Nonlinear Differ. Equ. Appl. (2018) 25:10

1021-9722/18/020001-18
published online February 23, 2018
https://doi.org/10.1007/s00030-018-0503-0

Nonlinear Differential Equations and Applications NoDEA

A new approach to the Cauchy and Goursat problems for the nonlinear Wheeler–DeWitt equation

João-Paulo Dias and Philippe G. LeFloch

Abstract. We consider a nonlinear version of the Wheeler–DeWitt equation which was introduced by Cooper, Susskind, and Thorlacius in the context of two-dimensional quantum cosmology. We establish the existence of global solutions to the Cauchy problem and Goursat problems which, both, arise naturally in physics. Our method of proof is based on a nonlinear transformation of the Wheeler–DeWitt equation and on techniques introduced by Baez and collaborators and by Tsutsumi for nonlinear wave equations.

Mathematics Subject Classification. 83F05, 74J30, 83C47.

1. Introduction

Objective of this paper

The *Wheeler–DeWitt equation* provides a simple, yet challenging model which describes a homogeneous isotropic Universe filled with a scalar field y with mass m. This equation arose from an early attempt to combine ideas from quantum mechanics and general relativity. The Wheeler–DeWitt is a linear, but singular wave equation which reads as follows [8–10]:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{p}{x}\frac{\partial\psi}{\partial x} - \frac{1}{x^2}\frac{\partial^2\psi}{\partial y^2} + m^2x^4y^2\psi - x^2\psi = 0, \tag{1.1}$$

in which the independent variable $x \in (0, +\infty)$ represents a scale factor and the scalar field y is viewed as an independent variable. Moreover, $p \in \mathbb{R}$ is a factor-ordering coefficient due to quantization, and the unknown function $\psi = \psi(x, y) \in \mathbb{C}$ is the so-called *wave function of the Universe* for the mini-superspace model under consideration.

A mathematical study of the corresponding Cauchy problem with prescribed initial condition at $y = 0$, say

$$\psi(x, 0) = \psi_0(x), \qquad \frac{\partial\psi}{\partial y}(x, 0) = \psi_1(x), \tag{1.2}$$

was initiated by Dias and Figueira [4] in two simplified cases: they treated the case $x \in (0, R)$ with $R > 0$ as well as the massless case $m = 0$ in the whole interval $x \in (0, +\infty)$ by introducing a suitable transformation of the equation [6,7].

On the other hand, more recently for a modeling effects arising in quantum cosmology, Cooper et al. [3] introduced a *nonlinear Wheeler–DeWitt equation*, namely

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{p}{x}\frac{\partial \psi}{\partial x} - \frac{1}{x^2}\frac{\partial^2 \psi}{\partial y^2} + m^2 x^4 y^2 \psi - x^2 \psi + \lambda(x)\,|\psi|^r \psi = 0, \tag{1.3}$$

in which the function $\lambda = \lambda(x) \in \mathbb{R}$ is prescribed and $r \geq 1$ is a parameter. This model was found to provide a better description of some phenomena in quantum cosmology. (We also refer [12] for an alternative nonlinear model.) In Dias and Figueira [5], this nonlinear model was also consider in a simplified case, that is, $x \in (0, R)$ with $R > 0$, and the Cauchy problem was solved for general data $\Big(\psi(x, 0), \frac{\partial \psi}{\partial y}(x, 0)\Big)$ and for the function $\lambda(x) = \lambda\, x^{q-2}$, $q \geq \frac{1}{2}\, rp$ with $\lambda \in \mathbb{R}$.

In the present work, we pursue this analysis further and rely on the transformation introduced in [6,7] (in the linear case) in order to study the *nonlinear* equation (1.3) in the *whole* interval $x > 0$. Specifically, we assume that the nonlinearity of the Wheeler–DeWitt equation satisfies the conditions

$$r \geq 2, \qquad \lambda(x) = \lambda\, x^{q-2}, \quad q = \frac{p-1}{2}\, r, \quad \lambda \in \mathbb{R}. \tag{1.4}$$

By setting

$$z = \log x, \qquad x \in (0, +\infty) \tag{1.5}$$

and in view of

$$u(z, y) = x^{\frac{p-1}{2}}\, \psi(x, y) = e^{\frac{p-1}{2}\, z}\, \psi(e^z, y), \tag{1.6}$$

we arrive at the following terminology.

Definition 1.1. The **reduced nonlinear Wheeler–DeWitt equation** by definition is

$$\frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial y^2} - \frac{1}{4}(p-1)^2\, u + \left(m^2 y^2 e^{6z} - e^{4z}\right) u + \lambda\, |u|^r u = 0. \tag{1.7}$$

in which $u = u(z, y)$ is a complex-valued function defined over $(z, y) \in \mathbb{R}^2$.

Observe that the principal part of (1.7) decomposes into two parts, i.e.

- the $1 + 1$ Klein-Gordon operator, that is,

$$\frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial y^2} - \frac{1}{4}(p-1)^2\, u \tag{1.8}$$

- and a potential term of exponential type

$$\left(m^2 y^2 e^{6z} - e^{4z}\right) u. \tag{1.9}$$

In addition, the nonlinear version of the Wheeler–DeWitt equation under consideration also involves the nonlinear term

$$\lambda\,|u|^r u. \tag{1.10}$$

Our objective in this paper is establishing a well-posedness theory for the Cauchy problem and for the Goursat problem by extending the methods introduced originally by Baez et al. [1,2] and Tsutsumi [14] for nonlinear wave equations.

Main result of this paper

First of all, in Sect. 2, we study the massless case $m = 0$ and consider the Cauchy problem for the equation (1.7) with data

$$\left(u(x,0), \frac{\partial u}{\partial y}(x,0)\right) = (u_0(x), v_0(x)) \in X \times H^1_V, \tag{1.11}$$

where (in this case y is regarded as our "time" variable)

$$\begin{aligned} V(z) &= e^{4z},\\ H^1_V &= \left\{u \in H^1(\mathbb{R}) \,/\, V^{1/2}u \in L^2(\mathbb{R})\right\},\\ X &= \left\{u \in H^1_V \,/\, \frac{d^2u}{dz^2} - Vu \in L^2(\mathbb{R})\right\}, \end{aligned} \tag{1.12}$$

endowed with their natural norms. Here, we will be able to rely on rather standard techniques for nonlinear Klein–Gordon equations (see for instance [13] and the references therein). Considering next a particular class of initial data and provided $\lambda < 0$, we study the sequence [1]

$$v_p(z,y) = e^{ic_p^2 y}\, u_p(z, c_p y), \tag{1.13}$$

where $c_p = \frac{1}{2}(p-1)$ (with $p \neq 1$) and u_p is the solution to the corresponding Cauchy problem and, when $p \to \infty$, we prove that the functions v_p converge in the topology $C([-T,T]; L^2(\mathbb{R}))$, $\forall T > 0$, toward a function

$$\widetilde{v}(\widehat{z},\widehat{y}) \in C(\mathbb{R}; L^2(\mathbb{R})) \cap L^\infty_{\text{loc}}(\mathbb{R}; X) \tag{1.14}$$

such that

$$\frac{\partial \widetilde{v}}{\partial y} \in L^\infty_{\text{loc}}(\mathbb{R}; L^2(\mathbb{R})), \qquad \widetilde{v}(\widehat{z},0) = \lim_{p\to\infty} u_p(\widehat{z},0) \text{ in } L^2(\mathbb{R}), \tag{1.15}$$

and, moreover, this function is nothing but a solution to the **nonlinear Schrodinger equation**

$$i\,\frac{\partial \widetilde{v}}{\partial y} + \frac{1}{2}\left(\frac{\partial^2 \widetilde{v}}{\partial z^2} - V\widetilde{v} + \lambda\,|\widetilde{v}|^r\,\widetilde{v}\right) = 0. \tag{1.16}$$

Our technique of proof for this latter statement is an adaptation of the method developed by Tsutsumi [14] for two space dimensions and $V = 0$. Importantly, our result validates a heuristic given by physicists about the Wheeler–DeWitt equation.

[1] Here, $i = \sqrt{-1}$.

Next, in Sect. 3 we study the periodic Goursat problem associated with the Wheeler–DeWitt equation (1.7), and establish the existence of 2π-periodic solutions ((in the y variable) which is now regarded as the "space" variable), when with data are prescribed on the characteristic cone

$$C_0 = \left\{(z, y) \,/\, z = |y|,\ |y| \leq \pi\right\}. \tag{1.17}$$

Our technique of proof is an adaptation of the method developed by Baez et al. [1,2] and begins by reducing the problem under consideration to a more convenient Cauchy problem for an evolution equation. When $m \neq 0$, in the equation (1.7) we need to replace the function $m^2 y^2 e^{6z}$ by $m^2 \theta(y^2)\, e^{6z}$ where $\theta(y^2)$ is the 2π-periodic extension of the function y^2 in $[-\pi, \pi]$. In order to obtain smooth local (in the variable z) solutions, we restrict the Goursat data accordingly, and to obtain global (in z) solutions we take $\lambda > 0$.

Two cases are of particular interest and are covered by our theorems in this section:

- Case $m^2 y^2 = k^2$ (a positive constant). This is a simplification which is often made in the physical applications, for instance in the study of tunneling solutions; cf. [8].
- Case $c_p = 0$, that is $p = 1$. The spatial curvature term e^{4z} is also neglected in the study of inflationary solutions, cf. again [8].

2. The Cauchy problem for the massless case

In this section we extend to the nonlinear equation (1.7), in the particular case $m = 0$, the existence results for the Cauchy problem and the singular limit when $p \to \infty$ obtained in [6] and [7]. We write the equation (1.7) for $m = 0$:

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} + \frac{1}{4}(p-1)^2 u + V u + \lambda\, |u|^r u = 0, \tag{2.1}$$

with $V(z) = e^{4z} u$, $(z, y) \in \mathbb{R}^2$.

We will study the Cauchy problem for initial data $u(z, 0)$, $\frac{\partial u}{\partial y}(z, 0)$, for $z \in \mathbb{R}$. For this purpose we introduce, as in [6], the space (in z)

$$H^1_V = \left\{v \in H^1(\mathbb{R}) \,/\, V^{1/2} v \in L^2(\mathbb{R})\right\} \tag{2.2}$$

with norm

$$\|v\|_{H^1_V} = \left(\|v\|^2_{H^1} + \|V^{1/2} v\|^2_2\right)^{1/2},$$

where $\|\cdot\|_p$ denotes the standard L^p norm. Let

$$X = \left\{v \in H^1_V \,\Big|\, \frac{\partial^2 v}{\partial z^2} - V v \in L^2\right\},$$

and $H = H^1_V \times L^2$, $D(A) = X \times H^1_V$, $A\colon D(A) \subset H \to H$ defined by

$$A \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{\partial^2}{\partial z^2} - V - c_p^2 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{2.3}$$

with $c_p = \frac{1}{2}\,(p-1)$.

With $v = \frac{\partial u}{\partial y}$ the equation (2.1) can be written in the first-order form

$$\frac{\partial}{\partial y}\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix} + J\begin{pmatrix} u \\ v \end{pmatrix}, \qquad J\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ -\lambda\,|u|^r u \end{pmatrix}. \tag{2.4}$$

The operator A is skew-self-adjoint in H (cf. [6], Theorem 1) and so generates a unitary group of operators in H. We take the initial (in y) data

$$\left(u_0(\widehat{z}) = u(\widehat{z}, 0),\ v_0(\widehat{z}) = \frac{\partial u}{\partial y}(\widehat{z}, 0)\right) \in D(A).$$

We study first the existence of a local (in y) solution to the Cauchy problem

$$u \in C\left([0, y_0]; X\right) \cap C^1\left([0, y_0]; H^1_V\right) \cap C^2\left([0, y_0], L^2\right). \tag{2.5}$$

If $\varphi = \begin{pmatrix} u \\ v \end{pmatrix} \in D(A) = X \times H^1_V$ it is easy to see, since $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, that $J\varphi \in D(A)$, and if $\varphi_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$, $\varphi_2 = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \in D(A)$ we have

$$A(J\varphi_1 - J\varphi_2) = \begin{pmatrix} -\lambda\,|u_1|^r u_1 + \lambda\,|u_2|^r u_2 \\ 0 \end{pmatrix},$$

$$\begin{aligned}\left\|J(\varphi_1) - J(\varphi_2)\right\|_A^2 &= \left\|J(\varphi_1) - J(\varphi_2)\right\|^2_{H^1_V \times L^2} + \left\|AJ(\varphi_1) - AJ(\varphi_2)\right\|^2_{H^1_V \times L^2} \\ &= |\lambda|^2\ \left\||u_1|^r u_1 - |u_2|^r u_2\right\|_2^2 + |\lambda|^2\ \left\||u_1|^r u_1 - |u_2|^r u_2\right\|_{H^1}^2 \\ &\quad + |\lambda|^2\ \left\|V^{1/2}\left(|u_1|^r u_1 - |u_2|^r u_2\right)\right\|_2^2.\end{aligned}$$

We have

$$\left||u_1|^r u_1 - |u_2|^r u_2\right| \le c\left(|u_1|^r + |u_2|^r\right)\ |u_1 - u_2|$$

and so

$$\left\|V^{1/2}\left(|u_1|^r u_1 - |u_2|^r u_2\right)\right\|_2 \le c\left(\|\varphi_1\|_A^r + \|\varphi_2\|_A^r\right)\ \|\varphi_1 - \varphi_2\|_A$$

since $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. Moreover, for $r \ge 2$, it is not difficult to derive

$$\begin{aligned}\left\||u_1|^r u_1 - |u_2|^r u_2\right\|_{H^1} &\le c\left(\|u_1\|_{H^1}^r + \|u_2\|_{H^1}^r\right)\ \|u_1 - u_2\|_{H^1} \\ &\le c\left(\|\varphi_1\|_A^r + \|\varphi_2\|_A^r\right)\ \|\varphi_1 - \varphi_2\|_A.\end{aligned}$$

Hence,

$$\left\|J(\varphi_1) - J(\varphi_2)\right\|_A \le c\left(\|\varphi_1\|_A^r + \|\varphi_2\|_A^r\right)\ \|\varphi_1 - \varphi_2\|_A.$$

In view of Theorem X.72 in [13], we conclude the following.

Theorem 2.1. *For* $\varphi_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in D(A)$, *there exists a* $y_0 > 0$ *and a unique function* $\varphi(\widehat{y}) = (u(\widehat{y}), v(\widehat{y}))$, $y \in [0, y_0]$, *such that* $\varphi \in C([0, y_0]; D(A)) \cap C^1([0, y_0]; H)$ *and* $\frac{\partial}{\partial y}\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix} + J\begin{pmatrix} u \\ v \end{pmatrix}$, $y \in [0, y_0]$, $\varphi(0) = \varphi_0$.

Returning to the Cauchy problem for (2.1) we deduce the following result:

Corollary 2.1. *For $(u_0, v_0) \in D(A)$, there exists a $y_0 > 0$ and a unique function $u(\widehat{y}) \in C([0, y_0]; X) \cap C^1([0, y_0]; H) \cap C^2([0, y_0]; L^2)$ satisfying (2.1) for $y \in [0, y_0]$ and $u(0) = u_0$, $\frac{\partial u}{\partial y}(0) = v_0$.*

Moreover, we have the energy conservation law

$$\begin{aligned} E(y) = \frac{1}{2} \int_{\mathbb{R}} \left| \frac{\partial u}{\partial y} \right|^2 dz + \frac{1}{4} (p-1)^2 \int_{\mathbb{R}} |u|^2 dz + \frac{1}{2} \int_{\mathbb{R}} V |u|^2 dz \\ + \frac{\lambda}{n+2} \int_{\mathbb{R}} |u|^{r+2} dz = E(0), \qquad y \in [0, y_0]. \end{aligned} \tag{2.6}$$

Now, let us assume $\lambda > 0$. From (2.6) we establish the existence of a local solution to the Cauchy problem, if $p \neq 1$, setting $\varphi(y) = \left(u(y), \frac{\partial u}{\partial y}(y) \right)$, $y \in [0, y_0]$, $H = H^1_V \times L^2$,

$$\| AJ(\varphi) \|_H \leq c \, \| \varphi \|_H^{r+1} \leq c. \tag{2.7}$$

Hence, from the semigroup integral formula we deduce

$$\begin{aligned} \| A\varphi(y) \|_H &\leq \| A\varphi(0) \|_H + \int_0^y \| AJ(\varphi(s)) \|_H \, ds \\ &\leq \| A\varphi_0 \|_H + c \int_0^y \| \varphi(s) \|_H^{r+1} ds \\ &\leq \| A\varphi_0 \|_H + cy \end{aligned}$$

and so, by Gronwall's inequality,

$$\| A\varphi(y) \|_H \leq \| A\varphi(0) \|_H \, e^{cy}.$$

We can thus state the following result.

Theorem 2.2. *Assuming $\lambda > 0$, $p \neq 1$ and $(u_0, v_0) \in D(A)$, there is a unique function $u \in C((0, +\infty); X) \cap C^1((0, +\infty); H^1_V) \cap C^2((0, +\infty); L^2)$ satisfying (2.1) and $u(0) = u_0$, $\frac{\partial u}{\partial y}(0) = v_0$.*

Now, consider, for each $p \neq 1$ and with $\lambda > 0$, $\left(u_p(0), \frac{\partial u_p}{\partial y}(0) \right) \in D(A)$, the unique solution $u_p \in C((0, +\infty); X) \cap C^1((0, +\infty); H^1_V) \cap C^2((0, +\infty); L^2)$ of the previous Cauchy problem.

Let us introduce (cf. [7]) the function

$$v_p(z, y) = e^{i c_p^2 y} \, u_p(z, c_p y), \qquad c_p = \frac{1}{2}(p-1) \tag{2.8}$$

and

$$\varepsilon_p^2 = \frac{1}{2 \, c_p^2} = \frac{2}{(p-1)^2}. \tag{2.9}$$

We have

$$\begin{aligned} & i \frac{\partial v_p}{\partial y} - \varepsilon_p^2 \frac{\partial^2 v_p}{\partial y^2} + \frac{1}{2} \left(\frac{\partial^2 v_p}{\partial z^2} - V v_p - \lambda \, |v_p|^r \, v_p \right) = 0, \\ & v_p(z, 0) = v_{0p}(z) = u_{0p}(z), \quad \frac{\partial v_p}{\partial y}(z, 0) = v_{1p}(z) = c_p (i u_{0p} + u_{1p})(z), \end{aligned} \tag{2.10}$$

where $u_{0p} = u_p(0)$, $u_{1p} = \frac{\partial u_p}{\partial y}(0)$. We assume

$$\begin{aligned} &\{u_{0p}\}_p \text{ bounded in } X, \quad \{u_{1p}\}_p \text{ bounded in } H^1_V, \\ &u_{0p} \underset{p\to\infty}{\longrightarrow} \widetilde{v}_0 \text{ in } L^2(\mathbb{R}), \quad \text{with } \widetilde{v}_0 \in X. \end{aligned} \tag{2.11}$$

Using the technique in [14], we want to extend Theorem 3 in [7] to obtain the following result:

Theorem 2.3. *Assume* (2.10) *and* (2.11). *Then, there exists a unique function* $\widetilde{v} \in C(\mathbb{R}_+; L^2) \cap L^\infty_{\text{loc}}(\mathbb{R}_+; X)$, *such that* $v_y \in L^\infty_{\text{loc}}(\mathbb{R}_+, L^2)$, *solution to the Cauchy problem*

$$\begin{aligned} &i\frac{\partial \widetilde{v}}{\partial y} + \frac{1}{2}\left(\frac{\partial^2 \widetilde{v}}{\partial z^2} - V\widetilde{v} - \lambda|\widetilde{v}|^r\widetilde{v}\right) = 0, \qquad (\lambda > 0, \quad r \geq 2), \\ &\widetilde{v}(0) = \widetilde{v}_0. \end{aligned} \tag{2.12}$$

Moreover, for each $T > 0$, *we have (with* v_p *solution to* (2.10)*),*

$$v_p \underset{p\to\infty}{\longrightarrow} \widetilde{v} \quad \text{in } C([0,T]; L^2(\mathbb{R})) \quad \text{(with the sup norm).}$$

Proof. To simplify the notation, we will replace v_p by v and ε_p^2 by ε^2 (if necessary) and we assume $\lambda = 1$.

Multiplying the equation in (2.10) by $\overline{v}_y$ (complex conjugate of v_y), integrating in $\mathbb{R}$ (in z) and taking the real part, we obtain (denoting by v_y the derivative $\frac{\partial v}{\partial y}, \ldots$),

$$\frac{1}{2}\,\varepsilon^2\frac{d}{dy}\int |v_y|^2 dz + \frac{1}{4}\frac{d}{dy}\int |v_z|^2 dz + \frac{1}{4}\frac{d}{dy}\int V|v|^2 dz + \frac{1}{2}\frac{1}{r+2}\frac{d}{dy}\int |v|^{r+2} dz = 0. \tag{2.13}$$

Hence, with c independent of p and y,

$$\varepsilon_p \|(v_p)_y\|_2 \leq c, \tag{2.14}$$

$$\|(v_p)_z\|_2 \leq c, \tag{2.15}$$

$$\|v_p\|_{r+2} \leq c, \tag{2.16}$$

$$\|V^{1/2}\, v_p\|_2 \leq c. \tag{2.17}$$

Multiplying the equation in (2.10) by $\overline{v}_p$, integrating in $\mathbb{R}$, and taking the imaginary part, we obtain

$$\varepsilon^2 \,\mathrm{Im} \int v_{yy}\overline{v}\, dz - \mathrm{Re} \int v_y \overline{v}\, dz = 0,$$

and since

$$\mathrm{Im} \int v_{yy}\overline{v}\, dz = \mathrm{Im}\left(\frac{d}{dy}\int v_y\overline{v}\, dz - \int v_y\overline{v}_y\, dz\right) = \mathrm{Im}\,\frac{d}{dy}\int v_y\overline{v}\, dz,$$

we find

$$\begin{aligned} &\varepsilon^2\,\frac{d}{dy}\,\mathrm{Im}\int v_y\overline{v}\, dz - \frac{d}{dy}\frac{1}{2}\int |v|^2 dz = 0, \\ &\int |v|^2 dz \leq 2\,\varepsilon^2\, \|v_y\|_2\, \|v\|_2 + c, \end{aligned}$$

and so, by (2.14),

$$\|v_p\|_2 \le c. \tag{2.18}$$

Now (the calculations can be justified by a suitable regularization technique) we take the y derivative in the equation in (2.10) to obtain, by multiplying by $(\overline{v}_p)_y$, integrating and taking the imaginary part:

$$\begin{aligned}
&\operatorname{Im}\varepsilon^2 \int \frac{\partial^3 v}{\partial y^3}\,\overline{v}_y\,dz - \operatorname{Re}\int v_{yy}\overline{v}_y\,dz - \operatorname{Im}\frac{1}{2}\int \frac{\partial^3 v}{\partial y\,\partial^2 z}\,\overline{v}_y\,dz \\
&\quad + \operatorname{Im}\frac{1}{2}\int V v_y\overline{v}_y\,dz + \operatorname{Im}\frac{1}{2}\int (|v|^r v)_y\,\overline{v}_y\,dz = 0, \\
&\operatorname{Im}\frac{d}{dy}\,\varepsilon^2\int v_{yy}\overline{v}_y\,dz - \frac{1}{2}\frac{d}{dy}\int |v_y|^2 dz + \frac{r}{2}\operatorname{Im}\int |v|^{r-2}\operatorname{Re}(v\overline{v}_y)\,v\overline{v}_y\,dz = 0,
\end{aligned}$$

and so, by (2.15), (2.18) and (2.11), for $y > 0$ we find

$$\int |v_y|^2\,dz \le c + \varepsilon^2\,\|v_{yy}\|_2\,\|v_y\|_2 + c\int_0^y\!\!\int |v_y|^2\,dz\,d\tau,$$

and so

$$\int |(v_p)_y|^2\,dz \le c + \left(\varepsilon_p^4\,\|(v_p)_{yy}\|_2^2\right) + c\int_0^y\!\!\int |(v_p)_y|^2\,dz\,d\tau. \tag{2.19}$$

Now, we take again the y derivative in (2.10), multiply by $(\overline{v}_p)_{yy}$, integrate in $\mathbb{R}$, and take the real part:

$$\varepsilon^2\frac{d}{dy}\int |v_{yy}|^2 dz + \frac{1}{2}\frac{d}{dy}\int |v_{yz}|^2 dz + \frac{1}{2}\frac{d}{dy}\int V|v_y|^2 dz \frac{1}{2}\operatorname{Re}\int (|v|^r v)_y\,\overline{v}_{yy}\,dz = 0. \tag{2.20}$$

We have (cf. [14], pg. 640):

$$\begin{aligned}
2\operatorname{Re}\left((|v|^r v)_y\,\overline{v}_{yy}\right) &= \frac{r}{2}\,|v|^{r-2}\,\frac{d}{dy}(v\overline{v}_y + \overline{v}v_y)^2 + |v|^r\,\frac{d}{dy}|v_y|^2 \\
&\quad - r|v|^{r-2}\left(v|v_y|^2\overline{v}_y + \overline{v}|v_y|^2 v_y\right).
\end{aligned} \tag{2.21}$$

Hence, by (2.20) and (2.21), we obtain, by applying the Gagliardo–Nirenberg inequality:

$$\begin{aligned}
&\varepsilon^4\frac{d}{dy}\int |v_{yy}|^2 dz + \frac{1}{2}\varepsilon^2\frac{d}{dy}\int |v_{yz}|^2 dz + \frac{1}{2}\varepsilon^2\frac{d}{dy}\int V|v_y|^2 dz \\
&\quad + \frac{\varepsilon^2}{8}\frac{d}{dy}\int |v|^{r-2}\left(\operatorname{Re}(v\overline{v}_y)\right)^2 dz + \frac{\varepsilon^2}{4}\frac{d}{dy}\int |v|^r\,|v_y|^2 dz \\
&\quad \le c\,\varepsilon^2\int |v|^{r-1}\,|v_y|^3\,dz \;\le\; c\,\varepsilon^2\,\|v\|_\infty^{r-1}\,\|v_y\|_2^{5/2}\,\|v_{yz}\|_2^{1/2} \\
&\quad \le c\,\|v\|_\infty^{r-1}\left(\varepsilon^2\,\|v_{yz}\|_2^2\right)^{1/4}\|v_y\|_2^{5/2}\,\varepsilon^{3/2}.
\end{aligned}$$

Hence, in view of (2.15), (2.18), (2.11) and (2.19), we get

$$\begin{aligned}
&\varepsilon^4\int |v_{yy}|^2 dz + \varepsilon^2\int |v_{yz}|^2 dz + \varepsilon^2\int V|v_y|^2 dz \\
&\quad + \varepsilon^2\int |v|^{r-2}\left(\operatorname{Re}(v\overline{v}_y)\right)^2 dz + \frac{1}{2}\int |v_y|^2 dz
\end{aligned}$$

$$\le c + \frac{1}{2}\left(\varepsilon^4 \|v_{yy}\|_2^2\right) + c\int_0^y \|v_y\|_2^{5/2}\,\varepsilon^{3/2}\left(\varepsilon^2\|v_{yz}\|_2^2\right)^{1/4} d\tau$$
$$+ c\int_0^y\!\!\int |v_y|^2\,dz\,d\tau.$$

Therefore, by (2.14), we obtain

$$\begin{aligned}
&\frac{1}{2}\,\varepsilon^4\int |v_{yy}|^2 dz + \varepsilon^2\int |v_{yz}|^2 dz + \frac{1}{2}\int |v_y|^2 dz + \varepsilon^2\int V|v_y|^2 dz\\
&\le c + c\int_0^y \|v_y\|_2\left(\varepsilon^2\|v_{yz}\|_2^2\right)^{1/4} d\tau + c\int_0^y\!\!\int |v_y|^2\,dz\,d\tau\\
&\le c + c\int_0^y\!\!\int |v_y|^2\,dz\,d\tau + c\int_0^y \left(\varepsilon^2\|v_{yz}\|_2^2\right)^{1/2} d\tau\\
&\le c + c\int_0^y\!\!\int |v_y|^2\,dz\,d\tau + c\int_0^y \varepsilon^2\int |v_{yz}|^2\,dz\,d\tau + c\,y.
\end{aligned}$$

We conclude that, by applying the Gronwall inequality, and for fixed $T > 0$, and $y \in [0, T]$,

$$\varepsilon_p^2\int |(v_p)_{yz}|^2 dz + \int |(v_p)_y|^2 dz \le c(T), \tag{2.22}$$

$$\varepsilon_p^4\int |(v_p)_{yy}|^2 dz + \varepsilon_p^2\int V|(v_p)_y|^2 dz \le c(T). \tag{2.23}$$

We have, in particular, from (2.14), (2.15), (2.17), (2.18), (2.22),

$$\begin{aligned}
v_p \in L^\infty(\mathbb{R}_+; H^1), &\quad V^{1/2}v_p \in L^\infty(\mathbb{R}_+; L^2),\\
(v_p)_y \in L^\infty(]0, T[; L^2), &\quad \varepsilon_p(v_p)_y \in L^\infty(\mathbb{R}_+; L^2).
\end{aligned} \tag{2.24}$$

From (2.10) (cf. [14], pg. 642, for similar computations with $V \equiv 0$) we deduce

$$\varepsilon_p^2\,\frac{\partial^2 v_p}{\partial y^2} - \varepsilon_q^2\,\frac{\partial^2 v_q}{\partial y^2} - i\,\frac{\partial}{\partial y}(v_p - v_q) - \frac{1}{2}\,\frac{\partial^2}{\partial z^2}(v_p - v_q)$$
$$+\frac{1}{2}V(v_p - v_q) + \frac{1}{2}|v_p|^r v_p - \frac{1}{2}|v_q|^r v_q = 0.$$

Multiplying the previous equation by $\overline{v_p - \widetilde{v}_q}$, taking the imaginary part and integrating in z, we see that for any fixed $T > 0$ and all $y \in [0, T]$,

$$\begin{aligned}
&\frac{d}{dy}\|v_p - v_q\|_2^2 + 2\ \mathrm{Im}\int (\overline{v_p - v_q})\left(\varepsilon_p^2\frac{\partial^2 v_p}{\partial y^2} - \varepsilon_q^2\,\frac{\partial^2 v_q}{\partial y^2}\right) dz\\
&= -\,\mathrm{Im}\int (\overline{v_p - v_q})\left(|v_p|^r v_p - |v_q|^r v_q\right) dz,\\
&\frac{d}{dy}\|v_p - v_q\|_2^2 + 2\,\varepsilon_p^2\ \mathrm{Im}\int (\overline{v_p - v_q})\left(\frac{\partial^2 v_p}{\partial y^2} - \frac{\partial^2 v_q}{\partial y^2}\right) dz\\
&+ 2(\varepsilon_p^2 - \varepsilon_q^2)\ \mathrm{Im}\int (\overline{v_p - v_q})\,\frac{\partial^2 v_q}{\partial y^2}\,dz\\
&= \frac{d}{dy}\|v_p - v_q\|_2^2 + 2(\varepsilon_p^2 - \varepsilon_q^2)\,\frac{d}{dy}\ \mathrm{Im}\int (\overline{v_p - v_q})\,\frac{\partial v_q}{\partial y}\,dz
\end{aligned}$$

$$
\begin{aligned}
&- 2(\varepsilon_p^2 - \varepsilon_q^2)\ \mathrm{Im} \int \frac{d}{dy} \left(\overline{v_p - v_q}\right) \frac{\partial v_q}{\partial y}\, dz \\
&= - \mathrm{Im} \int \left(\overline{v_p - v_q}\right) \left(|v_p|^r v_p - |v_q|^r v_q\right) dz.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
&\frac{d}{dy} \|v_p - v_q\|_2^2 + 2(\varepsilon_p^2 - \varepsilon_q^2) \frac{d}{dy} \mathrm{Im} \int \left(\overline{v_p - v_q}\right) \frac{\partial v_q}{\partial y}\, dz \\
&\quad \leq (\varepsilon_p^2 + \varepsilon_q^2)\, c(T) + \left(\|v_p\|_\infty^{r-1} + \|v_q\|_\infty^{r-1}\right) \|v_p - v_q\|_2^2 \\
&\quad \leq c(T)\, (\varepsilon_p^2 + \varepsilon_q^2) + c\, \|v_p - v_q\|_2^2.
\end{aligned}
$$

For $y \in [0, T]$, we have

$$
\|v_p(y) - v_q(y)\|_2^2 \leq (\varepsilon_p^2 + \varepsilon_q^2)\, c(T) + c \int_0^y \|v_p(\tau) - v_q(\tau)\|_2^2\, d\tau.
$$

Applying Gronwall's inequality, we deduce that $\{v_p\}$ is a Cauchy sequence in $L^\infty(]0, T[; L^2(\mathbb{R}))$ and so, by a suitable diagonalization method, there exists a function $\widetilde{v} \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}))$ and a subsequence $v_p \underset{p\to\infty}{\longrightarrow} \widetilde{v}$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}))$. By the previous estimates, it is easy to prove that

$$
\begin{aligned}
&\widetilde{v} \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; H^1(\mathbb{R})), \quad \widetilde{v}_y \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R})), \\
&V^{1/2}\, \widetilde{v} \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R})), \quad \text{and so } \ \widetilde{v} \in C(\mathbb{R}_+; L^2(\mathbb{R})).
\end{aligned}
$$

By applying Aubin's Lemma (cf. [11]) there exists a subsequence $v_p \underset{p\to\infty}{\longrightarrow} \widetilde{v}$ a.e. in $\mathbb{R}_+ \times \mathbb{R}$. Hence, $\widetilde{v}$ satisfies (2.12) in the sense of distributions and $\widetilde{v} \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; X)$. Finally, by standard methods, it is easy to see that $\widetilde{v}$ is the unique function with these properties that satisfies the Cauchy problem (2.12), and this achieves the proof of Theorem 2.3. □

Remark 2.1. If, for $\lambda > 0$, $p \neq 1$, u is the solution to the Cauchy problem for the equation (2.1) under the conditions in Theorem 2.2, we can multiply (2.1) by $\frac{\partial \overline{u}}{\partial z}$, then take the real part, and integrate over $\mathbb{R}$ (in z),

$$
0 = \mathrm{Re}\, \frac{d}{dy} \int \frac{\partial u}{\partial y} \frac{\partial \overline{u}}{\partial z}\, dz + \mathrm{Re} \int V u\, \frac{\partial \overline{u}}{\partial z}\, dz = \mathrm{Re}\, \frac{d}{dy} \int \frac{\partial u}{\partial y} \frac{\partial \overline{u}}{\partial z}\, dz - 2 \int V |u|^2\, dz,
$$

and so, by the energy conservation (2.6), we obtain

$$
\int_0^y \int V |u|^2\, dz\, d\tau \leq c, \tag{2.25}
$$

c independent of $y > 0$, and so we have the decay property

$$
\int_y^{y+1} \int V |u|^2\, dz\, d\tau \underset{y\to\infty}{\longrightarrow} 0.
$$

3. The periodic Goursat problem

In this section, we investigate the nonlinear Wheeler–DeWitt equation by following the approach and techniques developed in [1]. The difference is that, in our case, we have the following additional term in (1.7):

$$f(z,y)\,u, \quad \text{with} \quad f(z,y) = m^2 y^2 e^{6z} - e^{4z} - c_p^2, \tag{3.1}$$

where $c_p = \frac{1}{2}(p-1)$ and u is complex valued. With (in this case z will be the "time" variable)

$$\Box\, u = \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial y^2} \quad \text{and} \quad g(u) = \lambda\, |u|^r u \tag{3.2}$$

the equation (1.7) can be rewritten as

$$\Box\, u + f(z,y)\, u + g(u) = 0. \tag{3.3}$$

As in [1], we introduce the characteristic cones

$$C_z = \left\{(\tau, y) \,/\, \tau = z + |y|,\ |y| \le \pi\right\} \tag{3.4}$$

and the functions ϕ_+, ϕ_-, ϕ, $c_0(z)$ and $c_\pi(z)$, for a 2π periodic (in y) solution u of equation (3.3), defined by

$$\begin{aligned} \phi_+(z,y) &= u(z+y,y), \quad 0 \le y \le \pi,\\ \phi_-(z,y) &= u(z+y,-y), \quad 0 \le y \le \pi,\\ \phi(z,y) &= \left(\phi_+(z,y), \phi_-(z,y)\right),\\ c_0(z) &= u(z,0) = \phi_+(z,0) = \phi_-(z,0),\\ c_\pi(z) &= u(z+\pi,\pi) = \phi_+(z,\pi) = \phi_-(z,\pi). \end{aligned} \tag{3.5}$$

We then recall the following; see Lemma 1 in [1].

Lemma 3.1. *If u is a C^2 solution to the equation $\Box\, u = h(z,y)$ on a neighborhood U of C_z, when h is a continuous function in U, then*

$$\begin{aligned} \partial^2_{zy}\phi_+(z,y) &= \frac{1}{2}\left(\partial^2_y \phi_+(z,y) + h_+(z,y)\right),\\ \partial^2_{zy}\phi_-(z,y) &= \frac{1}{2}\left(\partial^2_y \phi_-(z,y) + h_-(z,y)\right), \end{aligned}$$

where $h_+(z,y) = h(z+y,y)$, $h_-(z,y) = h(z+y,-y)$, $0 \le y \le \pi$.

Now, if u is a C^2 function, 2π-periodic in y, solution to (3.3), we can multiply by $\frac{\partial \overline{u}}{\partial z}$, take the real part, and integrate in y:

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial z}\int_{-\pi}^{\pi}\left(\left|\frac{\partial u}{\partial z}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2\right) dy + \frac{\lambda}{r+2}\frac{\partial}{\partial z}\int_{-\pi}^{\pi} |u|^{r+2} dy\\ &\quad + \frac{1}{2}\frac{\partial}{\partial z}\int_{-\pi}^{\pi} f(z,y)\,|u|^2 dy - \frac{1}{2}\int_{-\pi}^{\pi}\frac{\partial f}{\partial z}(z,y)\,|u|^2 dy = 0. \end{aligned} \tag{3.6}$$

In addition, with $z' = z + y$, it is not difficult to deduce

$$
\begin{aligned}
&\frac{1}{2}\int_0^\pi \left(|\partial_y\phi_+|^2(z,y) + |\partial_y\phi_-|^2(z,y)\right) dy \\
&\quad = \frac{1}{2}\int_{-\pi}^{\pi} \left(\left|\frac{\partial u}{\partial y}\right|^2 (z',y) + \left|\frac{\partial u}{\partial z}\right|^2 (z',y) \right) dy \\
&\qquad + \mathrm{Re} \int_{-\pi}^{\pi} \frac{\partial u}{\partial z}(z',y)\,\frac{\partial \overline{u}}{\partial y}(z',y)\,dy
\end{aligned} \tag{3.7}
$$

and

$$
\begin{aligned}
&\frac{\partial}{\partial z}\,\mathrm{Re}\int_{-\pi}^{\pi} \frac{\partial u}{\partial z}(z',y)\,\frac{\partial \overline{u}}{\partial y}(z',y)\,dy \\
&= \mathrm{Re}\int_{-\pi}^{\pi} \frac{\partial^2 u}{\partial z^2}(z',y)\,\frac{\partial \overline{u}}{\partial y}(z',y)\,dy + \mathrm{Re}\int_{-\pi}^{\pi} \frac{\partial u}{\partial z}(z',y)\,\frac{\partial^2 \overline{u}}{\partial y\,\partial z}(z',y)\,dy \\
&= \mathrm{Re}\int_{-\pi}^{\pi} \left(\frac{\partial^2 u}{\partial y^2}(z',y) - f(z',y)\,u(z',y) - \lambda\,|u|^r\,u(z',y) \right) \frac{\partial \overline{u}}{\partial y}(z',y)\,dy \\
&= -\int_{-\pi}^{\pi} \left(f(z',y)\,\frac{1}{2}\,\frac{\partial}{\partial y}|u(z',y)|^2 - \frac{1}{r+2}\,\frac{\partial}{\partial y}|u|^{r+2}(z',y) \right) dy \\
&= \frac{1}{2}\int_{-\pi}^{\pi} \frac{\partial f}{\partial y}(z',y)\,|u(z',y)|^2\,dy \\
&= \frac{1}{2}\int_0^{\pi} \frac{\partial f}{\partial y}(z',y)\,|\phi_+(z,y)|^2\,dy + \frac{1}{2}\int_0^{\pi} \frac{\partial f}{\partial y}(z',-y)\,|\phi_-(z,y)|^2\,dy.
\end{aligned} \tag{3.8}
$$

From (3.6), (3.7) and (3.8), with $z' = z + y$ we deduce

$$
\begin{aligned}
&\frac{1}{2}\,\frac{\partial}{\partial z}\int_0^\pi \left(|\partial_y\phi_+|^2(z,y) + |\partial_y\phi_-|^2(z,y)\right)\,dy \\
&\quad = -\frac{\lambda}{r+2}\,\frac{\partial}{\partial z}\int_0^\pi \left(|\phi_+|^{r+2}(z,y) + |\phi_-|^{r+2}(z,y)\right) dy \\
&\qquad - \frac{1}{2}\,\frac{\partial}{\partial z}\int_0^\pi \left(f(z',y)\,|\phi_+|^2(z,y) + f(z',-y)\,|\phi_-|^2(z,y)\right) dy \\
&\qquad + \frac{1}{2}\int_0^\pi \left(\frac{\partial f}{\partial z}(z',y)\,|\phi_+|^2(z,y) + \frac{\partial f}{\partial z}(z',-y)\,|\phi_-|^2(z,y) \right) dy \\
&\qquad + \frac{1}{2}\int_0^\pi \left(\frac{\partial f}{\partial y}(z',y)\,|\phi_+|^2(z,y) + \frac{\partial f}{\partial y}(z',-y)\,|\phi_-|^2(z,y) \right) dy
\end{aligned} \tag{3.9}
$$

and so, with $z' = z + y$, $\tau' = \tau + y$, we have

$$
\begin{aligned}
E(z) &= \frac{1}{2}\int_0^\pi \left(|\partial_y\phi_+|^2(z,y) + |\partial_y\phi_-|^2(z,y)\right) dy \\
&\quad + \frac{\lambda}{r+2}\int_0^\pi \left(|\phi_+|^{r+2}(z,y) + |\phi_-|^{r+2}(z,y)\right) dy \\
&\quad + \frac{1}{2}\int_0^\pi \left(f(z',y)\,|\phi_+|^2(z,y) + f(z',-y)\,|\phi_-|^2(z,y)\right) dy
\end{aligned} \tag{3.10}
$$

$$= E(0) + \frac{1}{2}\int_0^z\int_0^\pi \left(\frac{\partial f}{\partial z}(\tau', y)\,|\phi_+|^2(\tau, y) + \frac{\partial f}{\partial z}(\tau', -y)\,|\phi_-|^2(\tau, y)\right) dy\,dz$$
$$+ \frac{1}{2}\int_0^z\int_0^\pi \left(\frac{\partial f}{\partial y}(\tau', y)\,|\phi_+|^2(\tau, y) + \frac{\partial f}{\partial y}(\tau', -y)\,|\phi_-|^2(\tau, y)\right) dy\,d\tau,$$

and $E(z)$ is the energy associated to our problem.

Recall that $\phi(0, y) = (\phi_+(0, y), \phi_-(0, y))$, $0 \le y \le \pi$ are the Goursat data $(u(y, y), u(y, -y))$, $0 \le y \le \pi$. For each z, that is, for each cone C_z, in [1] it is introduced the Banach space $(0 \le y \le \pi)$:

$$H(C_z) = \left\{ \phi(z, \widehat{y}) = \big(\phi_+(z, \widehat{y}), \phi_-(z, \widehat{y})\big) = \big(u(z + \widehat{y}, \widehat{y}), u(z + \widehat{y}, -\widehat{y})\big) \;\Big|\; \right.$$
$$\text{with finite norm } \|\phi(z, \cdot)\|_{H(C_z)}$$
$$= \left(\int_0^\pi \left(|\partial_y \phi_+(z, y)|^2 + |\partial_y \phi_-(z, y)|^2\right) dy\right)^{1/2} + |c_0(z)|$$
$$\text{and such that } c_0(z) = \phi_+(z, 0) = \phi_-(z, 0),$$
$$\left. c_\pi(z) = \phi_+(z, \pi) = \phi_-(z, \pi) \right\}. \tag{3.11}$$

We have

$$H(C_z) \hookrightarrow (L^p(0, \pi))^2, \qquad \text{for } 1 \le p \le +\infty. \tag{3.12}$$

Now, to solve the Goursat 2π-periodic (in y) problem for the equation (3.3), with data

$$(u(y, y) = \phi_+(0, y), \quad u(y, -y) = \phi_-(0, y))$$

$(0 \le y \le \pi)$ in $H(C_0)$, we follow closely the idea in Section 4 of [1], which reduces the problem to an abstract Cauchy form

$$\phi(z) = T_z\phi(0) - k_z(fu + g(u)),$$

with k_z linear with values in C_z defined by:

$$\phi = (\phi_+, \phi_-), \quad \phi_+(0, 0) = \phi_-(0, 0), \quad \phi_+(0, \pi) = \phi_-(0, \pi),$$
$$\phi_+(z, y) = u(z + y, y), \quad \phi_-(z, y) = u(z + y, -y),$$
$$(T_z\phi(0))(z, y) = \left(\phi_+\left(0, \frac{z + 2y}{2}\right) + \phi_-\left(0, \frac{z}{2}\right) - c_0(0),\right.$$
$$\left.\phi_-\left(0, \frac{z + 2y}{2}\right) + \phi_+\left(0, \frac{z}{2}\right) - c_0(0)\right),$$

for $y \le \pi - \frac{z}{2}$,

$$(T_z\phi(0))(z, y) = \left(\phi_+\left(0, \frac{z + 2y}{2} - \pi\right) + \phi_-\left(0, \frac{z}{2}\right) - 2\,c_0(0) + c_\pi(0),\right.$$
$$\left.\phi_-\left(0, \frac{z + 2y}{2} - \pi\right) + \phi_+\left(0, \frac{z}{2}\right) - 2\,c_0(0) + c_\pi(0)\right),$$

for $\pi - \frac{z}{2} \le y \le \pi$,

where

$$
\begin{aligned}
c_0(z) &= \phi_+\left(0,\frac{z}{2}\right) + \phi_-\left(0,\frac{z}{2}\right) - c_0(0),\\
c_\pi(z) &= \phi_+\left(0,\frac{z}{2}\right) + \phi_-\left(0,\frac{z}{2}\right) - 2\,c_0(0) + c_\pi(0),\\
c_0(0) &= \phi_+(0,0) = \phi_-(0,0), \quad c_\pi(0) = \phi_+(0,\pi) = \phi_-(0,\pi),
\end{aligned}
$$

and, for a continuous function $h(z,y)$, 2π-periodic in y,

$$
\begin{aligned}
(k_z h)_\pm(y) &= \int_0^{(z+2y)/2}\int_0^{z/2} h\left(p+q,\pm(p-q)\right) dq\,dp, \quad \text{for } y \le \pi - \frac{z}{2},\\
(k_z h)_\pm(y) &= \int_0^{(z+2y)/2-\pi}\int_\pi^{y+\pi} h\left(p+q,\pm(p-q)\right) dq\,dp\\
&\quad + \int_0^{\pi}\int_0^{z/2} h\left(p+q,\pm(p-q)\right) dq\,dp\\
&\quad + \int_\pi^{(z+2y)/2}\int_{p-\pi}^{z/2} h\left(p+q,\pm(p-q)\right) dq\,dp, \quad \text{for } \pi - \frac{z}{2} \le y \le \pi.
\end{aligned} \tag{3.13}
$$

Let us introduce, with C_τ defined in (3.4),

$$
D_z = \bigcup C_\tau, \qquad \text{for } 0 \le \tau \le z.
$$

For u continuous, 2π-periodic in y, it is easy to see, cf. [1] and with c independent of $z \in [0,\pi]$,

$$
\|k_z(fu + g(u))\|_{H(z)} \le c\|fu + g(u)\|_{L^2(D_z)}. \tag{3.14}
$$

Hence, by applying the estimate (8) in [1] and for all ϕ satisfying (3.13) and $z \in [0,\pi]$, we find

$$
\|\phi(z)\|_{H(C_z)} \le (1 + c\sqrt{z})\,\|\phi(0)\|_{H(C_0)} + c\,\|fu + g(u)\|_{L^2(D_z)}. \tag{3.15}
$$

Now, for a function $\phi = (\phi_+,\phi_-)$ defined in C_z we say that $\phi \in L^p(C_z)$, $1 \le p \le +\infty$, if $\phi_+(z,\widehat{y})$ and $\phi_-(z,\widehat{y})$ belong to $L^p(0,\pi)$ and we put

$$
\|\phi\|_{L^p(C_z)} = \|\phi_+(z,\widehat{y})\|_{L^p(0,\pi)} + \|\phi_-(z,\widehat{y})\|_{L^p(0,\pi)}.
$$

We have $\|\phi\|_{L^\infty(C_z)} \le c\,\|\phi\|_{H(C_z)}$, for $\phi \in H(C_z)$. By setting in (3.13), $N_z(\phi) = -k_z(fu + g(u))$, we want to prove the following result which is a variant of Theorem 2 in [1].

Theorem 3.1. *If $\phi(0) \in H(C_0)$, then there is a $z_0 \in (0,\pi)]$ and a unique continuous function $\phi(\widehat{z}) = (\phi_+(\widehat{z}),\phi_-(\widehat{z}))\colon [0,z_0] \to \widetilde{H}_1(0,\pi) = \{\phi = (\phi_+,\phi_-) \in H^1(0,\pi)^2 \mid \phi_+(0) = \phi_-(0),\ \phi_+(\pi) = \phi_-(\pi)\}$ such that*

$$
\phi(z) = T_z\phi(0) + N_z(\phi), \qquad z \in [0,z_0]. \tag{3.16}
$$

Proof. Replacing an iteration method by a fixed point argument, we follow the lines of the proof of Theorem 2 in [1], which is a special case of the proof of

Theorem 13 in [2]. We have, with an increasing continuous function $\theta\colon \mathbb{R} \to \mathbb{R}$ with $\theta(0) = 1$,

$$\|N_z(\phi_1) - N_z(\phi_2)\|_{L^2(0,\pi)} \leq \theta(M)\ \|\phi_1(z) - \phi_2(z)\|_{\widetilde{H}^1(0,\pi)} \quad \text{if}\ \ \|\phi_1(z)\|_{\widetilde{H}^1(0,\pi)}, \|\phi_2(z)\|_{\widetilde{H}^1(0,\pi)} \leq M, \tag{3.17}$$

since $H^1((0,2\pi)) \hookrightarrow L^\infty((0,2\pi))$,

$$||a|^r a - |b|^r b| \leq c\left(|a|^{r-1} + |b|^{r-1}\right)|a-b|, \qquad a, b \in \mathbb{C},$$

and, in addition, we have, by (3.1), for $z, y \in [0,\pi]$,

$$|f(z,y)| \leq m^2\pi^2 e^{6\pi} + e^{4\pi} + c_p^2.$$

For $\phi \in C([0,z_0];\widetilde{H}^1(0,\pi)$ let us define $\widetilde{\phi} \in C([0,z_0],\widetilde{H}^1(0,\pi)$ by

$$\widetilde{\phi}(z) = T_z\phi(0) + N_z(\phi), \qquad z \in [0,z_0].$$

With $\phi_i \in C([0,z_0];\widetilde{H}^1(0,\pi)$, $i = 1,2$, $\phi_{i+}(z,\widehat{y}) = u_i(z+\widehat{y},\widehat{y})$ and $\phi_{i-}(z,\widehat{y}) = u_i(z+\widehat{y},-\widehat{y})$, in view of (3.14), (3.15), (3.17), with $X_1 = C([0,z_0];\widetilde{H}^1(0,\pi)$, we find

$$\begin{aligned}\|\widetilde{\phi}_2 - \widetilde{\phi}_1\|_{X_1} \leq &(1 + c\sqrt{z_0})\,\|\phi_2(0) - \phi_1(0)\|_{\widetilde{H}^1(0,\pi)} \\ &+ c\,\theta(M)\int_0^{z_0} \|\phi_2(z) - \phi_1(z)\|_{\widetilde{H}^1(0,\pi)}\,dz \\ \leq &(1 + c\sqrt{z_0})\,\|\phi_2(0) - \phi_1(0)\|_{\widetilde{H}^1(0,\pi)} + c\,\theta(M)\,z_0\,\|\phi_1 - \phi_2\|_X \\ &\text{if}\ \ \|\phi_1\|_{X_1}, \|\phi_2\|_{X_1} \leq M \quad (X_1 \text{ endowed with the sup norm}).\end{aligned} \tag{3.18}$$

If we choose $M \geq \|\phi_1(0)\|_{\widetilde{H}^1(0,\pi)} + 1$, then from (3.18) with $\phi_2 \equiv 0$,

$$\begin{aligned}\|\widetilde{\phi}_1\|_{X_1} &\leq (1 + c\sqrt{z_0})\,\|\phi_1(0)\|_{\widetilde{H}^1(0,\pi)} + c\,\theta(M)\,z_0\,M \\ &\leq (1 + c\sqrt{z_0})\,(M-1) + c\,\theta(M)\,z_0\,M \ \leq\ M\end{aligned}$$

for $z_0 \leq z(M)$.

From (3.18) we also derive, for ϕ_1, ϕ_2 such that $\phi_1(0) = \phi_2(0)$,

$$\|\widetilde{\phi}_2 - \widetilde{\phi}_1\|_{X_1} \leq c\,\theta(M)\,z_0\,\|\phi_2 - \phi_1\|_{X_1} \leq \frac{1}{2}\,\|\phi_2 - \phi_1\|_X$$

for $z_0 \leq z_1(M)$. Then, for $z_0 \leq \min(z(M), z_1(M)))$, the map $\phi \to \widetilde{\phi}$ is a strict contraction in the subspace

$$\left\{\phi \in C([0,z_0];\widetilde{H}^1(0,\pi)\,/\ \ \phi(0) = \phi_1(0),\ \ \|\phi\|_{X_1} \leq M\right\}$$

which is a Banach space. Hence, there is a unique fixed point, and the theorem is proved. □

In order to prove a global (in z) existence result for the equation (3.16) we need to extend (3.10) which was proved for $u \in C^2(D_{z_0})$ that is for $\phi = (\phi_+, \phi_-) \in (C^2([0,\pi]))^2$. In that case we must assume $r \geq 2$ in (1.7), to extend Theorem 4 in [1] to our case:

Theorem 3.2. *Assume the hypothesis of Theorem 3.1. Then the associated function u in $C^2(D_{z_0})$ is a solution to equation (3.3) if and only if $\phi_\pm(0) \in C^2([0,\pi])$ and satisfy the following nonlinear conditions:*

$$\begin{aligned}
&\partial_y \phi_\pm(0,\pi) - \partial_y \phi_\pm(0,0) + \int_0^\pi \left(f(y,y)\,\phi_\mp(0,y) + g(\phi_\mp(0,y))\right) dy = 0,\\
&\partial_y^2 \phi_\pm(0,\pi) - \partial_y^2 \phi_\pm(0,0)\\
&\quad = f(\pi,\pi)\,\phi_\pm(0,\pi) + g(\phi_\pm(0,\pi)) - f(0,0)\,\phi_\pm(0,0) - g(\phi_\pm(0,0))\\
&\qquad - 2\int_0^\pi \left(\frac{\partial f}{\partial z}(y,\mp y)\,\phi_\mp(0,y) + f(y,\mp y)\,\partial_z \phi_\mp(0,y) + \frac{\partial}{\partial z} g(\phi_\mp(0,y))\right) dy.
\end{aligned} \tag{3.19}$$

The proof of Theorem 3.2 is similar to the proof of Theorem 4 in [1]. In particular, for the second condition in (3.19) we must apply Lemma 3.1.

To prove that the solution ϕ obtained in Theorem 3.1 is global in z, we must extend to ϕ the energy formula (3.10), proved for $\phi \in C^2$. This can be made by an approximation method exactly as it was developed in the proof of Theorem 6 in [1]: we approximate $\phi(0) \in H(C_0) = \widetilde{H}^1(0,\pi)$ by a sequence $\{\phi_{n\pm}(0)\} \in H(C_0) \cap C^2([0,\pi])$ satisfying conditions (3.19). The corresponding solutions $\phi_n = (\phi_{n+}, \phi_{n-})$ satisfy (3.10) and, cf. a variant of Theorem 3 in [1], we obtain $\|\phi_n(z) - \phi(z)\|_{\widetilde{H}^1(0,\pi)} \underset{n\to\infty}{\longrightarrow} 0$. Hence, the energy formula (3.10) can be extended for $\phi \in C([0,z_0]; \widetilde{H}^1(0,\pi)$.

Now, let $\phi \in C([0,z_0]; \widetilde{H}^1(0,\pi)$ be the unique solution to (3.16) for a given $\phi(0) \in \widetilde{H}^1(0,\pi)$. Let u be the associated function such that $\phi = (\phi_+, \phi_-)$, $\phi_+(z,y) = u(z+y,y)$, $\phi_-(z,y) = u(z+y,-y)$, $y \in [0,\pi]$ and assume $\lambda > 0$. From (3.10) we deduce that

$$\begin{aligned}
&\frac{1}{2}\int_0^\pi \left(|\partial_y \phi_+|^2 + |\partial_y \phi_-|^2\right)(z,y)\,dy + \frac{\lambda}{r+2}\int_0^\pi \left(|\phi_+|^{r+2} + |\phi_-|^{r+2}\right)(z,y)\,dy\\
&\quad \le c(\varepsilon) + \varepsilon \int_0^\pi \left(|\phi_+|^{r+2} + |\phi_-|^{r+2}\right)(z,y)\,dy + E(0)\\
&\qquad + c\,z + c\int_0^z\int_0^\pi \left(|\phi_+|^{r+2} + |\phi_+|^{r+2}\right)(\tau,y)\,dy\,d\tau, \qquad \text{for each } \varepsilon > 0.
\end{aligned}$$

We can choose $\varepsilon < \frac{\lambda}{r+2}$ and, by applying Gronwall's inequality, it follows that

$$\begin{aligned}
&\frac{1}{2}\int_0^\pi \left(|\partial_y \phi_+|^2 + |\partial_y \phi_-|^2\right)(z,y)\,dy\\
&\qquad + \frac{\lambda}{r+2}\int_0^\pi \left(|\phi_+|^{r+2} + |\phi_-|^{r+2}\right)(z,y)\,dy\\
&\le c(z),
\end{aligned}$$

c continuous in $(0,+\infty)$. Hence, $\|\phi(z)\|_{\widetilde{H}^1(0,\pi)} \le c_1(z)$, c_1 continuous in $(0,+\infty)$.

Hence, we can finally state the following result.

Theorem 3.3. *Assume $\lambda > 0$. Let $\phi_0 = (\phi_{0+}, \phi_{0-}) \in H(C_0) = \widetilde{H}^1(0,\pi)$ and u_0 its associated function such that $\phi_{0+}(y) = u_0(y,y)$, $\phi_{0-}(y) = u_0(y,-y)$,*

$y \in (0,\pi)[$. Then, there exists a unique function $\phi \in C((0,+\infty); \widetilde{H}^1(0,\pi)$ such that

$$\phi(z) = T_z\,\phi_0 + N_z(\phi), \qquad \phi(0) = \phi_0, \quad z \geq 0,$$

and the associated function $u(z,y)$ defined by

$$u(z+y,y) = \phi_+(z,y), \qquad u(z+y,-y) = \phi_-(z,y),$$

is a weak solution to the Goursat 2π-periodic (in y) problem for the equation (3.3) in $D = \{(\tau,y) \,/\, |y| \leq \tau \leq z+|y|,\, z \geq 0,\, |y| \leq \pi\}$ and $2\pi - D$ translations (with f replaced by $f(\widehat{z}, \theta(\widehat{y}))$, where $\theta(\widehat{y})$ is the 2π-periodic extension of $\widehat{y}^2$ defined in $[-\pi,\pi]$). In addition, if $\phi_0 \in C^2([0,\pi])$, then u is a classical solution to the Goursat 2π-periodic (in y) problem for the equation (3.3) in the same domain.

Acknowledgements

The authors are indebted to Rémi Carles, Armando Machado and Hugo Beirão da Veiga for useful discussions and suggestions. J.P. Dias was supported by the Grant UID/MAT/04561/3013 from the Fundação para a Ciência e a Tecnologia (FCT). P.G. LeFloch was supported by the grant ITN 642768 entitled "ModCompShock" under the European program MSCA-ITN-2014-ETN: Marie Skodowska-Curie Innovative Training Networks (ITN-ETN).

References

[1] Baez, J.C., Zhou, Z.: The global Goursat problem on $\mathbb{R} \times S^1$. J. Funct. Anal. **83**, 364–382 (1989)

[2] Baez, J.C., Segal, I.E., Zhou, Z.: The global Goursat problem and scattering for nonlinear wave equations. J. Funct. Anal. **93**, 239–269 (1990)

[3] Cooper, A., Susskind, L., Thorlacius, L.: Two-dimensional quantum cosmology. Nucl. Phys. B **263**, 132–162 (1991)

[4] Dias, J.P., Figueira, M.: The simplified Wheeler–DeWitt equation: the Cauchy problem and some spectral properties. Ann. Inst. Henri Poincaré: Phys. Théorique **54**, 17–26 (1991)

[5] Dias, J.P., Figueira, M.: The Cauchy problem for a nonlinear Wheeler–DeWitt equation. Ann. Inst. Henri Poincaré: Anal. Non-Linear **10**, 99–107 (1993)

[6] Dias, J.P., Figueira, M.: On a class of solutions for the simplified Wheeler–DeWitt equation with a massless single scalar field. Ricerche di Mat. **44**, 145–155 (1995)

[7] Dias, J.P., Figueira, M.: On a singular limit for a class of solutions of the simplified Wheeler–DeWitt equation with a massless single scalar field. Rendiconti di Mat. Serie VII **13**, 529–542 (1993)

[8] Gibbons, G.W., Grischchuk, L.P.: What is a typical wave function for the universe? Nucl. Phys. B **313**, 736–748 (1989)

[9] Hartle, J.B., Hawking, S.W.: Wave function of the Universe. Phys. Rev. D **28**, 2960–2975 (1983)

[10] Hawking, S.W.: The quantum state of the universe. Nucl. Phys. B **239**, 257–276 (1984)

[11] Lions, J.L.: Quelques Méthodes de Résolution des Problèmes aux Limites Non-Linéaires. Dunod, Paris (1969)

[12] Nguyen, L.H., Parwani, R.R.: Nonlinear quantum cosmology. Gen. Relativ. Gravit. **41**, 2543–2560 (2009)

[13] Reed, M., Simon, B.: Methods of Modern Mathematical Physics, Vol. II: Fourier Analysis, Self-Adjointness. Academic Press, London (1975)

[14] Tsutsumi, M.: Nonrelativistic approximation of nonlinear Klein–Gordon equations in two space dimensions. Nonlinear Ann. T.M.A. **8**, 637–643 (1984)

João-Paulo Dias
Center for Mathematics, Fundamental Applications, and Operations Research
Universidade de Lisboa
Campo Grande
1749-016 Lisbon
Portugal
e-mail: jpdias@fc.ul.pt

Philippe G. LeFloch
Laboratoire Jacques-Louis Lions, Centre National de la Recherche Scientifique
Sorbonne Université
4 Place Jussieu
75252 Paris
France
e-mail: contact@philippelefloch.org

Received: 11 August 2017.
Accepted: 15 February 2018.