|} p(s)\,\mathrm{d}s\ge\int_0^{|t|} s\,\mathrm{d}s=t^2/2$, that is $t^2\le 2P(t)$ for all $t\in\mathbb{R}$. Consequently,

$$\int_\Omega v^2\,\mathrm{d}x\le 2\int_\Omega P(v)\,\mathrm{d}x, \text{ for all } v\in\mathcal{L}_P(\Omega). \tag{3.9}$$

Taking $v=u/\|u\|_{(P)}$ with $u\ne 0$ in (3.9) and using (2.8) it yields

$$\|u\|_{L^2(\Omega)}\le\sqrt{2}\|u\|_{(P)} \text{ for all } u\in L_P(\Omega),$$

and the first assertions of this Lemma are readily deduced.

Now let $P \ll M$. Owing to the convexity of $P$ and $M$ we can derive the following estimates

$$P(t) \leq P(1)|t| \text{ for } |t| \leq 1, \text{ and } P(t) \leq \frac{P(1)}{M(1)} M(t) \text{ for } |t| \geq 1.$$

Then, taking $v \in \mathcal{L}_M(\Omega)$ we deduce

$$\begin{aligned} \int_\Omega v^2 \,\mathrm{d}x &\leq 2 \int_{\{|v|<1\}} P(v) \,\mathrm{d}x + 2 \int_{\{|v|\geq 1\}} P(v) \,\mathrm{d}x \\ &\leq 2P(1) \int_\Omega |v| \,\mathrm{d}x + \frac{P(1)}{M(1)} \int_\Omega M(v) \,\mathrm{d}x \\ &\leq C_1 \|v\|_{(M)} + C_2 \int_\Omega M(v) \,\mathrm{d}x. \end{aligned}$$

Making $v = u/\|u\|_{(M)}$, $u \neq 0$, in this last inequality and using (2.8) we finally deduce

$$\|u\|_{L^2(\Omega)} \leq C_3 \|u\|_{(M)} \text{ for all } u \in L_M(\Omega),$$

where $C_3 = (C_1 + C_2)^{1/2}$. □

**Remark 3.2.** Under the assumptions of Lemma 3.1, we have

$$L^2(0,T;H^{-1}(\Omega)) \hookrightarrow W^{-1,x} L_{\bar{P}}(Q_T) \hookrightarrow W^{-1,x} E_{\bar{M}}(Q_T).$$

Indeed, let $f \in L^2(0,T;H^{-1}(\Omega))$. Then, for some $f_\alpha \in L^2(Q_T)$, $|\alpha| \leq 1$, $f = \sum_{|\alpha|\leq 1} \nabla_x^\alpha f_\alpha$. Since $L^2(Q_T) \subset L_{\bar{P}}(Q_T) \subset E_{\bar{M}}(Q_T)$ we deduce that $f \in W^{-1,x} L_{\bar{P}}(Q_T) \hookrightarrow W^{-1,x} E_{\bar{M}}(Q_T)$.

**Remark 3.3.** If we take the $N$-function $P(t) = |t|^r/r$, $1 < r < +\infty$, we are in the case of the classical Lebesgue spaces $L^r$ and we have, $\bar{P}(t) = |t|^{r'}/r'$ with $\frac{1}{r} + \frac{1}{r'} = 1$ and $p(t) = t^{r-1}$. The condition $0 \leq t \leq p(t)$ is equivalent to $r \geq 2$ and the following continuous inclusions hold $L^r(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L^{r'}(\Omega)$, and also $W_0^{1,r}(\Omega) \hookrightarrow H_0^1(\Omega)$ and $H^{-1}(\Omega) \hookrightarrow W^{-1,r'}(\Omega)$.

## 4. Definition of a capacity solution

The definition of a capacity solution for problem (1.1) can be stated as follows.

**Definition 4.1.** A triplet $(u, \varphi, \Phi)$ is called a capacity solution of (1.1) if the following conditions are fulfilled:

$(C_1)$ $u \in \mathbf{W}$, $a(x,t,u,\nabla u) \in L_{\bar{M}}(Q_T)^d, \varphi \in L^\infty(Q_T)$, $\Phi \in L^2(Q_T)^d$,

$(C_2)$ $(u, \varphi, \Phi)$ verifies the system of differential equations

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div} a(x,t,u,\nabla u) = \operatorname{div}(\varphi \Phi) \text{ in } Q_T, \\ \operatorname{div} \Phi = 0 \text{ in } Q_T, \end{cases} \tag{4.1}$$

$(C_3)$ For every $S \in C_0^1(\mathbb{R})$, one has $S(u)\varphi - S(0)\varphi_0 \in L^2(0,T;H_0^1(\Omega))$, and

$$S(u)\Phi = \rho(u)[\nabla(S(u)\varphi) - \varphi \nabla S(u)],$$

$(C_4)$ $u(\cdot,0) = u_0$ in $\Omega$.

Notice that, thanks to Lemma 2.9 and the regularity of $u$, we obtain in particular $u \in C([0,T]; L^1(\Omega))$ and thus the initial condition $(C_4)$ makes sense at least in $L^1(\Omega)$.

## 5. An existence result

This section is devoted to the proof of the following existence theorem which is the main result of this work.

**Theorem 5.1.** *Under the assumptions* (3.2)–(3.8)*, the system* (1.1) *admits a capacity solution.*

In order to prove this result, we will need to show the existence of a weak solution to a similar problem but with stronger assumptions, namely, there exists $c \in E_{\bar{M}}(Q_T)$, and two real numbers $\zeta > 0$ and $k \geq 0$, such that for almost every $(x,t) \in Q_T$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^d$,

$$|a(x,t,s,\xi)| \leq \zeta\Big[c(x,t) + \bar{M}^{-1}(M(k|\xi|))\Big], \tag{5.1}$$

$$\begin{cases} \rho \in C(\mathbb{R}) \ \text{ and there exist } \rho_1 \text{ and } \rho_2 \in \mathbb{R} \text{ such that} \\ 0 < \rho_1 \leq \rho(s) \leq \rho_2, \ \text{for all } s \in \mathbb{R}. \end{cases} \tag{5.2}$$

**Theorem 5.2.** *Assume* (3.2)–(3.8)*, with* (5.1) *and* (5.2) *instead of* (3.2) *and* (3.6)*, respectively. Then there exists a weak solution* $(u,\varphi)$ *to problem* (1.1)*, that is*

$$\begin{aligned} & u \in W_0^{1,x}L_M(Q_T) \cap C([0,T];L^2(\Omega)),\ a(x,t,u,\nabla u) \in L_{\bar{M}}(Q_T)^d, \\ & \varphi - \varphi_0 \in L^\infty(0,T;H_0^1(\Omega)) \cap L^\infty(Q_T), \\ & u(\cdot,0) = u_0 \ \textit{in } \Omega, \\ & \int_0^t \left\langle \frac{\partial u}{\partial t}, \phi \right\rangle + \int_0^t\int_\Omega a(x,t,u,\nabla u)\nabla\phi = -\int_0^t\int_\Omega \rho(u)\varphi\nabla\varphi\nabla\phi, \\ & \ \textit{for all } \phi \in W_0^{1,x}L_M(Q_T),\ \textit{for all } t \in [0,T], \\ & \int_\Omega \rho(u)\nabla\varphi\nabla\psi = 0,\ \textit{for all } \psi \in H_0^1(\Omega),\ \textit{a.e.}\ \ t \in (0,T). \end{aligned}$$

*Proof.* So as to prove the existence of a weak solution, Schauder's fixed point theorem will be applied together with the existence and uniqueness result of a weak solution to a parabolic equation.

For every $\omega \in E_P(Q_T)$ and almost everywhere $t \in (0,T)$, we consider the elliptic problem

$$\begin{cases} \operatorname{div}(\rho(\omega)\nabla\varphi) = 0 \text{ in } \Omega, \\ \varphi = \varphi_0 \text{ on } \partial\Omega \times (0,T). \end{cases} \tag{5.3}$$

Thanks to Lax-Milgram's theorem, (5.3) has an unique solution $\varphi(t) \in H^1(\Omega)$, in fact, $\varphi$ is measurable in $t$ with values in $H^1(\Omega)$ [3]. In that case, it is $\varphi \in L^\infty(0,T;H^1(\Omega))$. Indeed, by the maximum principle we have

$$||\varphi||_{L^\infty(Q_T)} \leq ||\varphi_0||_{L^\infty(Q_T)}. \tag{5.4}$$

Using $\varphi - \varphi_0 \in H_0^1(\Omega)$ as a test function in (5.3) we get,

$$\int_\Omega \rho(\omega)\nabla\varphi\nabla(\varphi - \varphi_0) = 0,$$

hence

$$\rho_1 \int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x \leq \int_\Omega \rho(\omega)|\nabla\varphi||\nabla\varphi_0| \,\mathrm{d}x \leq \rho_2 \int_\Omega |\nabla\varphi||\nabla\varphi_0| \,\mathrm{d}x.$$

By the Cauchy-Schwarz inequality, we obtain

$$\int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x \leq C(\rho_1, \rho_2, \varphi_0) = C, \text{ a.e. } t \in (0,T). \tag{5.5}$$

Notice that the right hand side in the original parabolic equation is $\rho(u)|\nabla\varphi|^2 \in L^1(\Omega \times (0,T))$. Thanks to the elliptic equation, this term also belongs to the space $L^2(0,T;H^{-1}(\Omega))$. Indeed, let $\phi \in \mathcal{D}(\Omega)$ and take $\xi = \phi\varphi$ as a test function in (5.3). We have, for a.e. $t \in [0,T]$,

$$\int_\Omega \rho(\omega)\nabla\varphi\nabla(\phi\varphi) \,\mathrm{d}x = 0,$$

that is

$$\int_\Omega \rho(\omega)|\nabla\varphi|^2\phi \,\mathrm{d}x = -\int_\Omega \rho(\omega)\varphi\nabla\varphi\nabla\phi \,\mathrm{d}x = \langle \operatorname{div}(\rho(\omega)\varphi\nabla\varphi), \phi\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}.$$

This means that

$$\rho(\omega)|\nabla\varphi|^2 = \operatorname{div}(\rho(\omega)\varphi\nabla\varphi) \text{ in } \mathcal{D}'(\Omega) \text{ and a.e. in } [0,T]. \tag{5.6}$$

Since $\rho(\omega)\varphi\nabla\varphi \in L^2(Q_T)^d$ we finally deduce the regularity

$$\operatorname{div}(\rho(\omega)\varphi\nabla\varphi) \in L^2(0,T;H^{-1}(\Omega)).$$

The identity (5.6) is one of the keys that allows us to solve the classical thermistor problem and the introduction of the notion of a capacity solution as well.

Now we introduce the following parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div} a(x,t,\omega,\nabla u) = \operatorname{div}(\rho(\omega)\varphi\nabla\varphi) \text{ in } Q_T, \\ u = 0 \text{ on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0 \text{ in } \Omega. \end{cases} \tag{5.7}$$

The variational formulation of the parabolic equation is given as follows.

$$\begin{cases} u \in W_0^{1,x}L_M(Q_T) \cap C([0,T];L^2(\Omega)), \ a(x,t,\omega,\nabla u) \in L_{\bar{M}}(Q_T), \\ \displaystyle\int_0^t \left\langle \frac{\partial u}{\partial t}, \phi \right\rangle + \int_0^t\int_\Omega a(x,t,\omega,\nabla u)\nabla\phi = -\int_0^t\int_\Omega \rho(\omega)\varphi\nabla\varphi\nabla\phi, \\ \qquad \text{for all } \phi \in W_0^{1,x}L_M(Q_T), \text{ for all } t \in [0,T], \\ u(\cdot,0) = u_0 \text{ in } \Omega. \end{cases} \tag{5.8}$$

Notice that $\operatorname{div}(\rho(\omega)\varphi\nabla\varphi) \in L^2(0,T;H^{-1}(\Omega)) \hookrightarrow W^{-1,x}E_{\bar{M}}(Q_T)$ due to (5.3), (5.4), (5.5), Lemma 3.1 and Remark 3.2.

The existence of a solution to (5.8) is obtained by a straightforward application of Theorem 1, p. 107 in [10]. Also we can easily check that the solution of (5.8) is unique [2] Now, we show that $|\nabla u| \in \mathcal{L}_M(Q_T)$, and the estimates

$$\int_0^T \int_\Omega M(|\nabla u|)\,\mathrm{d}x\,\mathrm{d}t \le C(u_0, \varphi_0, \alpha, T, \rho_2) = C_0, \tag{5.9}$$

$$\|a(x,t,\omega,\nabla u)\|_{\bar{M},Q_T} \le C_1, \tag{5.10}$$

where $C_1$ only depends on data, but not on $\omega$. Indeed, let $\lambda > 0$ such that $|\nabla u|/\lambda \in \mathcal{L}_M(Q_T)$. Since $\varphi \in L^2(0,T;H^1(\Omega)) \subset W^{1,x}L_{\bar{M}}(Q_T)$, there exists $\mu > 0$ such that $\frac{2}{\alpha\mu}\rho_2||\varphi_0||_{L^\infty(Q_T)}|\nabla\varphi| \in \mathcal{L}_{\bar{M}}(Q_T)$. By taking $\phi = u$ as a test function in (5.8), from (3.4), (3.5), (5.2), (5.4) and Young's inequality, we obtain

$$\frac{\alpha}{\lambda\mu}\int_0^T\!\!\int_\Omega M(|\nabla u|)\,\mathrm{d}x\,\mathrm{d}t \le \frac{1}{\lambda\mu}\int_0^T\!\!\int_\Omega a(x,t,\omega,\nabla u)\nabla u\,\mathrm{d}x\,\mathrm{d}t \le \frac{1}{2\lambda\mu}||u_0||^2_{L^2(\Omega)}$$
$$+\frac{\alpha\mu}{2}\int_0^T\!\!\int_\Omega \bar{M}\left(\frac{2}{\alpha\mu}\rho_2||\varphi_0||_{L^\infty(Q_T)}|\nabla\varphi|\right)\mathrm{d}x\,\mathrm{d}t + \frac{\alpha}{2\mu}\int_0^T\!\!\int_\Omega M(|\nabla u|/\lambda)\,\mathrm{d}x\,\mathrm{d}t.$$

This shows that $|\nabla u| \in \mathcal{L}_M(Q_T)$ and, consequently, the estimate (5.9) is derived by just taking $\lambda = 1$ in this last inequality. In order to obtain (5.10), first notice that from the last inequality we also have

$$\int_0^T \int_\Omega a(x,t,\omega,\nabla u)\nabla u\,\mathrm{d}x\,\mathrm{d}t \le \alpha C_0. \tag{5.11}$$

Then, owing to (3.4), for any $\phi \in W_0^{1,x}E_M(Q_T)$ such that $\|\nabla\phi\|_{M,Q_T} = 1/(k+1)$ it yields

$$0 \le \int_0^T\!\!\int_\Omega (a(x,t,\omega,\nabla u) - a(x,t,\omega,\nabla\phi))(\nabla u - \nabla\phi)\,\mathrm{d}x\,\mathrm{d}t,$$

and thus, using (5.11) and Young's inequality,

$$\begin{aligned}
&\int_0^T\!\!\int_\Omega a(x,t,\omega,\nabla u)\nabla\phi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\le \int_0^T\!\!\int_\Omega a(x,t,\omega,\nabla u)\nabla u\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\!\!\int_\Omega a(x,t,\omega,\nabla\phi)(\nabla u - \nabla\phi)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\le \alpha C_0 + \int_0^T\!\!\int_\Omega |a(x,t,\omega,\nabla\phi)\nabla u|\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_\Omega a(x,t,\omega,\nabla\phi)\nabla\phi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\le \alpha C_0 + 2\zeta\int_0^T\!\!\int_\Omega \left[\bar{M}\left(\frac{a(x,t,\omega,\nabla\phi)}{2\zeta}\right) + M(|\nabla u|)\right]\mathrm{d}x\,\mathrm{d}t\\
&\qquad+2\zeta\int_0^T\!\!\int_\Omega \left[\bar{M}\left(\frac{a(x,t,\omega,\nabla\phi)}{2\zeta}\right) + M(|\nabla\phi|)\right]\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

where $\zeta$ is the constant appearing in (5.1). Since

$$\bar{M}\left(\frac{a(x,t,\omega,\nabla\phi)}{2\zeta}\right) \le \frac{1}{2}(\bar{M}(c(x,t)) + M(k|\nabla\phi|)) \text{ a.e. in } Q_T,$$

then, using (2.7)

$$\int_0^T\int_\Omega \bar{M}\left(\frac{a(x,t,\omega,\nabla\phi)}{3\zeta}\right)\mathrm{d}x\,\mathrm{d}t \le \frac{1}{2}\int_0^T\int_\Omega \bar{M}(c(x,t))\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2} = C_2.$$

Notice that $C_2$ only depends on data (but not on $\omega$). Therefore, gathering all these estimates, we deduce for all $\phi \in W_0^{1,x}E_M(Q_T)$ such that $\|\nabla\phi\|_{M,Q_T} = 1/(k+1)$

$$\int_0^T\int_\Omega a(x,t,\omega,\nabla u)\nabla\phi\,\mathrm{d}x\,\mathrm{d}t \le C_1,$$

which finally yields the estimate (5.10) by considering the dual norm on $L_{\bar{M}}(Q_T)$.

Also from (3.2), (5.2), (5.4), (5.5) and (5.10) we obtain

$$\frac{\partial u}{\partial t} \in W^{-1,x}L_{\bar{M}}(Q_T) \text{ and } \left\|\frac{\partial u}{\partial t}\right\|_{W^{-1,x}L_{\bar{M}}(Q_T)} \le C_3, \tag{5.12}$$

where, again, $C_3$ is a constant depending only on data, but not on $\omega$.

We may define the operator $G\colon \omega \in E_P(Q_T) \longrightarrow G(\omega) = u \in \mathbf{W}$, with $u$ being the unique solution to (5.8). From Lemma 2.10, and Lemma 2.11 with $Y = L^1(\Omega)$, we have that $\mathbf{W} \hookrightarrow E_P(Q_T)$ with compact embedding. Consequently, $G$ maps $E_P(Q_T)$ into itself and, due to the estimates (5.9) and (5.12), $G$ is a compact operator. Moreover, from (5.9) we have, for $R > 0$ large enough $G(B_R) \subset B_R$ where $B_R = \{v \in E_P(Q_T) \,/\, ||v||_{L_P(Q_T)} \le R\}$.

To complete the proof, it remains to show that $G$ is a continuous operator. Thus, let $(\omega_n) \subset B_R$ be a sequence such that $\omega_n \to \omega$ strongly in $E_P(Q_T)$ and consider the corresponding functions to $\omega_n$, that is, $u_n = G(\omega_n)$ and $\varphi_n$ and put $F_n = \rho(\omega_n)\varphi_n\nabla\varphi_n$ and $F = \rho(\omega)\varphi\nabla\varphi$. We have to show that

$$u_n \to u = G(\omega) \quad \text{strongly in} \quad E_P(Q_T).$$

Owing to $P \ll M$ and (5.9), we have $\nabla u \in E_P(Q_T)^d$. Since the inclusion $L_P(Q_T) \subset L^2(Q_T)$ is continuous, we also have $\omega_n \to \omega$ strongly in $L^2(Q_T)$ and thus, we may extract a subsequence, still denoted in the same way, such that $\omega_n \to \omega$ a.e. in $Q_T$. Then, it is an easy task to show that $\varphi_n \to \varphi$ strongly in $L^2(0,T;H^1(Q_T))$ and, consequently, also for another subsequence denoted in the same way, $F_n \to F$ strongly in $L^2(Q_T)$.

On the other hand, since $(\omega_n) \subset L_P(Q_T)$ is bounded, in virtue of the estimates obtained above, we deduce, again modulo a subsequence,

$$u_n \to U \text{ in } E_P(Q_T), \text{ for some } U \in E_P(Q_T), \tag{5.13}$$

$$\nabla u_n \to \nabla U \text{ weakly in } L^2(Q_T)^d, \tag{5.14}$$

By subtracting the respective equations of (5.8) for $u_n$ and $u$, and taking $\phi = u_n - u$ as a test function, for all $t \in [0,T]$, we obtain

$$\begin{aligned}\frac{1}{2}||u_n(t)-u(t)||^2_{L^2(\Omega)} \\ +\int_0^t\int_\Omega (a(x,s,\omega_n,\nabla u_n)-a(x,s,\omega,\nabla u))\nabla(u_n-u) \\ = -\int_0^t\int_\Omega (F_n-F)\nabla(u_n-u). \qquad (5.15)\end{aligned}$$

By using (3.4), we get

$$\begin{aligned}(a(x,s,\omega_n,\nabla u_n)-a(s,t,\omega,\nabla u))\nabla(u_n-u) \geq \alpha M(|\nabla(u_n-u)|) \\ +(a(x,s,\omega_n,\nabla u)-a(x,s,\omega,\nabla u))\nabla(u_n-u).\end{aligned}$$

Let $h_n = a(x,s,\omega_n,\nabla u) - a(x,s,\omega,\nabla u)$ and $g_n = \nabla(u_n-u)$. Then, $|h_n| \to 0$ a.e. in $Q_T$. For a given positive number $\lambda_0$, to be chosen later, we have

$$\int_0^t\int_\Omega |h_n g_n| = \int_{\{|g_n|\leq\lambda_0\}} |h_n g_n| + \int_{\{|g_n|>\lambda_0\}} |h_n g_n|. \qquad (5.16)$$

For the first term of the right hand side of (5.16), we have

$$\int_{\{|g_n|\leq\lambda_0\}} |h_n g_n| \leq \lambda_0 \int_{Q_T} |h_n| = \lambda_0 \int_{\{|h_n|\leq 4\zeta\}} |h_n| + \lambda_0 \int_{\{|h_n|>4\zeta\}} |h_n|.$$

The first of these last integrals converges trivially to zero. As for second one, using the fact that $\frac{|h_n|}{4\zeta} > 1$ on the set $\{|h_n| > 4\zeta\}$ and (3.9), it yields

$$\lambda_0 \int_{\{|h_n|>4\zeta\}} |h_n| \leq 4\zeta\lambda_0 \int_{\{|h_n|>4\zeta\}} \left(\frac{|h_n|}{4\zeta}\right)^2 \leq 8\zeta\lambda_0 \int_{Q_T} P\left(\frac{|h_n|}{4\zeta}\right).$$

In virtue of (3.3), we deduce

$$P\left(\frac{|h_n|}{4\zeta}\right) \leq \frac{1}{4}\left(P(e)+P(\omega_n)+P(\omega)+kM(|\nabla u|)\right),$$

and since $P(\omega_n) \to P(\omega)$ strongly in $L^1(Q_T)$, by Lebesgue's dominated theorem it yields that

$$\lim_{n\to\infty} \int_{Q_T} P\left(\frac{|h_n|}{4\zeta}\right) = 0,$$

and consequently

$$\lim_{n\to\infty} \int_{\{|g_n|\leq\lambda_0\}} |h_n g_n| = 0.$$

For the second term of the right hand side of (5.16), we use Young's inequality and (3.9). It yields,

$$\begin{aligned}\int_{\{|g_n|>\lambda_0\}} |h_n g_n| &\leq \frac{1}{2\alpha}\int_{Q_T} |h_n|^2 + \frac{\alpha}{2}\int_{\{|g_n|>\lambda_0\}} |g_n|^2 \\ &\leq \frac{(4\zeta)^2}{\alpha}\int_{Q_T} P\left(\frac{|h_n|}{4\zeta}\right) + \alpha\int_{\{|g_n|>\lambda_0\}} P(|g_n|).\end{aligned}$$

It has been already shown that the first of these terms converges to zero. As for the second one, since $P \ll M$, we can take $\lambda_0$ large enough such that $P(s) \leq M(s)$ for $|s| > \lambda_0$, and then,

$$\alpha \int_{\{|g_n|>\lambda_0\}} P(|g_n|) \leq \alpha \int_0^t \int_\Omega M(|g_n|) = \alpha \int_0^t \int_\Omega M(|\nabla(u_n - u)|).$$

Consequently, for some sequence $(\epsilon_n) \subset \mathbb{R}$, $\epsilon_n \to 0$, we have the following estimate

$$\frac{1}{2}||u_n(t) - u(t)||^2_{L^2(\Omega)} \leq -\int_0^t \int_\Omega (F_n - F)\nabla(u_n - u)\,\mathrm{d}x\,\mathrm{d}s + \epsilon_n,$$

and integrating this inequality over $[0, T]$, we have

$$\frac{1}{2}||u_n - u||^2_{L^2(Q_T)} \leq -\int_0^T \int_\Omega (T-t)(F_n - F)\nabla(u_n - u)\,\mathrm{d}x\,\mathrm{d}t + T\epsilon_n. \quad (5.17)$$

The first term of right hand side in (5.17) converges to zero since $F_n \to F$ strongly in $L^2(Q_T)^d$ and $(T-t)(\nabla u_n - \nabla u)$ is bounded in $L^2(Q_T)^d$. In conclusion, $u_n \to u$ strongly in $L^2(Q_T)$. Since this limit does not depend upon the subsequence one may extract, it is in fact the whole sequence $(u_n)$ which converges to $u$ strongly in $L^2(Q_T)$. On the other hand, in virtue of (5.13), we also have $u_n \to U$ strongly in $L^2(Q_T)$, so that $u = U$ and we can rewrite (5.13) to give $u_n \to u$ strongly in $E_P(Q_T)$. This shows that $G$ is continuous and this ends the proof of Theorem 5.2. □

**Remark 5.3.** It can be easily shown that we can rid of the assumption (3.3) in Theorem 5.2 when $\bar{M}$ verifies the $\Delta_2$-condition. Also in the case $a(x,t,s,\xi) = \tilde{a}(x,t,\xi)$.

**Proof of Theorem 5.1**

The proof is divided into several steps, first we introduce a sequence of approximate problems and derive a priori estimates for the approximate problem and we show two intermediate results, namely the strong convergence in $L^1(Q_T)$ of both $\nabla u_n$ and $\varphi_n$, where $(u_n, \varphi_n)$ is a weak solution to the approximate problem of (1.1).

**Step 1**. For every $n \in \mathbb{N}$, we introduce the following regularization of the data,

$$\rho_n(s) = \rho(s) + \frac{1}{n}, \quad (5.18)$$

$$a_n(x,t,s,\xi) = a(x,t,T_n(s),\xi), \quad (5.19)$$

and consider the approximate system given as

$$\frac{\partial u_n}{\partial t} - \operatorname{div}\Big(a_n(x,t,u_n,\nabla u_n)\Big) = \rho_n(u_n)|\nabla\varphi_n|^2 \ \text{ in } Q_T, \quad (5.20)$$

$$\operatorname{div}(\rho_n(u_n)\nabla\varphi_n) = 0 \ \text{ in } Q_T, \quad (5.21)$$

$$u_n = 0 \ \text{ on } (0,T)\times\partial\Omega, \quad (5.22)$$

$$\varphi_n = \varphi_0 \ \text{ on } (0,T)\times\partial\Omega, \quad (5.23)$$

$$u_n(\cdot,0) = u_0 \ \text{ in } \Omega. \quad (5.24)$$

From (3.2) we deduce

$$\begin{aligned}|a(x,t,T_n(s),\xi)| &\le \zeta\Big[c(x,t) + \bar{M}^{-1}(P(k|T_n(s)|)) + \bar{M}^{-1}(M(k|\xi|))\Big]\\ &\le \zeta\Big[c_n(x,t) + \bar{M}^{-1}(M(k|\xi|))\Big],\end{aligned}$$

where $c_n \in E_{\bar{M}}(Q_T)$ is given by $c_n(x,t) = c(x,t) + \bar{M}^{-1}(P(kn))$. Also, in view of (3.6), we have that

$$n^{-1} \le \rho_n(s) \le \rho_3 + 1 = \rho_4, \text{ for all } s \in \mathbb{R}. \tag{5.25}$$

Thus, we can apply Theorem 5.2 to deduce the existence of a weak solution $(u_n, \varphi_n)$ to the system (5.20)–(5.24).

By the maximum principle we have

$$||\varphi_n||_{L^\infty(Q_T)} \le ||\varphi_0||_{L^\infty(Q_T)}, \tag{5.26}$$

hence there exists a function $\varphi \in L^\infty(Q_T)$ and a subsequence, still denoted in the same way, such that

$$\varphi_n \to \varphi \text{ weakly-} * \text{ in } L^\infty(Q_T). \tag{5.27}$$

Now let multiply (5.21) by $\varphi_n - \varphi_0 \in L^2(0,T;H_0^1(\Omega))$ and integrate over $Q_T$. We get

$$\int_0^T \int_\Omega \rho_n(u_n)\nabla\varphi_n\nabla(\varphi_n - \varphi_0)\,\mathrm{d}x\,\mathrm{d}t = 0,$$

hence

$$\int_0^T \int_\Omega \rho_n(u_n)|\nabla\varphi_n|^2\,\mathrm{d}x\,\mathrm{d}t \le C_1, \text{ for all } n \ge 1, \tag{5.28}$$

where $C_1 = C_1(\bar{\rho}, \|\varphi_0\|_{L^2(0,T;H^1(\Omega))})$. Consequently, the sequence $(\rho_n(u_n)\nabla\varphi_n)$ is bounded in $L^2(Q_T)$. Thus, there exists a function $\Phi \in L^2(Q_T)^d$ and a subsequence, still denoted in the same way, such that

$$\rho_n(u_n)\nabla\varphi_n \to \Phi \text{ weakly in } (L^2(Q_T))^d. \tag{5.29}$$

This weak limit function $\Phi \in (L^2(Q_T))^d$ is in fact the third component of the triplet appearing in the Definition 4.1 of a capacity solution.

Taking $u_n$ as a test function in (5.20), for all $t \in [0,T]$, we obtain

$$\begin{aligned}&\frac{1}{2}||u_n(t)||^2_{L^2(\Omega)} + \int_0^t \int_\Omega a(x,t,T_n(u_n),\nabla u_n)\nabla u_n\,\mathrm{d}x\,\mathrm{d}t\\ &\quad = \frac{1}{2}||u_0||^2_{L^2(\Omega)} - \int_0^t \int_\Omega \rho_n(u_n)\varphi_n\nabla\varphi_n\nabla u_n\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{5.30}$$

From (3.4), (3.5), (5.26) and (5.25), we get

$$\alpha\int_0^t \int_\Omega M(|\nabla u_n|)\,\mathrm{d}x\,\mathrm{d}t \le \frac{1}{2}||u_0||^2_{L^2(\Omega)} + \int_0^t \int_\Omega ||\varphi_0||_{L^\infty(Q_T)}\rho_2\nabla\varphi_n\nabla u_n\,\mathrm{d}x\,\mathrm{d}t, \tag{5.31}$$

and in virtue of Young's inequality, we may deduce, for all $t \in [0,T]$,

$$\int_0^t \int_\Omega M(|\nabla u_n|)\,\mathrm{d}x\,\mathrm{d}t \le C, \tag{5.32}$$

where $C$ is a positive constant not depending on $n$. It follows that the sequence $(u_n)$ is bounded in $W_0^{1,x}L_M(Q_T)$. Consequently, there exist a subsequence of $(u_n)$, still denoted in the same way, and a function $u \in W_0^{1,x}L_M(Q_T)$ such that:

$$u_n \rightharpoonup u \text{ in } W_0^{1,x}L_M(Q_T) \text{ for } \sigma(\Pi L_M, \Pi E_{\bar{M}}). \tag{5.33}$$

On the other hand, Let $\phi \in W_0^{1,x}E_M(Q_T)^d$ be arbitrary with $\|\nabla\phi\|_{(M)} = 1/(k+1)$. In view of the monotonicity of $a_n$, one easily has

$$\begin{cases} \displaystyle\int_{Q_T} a_n(x,t,u_n,\nabla u_n)\nabla\phi \leq \int_{Q_T} a_n(x,t,u_n,\nabla u_n)\nabla u_n \\ \qquad \displaystyle - \int_{Q_T} a_n(x,t,u_n,\nabla\phi)(\nabla u_n - \nabla\phi) \\ \qquad \displaystyle \leq C + \int_{Q_T} |a_n(x,t,u_n,\nabla\phi)\nabla u_n| + \int_{Q_T} a_n(x,t,u_n,\nabla\phi)\nabla\phi. \end{cases} \tag{5.34}$$

We can show that the two last integrals in (5.34) are bounded with respect to $n$. Indeed, for the first one, by Young's inequality

$$\int_{Q_T} |a_n(x,t,u_n,\nabla\phi)\nabla u_n| \leq 3\zeta \int_{Q_T} \left[ \bar{M}\left( \frac{a(x,t,T_n(u_n),\nabla\phi)}{3\zeta} \right) + M(|\nabla u_n|) \right],$$

using (3.2) we have

$$3\zeta\bar{M}\left( \frac{a(x,t,T_n(u_n),\nabla\phi)}{3\zeta} \right) \leq \zeta\left( \bar{M}(c(x,t)) + P(kT_n(u_n)) + M(k\nabla\phi) \right),$$

since $(u_n)$ is bounded in $W_0^{1,x}L_M(Q_T)$, and owing to Poincare's inequality, there exists $\lambda > 0$ such that $\int_{Q_T} M(u_n/\lambda) \leq 1$ for all $n \geq 1$. Also, since $P \ll M$, there exists $s_0 > 0$ such that $P(ks) \leq P(ks_0) + M(s/\lambda)$ for all $s \in \mathbb{R}$. Consequently,

$$\begin{aligned} 3\zeta \int_{Q_T} \bar{M}\left( \frac{a(x,t,T_n(u_n),\nabla\phi)}{3\zeta} \right) &\leq \zeta \Bigg( \int_{Q_T} \bar{M}(c(x,t)) + |Q_T|P(ks_0) \\ &\quad + \int_{Q_T} M(u_n/\lambda) + \int_{Q_T} M(k\nabla\phi) \Bigg) \leq C, \end{aligned}$$

and thus $\int_{Q_T}|a_n(x,t,u_n,\nabla\phi)\nabla u_n| \leq C$, for all $n \geq 1$ and $\phi \in W_0^{1,x}E_M(Q_T)^d$ such that $\|\nabla\phi\|_{(M)} = 1/(k+1)$. On the other hand, the second integral in (5.34), namely $\int_{Q_T} a_n(x,t,u_n,\nabla\phi)\nabla\phi$ can be dealt in the same way so that it is easy to check that it is also bounded. Gathering all these estimates, and using the dual norm, one easily deduce that

$$(a_n(x,t,u_n,\nabla u_n)) \text{ is bounded in } L_{\bar{M}}(Q_T)^d. \tag{5.35}$$

Thus, up to a subsequence, still denoted in the same way, there exists $\delta \in L_{\bar{M}}(Q_T)^d$ such that

$$a_n(x,t,u_n,\nabla u_n) \rightharpoonup \delta \text{ in } L_{\bar{M}}(Q_T)^d \text{ for } \sigma(\Pi L_{\bar{M}}, \Pi E_M). \tag{5.36}$$

Finally, since both sequences $(\operatorname{div} a_n(x,t,u_n,\nabla u_n))$ and $(\operatorname{div}(\rho_n(u_n)\varphi_n\nabla\varphi_n))$ are bounded in the space $W^{-1,x}L_{\bar{M}}(Q_T)$ then, according to (5.20), we have

$$\left(\frac{\partial u_n}{\partial t}\right) \text{ is bounded in } W^{-1,x}L_{\bar{M}}(Q_T). \tag{5.37}$$

Consequently, $(u_n)\subset \mathbf{W}$ is bounded and, since the embedding $\mathbf{W}\hookrightarrow E_P(Q_T)$ is compact, for a subsequence, still denoted in the same way, we have

$$u_n \to u \text{ strongly in } E_P(Q_T) \text{ and a.e. in } Q_T, \tag{5.38}$$

where $u\in W_0^{1,x}L_M(Q_T)$ is also the limit function appearing in (5.33).

**Step 2**. Introduction of regularized sequences and the almost everywhere convergence of the gradients.

We first introduce two smooth sequences, namely, $(v_j)\subset\mathcal{D}(Q_T)$ and $(\psi_i)\subset\mathcal{D}(\Omega)$ such that

1. $v_j\to u$ in $W_0^{1,x}L_M(Q_T)$ for the modular convergence;
2. $v_j\to u$ and $\nabla v_j\to\nabla u$ and almost everywhere in $Q_T$;
3. $\psi_i\to u_0$ strongly in $L^2(\Omega)$;
4. $\|\psi_i\|_{L^2(\Omega)}\le 2\|u_0\|_{L^2(\Omega)}$, for all $i\ge 1$.

For a fixed positive real number $K$, we consider the truncation function at height $K$, $T_K$, defined in (2.11). Then, for every $K,\mu>0$ and $i,j\in\mathbb{N}$, we introduce the function $w^i_{\mu,j}\in W_0^{1,x}L_M(Q_T)$ (to simplify the notation, we drop out the index $K$) defined as $w^i_{\mu,j}=T_K(v_j)_\mu+e^{-\mu t}T_K(\psi_i)$, where $T_K(v_j)_\mu$ is the mollification with respect to time of $T_K(v_j)$ given in (2.12). From Lemma (2.7), we know that

$$\begin{aligned} &\frac{\partial w^i_{\mu,j}}{\partial t}=\mu(T_K(v_j)-w^i_{\mu,j}),\ w^i_{\mu,j}(\cdot,0)=T_K(\psi_i),\\ &|w^i_{\mu,j}|\le K \text{ a.e in } Q_T, \end{aligned} \tag{5.39}$$

$$w^i_{\mu,j}\to w^i_\mu \overset{\text{def}}{=} T_K(u)_\mu+e^{-\mu t}T_K(\psi_i) \text{ in } W_0^{1,x}L_M(Q_T), \tag{5.40}$$

for the modular convergence as $j\to\infty$.

$$T_K(u)_\mu+e^{-\mu t}T_K(\psi_i)\to T_K(u) \text{ in } W_0^{1,x}L_M(Q_T), \tag{5.41}$$

for the modular convergence as $\mu\to\infty$. Since we may consider subsequences in (5.39)–(5.41), we will assume without loss of generality that the convergences (5.40) and (5.41) also hold almost everywhere in $Q_T$.

We will establish the following proposition.

**Proposition 5.4.** *Let $(u_n,\varphi_n)$ be a solution of the approximate problem* (5.20)–(5.24). *Then,*

$$\nabla u_n\to\nabla u \text{ a.e. in } Q_T, \tag{5.42}$$

*as $n$ tends to $+\infty$.*

*Proof.* In the sequel and throughout the paper, $\chi^j_s$ and $\chi_s$ will denote, respectively, the characteristic functions of the sets

$$Q^j_s=\Big\{(x,t)\in Q_T\,/\,|\nabla T_K(v_j)|\le s\Big\} \text{ and } Q_s=\Big\{(x,t)\in Q_T\,/\,|\nabla T_K(u)|\le s\Big\}.$$

We also introduce the primitive of the truncation function $T_K$ vanishing at the origin, $\Theta_K$, that is

$$\Theta_K(t) = \int_0^t T_K(s)ds = \begin{cases} t^2/2 & \text{if } |t| \le K, \\ K|t| - K^2/2 & \text{if } |t| > K. \end{cases} \tag{5.43}$$

It is straightforward to show that $0 \le \Theta_K(t) \le K|t|$ for all $t \in \mathbb{R}$.

We will also make use of the following notation for vanishing sequences: $\epsilon(n)$ means a sequence such that $\lim_{n\to\infty} \epsilon(n) = 0$ or $\limsup_{n\to\infty} \epsilon(n) = 0$; $\epsilon(n, j)$ is a term such that $\lim_{j\to\infty} \lim_{n\to\infty} \epsilon(n, j) = 0$ where any occurrence of lim may be substituted by lim sup. And so on for $\epsilon(n, j, \mu)$, etc.

For any $\mu, \nu > 0$ and $i, j, n \ge 1$ we may use the admissible test function $\varphi^{\mu,i}_{n,j,\nu} = T_\nu(u_n - w^i_{\mu,j})$ in (5.20). This leads to

$$\begin{cases} \left\langle \dfrac{\partial u_n}{\partial t}, \varphi^{\mu,i}_{n,j,\nu} \right\rangle + \displaystyle\int_{Q_T} a_n(x, t, u_n, \nabla u_n)\nabla T_\nu(u_n - w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t \\ \qquad = \displaystyle\int_{Q_T} \rho_n(u_n)|\nabla\varphi_n|^2 \varphi^{\mu,i}_{n,j,\nu}\,\mathrm{d}x\,\mathrm{d}t. \end{cases} \tag{5.44}$$

By using (5.28), we get

$$\left\langle \frac{\partial u_n}{\partial t}, \varphi^{\mu,i}_{n,j,\nu} \right\rangle + \int_{Q_T} a_n(x, t, u_n, \nabla u_n)\nabla T_\nu(u_n - w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t \le C_1\nu. \tag{5.45}$$

As far as the parabolic term is concerned, we have

$$\begin{aligned} \left\langle \frac{\partial u_n}{\partial t}, T_\nu(u_n - w^i_{\mu,j}) \right\rangle &= \left\langle \frac{\partial u_n}{\partial t} - \frac{\partial w^i_{\mu,j}}{\partial t}, T_\nu(u_n - w^i_{\mu,j}) \right\rangle \\ &\quad + \left\langle \frac{\partial w^i_{\mu,j}}{\partial t}, T_\nu(u_n - w^i_{\mu,j}) \right\rangle. \end{aligned} \tag{5.46}$$

The first term of the right hand side in (5.46) can be written as

$$\begin{aligned} &\left\langle \frac{\partial u_n}{\partial t} - \frac{\partial w^i_{\mu,j}}{\partial t}, T_\nu(u_n - w^i_{\mu,j}) \right\rangle \\ &\quad = \int_\Omega \Theta_\nu(u_n(T) - w^i_{\mu,j}(T)) - \int_\Omega \Theta_\nu(u_0 - T_K(\psi_i)). \end{aligned}$$

Since $0 \le \int_\Omega \Theta_\nu(u_0 - T_K(\psi_i)) \le \nu \int_\Omega |u_0 - T_K(\psi_i)| \le \nu|\Omega|^{1/2}(\int_\Omega |u_0 - T_K(\psi_i)|^2)^{1/2} \le 3\|u_0\|_{L^2(\Omega)}|\Omega|^{1/2}\nu = C_2\nu$, we deduce that, for all $i, j, n \ge 1$ and $\mu, n, K > 0$, it is

$$\left\langle \frac{\partial u_n}{\partial t} - \frac{\partial w^i_{\mu,j}}{\partial t}, T_\nu(u_n - w^i_{\mu,j}) \right\rangle \ge -C_2\nu. \tag{5.47}$$

As for the second term of the right hand side in (5.46) we have

$$\left\langle \frac{\partial w^i_{\mu,j}}{\partial t}, T_\nu(u_n - w^i_{\mu,j}) \right\rangle = \mu \int_{Q_T} (T_K(v_j) - w^i_{\mu,j})T_\nu(u_n - w^i_{\mu,j}). \tag{5.48}$$

Passing to the limit first in $n \to \infty$, then in $j \to \infty$, it yields

$$\lim_{j\to\infty}\lim_{n\to\infty}\left\langle \frac{\partial w^i_{\mu,j}}{\partial t}, T_\nu(u_n - w^i_{\mu,j})\right\rangle = \mu\int_{Q_T}(T_K(u) - w^i_\mu)T_\nu(u - w^i_\mu).$$

Owing to (5.39) and (5.40) we have $|w^i_\mu| \le K$ almost everywhere in $Q_T$. Also, since $sT_\nu(s) \ge 0$ for all $s \in \mathbb{R}$, we deduce, for all $\mu, \nu, K > 0$ and $i \ge 1$,

$$\lim_{j\to\infty}\lim_{n\to\infty}\left\langle \frac{\partial w^i_{\mu,j}}{\partial t}, T_\nu(u_n - w^i_{\mu,j})\right\rangle \ge 0. \tag{5.49}$$

Gathering (5.46), (5.47) and (5.49) we finally obtain, for all $\mu, \nu, K > 0$ and $i \ge 1$, the following estimate for the parabolic term

$$\liminf_{j\to\infty}\liminf_{n\to\infty}\left\langle \frac{\partial u_n}{\partial t}, T_\nu(u_n - w^i_{\mu,j})\right\rangle \ge -C_2\nu. \tag{5.50}$$

It remains to analyze the diffusion term of (5.44). We have

$$\begin{aligned}
&\int_{Q_T} a_n(x,t,u_n,\nabla u_n)\nabla T_\nu(u_n - w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_{\{|u_n - w^i_{\mu,j}|\le\nu\}} a_n(x,t,u_n,\nabla u_n)\nabla(u_n - w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_{\{|u_n|>K\}\cap\{|u_n - w^i_{\mu,j}|\le\nu\}} a_n(x,t,u_n,\nabla u_n)\nabla(u_n - w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_{\{|u_n|\le K\}\cap\{|u_n - w^i_{\mu,j}|\le\nu\}} a_n(x,t,u_n,\nabla u_n)\nabla(u_n - w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_{\{|T_K(u_n) - w^i_{\mu,j}|\le\nu\}} a_n(x,t,T_K(u_n),\nabla T_K(u_n))(\nabla T_K(u_n) - \nabla w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_{\{|u_n|>K\}\cap\{|u_n - w^i_{\mu,j}|\le\nu\}} a_n(x,t,u_n,\nabla u_n)\nabla u_n\,\mathrm{d}x\,\mathrm{d}t\\
&\quad - \int_{\{|u_n|>K\}\cap\{|u_n - w^i_{\mu,j}|\le\nu\}} a_n(x,t,u_n,\nabla u_n)\nabla w^i_{\mu,j}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

By (3.4) and (3.5) we have

$$\begin{aligned}
&\int_{\{|u_n|>K\}\cap\{|u_n - w^i_{\mu,j}|\le\nu\}} a_n(x,t,u_n,\nabla u_n)\nabla u_n\,\mathrm{d}x\,\mathrm{d}t\\
&\quad \ge \alpha\int_{\{|u_n|>K\}\cap\{|u_n - w^i_{\mu,j}|\le\nu\}} M(|\nabla u_n|)\,\mathrm{d}x\,\mathrm{d}t \ge 0,
\end{aligned}$$

which implies that

$$\begin{cases}\displaystyle\int_{Q_T} a_n(x,t,u_n,\nabla u_n)\nabla T_\nu(u_n-w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t\\ \quad\displaystyle\geq \int_{\{|T_K(u_n)-w^i_{\mu,j}|\leq\nu\}} a_n(x,t,T_K(u_n),\nabla T_K(u_n))(\nabla T_K(u_n)-\nabla w^i_{\mu,j})\\ \quad\displaystyle-\int_{\{|u_n|>K\}\cap\{|u_n-w^i_{\mu,j}|\leq\nu\}} a_n(x,t,u_n,\nabla u_n)\nabla w^i_{\mu,j}\,\mathrm{d}x\,\mathrm{d}t.\end{cases} \tag{5.51}$$

On one hand, let us observe that for any $K>0$, and for $n$ large enough, namely $n>K+\nu\geq K$, we have,

$$a_n(x,t,T_K(u_n),\nabla T_K(u_n)) = a(x,t,T_K(u_n),\nabla T_K(u_n)). \tag{5.52}$$

On the other hand, from (5.39), we have $|w^i_{\mu,j}|\leq K$ a.e. in $Q_T$, then in the set $\{|u_n-w^i_{\mu,j}|\leq\nu\}$, we have $|u_n|\leq|u_n-w^i_{\mu,j}|+|w^i_{\mu,j}|\leq\nu+K$. Then for $n>\nu+K$, we obtain,

$$\begin{cases}\displaystyle\int_{\{|u_n|>K\}\cap\{|u_n-w^i_{\mu,j}|\leq\nu\}} a_n(x,t,u_n,\nabla u_n)\nabla w^i_{\mu,j}\,\mathrm{d}x\,\mathrm{d}t\\ \quad\displaystyle=\int_{\{|u_n|>K\}\cap\{|u_n-w^i_{\mu,j}|\leq\nu\}} a(x,t,T_{\nu+K}(u_n),\nabla T_{\nu+K}(u_n))\nabla w^i_{\mu,j}\,\mathrm{d}x\,\mathrm{d}t.\end{cases} \tag{5.53}$$

From (5.52) and (5.53), (5.51) becomes

$$\begin{cases}\displaystyle\int_{Q_T} a_n(x,t,u_n,\nabla u_n)\nabla T_\nu(u_n-w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t\\ \quad\displaystyle\geq \int_{\{|T_K(u_n)-w^i_{\mu,j}|\leq\nu\}} a(x,t,T_K(u_n),\nabla T_K(u_n))(\nabla w^i_{\mu,j}T_K(u_n)-\nabla w^i_{\mu,j})\\ \quad\displaystyle-\int_{\{|u_n|>K\}\cap\{|u_n-w^i_{\mu,j}|\leq\nu\}} a(x,t,T_{\nu+K}(u_n),\nabla T_{\nu+K}(u_n))\nabla w^i_{\mu,j}.\end{cases} \tag{5.54}$$

We put

$$J_1=\int_{\{|u_n|>K\}\cap\{|u_n-w^i_{\mu,j}|\leq\nu\}} a(x,t,T_{\nu+K}(u_n),\nabla T_{\nu+K}(u_n))\nabla w^i_{\mu,j}\,\mathrm{d}x\,\mathrm{d}t.$$

Since $(a(x,t,T_{K+\nu}(u_n),\nabla T_{K+\nu}(u_n)))$ is bounded in $(L_{\overline{M}}(Q_T))^d$, we have,

$$a(x,t,T_{K+\nu}(u_n),\nabla T_{K+\nu}(u_n))\rightharpoonup l_{K+\nu}$$

weakly in $L_{\overline{M}}(Q_T)$ in $\sigma(\Pi L_{\overline{M}},\Pi E_M)$ as $n$ tends to infinity and since

$$\nabla w^i_{\mu,j}\chi_{\{|u_n|>K\}\cap\{|u_n-w^i_{\mu,j}|\leq\nu\}}\to\nabla w^i_{\mu,j}\chi_{\{|u|>K\}\cap\{|u-w^i_{\mu,j}|\leq\nu\}}$$

strongly in $(E_M(Q_T))^d$ as $n$ tends to infinity, we have,

$$\int_{\{|u_n|>K\}\cap\{|u_n-w^i_{\mu,j}|\le\nu\}} a(x,t,T_{\nu+K}(u_n),\nabla T_{\nu+K}(u_n))\nabla w^i_{\mu,j}\,\mathrm{d}x\,\mathrm{d}t \to \int_{\{|u|>K\}\cap\{|u-w^i_{\mu,j}|\le\nu\}} l_{K+\nu}\nabla w^i_{\mu,j}\,\mathrm{d}x\,\mathrm{d}t$$

as $n$ goes to infinity.

Using Lemma 2.2 with the convergences (5.40), (5.41), together with the almost everywhere convergence, and letting first $j$ then $\mu$ tend to infinity, we obtain (notice that the index $i$ disappears in this process)

$$\int_{\{|u|>K\}\cap\{|u-w^i_{\mu,j}|\le\nu\}} l_{K+\nu}\nabla w^i_{\mu,j} \to \int_{\{|u|>K\}\cap\{|u-T_K(u)|\le\nu\}} l_{K+\nu}\nabla T_K(u)=0,$$

since $\nabla T_K(u)=0$ in the set $\{|u|>K\}$. This gives

$$J_1=\epsilon(n,j,\mu,i). \tag{5.55}$$

Using (5.50), (5.54) and (5.55) in (5.45), we obtain

$$\int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}} a(x,t,T_K(u_n),\nabla T_K(u_n))(\nabla T_K(u_n)-\nabla w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t \le C\nu+\epsilon(n,j,\mu,i). \tag{5.56}$$

where $C=(C_1+C_2)$.

On the other hand, note that

$$\begin{cases} \displaystyle\int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}} a(x,t,T_K(u_n),\nabla T_K(u_n))(\nabla T_K(u_n)-\nabla w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t \\ \displaystyle = \int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}} a(x,t,T_K(u_n),\nabla T_K(u_n))(\nabla T_K(u_n)-\nabla T_K(v_j)\chi^s_j) \\ \displaystyle \quad + \int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}} a(x,t,T_K(u_n),\nabla T_K(u_n))(\nabla T_K(v_j)\chi^s_j-\nabla w^i_{\mu,j}) \\ = J_2+J_3. \end{cases} \tag{5.57}$$

The integral term $J_3$ tends to 0 as first $n$, then $j$, $\mu$, $i$ and $s$ go to $\infty$. Indeed, since,

$$a(x,t,T_K(u_n),\nabla T_K(u_n)) \rightharpoonup l_K \text{ weakly in } (L_{\overline{M}}(Q_T))^d,$$

and since,

$$(\nabla T_K(v_j)\chi^s_j-\nabla w^i_{\mu,j})\chi_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}} \to (\nabla T_K(v_j)\chi^s_j-\nabla w^i_{\mu,j})\chi_{\{|T_K(u)-w^i_{\mu,j}|\le\nu\}}$$

strongly in $(E_{\overline{M}}(Q_T))^d$ as $n\to\infty$, then

$$\lim_{n\to\infty} J_3=\int_{\{|T_K(u)-w^i_{\mu,j}|\le\nu\}} l_K\cdot(\nabla T_K(v_j)\chi^s_j-\nabla w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t.$$

Letting $j$, $\mu$, $i$ and $s$, in this order, tend to infinity we readily deduce that

$$J_3 = \epsilon(n, j, \mu, i, s). \tag{5.58}$$

Consequently, from (5.56), (5.57) and (5.58), one has

$$\begin{cases} J_2 = \displaystyle\int_{\{|T_K(u_n)-w^i_{\mu,j}|\leq \nu\}} a(x,t,T_K(u_n),\nabla T_K(u_n))(\nabla T_K(u_n) - \nabla T_K(v_j)\chi^s_j) \\ \qquad \leq C\nu + \epsilon(n,i,j,\mu,s). \end{cases} \tag{5.59}$$

Let $M_n$ be the following non-negative expression

$$M_n = (a(x,t,T_K(u_n),\nabla T_K(u_n)) - a(x,t,T_K(u_n),\nabla T_K(u))) \\ \cdot(\nabla T_K(u_n) - \nabla T_K(u)),$$

then for any $0 < \theta < 1$, we write

$$I_{n,r} = \int_{Q_r} M_n^\theta \,\mathrm{d}x\,\mathrm{d}t.$$

We have

$$\int_{Q_r} M_n^\theta = \int_{Q_r} M_n^\theta \chi_{\{|T_K(u_n)-w^i_{\mu,j}|\leq\nu\}} + \int_{Q_r} M_n^\theta \chi_{\{|T_K(u_n)-w^i_{\mu,j}|>\nu\}}. \tag{5.60}$$

Using Hölder's inequality the second term of the right-side hand is less than,

$$\left(\int_{Q_r} M_n \,\mathrm{d}x\,\mathrm{d}t\right)^\theta \cdot \left(\int_{Q_r} \chi_{\{|T_K(u_n)-w^i_{\mu,j}|>\nu\}} \,\mathrm{d}x\,\mathrm{d}t\right)^{1-\theta}.$$

Note that,

$$\begin{aligned}\int_{Q_r} M_n \,\mathrm{d}x\,\mathrm{d}t &= \int_{Q_r} a(x,t,T_K(u_n),\nabla T_K(u_n))\nabla T_K(u_n)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \int_{Q_r} a(x,t,T_K(u_n),\nabla T_K(u_n))\nabla T_K(u)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_{Q_r} a(x,t,T_K(u_n),\nabla T_K(u))\nabla T_K(u)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \int_{Q_r} a(x,t,T_K(u_n),\nabla T_K(u))\nabla T_K(u_n)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Since $(a(x,t,T_K(u_n),\nabla T_K(u_n)))$ is bounded in $(L_{\overline{M}}(Q_T))^d$, $(\nabla T_K(u_n))$ is bounded in $(L_M(Q_T))^d$ and $(a(x,t,T_K(u_n),\nabla T_K(u)))$ is bounded in $L^\infty(Q_r)$, we have $(M_n)$ is bounded in $L^1(Q_r)$.

It follows that there exists a constant $C_3 > 0$ such that

$$\int_{Q_r} M_n^\theta \chi_{\{|T_K(u_n)-w^i_{\mu,j}|>\nu\}} \,\mathrm{d}x\,\mathrm{d}t \leq C_3 \,\mathrm{meas}\{|T_K(u_n) - w^i_{\mu,j}| > \nu\}^{1-\theta}. \tag{5.61}$$

Using again Hölder's inequality, it yields

$$\int_{Q_r} M_n^\theta \chi_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}}\,\mathrm{d}x\,\mathrm{d}t \le \left(\int_{Q_r} 1\,\mathrm{d}x\,\mathrm{d}t\right)^{1-\theta}\left(\int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}\cap Q_r} M_n\,\mathrm{d}x\,\mathrm{d}t\right)^\theta \le C_4\left(\int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}\cap Q_r} M_n\,\mathrm{d}x\,\mathrm{d}t\right)^\theta. \tag{5.62}$$

From (5.61) and (5.62), we obtain

$$I_{n,r} \le C_3\,\mathrm{meas}\{|T_K(u_n)-w^i_{\mu,j}|>\nu\}^{1-\theta} + C_4\left(\int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}\cap Q_r} M_n\,\mathrm{d}x\,\mathrm{d}t\right)^\theta. \tag{5.63}$$

On the other hand, we have for every $s \ge r$, $r>0$

$$\begin{aligned}
&\int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}\cap Q_r} M_n\,\mathrm{d}x\,\mathrm{d}t \le \int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}\cap Q_s} M_n\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}\cap Q_s} [a(x,t,T_K(u_n),\nabla T_K(u_n)) - a(x,t,T_K(u_n),\nabla T_K(u)\chi_s)]\\
&\qquad\cdot[\nabla T_K(u_n) - \nabla T_K(u)\chi_s]\,\mathrm{d}x\,\mathrm{d}t\\
&\le \int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}} [a(x,t,T_K(u_n),\nabla T_K(u_n)) - a(x,t,T_K(u_n),\nabla T_K(u)\chi_s)]\\
&\qquad\cdot[\nabla T_K(u_n) - \nabla T_K(u)\chi_s]\,\mathrm{d}x\,\mathrm{d}t\\
&\le \int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}} [a(x,t,T_K(u_n),\nabla T_K(u_n)) - a(x,t,T_K(u_n),\nabla T_K(v_j)\chi_j^s)]\\
&\qquad\cdot[\nabla T_K(u_n) - \nabla T_K(v_j)\chi_j^s]\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}} a(x,t,T_K(u_n),\nabla T_K(u_n))\cdot[\nabla T_K(v_j)\chi_j^s - \nabla T_K(u)\chi_s]\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}} [a(x,t,T_K(u_n),\nabla T_K(v_j)\chi_j^s) - a(x,t,T_K(u_n),\nabla T_K(u)\chi^s)]\\
&\qquad\cdot\nabla T_K(u_n)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad - \int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}} a(x,t,T_K(u_n),\nabla T_K(v_j)\chi_j^s)\cdot\nabla T_K(v_j)\chi_j^s\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}} a(x,t,T_K(u_n),\nabla T_K(u)\chi_s)\cdot\nabla T_K(u)\chi_s\,\mathrm{d}x\,\mathrm{d}t\\
&= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

We will take the limit first in $n$ then in $j$, $\mu$, $i$ and $s$ as they tend to infinity in these last five integrals.

Starting with $I_1$, we have

$$\begin{aligned} I_1 &= \int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}} (a(x,t,T_K(u_n),\nabla T_K(u_n)) - a(x,t,T_K(u_n),\nabla T_K(v_j)\chi^s_j)) \\ &\quad \cdot(\nabla T_K(u_n) - \nabla T_K(v_j)\chi^s_j)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}} a(x,t,T_K(u_n),\nabla T_K(u_n))\cdot(\nabla T_K(u_n) - \nabla T_K(v_j)\chi^s_j) \\ &\quad - \int_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}} a(x,t,T_K(u_n),\nabla T_K(v_j)\chi^s_j) \\ &\quad \cdot(\nabla T_K(u_n) - \nabla T_K(v_j)\chi^s_j) \\ &= J_2 - J_3. \end{aligned}$$

Since the sequence $(a(x,t,T_K(u_n),\nabla T_K(v_j)\chi^s_j)\chi_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}})_n$ converges to $a(x,t,T_K(u),\nabla T_K(v_j)\chi^s_j)\chi_{\{|T_K(u)-w^i_{\mu,j}|\le\nu\}}$ strongly in $(E_{\overline{M}}(Q_T))^d$ and $(\nabla T_K(u_n))$ converges to $\nabla T_K(u)$ weakly in $(L_M(Q_T))^d$ for $\sigma(\Pi L_M, \Pi E_{\overline{M}})$, we then have

$$J_3 = \int_{\{|T_K(u)-w^i_{\mu,j}|\le\nu\}} a(x,t,T_K(u),\nabla T_K(v_j)\chi^s_j)(\nabla T_K(u) - \nabla T_K(v_j)\chi^s_j) + \epsilon(n).$$

Using the almost everywhere convergence of $w^i_{\mu,j}$ and since $(\nabla T_K(v_j)\chi^s_j)_j$ converges to $\nabla T_K(u)\chi_s$ strongly in $(E_M(Q_T))^d$, $(a(x,t,T_K(u),\nabla T_K(v_j)\chi^s_j))_j$ converges to $a(x,t,T_K(u),\nabla T_K(u)\chi_s)$ strongly in $(L_{\overline{M}}(Q_T))^d$, we deduce

$$\begin{aligned} J_3 &= \int_{Q_T} a(x,t,T_K(u),\nabla T_K(u)\chi_s)(\nabla T_K(u) - \nabla T_K(u)\chi_s)\,\mathrm{d}x\,\mathrm{d}t + \epsilon(n,j,\mu,i) \\ &= \epsilon(n,j,\mu,i,s). \end{aligned}$$

Gathering all these estimates, taking into account (5.59), we obtain

$$I_1 \le C\nu + \epsilon(n,j,\mu,i,s) = \epsilon(n,j,\mu,i,s,\nu). \tag{5.64}$$

As for $I_2$, since $(a(x,t,T_K(u_n),\nabla T_K(u_n)))_n$ converges to $l_K$ weakly in the space $(L_{\overline{M}}(Q_T))^d$ for $\sigma(\Pi L_{\overline{M}}, \Pi E_M)$ and, in its turn, the sequence $((\nabla T_K(v_j)\chi^s_j - \nabla T_K(u)\chi_s)\chi_{\{|T_K(u_n)-w^i_{\mu,j}|\le\nu\}})_n$ converges to $(\nabla T_K(v_j)\chi^s_j - \nabla T_K(u)\chi_s)\chi_{\{|T_K(u)-w^i_{\mu,j}|\le\nu\}}$ strongly in $(E_M(Q_T))^d$, we obtain

$$I_2 = \int_{\{|T_K(u)-w^i_{\mu,j}|\le\nu\}} l_K(\nabla T_K(v_j)\chi^s_j - \nabla T_K(u)\chi_s)\,\mathrm{d}x\,\mathrm{d}t + \epsilon(n).$$

By letting now $j\to\infty$, and using Lebesgue's theorem, we deduce then that

$$I_2 = \epsilon(n,j). \tag{5.65}$$

Similar tools as above yield

$$I_3 = \epsilon(n,j). \tag{5.66}$$

$$I_4 = -\int_{Q_T} a(x,t,T_K(u),\nabla T_K(u)\chi_s)\nabla T_K(u)\chi_s + \epsilon(n,j,\mu,i,s). \tag{5.67}$$

$$I_5 = \int_{Q_T} a(x,t,T_K(u),\nabla T_K(u)\chi_s)\nabla T_K(u)\chi_s + \epsilon(n,j,\mu,i,s). \tag{5.68}$$

Combining (5.63)–(5.68), we get

$$I_{n,r} \leq C_4\epsilon(n,j,\mu,i,s,\nu)^\theta + C_3 \operatorname{meas}\{|T_K(u_n) - w^i_{\mu,j}| > \nu\}^{1-\theta}. \tag{5.69}$$

Consequently, when we take the limsup first in $n$, then in $j$, $\mu$, $i$, $s$ and $\nu$ in (5.69), we obtain

$$\begin{cases} \limsup_{n\to\infty} \int_{Q_r} \Big( (a(x,t,T_K(u_n),\nabla T_K(u_n)) - a(x,t,T_K(u_n),\nabla T_K(u))) \\ \qquad \cdot(\nabla T_K(u_n) - \nabla T_K(u))\Big)^\theta \mathrm{d}x\,\mathrm{d}t = 0. \end{cases}$$

According to (3.4) this last expression implies that

$$\lim_{n\to\infty} \int_{Q_r} M(\nabla T_K(u_n) - \nabla T_K(u))^\theta \mathrm{d}x\,\mathrm{d}t = 0.$$

hence, for a subsequence, $\nabla T_K(u_n) \to \nabla T_K(u)$ almost everywhere in $Q_r$. Since $r > 0$ is arbitrary, we may deduce that, maybe for another subsequence, $\nabla T_K(u_n) \to \nabla T_K(u)$ almost everywhere in $Q_T$. Finally, since $K > 0$ is arbitrary, it yields, still for a subsequence,

$$\nabla u_n \to \nabla u \quad \text{a.e in } Q_T. \tag{5.70}$$

This ends the proof of Proposition 5.4. □

**Remark 5.5.** A straightforward consequence of Proposition 5.4 is that, owing to (5.36), $\delta = a(x,t,u,\nabla u)$ that is,

$$\nabla a_n(x,t,u_n\nabla u_n) \rightharpoonup a(x,t,u,\nabla u) \text{ in } L_{\bar{M}}(Q_T)^d \text{ for } \sigma(\Pi L_{\bar{M}}, \Pi E_M). \tag{5.71}$$

**Step 3**. In this step, we will show that $\varphi_n \to \varphi$ strongly in $L^1(Q_T)$ modulo a subsequence.

The strongly convergence of $(\varphi_n)$ in $L^1(Q_T)$ is based in the next result which generalizes that of González Montesinos and Ortegón Gallego in [14], Lemma 4 (see also [25]).

**Lemma 5.6.** *Let $P$ be an $N$-function which admits the representation: $P(t) = \int_0^t p(s)\,\mathrm{d}s$ with $t \leq p(t)$, $(u_n)$ is a bounded sequence in $W^{1,x}L_M(Q_T)$ such that $u_n \to u$ strongly in $E_P(Q_T)$. Then there exists a subsequence $(u_{n(k)}) \subset (u_n)$ such that, for every $\epsilon > 0$, there exists a constant value $\boldsymbol{M} = \boldsymbol{M}(\epsilon)$ and a* *function* $\psi \in L^1(0,T;W^{1,1}(\Omega))$ *satisfying the following properties:*

$$0 \le \psi \le 1. \tag{5.72}$$

$$||\psi - 1||_{L^1(Q_T)} + ||\nabla\psi||_{L^1(Q_T)} \le \epsilon. \tag{5.73}$$

$$|u|, |u_{n(k)}| \le \boldsymbol{M} \text{ on } \{\psi > 0\} \text{ for all } k \ge 1. \tag{5.74}$$

*Proof.* According to lemmas 2.3 and 3.1 we deduce the the following continuous inclusions:

$$L_P(Q_T) \hookrightarrow L_{\bar{P}}(Q_T) \hookrightarrow L_{\bar{M}}(Q_T).$$

Since $(u_n)$ is relatively compact in $E_P(Q_T)$, we can extract a subsequence $(u_{n(k)}) \subset (u_n)$ such that :

$$\sum_{k=1}^{\infty} ||u_{n(k)} - u||_{L_{\bar{M}}(Q_T)} \le 1. \tag{5.75}$$

Fix $K > 0$ to be chosen later big enough and introduce the function $\gamma$ given by

$$\gamma = (|u| - K)^+ + \sum_{k=1}^{\infty}(|u_{n(k)} - u| - K)^+. \tag{5.76}$$

Then putting $v_k = u_{n(k)} - u$, $k \ge 1$, and $v_0 = u$, we have

$$\begin{aligned}
&\int_{Q_T}(|v_k| - K)^+ + \int_{Q_T} |\nabla(|v_k| - K)^+| \\
&= \int_{\{|v_k|>K\}} (|v_k| - K)^+ \frac{|v_k|}{|v_k|} + \int_{\{|v_k|>K\}} |\nabla(|v_k| - K)^+| \frac{|v_k|}{|v_k|} \\
&\le \frac{1}{K}(||v_k||_{L_M(Q_T)} + ||\nabla v_k||_{L_M(Q_T)})||v_k||_{L_{\bar{M}}(Q_T)}.
\end{aligned}$$

Summing up these inequalities, bearing in mind that $(u_{n(k)})$ and $(v_k)$ are bounded in $W^{1,x}L_M(Q_T)$ and (5.76), we deduce

$$\begin{aligned}
&\sum_{k=0}^{\infty}(||(|v_k| - K)^+||_{L^1(Q_T)} + ||(|\nabla v_k| - K)^+||_{L^1(Q_T)}) \\
&\le \frac{C_0}{K}\sum_{k=0}^{\infty} ||v_k||_{L_{\bar{M}}(Q_T)} = \frac{C_0}{K}\left(||u||_{L_{\bar{M}}(Q_T)} + \sum_{k=1}^{\infty} ||u_{n(k)} - u||_{L_{\bar{M}}(Q_T)}\right) \\
&\le \frac{C_0}{K}(||u||_{L_{\bar{M}}(Q_T)} + 1) = \frac{C}{K}.
\end{aligned}$$

Hence

$$||\gamma||_{L^1(0,T;W^{1,1}(\Omega))} \le \frac{C}{K}.$$

It is straightforward to check that the function $\psi = (1-\gamma)^+$ verifies the asserted condition (5.72)–(5.74) for $K \ge C/\epsilon$ and $\mathbf{M} = K + 1$. □

The next two results analyze the behavior of certain subsequences of $(\varphi_n)$. They will allow us, together with the convergences deduced in the previous steps, to pass to the limit in the approximate problems (5.20)–(5.24) in order to show the existence of a capacity solution to the system (1.1).

**Lemma 5.7.** ([14]) *Let $(u_n, \varphi_n)$ be a weak solution to the system (5.20)–(5.24), $u \in E_P(Q_T)$ and $\varphi \in L^\infty(Q_T)$ the limit functions appearing, respectively, in (5.27) and (5.38). Then, for any function $S \in C_0^1(\mathbb{R})$, there exists a subsequence, still denoted in the same way, such that*

$$S(u_n)\varphi_n \rightharpoonup S(u)\varphi \text{ weakly in } L^2(0,T;H^1(\Omega)). \tag{5.77}$$

*Moreover, if $0 \leq S \leq 1$, then there exists a constant $C > 0$, independent of $S$, such that*

$$\limsup_{n\to\infty} \int_{Q_T} \rho_n(u_n)|\nabla[S(u_n)\varphi_n - S(u)\varphi]|^2 \leq C||S'||_\infty(1 + ||S'||_\infty). \tag{5.78}$$

**Lemma 5.8.** *There exists a subsequence $(\varphi_{n(k)}) \subset (\varphi_n)$ such that*

$$\lim_{k\to\infty} \int_{Q_T} |\varphi_{n(k)} - \varphi| = 0. \tag{5.79}$$

*Proof.* The proof of this result is almost identical to that of Lemma 4.8 in [14]. For the sake of completeness, we include it here.

Since the conditions of Lemma 5.6 are fulfilled by a suitable subsequence $(u_{n(k)})$, we have for every $\epsilon > 0$ there exists $\mathbf{M} > 0$ and $\psi \in L^1(0,T;W^{1,1}(\Omega))$ such that (5.72)–(5.74) are satisfied. By (5.74), there exists $C_{\mathbf{M}} > 0$ such that

$$\xi_k \overset{\text{def}}{=} \rho_{n(k)}(u_{n(k)}) \geq C_{\mathbf{M}} \text{ on } \{\psi > 0\}, \text{ for all } k \geq 1. \tag{5.80}$$

We consider a sequence of regular functions $(S_m) \subset C_0^1(\mathbb{R})$ such that

$$0 \leq S_m \leq 1, \quad S_m = 1 \text{ in } [-\mathbf{M},\mathbf{M}], \text{ for all } k \geq 1. \tag{5.81}$$

$$||S_m'||_{L^\infty(\mathbb{R})} \leq \frac{1}{m}, \text{ for all } m \geq 1. \tag{5.82}$$

From (5.74) and (5.81), we write

$$\int_{Q_T} |\varphi_{n(k)} - \varphi| = \int_{\{\psi>0\}} |S_m(u_{n(k)})\varphi_{n(k)} - S_m(u)\varphi| + \int_{\{\psi=0\}} |\varphi_{n(k)} - \varphi|.$$

Inserting $\pm\psi|S_m(u_{n(k)})\varphi_{n(k)} - S_m(u)\varphi|$ in the first integral above and $-\psi|\varphi_{n(k)} - \varphi| = 0$ in the second one, then owing to (5.26), (5.27), (5.72) and using Poincaré's inequality, we obtain

$$\int_{Q_T} |\varphi_{n(k)} - \varphi| = \int_{\{\psi>0\}} \psi |S_m(u_{n(k)})\varphi_{n(k)} - S_m(u)\varphi|$$
$$+ \int_{\{\psi>0\}} (1-\psi)|S_m(u_{n(k)})\varphi_{n(k)} - S_m(u)\varphi| + \int_{\{\psi=0\}} (1-\psi)|\varphi_{n(k)} - \varphi|$$
$$\leq C_0 \int_{Q_T} |\nabla(\psi(S_m(u_{n(k)})\varphi_{n(k)} - S_m(u)\varphi))| + 2\|\varphi_0\|_{L^\infty(Q_T)} \int_{Q_T} |1-\psi|$$
$$\leq 2C_0 \|\varphi_0\|_{L^\infty(Q_T)} \int_{Q_T} |\nabla\psi| + C_0 \int_{Q_T} |\nabla(S_m(u_{n(k)}))\varphi_{n(k)} - S_m(u)\varphi)|$$
$$+ 2\|\varphi_0\|_{L^\infty(Q_T)} \int_{Q_T} |1-\psi|,$$

Putting $C^* = 2||\varphi_0||_{L^\infty(\Omega)} \max(C_0, 1)$, $K_{\mathbf{M}} = C_0 C_{\mathbf{M}}^{-1/2} |\Omega|^{1/2} T^{1/2}$ and taking into account (5.73) and (5.80), we deduce

$$\int_{Q_T} |\varphi_{n(k)} - \varphi| \leq C^*\epsilon + C_0 \int_{Q_T} \xi_k^{-1/2} \xi_k^{1/2} |\nabla(S_m(u_{n(k)}))\varphi_{n(k)} - S_m(u)\varphi)|$$
$$C^*\epsilon + K_{\mathbf{M}} \left( \int_{Q_T} \xi_k |\nabla(S_m(u_{n(k)}))\varphi_{n(k)} - S_m(u)\varphi|^2 \right)^{1/2},$$

Owing to (5.78) and (5.82), we obtain

$$\limsup_{k\to\infty} \int_{Q_T} |\varphi_{n(k)} - \varphi| \leq C^*\epsilon + K_M \Big( C||S_m'||_\infty (1 + ||S_m'||_\infty) \Big)^{1/2}$$
$$\leq C^*\epsilon + K_M C^{1/2} \left[ \frac{1}{m} \Big( 1 + \frac{1}{m} \Big) \right]^{1/2}.$$

And since $\epsilon > 0$ and $m \geq 1$ are arbitrary, we derive the desired result. □

**Step 5**. Passing to the limit.

According to (5.27), (5.29), (5.33), (5.35) and (5.37), it is straightforward that the condition $(C_1)$ of Definition 1 is fulfilled. The convergences in Proposition 5.4 and Lemma 5.8 lead us to $(C_2)$ of Definition 1, and in order to obtain the condition $(C_3)$, using Proposition 5.4 and Lemma 5.8 again with (5.77), it is enough to let $k$ goes to infinity in the following expression

$$S(u_{n(k)})\rho_{n(k)}(u_{n(k)})\nabla\varphi_{n(k)} = \rho_{n(k)}(u_{n(k)})[\nabla(S(u_{n(k)})\varphi_{n(k)}) - \varphi_{n(k)}\nabla S(u_{n(k)})]$$

**Step 6**. Regularity of $u$.

Finally, it remains to establish the regularity $u \in C([0,T]; L^1(\Omega))$. Though this is a straightforward consequence of Lemma 2.9, since $u \in \mathbf{W} \subset W \subset C([0,T]; L^1(\Omega))$, it is interesting to show this property from the results deduced on the previous steps about the (sub)sequences $(u_n)$ and $(\varphi_n)$ and how the notion of capacity solution is used along this proof. To this end, we go back to the expression (5.44) but the integration in time happens in the interval $(0,\tau)$ for any $\tau \in (0,T]$, namely (see [12])

$$\left\langle \frac{\partial u_n}{\partial t}, T_\nu(u_n - w^i_{\mu,j}) \right\rangle_{Q_\tau} = \int_{Q_\tau} a_n(x,t,u_n,\nabla u_n)(\nabla w^i_{\mu,j} - \nabla u_n)\chi_{\{|u_n - w^i_{\mu,j}|\le \nu\}} - \int_{Q_\tau} \rho_n(u_n)\varphi_n \nabla\varphi_n \nabla T_\nu(u_n - w^i_{\mu,j}). \tag{5.83}$$

where $\nu \in (0,1]$, $Q_\tau = (0,\tau)\times\Omega$ and $\langle\cdot,\cdot\rangle_{Q_\tau}$ is the duality product between $W^{-1,x}L_{\bar M}(Q_\tau)$ and $W^{1,x}_0 L_M(Q_\tau)$. We will consider the necessary subsequences to assure the almost everywhere convergence in $Q_T$ of $\varphi_n \to \varphi$, $u_n \to u$, $\nabla u_n \to \nabla u$, and also for $(T_\nu(u_n - w^i_{\mu,j}))$, etc. From (5.71) we readily obtain

$$\lim_{n\to\infty} \int_{Q_\tau} a_n(x,t,u_n,\nabla u_n)\nabla w^i_{\mu,j}\chi_{\{|u_n - w^i_{\mu,j}|\le \nu\}} = \int_{Q_\tau} a(x,t,u,\nabla u)\nabla w^i_{\mu,j}\chi_{\{|u - w^i_{\mu,j}|\le \nu\}}.$$

Also, by Fatou's lemma we get

$$\lim_{n\to\infty} \int_{Q_\tau} a(x,t,u,\nabla u)\nabla u\chi_{\{|u - w^i_{\mu,j}|\le \nu\}} \le \liminf_{n\to\infty} \int_{Q_\tau} a_n(x,t,u_n,\nabla u_n)\nabla u_n\chi_{\{|u_n - w^i_{\mu,j}|\le \nu\}}.$$

Then, passing to the limit in these two expressions, first in $j$, then in $\mu$, $i$ and $K$, we deduce, uniformly in $\tau$, that

$$\int_{Q_\tau} a_n(x,t,u_n,\nabla u_n)(\nabla w^i_{\mu,j} - \nabla u_n)\chi_{\{|u_n - w^i_{\mu,j}|\le \nu\}} \le \epsilon(n,j,\mu,i,K). \tag{5.84}$$

The analysis of the term $\int_{Q_\tau} \rho_n(u_n)\varphi_n\nabla\varphi_n\nabla T_\nu(u_n - w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t$ is more involved. Here the difficulty relies on the fact that the sequence $(\rho_n(u_n)|\nabla\varphi_n|^2)$ does not converge, in general, strongly in $L^1(Q_T)$. In order to deal with this situation, we are going to make use of the properties already shown for a capacity solution. Indeed, we first notice that $\nabla T_\nu(u_n - w^i_{\mu,j}) = 0$ in the set $\{|u_n| \le K+\nu\} \subset \{|u_n| \le K+1\}$. Then we consider a sequence of functions $S_K \subset C^1_0(\mathbb{R})$ such that

$$0 \le S_K \le 1, \quad S_K = 1 \text{ in } [-(K+1), K+1], \text{ for all } K > 0.$$
$$||S'_K||_{L^\infty(\mathbb{R})} \le \frac{1}{K+1}, \text{ for all } K > 0.$$

We have

$$\begin{aligned}
&\int_{Q_\tau} \rho_n(u_n)\varphi_n \nabla\varphi_n \nabla T_\nu(u_n - w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_{Q_\tau} \rho_n(u_n)\varphi_n \nabla[S_K(u_n)\varphi_n]\nabla T_\nu(u_n - w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_{Q_\tau} \rho_n(u_n)\varphi_n \nabla[S_K(u_n)\varphi_n - S(u)\varphi]\nabla T_\nu(u_n - w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_{Q_\tau} \rho_n(u_n)\varphi_n \nabla[S_K(u)\varphi]\nabla T_\nu(u_n - w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t = L_1 + L_2.
\end{aligned}$$

According to the almost everywhere convergence of $(u_n)$ and $(\varphi_n)$ together with (5.26) and (5.33), we readily deduce that

$$\lim_{n\to\infty} L_2 = \int_{Q_\tau} \rho(u)\varphi\nabla[S_K(u)\varphi]\nabla T_\nu(u - w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t,$$

and using the identity $(C_3)$, already shown in the previous step, namely, $\rho(u)\nabla[S_K(u)\varphi] = S_K(u)\Phi + \varphi\nabla S_K(u)$, we can easily obtain the estimate $L_2 = \epsilon(n, j, \mu, i, K)$.

As for the term $L_1$, we use (5.78) to get, for some constant $C > 0$,

$$\begin{aligned}
|L_1|^2 &\le \left(\int_{Q_\tau} \rho_n(u_n)|\nabla[S_K(u_n)\varphi_n - S(u)\varphi]|^2\,\mathrm{d}x\,\mathrm{d}t\right) \\
&\qquad \cdot \left(\int_{Q_\tau} \rho_n(u_n)|\varphi_n|^2|\nabla T_\nu(u_n - w^i_{\mu,j})|^2\,\mathrm{d}x\,\mathrm{d}t\right) \le \frac{C}{K+1},
\end{aligned}$$

and thus it is also $L_1 = \epsilon(n, j, \mu, i, K)$.

Consequently, we get, for any fixed $\nu \in (0, 1]$ and uniformly in $\tau \in [0, T]$,

$$\int_{Q_\tau} \rho_n(u_n)\varphi_n\nabla\varphi_n\nabla T_\nu(u_n - w^i_{\mu,j})\,\mathrm{d}x\,\mathrm{d}t \le \epsilon(n, j, \mu, i, K). \tag{5.85}$$

Gathering (5.83), (5.84) and (5.85) we get the estimate

$$\left\langle \frac{\partial u_n}{\partial t}, T_\nu(u_n - w^i_{\mu,j})\right\rangle_{Q_\tau} \le \epsilon(n, j, \mu, i, K). \tag{5.86}$$

Then we write, as in (5.46)–(5.49),

$$\begin{aligned}
&\int_\Omega \Theta_\nu(u_n(x,\tau) - w^i_{\mu,j}(x,\tau))\,\mathrm{d}x \\
&\quad = \left\langle \frac{\partial(u_n - w^i_{\mu,j})}{\partial t}, T_\nu(u_n - w^i_{\mu,j})\right\rangle_{Q_\tau} + \int_\Omega \Theta_\nu(u_0 - T_K(\psi_i))\,\mathrm{d}x \\
&\quad = \left\langle \frac{\partial u_n}{\partial t}, T_\nu(u_n - w^i_{\mu,j})\right\rangle_{Q_\tau} - \left\langle \frac{\partial w^i_{\mu,j}}{\partial t}, T_\nu(u_n - w^i_{\mu,j})\right\rangle_{Q_\tau} \\
&\qquad + \int_\Omega \Theta_\nu(u_0 - T_K(\psi_i))\,\mathrm{d}x.
\end{aligned}$$

Consequently, owing to (5.49) and (5.86), it yields, for every fixed $\nu \in (0,1]$ and uniformly in $\tau \in [0,T]$,

$$\int_\Omega \Theta_\nu(u_n(x,\tau) - w^i_{\mu,j}(x,\tau))\,\mathrm{d}x \le \epsilon(n,j,\mu,i,K),$$

and using the convexity of the function $\Theta_\nu$ we may also derive the following estimate

$$\begin{aligned}\int_\Omega \Theta_\nu\Big(\tfrac{1}{2}(u_n(x,\tau) - u_m(x,\tau))\Big)\,\mathrm{d}x &\le \frac{1}{2}\int_\Omega \Theta_\nu(u_n(x,\tau) - w^i_{\mu,j}(x,\tau))\,\mathrm{d}x\\ &\quad + \frac{1}{2}\int_\Omega \Theta_\nu(u_m(x,\tau) - w^i_{\mu,j}(x,\tau))\,\mathrm{d}x\\ &\le \epsilon(n,j,\mu,i,K) + \epsilon(m,j,\mu,i,K),\end{aligned}$$

and thus, for any fixed $\nu > 0$ and uniformly in $\tau \in [0,T]$, we have

$$\int_\Omega \Theta_\nu\Big(\tfrac{1}{2}(u_n(x,\tau) - u_m(x,\tau))\Big)\,\mathrm{d}x \le \epsilon(n) + \epsilon(m). \tag{5.87}$$

Consequently, using the definition of $\Theta_\nu$ and (5.87), for all $\tau \in [0,T]$, it is

$$\begin{aligned}&\int_\Omega \tfrac{1}{2}|u_n(x,\tau) - u_m(x,\tau)|\,\mathrm{d}x\\ &\quad\le \int_{\{|u_n(x,\tau)-u_m(x,\tau)|\le 2\nu\}} \tfrac{1}{2}|u_n(x,\tau) - u_m(x,\tau)|\,\mathrm{d}x\\ &\qquad + \int_{\{|u_n(x,\tau)-u_m(x,\tau)|> 2\nu\}} \tfrac{1}{2}|u_n(x,\tau) - u_m(x,\tau)|\,\mathrm{d}x\\ &\quad\le |\Omega|\nu + \frac{1}{\nu}\int_{\{|u_n(x,\tau)-u_m(x,\tau)|> 2\nu\}} \tfrac{\nu}{2}|u_n(x,\tau) - u_m(x,\tau)|\,\mathrm{d}x\\ &\quad= |\Omega|\nu + \frac{1}{\nu}\int_{\{|u_n(x,\tau)-u_m(x,\tau)|> 2\nu\}} \left[\Theta_\nu\Big(\tfrac{1}{2}|u_n(x,\tau) - u_m(x,\tau)|\Big) + \frac{\nu^2}{2}\right]\mathrm{d}x\\ &\quad= \frac{3}{2}|\Omega|\nu + \frac{1}{\nu}(\epsilon(n) + \epsilon(m)).\end{aligned}$$

This last estimate shows that $(u_n)$ is a Cauchy sequence in $C([0,T]; L^1(\Omega))$ and, in particular, its limit $u$ lies in this space.

This completes the proof of the Main Theorem.

**Remark 5.9.** According to the proof of the existence result given in the Main Theorem, this result also holds if the assumption (3.8), namely, $u_0 \in L^2(\Omega)$ is changed to $u_0 \in L^1(\Omega)$. Indeed, it is enough to rewrite the initial condition (5.24) in the approximate system as follows: $u_n(\cdot,0) = T_n(u_0)$ in $\Omega$.

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