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Positive solutions of an elliptic Neumann problem with a sublinear indefinite nonlinearity

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Abstract. Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded and smooth domain and $a : \Omega \to \mathbb{R}$ be a sign-changing weight satisfying $\int_\Omega a < 0$. We prove the existence of a *positive* solution u_q for the problem

$$(P_{a,q}) \qquad \begin{cases} -\Delta u = a(x)u^q & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

if $q_0 < q < 1$, for some $q_0 = q_0(a) > 0$. In doing so, we improve the existence result previously established in Kaufmann et al. (J Differ Equ 263:4481–4502, 2017). In addition, we provide the asymptotic behavior of u_q as $q \to 1^-$. When Ω is a ball and a is radial, we give some explicit conditions on q and a ensuring the existence of a positive solution of $(P_{a,q})$. We also obtain some properties of the set of q's such that $(P_{a,q})$ admits a solution which is positive on $\overline{\Omega}$. Finally, we present some results on nonnegative solutions having *dead cores*. Our approach combines bifurcation techniques, *a priori* bounds and the sub-supersolution method.

Mathematics Subject Classification. 35J25, 35J61.

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1. Introduction

Let Ω be a bounded and smooth domain of $\mathbb{R}^N$ with $N \geq 1$, and $0 < q < 1$. The purpose of this article is to discuss the existence of *positive* solutions for the problem

$$(P_{a,q}) \qquad \begin{cases} -\Delta u = a(x)u^q & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ is the usual Laplacian in $\mathbb{R}^N$, and ν is the outward unit normal to $\partial\Omega$. Throughout this article, unless otherwise stated, we assume that $r > N$

and $a \in L^r(\Omega)$ is such that

$$(H_0) \qquad a \text{ changes sign and } \int_\Omega a < 0.$$

We set $a^\pm := \max(\pm a, 0)$. Note that the change of sign in a means that $|\text{supp } a^\pm| > 0$, where $|A|$ stands for the Lebesgue measure of $A \subset \mathbb{R}^N$. By a *nonnegative solution* of $(P_{a,q})$ we mean a function $u \in W^{2,r}(\Omega)$ (and hence $u \in \mathcal{C}^1(\overline{\Omega})$) that satisfies the equation for the weak derivatives and the boundary condition in the usual sense, and such that $u \geq 0$ in Ω. If, in addition, $u > 0$ in Ω, then we call it a *positive solution* of $(P_{a,q})$. In this case, we shall also say that (q, u) is a *positive solution* of $(P_{a,q})$. Let us denote by $\mathcal{P}^\circ$ the interior of the positive cone of $\mathcal{C}^1(\overline{\Omega})$, i.e.

$$\mathcal{P}^\circ := \left\{u \in \mathcal{C}^1(\overline{\Omega}) : u > 0 \text{ on } \overline{\Omega}\right\}.$$

We observe that a positive solution of $(P_{a,q})$ *need not* belong to $\mathcal{P}^\circ$ (see e.g. Remark 1.5 below).

Very few works have been devoted to $(P_{a,q})$, the first and main one being [4], where the following results were established (see Theorem 2.1 and Lemmas 2.1 and 3.1 therein):

Theorem 1.0. (Bandle–Pozio–Tesei [4]) *Let $0 < q < 1$ and a be a sign-changing Hölder continuous function on $\overline{\Omega}$. Then, the following three assertions hold:*

- *(i) If $(P_{a,q})$ has a positive solution then $\int_\Omega a < 0$.*
- *(ii) If $\int_\Omega a < 0$, then $(P_{a,q})$ has at least one nontrivial nonnegative solution.*
- *(iii) $(P_{a,q})$ has at most one solution in $\mathcal{P}^\circ$.*

Denoting by $\Omega_+ = \Omega_+(a)$ the largest open subset of Ω where $a > 0$ *a.e.* let us consider the following two conditions:

(H_1) Ω_+ has finitely many connected components and $\left|(\text{supp } a^+) \setminus \Omega_+\right| = 0$,

(H_+) $\partial\Omega_+$ satisfies the inner sphere condition with respect to Ω_+.

Remark 1.1.

- (i) One may easily see that Theorem 1.0 (i) and (iii) still hold if $a \in L^r(\Omega)$, with $r > N$, cf. the proofs of [4, Lemma 2.1, and Lemma 3.1]. Moreover, we deduce that so does Theorem 1.0 (ii), by using a variational approach as the one in the proof of [16, Corollary 1.8].
- (ii) If a is Hölder continuous, Ω_+ has finitely many connected components and (H_+) holds, then [4, Theorem 3.1] shows, in particular, that $(P_{a,q})$ has at most one nonnegative solution u such that $u > 0$ in Ω_+. This result can be extended to $a \in L^r(\Omega)$, $r > N$, with the same proof, assuming now (H_1) and (H_+).

Let us mention that some of the above results were extended in [1] to a problem that is a linear perturbation of $(P_{a,q})$. However, no *sufficient* conditions for the existence of *positive* solutions have been provided in [1,4]. Let us point out that, due to the non-Lipschitzian character of u^q at $u = 0$ and the change of sign in a, the strong maximum principle does not apply to $(P_{a,q})$.

As a consequence, one cannot derive the positivity of nontrivial nonnegative solutions of $(P_{a,q})$.

To the best of our knowledge, the first existence result on positive solutions of $(P_{a,q})$ has been proved in our recent work [16]. We recall it now. Under (H_1), we showed that *every* nontrivial nonnegative solution of $(P_{a,q})$ belongs to $\mathcal{P}^\circ$ if q is close enough to 1 (see [16, Theorem 1.7]). This positivity result was proved via a continuity argument inspired by [14, Theorem 4.1] (see also [15]), which is based on the fact that the strong maximum principle applies to $(P_{a,q})$ if $q = 1$. As a consequence, assuming in addition (H_0), we deduced that if q is close enough to 1 then $(P_{a,q})$ has a unique nontrivial nonnegative solution, which belongs to $\mathcal{P}^\circ$ (see [16, Corollary 1.8]). Let us mention that, in general, uniqueness of nonnegative solutions for $(P_{a,q})$ *does not* hold (see e.g. the proof of Theorem 1.4 (ii) below).

Regarding the Dirichlet counterpart of $(P_{a,q})$, we refer to [3,18] for the existence of nontrivial nonnegative solutions, and to [10,11,13,16] for the existence of a positive solution. Finally, let us mention, as already pointed out in [1,3,4], that problems like $(P_{a,q})$ and its Dirichlet counterpart naturally arise in population dynamics models, cf. [12,17].

Our purpose in this article is to carry on the investigation of $(P_{a,q})$, refining and extending the existence results on positive solutions established in [16]. In particular, following a different approach to the one in [16], we shall remove (H_1) and prove that under (H_0) the problem $(P_{a,q})$ has a solution $u_q \in \mathcal{P}^\circ$ for q close to 1. As a byproduct, we deduce that (H_0) is necessary and sufficient for the existence of a positive solution of $(P_{a,q})$ for *some* $q \in (0,1)$, see Corollary 1.3. Moreover, we shall provide the stability properties of u_q and its asymptotic behavior as $q \to 1^-$ (see Theorem 1.2 below). Note that the stability analysis for solutions in $\mathcal{P}^\circ$ of $(P_{a,q})$ is not easily carried out for $q \in (0,1)$ in general (see Remark 2.6 (ii)).

Under (H_0), let us denote by $\mu_1(a)$ the first positive eigenvalue of the problem

$$(E_{\mu,a}) \qquad \begin{cases} -\Delta\phi = \mu a(x)\phi & \text{in } \Omega, \\ \frac{\partial\phi}{\partial\nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

and by $\phi_1 = \phi_1(a)$ the associated positive eigenfunction satisfying $\int_\Omega \phi_1^2 = 1$. It is well known that $\mu_1(a)$ is simple, and $\phi_1 \in \mathcal{P}^\circ$.

We shall look at q as a bifurcation parameter in $(P_{a,q})$. As a matter of fact, note that if $\mu_1(a) = 1$, then $u = t\phi_1$ solves $(P_{a,1})$, i.e. $(P_{a,q})$ has the trivial line Γ_1 of solutions in $\mathcal{P}^\circ$, where

$$\Gamma_1 := \{(q,u) = (1, t\phi_1) : t > 0\}.$$

We shall obtain, for q close to 1, a curve of solutions in $\mathcal{P}^\circ$ bifurcating from Γ_1 (see Fig. 1).

Let us recall that a solution $u \in \mathcal{P}^\circ$ of $(P_{a,q})$ is said to be *asymptotically stable* (respect. *unstable*) if $\gamma_1(q,u) > 0$ (respect. < 0), where $\gamma_1(q,u)$ is the

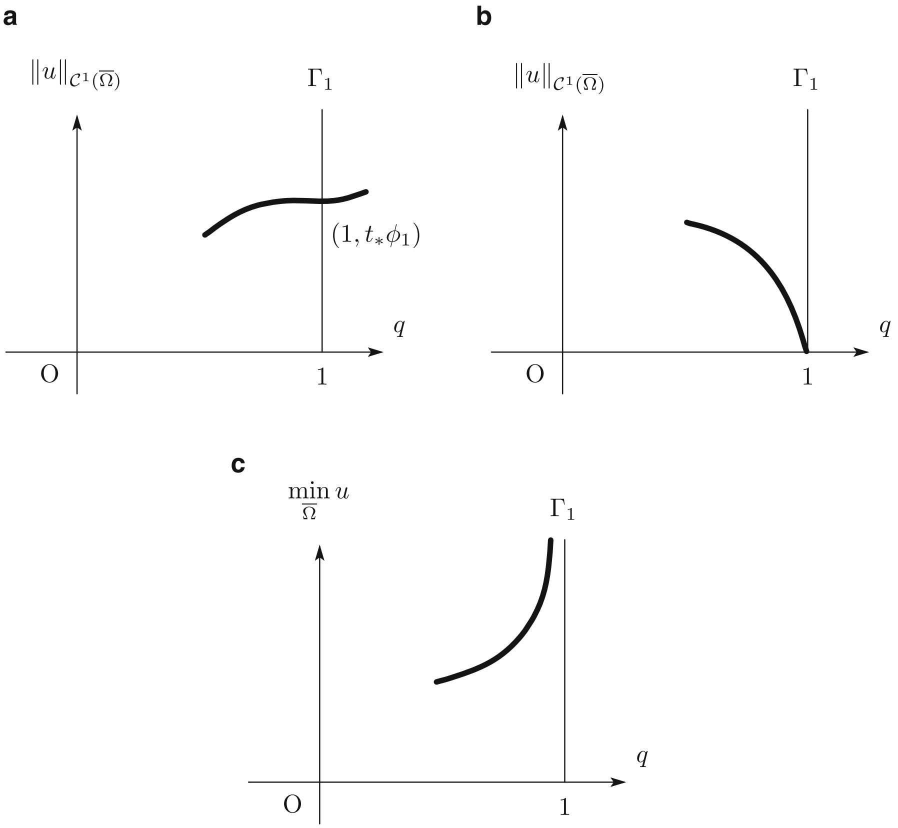

FIGURE 1. Bifurcating solutions in $\mathcal{P}^\circ$ from the trivial line Γ_1. **a** Case $\mu_1(a) = 1$, **b** Case $\mu_1(a) > 1$ and **c** Case $\mu_1(a) < 1$

first eigenvalue of the linearized eigenvalue problem at u, namely,

$$\begin{cases} -\Delta\varphi = qa(x)u^{q-1}\varphi + \gamma\varphi & \text{in } \Omega, \\ \frac{\partial\varphi}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

In addition, u is said to be *weakly stable* if $\gamma_1(q, u) \geq 0$.

Set

$$t_* := \exp\left[-\frac{\int_\Omega a(x)\,\phi_1^2 \log\phi_1}{\int_\Omega a(x)\,\phi_1^2}\right]. \tag{1.1}$$

We are now in position to state our main results.

Theorem 1.2. *Assume* (H_0)*. Then there exists* $q_0 = q_0(a) \in (0, 1)$ *such that* $(P_{a,q})$ *has a solution* $u_q \in \mathcal{P}^\circ$ *for* $q_0 < q < 1$*. Moreover,* u_q *is asymptotically stable and satisfies the asymptotics*

$$u_q \sim \mu_1(a)^{-\frac{1}{1-q}}\, t_*\phi_1 \quad as \quad q \to 1^-,$$

i.e. $\mu_1(a)^{\frac{1}{1-q}} u_q \to t_*\phi_1$ *in* $\mathcal{C}^1(\overline{\Omega})$ *as* $q \to 1^-$*. More specifically:*

(i) *If* $\mu_1(a) = 1$, *then* $u_q \to t_*\phi_1$ *in* $\mathcal{C}^1(\overline{\Omega})$ *as* $q \to 1^-$.
(ii) *If* $\mu_1(a) > 1$, *then* $u_q \to 0$ *in* $\mathcal{C}^1(\overline{\Omega})$ *as* $q \to 1^-$.
(iii) *If* $\mu_1(a) < 1$, *then* $\min_{\overline{\Omega}} u_q \to \infty$ *as* $q \to 1^-$.

As a consequence of Theorem 1.0 (i) (see Remark 1.1 (i)) and Theorem 1.2, we derive the following:

Corollary 1.3. $(P_{a,q})$ *has a positive solution (or a solution in* $\mathcal{P}^\circ$*) for some* $q \in (0,1)$ *if and only if* (H_0) *holds.*

We shall prove Theorem 1.2 using a bifurcation technique based on the Lyapunov–Schmidt reduction, which yields the existence of bifurcating solutions in $\mathcal{P}^\circ$ from Γ_1 provided that $\mu_1(a) = 1$. By a suitable rescaling, we deduce then the results for the case $\mu_1(a) \neq 1$. Let us also point out that, in general, it is hard to give a lower estimate for $q_0\,(a)$, see Remark 1.5 below.

Now, our next results concern the sets

$$\mathcal{A} = \mathcal{A}_a := \{q \in (0,1) : \text{any nontrivial nonnegative solution of } (P_{a,q}) \text{ lies in } \mathcal{P}^\circ\}$$

and

$$\mathcal{I} = \mathcal{I}_a := \{q \in (0,1) : (P_{a,q}) \text{ has a solution } u \in \mathcal{P}^\circ\}.$$

We observe that if (H_0) holds then $(P_{a,q})$ has a nontrivial nonnegative solution for any $0 < q < 1$ (see e.g. the proof of [16, Corollary 1.8]), so that $\mathcal{A} \subseteq \mathcal{I}$. In [16, Theorem 1.9], we proved, under (H_0) and (H_1), that $\mathcal{A} = (q_a, 1)$ for some $q_a \in [0,1)$.

Let us now introduce a stronger assumption than (H_1):

$$(H_1') \qquad \Omega_+ \text{ is connected and } \left|(\text{supp } a^+) \setminus \Omega_+\right| = 0.$$

Note that (H_1') corresponds to (H_1) with Ω_+ consisting of a single connected component.

We shall complement [16, Theorem 1.9] as follows:

Theorem 1.4.

(i) *If* (H_0) *holds then* $\mathcal{I} = (q_i, 1)$ *for some* $q_i \in [0,1)$. *Moreover, if* (H_1') *and* (H_+) *hold, then for all* $q \in (0,1)$, *there exists a unique nontrivial nonnegative solution of* $(P_{a,q})$. *In particular,* $\mathcal{I} = \mathcal{A}$.
(ii) *There exists* $a \in \mathcal{C}(\overline{\Omega})$ *such that* $\mathcal{A}_a \subsetneq \mathcal{I}_a$. *More precisely, given* $\Omega := (x_0, x_1) \subset \mathbb{R}$ *and* $q \in (0,1)$, *there exists* $a \in \mathcal{C}(\overline{\Omega})$ *such that* $q \in \mathcal{I}_a \setminus \mathcal{A}_a$.

Remark 1.5. As a consequence of Theorem 1.4 (i), it follows that, given $q \in (0,1)$, there exists $a \in \mathcal{C}(\overline{\Omega})$ such that $q \notin \mathcal{I}_a$. Indeed, let $q \in (0,1)$, and define $\Omega := (0,\pi)$,

$$r := \frac{2}{1-q} \in (2,\infty), \quad \text{and} \quad a\,(x) := r^{1-\frac{2}{r}}\left(1 - r\cos^2 x\right) \quad \text{for } x \in \Omega.$$

One can check that $u\,(x) := \frac{\sin^r x}{r}$ is a (strictly positive in Ω) solution of

$$\begin{cases} -u'' = a(x)u^q & \text{in } \Omega, \\ u' = 0 & \text{on } \partial\Omega. \end{cases}$$

It follows that $q \notin \mathcal{A}_a$ because $u \notin \mathcal{P}^\circ$. Now, since a satisfies (H_1') and (H_+), we deduce from Theorem 1.4 (i) that u is the unique nontrivial nonnegative solution of $(P_{a,q})$, and $\mathcal{I}_a = \mathcal{A}_a$. Consequently, we have $q \notin \mathcal{I}_a$.

When Ω is a ball and a is radial, we shall exhibit some *explicit* conditions on q and a so that $(P_{a,q})$ admits a positive solution. This will be done via the well known sub-supersolutions method. In Theorem 1.6 below we give a condition that guarantees the existence of a positive solution (not necessarily in $\mathcal{P}^\circ$), while Theorem 1.8 provides us with a solution in $\mathcal{P}^\circ$.

Given $0 < R_0 < R$, we write

$$\begin{aligned} B_{R_0} &:= \left\{x \in \mathbb{R}^N : |x| < R_0\right\}, \\ A_{R_0,R} &:= \left\{x \in \mathbb{R}^N : R_0 < |x| < R\right\}, \\ \omega_{N-1} &:= \text{surface area of the unit sphere } \partial B_1 \text{ in } \mathbb{R}^N. \end{aligned}$$

If f is a radial function, we write (with a slight abuse of notation) $f(x) := f(|x|) := f(r)$. We first consider the case that supp a^+ is contained in B_{R_0} for some $R_0 \in (0, R)$.

Theorem 1.6. *Let $\Omega := B_R$ and $a \in L^\infty(\Omega)$ be a radial function such that $\int_\Omega a < 0$. Assume that there exists $R_0 > 0$ such that:*

- *$a \geq 0$ in B_{R_0};*
- *$a \leq 0$ in $A_{R_0,R}$;*
- *$r \to a(r)$ is differentiable and nonincreasing in (R_0, R), and*

$$\frac{1-q}{1+q} \int_{A_{R_0,R}} a^- \leq \int_{B_{R_0}} a^+. \tag{1.2}$$

Then $(P_{a,q})$ has a positive solution.

Remark 1.7. The condition (1.2) can also be formulated as

$$\frac{-\int_\Omega a}{\int_\Omega |a|} \leq q < 1. \tag{1.3}$$

In particular, we see that (1.2) is satisfied if q is close enough to 1. Note that if we replace a by

$$a_\delta = a^+ - \delta a^-, \quad \text{with} \quad \delta > \delta_0 := \frac{\int_\Omega a^+}{\int_\Omega a^-},$$

then the left-hand side in (1.3) approaches 1 as $\delta \to \infty$, so that this condition becomes very restrictive for a_δ as $\delta \to \infty$. On the other side, we have that $\int_\Omega a_\delta \to 0^-$ as $\delta \to \delta_0^+$, so that (1.3) becomes much less constraining for a_δ as $\delta \to \delta_0^+$. A similar argument will be used in Remark 4.5.

Next we consider the case that supp a^- is contained in B_{R_0} for some $R_0 \in (0, R)$.

Theorem 1.8. *Let* $\Omega := B_R$ *and* $a \in L^\infty(\Omega)$ *be a radial function such that* $\int_\Omega a < 0$. *Assume that there exists* $R_0 \in (0, R)$ *such that* $a \geq 0$ *in* $A_{R_0,R}$, *and*

$$\frac{1-q}{2q+N(1-q)}\omega_{N-1}R_0^N \left\|a^-\right\|_{L^\infty(B_{R_0})} < \int_{A_{R_0,R}} a^+. \tag{1.4}$$

Then $(P_{a,q})$ *has a solution* $u \in \mathcal{P}^\circ$.

Remark 1.9. Observe that unlike in Theorem 1.6, no differentiability nor monotonicity condition is imposed on a^- in Theorem 1.8. Note again that (1.4) is also clearly satisfied if q is close enough to 1.

Finally, we shall investigate the existence of nonnegative *dead core* solutions of $(P_{a,q})$. Following [3,4], the set $\{x \in \Omega : u(x) = 0\}$ is called the *dead core* of a nontrivial nonnegative solution u of $(P_{a,q})$. Let us mention that in the proof of Theorem 1.4 (ii) we shall see that, when $N = 1$, for any $q \in (0, 1)$ there exists a with $(P_{a,q})$ admitting a solution in $\mathcal{P}^\circ$ and also nonnegative solutions with nonempty dead cores.

Indeed, we give some *sufficient* conditions for the existence of dead core solutions of $(P_{a,q})$. We introduce the following condition:

(H_2) $b_1 \in L^\infty(\Omega), b_2 \in \mathcal{C}(\overline{\Omega}), b_1, b_2 \geq 0$ and supp $b_1 \cap \{x \in \Omega : b_2(x) > 0\} = \emptyset$.

Given a nonempty open subset $G \subseteq \Omega$ and $\sigma > 0$, we set

$$G_\sigma := \{x \in G : \mathrm{dist}(x, \partial G) > \sigma\}. \tag{1.5}$$

We call the set $\Omega \setminus \overline{\Omega_\sigma}$ a *tubular neighborhood* of $\partial\Omega$.

Theorem 1.10.

(i) *Let* $q \in (0, 1)$, *and assume that* (H_1') *holds and* Ω_+ *contains a tubular neighborhood of* $\partial\Omega$. *Then, every nontrivial nonnegative solution of* $(P_{a,q})$ *is positive on* $\partial\Omega$. *In particular, if* u *is a nontrivial nonnegative solution of* $(P_{a,q})$, *then either* $u \in \mathcal{P}^\circ$ *or* u *has a nonempty dead core.*

(ii) *Let* $a_\delta := b_1 - \delta b_2$, *with* $b_1, b_2 \not\equiv 0$ *satisfying* (H_2), *and* $\delta > 0$. *If we set* $G := \{x \in \Omega : b_2(x) > 0\}$ *then, given* $0 < \overline{q} < 1$ *and* $\sigma > 0$, *there exists* $\delta_0 = \delta_0(\sigma, \overline{q}) > 0$ *such that any nontrivial nonnegative solution of* $(P_{a_\delta,q})$ *with* $q \in (0, \overline{q}]$ *vanishes in* G_σ *if* $\delta \geq \delta_0$.

The rest of the paper is organized as follows: in Sect. 2 we establish some bifurcation results and stability properties for solutions in $\mathcal{P}^\circ$ of $(P_{a,q})$, whereas Sect. 3 is devoted to the proof of Theorems 1.2, 1.6 and 1.8. In Sect. 4 we prove Theorem 1.4 and some corollaries of it. Finally, Sect. 5 is concerned with the existence of dead core solutions and the proof of Theorem 1.10.

2. Bifurcation analysis

This section is devoted to the bifurcation analysis of $(P_{a,q})$, where q is the bifurcation parameter. First we establish, under (H_0) and (H_1), some *a priori* bounds for nontrivial nonnegative solutions of $(P_{a,q})$, which imply that no

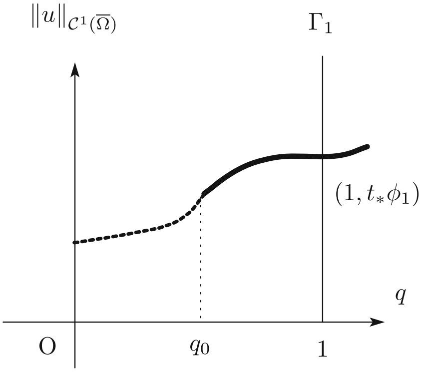

FIGURE 2. A bifurcation curve of the unique nontrivial nonnegative solution when $\mu_1(a) = 1$, conditions (H_1'), (H_+) hold, and Ω_+ contains a tubular neighborhood of $\partial\Omega$. Here the full curve represents solutions in $\mathcal{P}^\circ$, whereas the dotted curve represents dead core solutions

nontrivial nonnegative solutions bifurcate from zero or from infinity at any $q \in [0, 1)$. More precisely, we shall see that given $q \in [0, 1)$ there exists no sequence $\{q_n\} \subset (0, 1)$ such that $q_n \to q$ and (P_{a,q_n}) has a nontrivial nonnegative solution u_n satisfying $u_n \to 0$ in $\mathcal{C}(\overline{\Omega})$ or $\|u_n\|_\infty \to \infty$.

Proposition 2.1.

(i) Assume (H_1)*. Then, given* $q_1 \in (0, 1)$*, there exists* $C > 1$ *such that* $\|u\|_{L^\infty(\Omega)} > C^{-1}$ *for all nontrivial nonnegative solutions of* $(P_{a,q})$ *with* $q \in (0, q_1]$.

(ii) Assume (H_0)*. Then, given* $q_1 \in (0, 1)$*, there exists* $C > 1$ *such that* $\|u\|_{L^\infty(\Omega)} < C$ *for all nontrivial nonnegative solutions of* $(P_{a,q})$ *with* $q \in (0, q_1]$.

Proof.

(i) First we obtain an *a priori* bound from below. Assume by contradiction that there exist $0 < q_n \leq \overline{q} < 1$ and u_n nontrivial nonnegative solutions of (P_{a,q_n}) such that $u_n \to 0$ in $\mathcal{C}(\overline{\Omega})$. Then, thanks to (H_1), we may assume that $u_n \not\equiv 0$ in some fixed subdomain $\Omega' \subset \Omega_+$. By the strong maximum principle, we deduce that $u_n > 0$ in Ω'.
We fix $c > 0$ sufficiently large such that $\lambda_1(ca, \Omega') < 1$, where $\lambda_1(m, \Omega)$ denotes the first positive eigenvalue of the Dirichlet problem

$$\begin{cases} -\Delta\phi = \lambda m(x)\phi & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

and observe that $v_n := c^{\frac{1}{1-q_n}} u_n$ are nontrivial nonnegative solutions of (P_{ca,q_n}). We now apply [16, Lemma 2.5] to get a ball $B \subset \Omega'$ and a positive function ψ in B such that

$$v_n = c^{\frac{1}{1-q_n}} u_n \geq \psi \quad \text{in } B,$$

where ψ and B do not depend on n. It follows that $u_n \geq c^{-\frac{1}{1-q_n}} \psi$ in B, which provides a contradiction, since $q_n \leq \overline{q} < 1$.

(ii) We obtain now an *a priori* bound from above. Assume to the contrary that there exist $0 < q_n \leq \overline{q} < 1$ and u_n nontrivial nonnegative solutions of (P_{a,q_n}) such that $\|u_n\| := \|u_n\|_{H^1(\Omega)} \to \infty$. We set $v_n := \frac{u_n}{\|u_n\|}$, so that we may assume that $v_n \rightharpoonup v_0$ in $H^1(\Omega)$ and $v_n \to v_0$ in $L^s(\Omega)$ for $s \in [1, 2^*)$. From (P_{a,q_n}) we have that

$$\int_\Omega |\nabla v_n|^2 = \left(\int_\Omega a(x) v_n^{q_n+1} \right) \|u_n\|^{q_n - 1}.$$

Since $q_n \leq \overline{q} < 1$, it follows that $\int_\Omega |\nabla v_n|^2 \to 0$. Hence, we deduce that $v_n \to v_0$ in $H^1(\Omega)$, and v_0 is a positive constant. Finally, since $\int_\Omega a(x) v_n^{q_n+1} > 0$ we derive that $\int_\Omega a \geq 0$, which contradicts (H_0). By elliptic regularity, we have the desired conclusion. □

In view of Proposition 2.1, we see that, under (H_0) and (H_1), bifurcation from zero or from infinity can only occur at $q = 1$. As already mentioned, we shall look at q as a bifurcation parameter in $(P_{a,q})$, and then seek for bifurcating solutions in $\mathcal{P}^\circ$ from the trivial line $\Gamma_1 = \{(1, t\phi_1) : t > 0\}$ when $\mu_1(a) = 1$. To this end, we employ the Lyapunov–Schmidt reduction for $(P_{a,q})$, based on the positive eigenfunction ϕ_1. We set

$$A := -\Delta - a(x) \quad \text{and} \quad D(A) = W_N^{2,r}(\Omega) := \left\{ u \in W^{2,r}(\Omega) : \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}.$$

The usual decomposition of $D(A)$ is given by the formula

$$D(A) = \mathrm{Ker} A + X_2; \quad u = t\phi_1 + w,$$

where $t = \int_\Omega u\phi_1$, and $w = u - (\int_\Omega u\phi_1)\phi_1$. So, X_2 is characterized as

$$X_2 = \left\{ w \in W_N^{2,r}(\Omega) : \int_\Omega w\phi_1 = 0 \right\}.$$

On the other hand, put $Y := L^r(\Omega) = Y_1 + R(A)$, where

$$R(A) := \left\{ f \in L^r(\Omega) : \int_\Omega f\phi_1 = 0 \right\},$$

and

$$Y_1 = \langle \phi_1 \rangle := \{ s\phi_1 : s \in \mathbb{R} \}.$$

Let Q be the projection of Y to $R(A)$, given by

$$Q[f] := f - \left(\int_\Omega f\phi_1 \right) \phi_1.$$

We reduce $(P_{a,q})$ to the following coupled equations:

$$Q[Au] = Q[a(x)(u^q - u)],$$

$$(1-Q)[Au] = (1-Q)[a\,(x)\,(u^q - u)].$$

The first equation yields

$$-\Delta w - a(x)w = Q[a\,(x)\,\{(t\phi_1 + w)^q - (t\phi_1 + w)\}], \tag{2.1}$$

where we have used the fact that $\int_\Omega Au\phi_1 = \int_\Omega uA\phi_1 = 0$. The second equation implies that

$$\begin{aligned} 0 &= (1-Q)[a\,(x)\,(u^q - u)] \\ &= \left(\int_\Omega a(x)\,\{(t\phi_1 + w)^q - (t\phi_1 + w)\}\,\phi_1\right)\phi_1, \end{aligned}$$

and thus, that

$$0 = \int_\Omega a(x)\,\{(t\phi_1 + w)^q - (t\phi_1 + w)\}\,\phi_1. \tag{2.2}$$

Now, we see that $(q, t, w) = (1, t, 0)$ satisfies (2.1) and (2.2) for any $t > 0$. So, first we solve (2.1) with respect to w, around $(q, t, w) = (1, t_0, 0)$ for a fixed $t_0 > 0$. To this end, we introduce the mapping $F : (1-\delta, 1+\delta) \times (t_0 - d, t_0 + d) \times B_\rho(0) \to R(A)$ given by

$$F(q, t, w) := -\Delta w - a\,(x)\,w - Q[a\,(x)\,\{(t\phi_1 + w)^q - (t\phi_1 + w)\}],$$

where $B_\rho(w)$ is the ball in X_2 centered at w and with radius $\rho > 0$. It is clear that $F(1, t_0, 0) = 0$. Moreover, the Fréchet derivative $F_w(q, t, w) : X_2 \to R(A)$ is given by

$$F_w(q, t, w)\varphi = -\Delta\varphi - a\,(x)\,\varphi - Q[a\,(x)\,(q(t\phi_1 + w)^{q-1} - 1)\varphi].$$

We see that $F_w(1, t_0, 0)\varphi = -\Delta\varphi - a\,(x)\,\varphi$. Hence,

$$F_w(1, t_0, 0)\varphi = 0 \iff \varphi = c\phi_1 \quad \text{for some } c > 0.$$

Since $\varphi \in X_2$, it follows that $\int_\Omega (c\phi_1)\phi_1 = 0$, and thus $c = 0$. This means that $F_w(1, t_0, 0)$ is injective. It is also surjective from the fact that $\int_\Omega f\phi_1 = 0$ if and only if there exists φ such that

$$\begin{cases} -\Delta\varphi - a(x)\varphi = f & \text{in } \Omega, \\ \frac{\partial\varphi}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $F_w(1, t_0, 0)$ is continuous, from the bounded inverse theorem we infer that $F_w(1, t_0, 0)$ is an isomorphism. Hence, the implicit function theorem applies, and consequently, we have

$$\begin{aligned} &F(q, t, w) = 0,\ (q, t, w) \simeq (1, t_0, 0) \\ &\iff w = w(q, t),\ (q, t) \simeq (1, t_0) \text{ such that } w(1, t_0) = 0. \end{aligned}$$

We plug $w(q, t)$ into (2.2) to get the following bifurcation equation in $\mathbb{R}^2$:

$$\Phi(q, t) := \int_\Omega a(x)\{(t\phi_1 + w(q, t))^q - (t\phi_1 + w(q, t)\}\phi_1 = 0, \quad (q, t) \simeq (1, t_0).$$

We are now in position to prove the following result:

Theorem 2.2. *Assume* (H_0)*. If* $\mu_1(a) = 1$*, then the following assertions hold:*

(i) *Assume that* $(q_n, u_n) \in (0,1) \times \mathcal{P}^\circ$ *are solutions of* (P_{a,q_n}) *such that* $(q_n, u_n) \to (1, t\phi_1) \in \Gamma_1$ *in* $\mathbb{R} \times W^{2,r}(\Omega)$ *for some* $t > 0$*. Then, we have* $t = t_*$*, where* t_* *is given by* (1.1)*.*

(ii) *The set of positive solutions of* $(P_{a,q})$ *consists of* $\Gamma_1 \cup \Gamma_2$ *in a neighborhood of* $(q, u) = (1, t_*\phi_1)$ *in* $\mathbb{R} \times W^{2,r}(\Omega)$*, where*

$$\Gamma_2 := \{(q,\, t(q)\phi_1 + w(q, t(q))) : |q-1| < \delta_*\} \quad \textit{for some } \delta_* > 0.$$

Here $t(q)$ *and* $w(q, t(q))$ *are smooth with respect to* q *and satisfy* $t(1) = t_*$ *and* $w(1, t_*) = 0$*.*

Proof. Let us first verify assertion (i). Since $(q_n, u_n) \to (1, t\phi_1)$ in $\mathbb{R} \times W^{2,r}(\Omega)$ for some $t > 0$, we have $\Phi_q(1,t) = 0$ by the implicit function theorem. By direct computations, we get

$$\Phi_q(q,t) = \int_\Omega a(x) \left[(t\phi_1 + w)^q \left\{ \log(t\phi_1 + w) + \frac{q w_q}{t\phi_1 + w} \right\} - w_q \right] \phi_1. \quad (2.3)$$

Putting $q = 1$ and using that $w(1,t) = 0$, we find that

$$\begin{aligned} \Phi_q(1,t) &= \int_\Omega a(x) \left[(t\phi_1) \left\{ \log(t\phi_1) + \frac{w_q(1,t)}{t\phi_1} \right\} - w_q(1,t) \right] \phi_1 \\ &= t \int_\Omega a(x)\, \phi_1^2 \log(t\phi_1) \qquad (2.4) \\ &= t \left\{ (\log t) \int_\Omega a(x)\, \phi_1^2 + \int_\Omega a(x)\, \phi_1^2 \log \phi_1 \right\}. \end{aligned}$$

Thus

$$t = t_* := \exp\left[-\frac{\int_\Omega a(x)\, \phi_1^2 \log \phi_1}{\int_\Omega a(x)\, \phi_1^2} \right],$$

as claimed in assertion (i).

Next, we verify assertion (ii). To this end, we use the fact that the map $(q,t) \mapsto N(q,t) = t^q$ is analytic around $(q,t) = (1, t_*)$, and apply the implicit function theorem. We consider partial derivatives of Φ, and check that $\frac{\partial^k \Phi}{\partial t^k}(1,t) = 0$ and $\Phi_{qt}(1, t_*) > 0$. In fact, the case $k = 1$ is straightforward since Γ_1 is a trivial line of solutions of $(P_{a,q})$. Moreover, for $k \geq 2$, we have that

$$\frac{\partial^k \Phi}{\partial t^k}(q,t) = (q-1)\Phi_k(q,t) + \int_\Omega a(x) \left\{ q(t\phi_1 + w)^{q-1} \frac{\partial^k w}{\partial t^k}(q,t) - \frac{\partial^k w}{\partial t^k}(q,t) \right\} \phi_1$$

for some continuous function Φ_k of (q,t) at $(1,t)$, so that $\frac{\partial^k \Phi}{\partial t^k}(1,t) = 0$ for all $k \in \mathbb{N}$ and $t > 0$. Since $(q,t) \mapsto t^q = \exp[q \log t]$ is analytic at $(q,t) = (1,t)$, for any $t > 0$, a regularity result for the implicit function theorem (see e.g. [19]) ensures that so is $w(q,t)$ around $(1, t_*)$, and thus so is $\Phi(q,t)$. Combining this result with the fact that $\frac{\partial^k \Phi}{\partial t^k}(1,t) = 0$ for all $k \in \mathbb{N}$, we deduce that $\Phi(q,t)$ is given around $(1, t_*)$ by

$$\Phi(q,t) = (q-1)\hat{\Phi}(q,t), \quad \text{where}$$

$$\hat{\Phi}(q,t) = \frac{1}{2}\Phi_{qq}(1,t_*)(q-1) + \Phi_{qt}(1,t_*)(t-t_*)$$
$$+ \text{ higher order terms w.r.t. } (q-1) \text{ and } (t-t_*).$$

Therefore, applying the implicit function theorem to $\hat{\Phi}(q,t)$ at $(1,t_*)$, we infer that the set $\Phi(q,t)=0$ around $(1,t_*)$ is given completely by

$$q=1, \quad \text{and} \quad t=t(q) \quad \text{with} \quad t(1)=t_*,$$

provided that $\Phi_{qt}(1,t_*) \neq 0$, and thus, the desired conclusion follows.

It remains to check that $\Phi_{qt}(1,t_*) > 0$: by a direct computation from (2.3), we observe that

$$\Phi_{qt} = \int_\Omega a(x) \left[q(t\phi_1+w)^{q-1}(\phi_1+w_t)\left\{\log(t\phi_1+w) + \frac{qw_q}{t\phi_1+w}\right\} \right.$$
$$\left. +(t\phi_1+w)^q \left\{\frac{\phi_1+w_t}{t\phi_1+w} + \frac{qw_{qt}(t\phi_1+w)-qw_q(\phi_1+w_t)}{(t\phi_1+w)^2}\right\} - w_{qt}\right]\phi_1.$$

Letting $q=1$, it follows that

$$\Phi_{qt}(1,t) = \int_\Omega a(x)\left[(\phi_1+w_t(1,t))\log(t\phi_1) + (\phi_1+w_t(1,t))\right]\phi_1. \tag{2.5}$$

We differentiate (2.1) with respect to t, and we obtain that

$$-\Delta w_t - a(x)\, w_t = Q[a(x)\{q(t\phi_1+w)^{q-1}(\phi_1+w_t) - (\phi_1+w_t)\}].$$

Letting $q=1$ again, we deduce that

$$-\Delta w_t(1,t) - a(x)\, w_t(1,t) = 0, \quad \text{and } w_t(1,t) \in X_2.$$

Hence, $w_t(1,t)=0$, and thus, it follows from (2.5) that

$$\Phi_{qt}(1,t) = \int_\Omega a(x)\left[\phi_1\log(t\phi_1)+\phi_1\right]\phi_1$$
$$= \int_\Omega a(x)\,\phi^2\log(t\phi_1) + \int_\Omega a(x)\,\phi_1^2$$

When $t=t_*$, we know that $\int_\Omega a(x)\,\phi^2\log(t_*\phi_1)=0$ from (2.4), so that

$$\Phi_{qt}(1,t_*) = \int_\Omega a(x)\,\phi_1^2 > 0,$$

as desired. □

Next, as we did for $q<1$ close to 1, we show that the Lyapunov–Schmidt reduction is useful for the case $q>0$ close to 0. Indeed, we exhibit how to construct a such that $(P_{a,q})$ possesses a solution in $\mathcal{P}^\circ$ for $q>0$ arbitrarily close to 0. Consider the problem

$$(P_{a,0}) \qquad \begin{cases} -\Delta u = a(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

It is easy to check that $(P_{a,0})$ has a solution if and only if $\int_\Omega a = 0$, in which case all solutions are of the form $u+c$, where c is any constant and u is a particular solution.

Assume now that $a \not\equiv 0$ and $\int_\Omega a = 0$ (in particular, a changes sign). We set $X := \{u \in W^{2,r}(\Omega) : \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\}$, and write $X = \langle 1 \rangle + X_2$, where $\langle 1 \rangle$ is the set of constant functions and $X_2 := \{w \in W^{2,r}(\Omega) : \int_\Omega w = 0\}$. Let w_0 be the unique solution of $(P_{a,0})$ such that $w_0 \in X_2$, and $t_0 > 0$ be such that $u_0 = t_0 + w_0 > 0$ on $\overline{\Omega}$. Then $u_0 \in \mathcal{P}^\circ$ solves $(P_{a,0})$.

Given $\varepsilon, \delta > 0$ and $q \in (-\delta, \delta)$, we consider the following perturbation of $(P_{a,0})$:

$$(P_{a-\varepsilon,q}) \qquad \begin{cases} -\Delta u = (a(x) - \varepsilon)u^q & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that if ε is sufficiently small, then $a-\varepsilon$ changes sign, and $\int_\Omega (a(x)-\varepsilon) < 0$. Note also that $(P_{a-\varepsilon,q})$ admits $(q, \varepsilon, u) = (0, 0, t_0 + w_0)$ as a solution. Our aim is to look for positive solutions of $(P_{a-\varepsilon,q})$ in a neighborhood of $(0, 0, t_0 + w_0)$. Let $Y_2 := \{f \in L^r(\Omega) : \int_\Omega f = 0\}$ and Q be the usual projection of $L^r(\Omega)$ to Y_2, given by $Q[f] := f - \frac{1}{|\Omega|}\int_\Omega f$. Following the Lyapunov–Schmidt approach already used in Theorem 2.2, we reduce $(P_{a-\varepsilon,q})$ to the following coupled equations

$$Q[-\Delta u] = Q[(a - \varepsilon)u^q], \tag{2.6}$$

$$(1 - Q)[-\Delta u] = (1 - Q)[(a - \varepsilon)u^q]. \tag{2.7}$$

Associated with (2.6), we define the mapping

$$F : (-\delta_0, \delta_0) \times (-\varepsilon_0, \varepsilon_0) \times (t_0 - d_0, t_0 + d_0) \times B_{\rho_0}(w_0) \to Y_2$$

by

$$F(q, \varepsilon, t, w) := -\Delta w - Q[(a - \varepsilon)(t + w)^q],$$

where $B_{\rho_0}(w_0)$ is the ball in X_2 with center w_0 and radius ρ_0. We note that $F(0, 0, t_0, w_0) = 0$, since $Q[a] = a$. The Fréchet derivative $F_w(q, \varepsilon, t, w) : X_2 \to Y_2$ is given by

$$F_w(q, \varepsilon, t, w)\varphi = -\Delta\varphi - Q[(a - \varepsilon)q(t + w)^{q-1}\varphi].$$

Taking $(q, \varepsilon, t, w) = (0, 0, t_0, w_0)$, we see that $F_w(0, 0, t_0, w_0)\varphi = -\Delta\varphi$, so that $F_w(0, 0, t_0, w_0)$ is bijective, and the implicit function theorem applies. Consequently, we have

$$\begin{aligned} &F(q, \varepsilon, t, w) = 0, \quad (q, \varepsilon, t, w) \simeq (0, 0, t_0, w_0) \\ &\Longleftrightarrow w = w(q, \varepsilon, t), \quad (q, \varepsilon, t) \simeq (0, 0, t_0) \ \text{ with } w(0, 0, t_0) = w_0. \end{aligned}$$

Using $w(q, \varepsilon, t)$, we derive from (2.7) the equation

$$\Psi(q, \varepsilon, t) := \int_\Omega (a(x) - \varepsilon)(t + w(q, \varepsilon, t))^q = 0 \quad \text{in } \mathbb{R}^3.$$

Note that $\Psi(0, 0, t_0) = 0$. We prove now the following result:

Proposition 2.3. *Given $\varepsilon > 0$ sufficiently small, there exists $q_\varepsilon > 0$ such that $u_\varepsilon = t_0 + w(q_\varepsilon, \varepsilon, t_0) > 0$ on $\overline{\Omega}$, and $\Psi(q_\varepsilon, \varepsilon, t_0) = 0$. Moreover, $q_\varepsilon \to 0^+$ as $\varepsilon \to 0^+$. Consequently, $u_\varepsilon \in \mathcal{P}^\circ$ is a solution of $(P_{a-\varepsilon,q_\varepsilon})$.*

Proof. We apply the implicit function theorem for Ψ at $(0, 0, t_0)$. We observe that

$$\frac{\partial \Psi}{\partial q} = \int_\Omega (a(x) - \varepsilon)(t + w)^q \left\{ \log(t + w) + \frac{q}{t + w} \frac{\partial w}{\partial q} \right\},$$

and therefore

$$\frac{\partial \Psi}{\partial q}(0, 0, t_0) = \int_\Omega a(x) \log u_0.$$

Since u_0 is a solution in $\mathcal{P}^\circ$ of $(P_{a,0})$, we see that

$$0 < \int_\Omega \frac{|\nabla u_0|^2}{u_0} = \int_\Omega -\Delta u_0 \log u_0 = \int_\Omega a(x) \log u_0,$$

which implies that $\frac{\partial \Psi}{\partial q}(0, 0, t_0) > 0$. Hence, the implicit function theorem ensures that

$$\begin{aligned} \Psi(q, \varepsilon, t) = 0, \quad (q, \varepsilon, t) \simeq (0, 0, t_0) \Leftrightarrow q \\ = q(\varepsilon, t), \quad (\varepsilon, t) \simeq (0, t_0) \ \text{ with } \ q(0, t_0) = 0. \end{aligned}$$

Next we show that $\frac{\partial q}{\partial \varepsilon}(0, t_0) > 0$. To this end, we differentiate Ψ with respect to ε, obtaining that

$$\frac{\partial \Psi}{\partial \varepsilon} = \int_\Omega \left\{ -(t + w)^q + (a(x) - \varepsilon) q (t + w)^{q-1} \frac{\partial w}{\partial \varepsilon} \right\},$$

and so

$$\frac{\partial \Psi}{\partial \varepsilon}(0, 0, t_0) = -|\Omega| < 0.$$

It follows that

$$\frac{\partial q}{\partial \varepsilon}(0, t_0) = -\frac{\frac{\partial \Psi}{\partial \varepsilon}(0, 0, t_0)}{\frac{\partial \Psi}{\partial q}(0, 0, t_0)} = \frac{|\Omega|}{\int a(x) \log u_0} > 0.$$

Using the mean value theorem, we deduce that for $\varepsilon > 0$ small enough,

$$q(\varepsilon, t_0) = q(0, t_0) + \frac{\partial q}{\partial \varepsilon}(\theta \varepsilon, t_0) \varepsilon > 0,$$

for some $0 < \theta < 1$. Thus

$$q(\varepsilon, t_0) \to q(0, t_0) = 0 \quad \text{ as } \varepsilon \to 0^+,$$

as desired. □

Remark 2.4. Let us analyze the asymptotic behavior of nontrivial nonnegative solutions of $(P_{a,q})$ as $q \to 0^+$ under (H_0) and (H_1). From Proposition 2.1, we know that bifurcation from zero or from infinity does not occur as $q \to 0^+$. It is thus natural to investigate the limit of a sequence $\{u_n\}$ of nontrivial nonnegative solutions of (P_{a,q_n}) with $q_n \to 0^+$. Since $\{u_n\}$ is bounded in $L^\infty(\Omega)$, it follows, by elliptic regularity, that up to a subsequence, $u_n \to u_0$ in $\mathcal{C}^1(\overline{\Omega})$ with $u_0 \not\equiv 0$. We point out that u_0 must vanish in a nonempty subset of Ω with *positive measure* (in other words, u_0 has a nonempty dead core). Indeed, if $u_0 > 0$ a.e. in Ω, then, passing to the limit, we have that

$$\int_\Omega \nabla u_0 \nabla v = \int_\Omega a(x)v, \quad \forall v \in \mathcal{C}^1(\overline{\Omega}),$$

i.e. u_0 is a positive solution of $(P_{a,0})$. Integrating this equation, we deduce that $\int_\Omega a = 0$, which is a contradiction.

2.1. Stability properties

We conclude this section discussing the stability of the bifurcating positive solutions provided by Theorem 2.2 (ii).

Proposition 2.5. *Assume (H_0). If $\mu_1(a) = 1$, then the bifurcating positive solution $u(q) = t(q)\phi_1 + w(q, t(q))$ given by Theorem 2.2 (ii) is asymptotically stable (respect. unstable) for $q < 1$ (respect. $q > 1$).*

Proof. Consider

$$-\Delta\varphi_1(q) = qa(x)u(q)^{q-1}\varphi_1(q) + \gamma_1(q)\varphi_1(q), \tag{2.8}$$

where $\gamma_1(q) := \gamma_1(q, u(q))$, and $\varphi_1(q)$ is a positive eigenfunction associated to $\gamma_1(q)$. We see that $\gamma_1(1) = 0$ and $\varphi_1(1) = \phi_1$. To analyse $\gamma_1(q)$ for $q \neq 1$, we differentiate (2.8) with respect to q, to obtain that

$$-\Delta\varphi_1' = a\,(x)\,u^{q-1}\varphi_1 + qa(x)u^{q-1}\left(\log u + (q-1)\frac{u'}{u}\right)\varphi_1 + qa(x)u^{q-1}\varphi_1' \\ + \gamma_1'\varphi + \gamma_1\varphi_1'.$$

Letting $q = 1$ here, it follows that

$$A\varphi_1'(1) = \gamma_1'(1)\phi_1 + a(x)\{\phi_1 + \phi_1\log(t_*\phi_1)\},$$

and thus, by the divergence theorem,

$$0 = \int_\Omega A\varphi_1'(1)\phi_1 - \varphi_1'(1)A\phi_1 = \gamma_1'(1) + \int_\Omega a(x)\left(\phi_1 + \phi_1\log(t_*\phi_1)\right)\phi_1.$$

Since $\int_\Omega a(x)\phi_1^2\log(t_*\phi_1) = 0$, we obtain that

$$\gamma_1'(1) = -\int_\Omega a(x)\phi_1^2 < 0.$$

The desired conclusion follows from the fact that $\gamma_1(1) = 0$. □

Remark 2.6.

(i) The stability result of Proposition 2.5 also follows from [5, Theorem 1]. Even though this result assumes a to be smooth and the nonlinearity to be C^2 at 0, one may easily see that under our assumptions it also applies to solutions of $(P_{a,q})$ in $\mathcal{P}^\circ$. More generally, it shows that any such solution is asymptotically stable for *every* $0 < q < 1$.

(ii) When $q > 1$, we can deduce (by a well known approach) that *every* solution $u \in \mathcal{P}^\circ$ of $(P_{a,q})$ is unstable. Indeed, linearizing $(P_{a,q})$ at u we obtain $-\Delta\varphi = qa(x)u^{q-1}\varphi + \gamma\varphi$. The divergence theorem yields that

$$0 > \int_\Omega \frac{u}{\varphi_1} \sum_j \frac{\partial}{\partial x_j}\left(\varphi_1^2 \frac{\partial}{\partial x_j}\left(\frac{u}{\varphi_1}\right)\right) = (q-1)\int_\Omega a(x)u^{q+1} + \gamma_1 \int_\Omega u^2,$$

where $\gamma_1 = \gamma_1(q)$ and $\varphi_1 = \varphi_1(q)$. Hence, we obtain

$$\gamma_1 < \frac{(1-q)\int_\Omega a(x)u^{q+1}}{\int_\Omega u^2} < 0.$$

3. Proofs of Theorems 1.2, 1.6 and 1.8

Proof of Theorem 1.2: Let us first observe that by Theorem 2.2, there exists $q_0 = q_0(a) < 1$ such that $(P_{a,q})$ has a solution $u_q \in \mathcal{P}^\circ$ for $q_0 < q < 1$. Moreover, the proof of [4, Lemma 3.1] can be adapted to our setting, so that $(P_{a,q})$ has no other positive solution for $q_0 < q < 1$. We consider now the asymptotic behavior of u_q as $q \to 1^-$. Assertion (i) is a direct consequence of Theorem 2.2 (ii) and elliptic regularity.

Assume now that $\mu_1 = \mu_1(a) \neq 1$ and set $v := \mu_1^{\frac{1}{1-q}} u$. Note that if u solves $(P_{a,q})$ then v solves $(P_{\tilde{a},q})$, where $\tilde{a} := \mu_1 a$. Indeed,

$$-\Delta v = \mu_1^{\frac{1}{1-q}} a(x)\, u^q = \mu_1 a(x)\, v^q = \tilde{a}(x)\, v^q.$$

Moreover, we easily see that $\mu_1(\tilde{a}) = 1$. By item (i), we get a positive solution v_q of $(P_{\tilde{a},q})$ such that $v_q \to t_*(\tilde{a})\phi_1(\tilde{a})$, where $\phi_1(\tilde{a})$ is a positive eigenfunction of $(E_{1,\tilde{a}})$, which is nothing but $(E_{\mu_1,a})$, i.e. $\phi_1(\tilde{a}) = \phi_1(a)$ and $t_*(\tilde{a}) = t_*(a)$. In this way, we obtain a positive solution $u_q = \mu_1^{\frac{1}{q-1}} v_q$ of $(P_{a,q})$ for q close to 1. In particular, we see that if $\mu_1 > 1$ then $\mu_1^{\frac{1}{q-1}} \to 0$, so that $u_q \to 0$ in $\mathcal{C}^1(\overline{\Omega})$ as $q \to 1^-$. On the other hand, if $\mu_1 < 1$, then $\mu_1^{\frac{1}{q-1}} \to \infty$, so that $\min_{\overline{\Omega}} u_q \to \infty$ when $q \to 1^-$.

Finally, the asymptotic stability of u_q is a direct consequence of Proposition 2.5. □

When proving Theorems 1.6, 1.8 and 1.4, we shall repeatedly use the following remark:

Remark 3.1.

(i) Since $(P_{a,q})$ is homogeneous, we see that $(P_{a,q})$ has a nonnegative (respect. positive) solution if and only if, for any $\sigma > 0$ fixed, $(P_{\sigma a,q})$ has a nonnegative (respect. positive) solution.

(ii) Lemma 2.4 in [4] (which is proved using Proposition 2.1 therein) gives the existence of arbitrarily large supersolutions of $(P_{a,q})$ provided that $\int_\Omega a < 0$. Although it is assumed that a is Hölder continuous in [4], one can see that Lemma 2.4 and Proposition 2.1 still hold (with the same proof) if $a \in L^\infty(\Omega)$.

Proof of Theorem 1.6: We proceed in several steps. By Remark 3.1, it is enough to provide a positive (in Ω) weak subsolution for $(P_{b,q})$, where $b := \gamma a$ and $\gamma := 1/(1-q)$. Observe that $\gamma q = \gamma - 1$. We note also that, since $\int_{B_R} a < 0$, it holds that $R_0 < R$. Let us first define

$$C := \frac{1-q}{1+q},$$
$$w(r) := C \int_r^R \frac{1}{t^{N-1}} \int_t^R a^-(y)\, y^{N-1}\, dy\, dt := C\phi(r), \quad r \in [R_0, R].$$

Then, $w(R) = w'(R) = 0$ and $w(r) > 0$ for all $r \in [R_0, R)$. Also, a few computations show that

$$\phi'' + \frac{N-1}{r}\phi' = a^-(r). \tag{3.1}$$

Let now $z(r) := w^\gamma(r)$. We claim that

$$-\Delta z \le \gamma a(x)\, z^q \quad a.e. \text{ in } A_{R_0,R}. \tag{3.2}$$

Indeed, since z is radial, there holds

$$\Delta z = z'' + \frac{N-1}{r} z'$$
$$= \gamma(\gamma-1)\, w^{\gamma-2}(w')^2 + \gamma w^{\gamma-1} w'' + \frac{\gamma(N-1)}{r} w^{\gamma-1} w',$$

and also

$$-\gamma a(r)\, z^q \le \gamma a^-(r)\, w^{\gamma q} = \gamma a^-(r)\, w^{\gamma-1}.$$

Thus, in order to prove the claim it is enough to verify that

$$(\gamma-1)\frac{(w')^2}{w} + w'' + \frac{N-1}{r} w' \ge a^-(r).$$

Now, taking into account (3.1) and that $w = C\phi$, the above inequality is equivalent to

$$F(r) := (\gamma-1)(\phi')^2 \ge \left(\frac{1}{C} - 1\right) a^-(r)\, \phi := G(r). \tag{3.3}$$

We observe next that $F(R) = G(R) = 0$ and $F'(r) \le 0$ for all $r \in [R_0, R]$ (recall that $\phi' \le 0$). So, in order to check (3.3) it suffices to see that $F'(r) \le G'(r)$ for such r. Now,

$$F'(r) = 2(\gamma-1)\, \phi'\phi'',$$
$$G'(r) = \left(\frac{1}{C} - 1\right)\left((a^-(r))'\, \phi + a^-(r)\, \phi'\right) \ge \left(\frac{1}{C} - 1\right) a^-(r)\, \phi',$$

where we used the fact that a is differentiable and nonincreasing in $A_{R_0,R}$. Therefore, $F'(r) \le G'(r)$ provided that

$$2(\gamma-1)\, \phi'\phi'' \le \left(\frac{1}{C} - 1\right) a^-(r)\, \phi',$$

i.e.

$$2(\gamma-1)\left(-\frac{N-1}{r}\phi'+a^{-}(r)\right)\geq\left(\frac{1}{C}-1\right)a^{-}(r). \tag{3.4}$$

But (3.4) holds by our election of C. Indeed, since $\phi' \leq 0$, one only has to observe that $2(\gamma-1)=\frac{1}{C}-1$.

On the other side, let v be a solution of

$$\begin{cases} -\Delta v=\gamma a(x)v^{q} & \text{in } B_{R_0}, \\ v=z(R_0) & \text{on } \partial B_{R_0}. \end{cases} \tag{3.5}$$

Such v can be easily constructed by the sub and supersolutions method, since $a \geq 0$ in B_{R_0}. Moreover, v is radial. Indeed, this follows from either the fact that the sub and supersolutions can be chosen radial, or because the solution of (3.5) is unique (cf. [7]) and $v(Sx)$ is also a solution if S is an isometry of $\mathbb{R}^N$. Furthermore, it is also easy to check that $r \to v(r)$ is nonincreasing in $(0, R_0)$ because $a \geq 0$ in B_{R_0}. Hence, by the divergence theorem (as stated e.g. in [6], p. 742),

$$\begin{aligned} v'(R_0)\,\omega_{N-1}R_0^{N-1} &= \int_{B_{R_0}}\Delta v=-\int_{B_{R_0}}\gamma a v^{q} \\ &\leq -\gamma v^{q}(R_0)\int_{B_{R_0}}a=-\gamma w^{\gamma q}(R_0)\int_{B_{R_0}}a. \end{aligned} \tag{3.6}$$

On the other hand, recalling that $\gamma-1=\gamma q$, we obtain that

$$z'(R_0)=\gamma w^{\gamma-1}(R_0)\,w'(R_0)=-\gamma w^{\gamma q}(R_0)\frac{C}{R_0^{N-1}}\int_{R_0}^{R}a^{-}(y)\,y^{N-1}\,dy$$

and so

$$\begin{aligned} z'(R_0)\,\omega_{N-1}R_0^{N-1} &= -\gamma w^{\gamma q}(R_0)\,C\omega_{N-1}\int_{R_0}^{R}a^{-}(y)\,y^{N-1}\,dy \\ &= -\gamma w^{\gamma q}(R_0)\,C\int_{A_{R_0}}a^{-}. \end{aligned} \tag{3.7}$$

Next we observe that $v'(R_0)\leq z'(R_0)$. Indeed, taking into account (3.6), (3.7) and the definition of C, we see that this is true by (1.2).

To conclude the existence assertion, we define $u := z$ in $\overline{A}_{R_0,R}$ and $u := v$ in B_{R_0}. Then $u \in H_0^1(\Omega)\cap L^{\infty}(\Omega)$, $u > 0$ in Ω and $\frac{\partial u}{\partial \nu}=0$ on $\partial\Omega$. Moreover, recalling (3.2), (3.5) and that $v'(R_0)\leq z'(R_0)$, the divergence theorem yields that u is a weak subsolution of $(P_{b,q})$. □

Remark 3.2. As one can see from the proof of Theorem 1.6, the condition (1.2) guarantees the existence of a positive subsolution for the corresponding Dirichlet problem. Thus, since arbitrarily large supersolutions can be easily obtained in the Dirichlet case (see e.g. [10, Remark 1.1]), it follows that (1.2) ensures the existence of a positive solution for the analogous Dirichlet problem. Moreover, we point out that this condition substantially improves some of the results known in that case (see [10, Section 3]).

Proof of Theorem 1.8: Given $\varepsilon \in [0, q)$, we define

$$\gamma_\varepsilon := \frac{1-\varepsilon}{1-q}, \quad C_\varepsilon := \left(\frac{R_0^{2\varepsilon}}{2} \frac{\|a^-\|_\infty}{2(\gamma_\varepsilon - 1) + N} + \varepsilon \right)^{\frac{1}{1-\varepsilon}}.$$

We note that both $\varepsilon \mapsto \gamma_\varepsilon$ and $\varepsilon \mapsto C_\varepsilon$ are continuous, that

$$\varepsilon = \gamma_\varepsilon q - (\gamma_\varepsilon - 1), \quad \gamma_\varepsilon - 1 = \frac{q-\varepsilon}{1-q} > 0, \tag{3.8}$$

and that

$$2C_0 = \frac{\|a^-\|_\infty}{2(\gamma_0 - 1) + N} = \frac{1-q}{2q + N(1-q)} \|a^-\|_\infty. \tag{3.9}$$

Given $r \in [0, R_0]$ and $\delta \geq 0$, we set $u_{\delta,\varepsilon}(r) := C_\varepsilon r^2 + \delta$. We now observe that we can fix $\varepsilon > 0$ small enough such that

$$u'_{0,\varepsilon}(R_0) < u^\varepsilon_{0,\varepsilon}(R_0) \frac{1}{R_0^{N-1}} \int_{R_0}^{R} a(y) y^{N-1} \, dy. \tag{3.10}$$

Indeed, $u'_{0,\varepsilon}(R_0) = 2C_\varepsilon R_0$ and $u^\varepsilon_{0,\varepsilon}(R_0) = \left(C_\varepsilon R_0^2\right)^\varepsilon$, and so (3.10) holds if and only if

$$2C_\varepsilon^{1-\varepsilon} R_0^N < R_0^{2\varepsilon} \int_{R_0}^{R} a(y) y^{N-1} \, dy. \tag{3.11}$$

Now, by (1.4) and (3.9),

$$2C_0 R_0^N = \frac{1-q}{2q + N(1-q)} R_0^N \|a^-\|_\infty < \frac{\int_{A_{R_0,R}} a}{\omega_{N-1}} = \int_{R_0}^{R} a(y) y^{N-1} \, dy.$$

Thus, (3.11) (and consequently (3.10)) holds for $\varepsilon > 0$ sufficiently small.

Next, we note that, by definition,

$$2C_\varepsilon^{1-\varepsilon} > \frac{\|a^-\|_\infty R_0^{2\varepsilon}}{2(\gamma_\varepsilon - 1) + N}.$$

Therefore, for all $r \in [0, R_0]$,

$$4C_\varepsilon(\gamma_\varepsilon - 1) + 2NC_\varepsilon - \|a^-\|_\infty \left(C_\varepsilon r^2\right)^\varepsilon > 0. \tag{3.12}$$

In view of this inequality, we may fix $\delta > 0$ such that

$$\frac{4C_\varepsilon^2 r^2 (\gamma_\varepsilon - 1)}{C_\varepsilon r^2 + \delta} + 2NC_\varepsilon - \|a^-\|_\infty \left(C_\varepsilon r^2 + \delta\right)^\varepsilon > 0 \quad \forall r \in [0, R_0]. \tag{3.13}$$

Indeed, we pick first any $\delta_0 > 0$ small enough such that, for all $\delta \in (0, \delta_0]$,

$$2NC_\varepsilon > \|a^-\|_\infty \delta^\varepsilon.$$

Then there exists $r_0 = r_0(\delta_0) > 0$ such that

$$2NC_\varepsilon > \|a^-\|_\infty \left(C_\varepsilon r^2 + \delta\right)^\varepsilon \quad \forall r \in [0, r_0],$$

and thus (3.13) clearly holds for all $r \in [0, r_0]$. Suppose now that $r \in [r_0, R_0]$. Then, from (3.12) we derive that

$$\frac{4C_\varepsilon^2 r^2 (\gamma_\varepsilon - 1)}{C_\varepsilon r^2} + 2NC_\varepsilon - \|a^-\|_\infty \left(C_\varepsilon r^2\right)^\varepsilon > 0 \quad \forall r \in [r_0, R_0]. \tag{3.14}$$

Now, since by Dini's theorem the left-hand side of (3.13) converges to the left-hand side of (3.14) uniformly in $r \in [r_0, R_0]$ as $\delta \to 0^+$, then, decreasing δ_0 if necessary, we also see that (3.13) holds for all $r \in [r_0, R_0]$.

Finally, since $u_{0,\varepsilon}' = u_{\delta,\varepsilon}'$ and $u_{0,\varepsilon} < u_{\delta,\varepsilon}$, recalling (3.10), we get that

$$u_{\delta,\varepsilon}'(R_0) < u_{\delta,\varepsilon}^{\varepsilon}(R_0) \frac{1}{R_0^{N-1}} \int_{R_0}^{R} a(y) y^{N-1} \, dy. \tag{3.15}$$

We fix for the rest of the proof $\varepsilon, \delta > 0$ such that (3.13) and (3.15) hold.

Let $z_{\delta,\varepsilon}(r) := u_{\delta,\varepsilon}^{\gamma_\varepsilon}(r)$. Let us show that

$$\Delta z_{\delta,\varepsilon} \geq \gamma_\varepsilon \left\| a^- \right\|_\infty z_{\delta,\varepsilon}^q \quad \text{in } B_{R_0}. \tag{3.16}$$

Note that (3.16) implies that $-\Delta z_{\delta,\varepsilon} \leq \gamma_\varepsilon a(x) z_{\delta,\varepsilon}^q$ *a.e.* in B_{R_0}. We compute

$$\begin{aligned}\Delta z_{\delta,\varepsilon} &= \gamma_\varepsilon \left((\gamma_\varepsilon - 1) u_{\delta,\varepsilon}^{\gamma_\varepsilon - 2} \left| \nabla u_{\delta,\varepsilon} \right|^2 + u_{\delta,\varepsilon}^{\gamma - 1} \Delta u_{\delta,\varepsilon} \right) \\ &= \gamma_\varepsilon \left(4C_\varepsilon^2 r^2 (\gamma_\varepsilon - 1) u_{\delta,\varepsilon}^{\gamma_\varepsilon - 2} + 2NC_\varepsilon u_{\delta,\varepsilon}^{\gamma_\varepsilon - 1} \right).\end{aligned}$$

Thus, in order to prove (3.16) it is enough to see that

$$4C_\varepsilon^2 r^2 (\gamma_\varepsilon - 1) u_{\delta,\varepsilon}^{\gamma_\varepsilon - 2} + 2NC_\varepsilon u_{\delta,\varepsilon}^{\gamma_\varepsilon - 1} \geq \left\| a^- \right\|_\infty u_{\delta,\varepsilon}^{\gamma_\varepsilon q}.$$

Furthermore, since $\varepsilon = \gamma_\varepsilon q - (\gamma_\varepsilon - 1)$ (recall (3.8)), this is equivalent to

$$\frac{4C_\varepsilon^2 r^2 (\gamma_\varepsilon - 1)}{u_{\delta,\varepsilon}} + 2NC_\varepsilon \geq \left\| a^- \right\|_\infty u_{\delta,\varepsilon}^{\varepsilon}.$$

But taking into account the definition of $u_{\delta,\varepsilon}$, we see that the above inequality holds thanks to (3.13).

On the other side, let us define

$$\begin{aligned}\phi(r) &:= \int_r^R \frac{1}{t^{N-1}} \int_t^R a(y) y^{N-1} \, dy \, dt, \quad r \in [R_0, R], \\ K &:= \frac{\gamma_\varepsilon}{\gamma_0} \phi(R_0) + \left[u_{\delta,\varepsilon}(R_0) \right]^{\gamma_\varepsilon / \gamma_0}, \\ w(r) &:= K - \frac{\gamma_\varepsilon}{\gamma_0} \phi(r), \quad \text{and} \\ v(r) &:= w^{\gamma_0}(r).\end{aligned}$$

Note that $v(R_0) = z_{\delta,\varepsilon}(R_0)$, and observe also that

$$w'(r) = \frac{\gamma_\varepsilon}{\gamma_0} \frac{1}{r^{N-1}} \int_r^R a(y) y^{N-1} \, dy,$$

and hence $w'(R) = 0$ (and $v'(R) = 0$). We also infer that $w(r) > 0$ for $r \in [R_0, R]$ since $w(R_0) > 0$ and w is increasing. Moreover,

$$w'' + \frac{N-1}{r} w' = -\frac{\gamma_\varepsilon}{\gamma_0} a(r). \tag{3.17}$$

We prove now that

$$-\Delta v \leq \gamma_\varepsilon a(x) v^q \quad \text{a.e. in } A_{R_0,R}. \tag{3.18}$$

Indeed, since v is radial, (3.18) is equivalent to

$$\begin{aligned}-\Delta v &= -v'' - \frac{N-1}{r}v' \\ &= -\gamma_0\left(\left(\gamma_0-1\right)w^{\gamma_0-2}\left(w'\right)^2 + w^{\gamma_0-1}w'' + \frac{(N-1)}{r}w^{\gamma_0-1}w'\right) \\ &\le \gamma_\varepsilon a\left(r\right)w^{\gamma_0-1} = \gamma_\varepsilon a\left(r\right)w^{\gamma_0 q} = \gamma_\varepsilon a\left(r\right)v^q,\end{aligned}$$

and the above inequality clearly holds by (3.17).

We next verify that

$$z'_{\delta,\varepsilon}\left(R_0\right) \le v'\left(R_0\right). \tag{3.19}$$

We have that

$$\begin{aligned}z'_{\delta,\varepsilon}\left(R_0\right) &= \gamma_\varepsilon u_{\delta,\varepsilon}^{\gamma_\varepsilon-1}\left(R_0\right)u'_{\delta,\varepsilon}\left(R_0\right), \\ v'\left(R_0\right) &= \gamma_0 w^{\gamma_0-1}\left(R_0\right)w'\left(R_0\right),\end{aligned}$$

and so it suffices to check that

$$\gamma_\varepsilon u_{\delta,\varepsilon}^{\gamma_\varepsilon-1}\left(R_0\right)u'_{\delta,\varepsilon}\left(R_0\right) \le \gamma_0 w^{\gamma_0-1}\left(R_0\right)w'\left(R_0\right). \tag{3.20}$$

We observe now that, by definition, $w^{\gamma_0}\left(R_0\right) = u_{\delta,\varepsilon}^{\gamma_\varepsilon}\left(R_0\right)$, and hence

$$w^{\gamma_0-1}\left(R_0\right) = \left[u_{\delta,\varepsilon}\left(R_0\right)\right]^{\frac{\gamma_\varepsilon(\gamma_0-1)}{\gamma_0}}.$$

Therefore, (3.20) can be written as

$$\gamma_\varepsilon u'_{\delta,\varepsilon}\left(R_0\right) \le \gamma_0\left[u_{\delta,\varepsilon}\left(R_0\right)\right]^{\frac{\gamma_\varepsilon(\gamma_0-1)}{\gamma_0}-(\gamma_\varepsilon-1)}w'\left(R_0\right).$$

Now, $\gamma_0 - 1 = q/\left(1-q\right)$ and so, recalling the first equality in (3.8), we see that

$$\frac{\gamma_\varepsilon\left(\gamma_0-1\right)}{\gamma_0} - \left(\gamma_\varepsilon - 1\right) = \varepsilon.$$

Thus, we have to verify that

$$\gamma_\varepsilon u'_{\delta,\varepsilon}\left(R_0\right) \le \gamma_0 u_{\delta,\varepsilon}^{\varepsilon}\left(R_0\right)w'\left(R_0\right). \tag{3.21}$$

But

$$w'\left(R_0\right) = \frac{\gamma_\varepsilon}{\gamma_0}\frac{1}{R_0^{N-1}}\int_{R_0}^{R} a\left(y\right)y^{N-1}\,dy,$$

and so (3.21) follows immediately from (3.15).

Taking into account (3.16), (3.18) and (3.19), the proof can now be ended as the proof of Theorem 1.6. □

Remark 3.3. Note that if we take $\delta = 0$ in the proof of Theorem 1.8, then the subsolution vanishes at the origin. This is why we have to choose $\varepsilon > 0$ and we cannot pick $\varepsilon = 0$.

Remark 3.4. Although Theorems 1.6 and 1.8 hold in particular for $N = 1$, in this case one can obtain similar results without assuming that a is even. More precisely, if $\Omega := (\alpha, \beta)$ and $\mu \in \Omega$, a quick look at the proofs of the aforementioned theorems shows that one can replace B_{R_0} and $A_{R_0,R}$ by (α, μ) and (μ, β) respectively, in order to reach a similar conclusion.

4. Proof of Theorem 1.4 and some corollaries

4.1. Application of the implicit function theorem

In this subsection, we make good use of the implicit function theorem to show that $\mathcal{I}$ is open. Let $q_0 \in (0, 1)$ be such that $u_0 \in \mathcal{P}^\circ$ is a solution of (P_{a,q_0}). We set $U_0 := (0, 1) \times B_0$, where B_0 is an open ball in $W^{2,r}_N(\Omega)$, centered at u_0, and such that $B_0 \subset \mathcal{P}^\circ$ (this is possible since $W^{2,r}_N(\Omega) \subset \mathcal{C}^1(\overline{\Omega})$). We consider the nonlinear mapping

$$\mathcal{F} : U_0 \to L^r(\Omega); \quad \mathcal{F}(q, u) = -\Delta u - a(x)u^q.$$

We see that $\mathcal{F}$ and its Fréchet derivative $\mathcal{F}_u(q, u)$ are well defined. More precisely, $\mathcal{F}$ maps U_0 continuously to $L^r(\Omega)$, and $\mathcal{F}_u(q, u)$ is a bounded linear operator from $W^{2,r}_N(\Omega)$ to $L^r(\Omega)$.

Now, we have the following:

Proposition 4.1. *Let $u_0 \in \mathcal{P}^\circ$ be a solution of (P_{a,q_0}) with $q_0 \in (0, 1)$. Then the Fréchet derivative $\mathcal{F}_u(q_0, u_0)$ maps $W^{2,r}_N(\Omega)$ onto $L^r(\Omega)$ homeomorphically, and there exists a continuous curve $q \mapsto u(q)$ from $(q_0 - \delta_0, q_0 + \delta_0)$ to $W^{2,r}_N(\Omega)$, for some $\delta_0 > 0$, such that $u(q_0) = u_0$, $F(q, u(q)) = 0$, and $u(q) \in \mathcal{P}^\circ$ for $q \in (q_0 - \delta_0, q_0 + \delta_0)$. In particular, $\mathcal{I}$ is open.*

Proof. We show how to apply the implicit function theorem to (q_0, u_0) such that $\mathcal{F}(q_0, u_0) = 0$, with $q_0 \in (0, 1)$ and $u_0 \in \mathcal{P}^\circ$. Note that

$$\mathcal{F}_u(q_0, u_0)\varphi = -\Delta\varphi - q_0 a(x)u_0^{q_0-1}\varphi.$$

We claim that

$$\mathcal{F}_u(q_0, u_0) : W^{2,r}_N(\Omega) \to L^r(\Omega) \text{ is homeomorphic.} \tag{4.1}$$

To verify it, we study the eigenvalue problem

$$\mathcal{F}_u(q_0, u_0)\varphi = \gamma\varphi.$$

By $\gamma_1 = \gamma_1(q_0, u_0)$ we denote the smallest eigenvalue (which is simple) of this equation, and by φ_1 a positive eigenfunction belonging to $\mathcal{P}^\circ$, associated to γ_1. Then, arguing as in the proof of [5, Theorem 1], we shall show that $\gamma_1 > 0$.

Using the divergence theorem, we can deduce that

$$\int_\Omega (-\Delta u_0)q_0 u_0^{q_0-1}\varphi_1 + u_0^{q_0}\Delta\varphi_1 = \int_\Omega |\nabla u_0|^2 q_0(q_0 - 1)u_0^{q_0-2}\varphi_1. \tag{4.2}$$

Indeed, a direct computation yields

$$\int_\Omega \operatorname{div}\left(\nabla u_0 q_0 u_0^{q_0-1}\varphi_1\right) = \int_\Omega (\Delta u_0)q_0 u_0^{q_0-1}\varphi_1$$

$$+\int_\Omega q_0(q_0-1)|\nabla u_0|^2 u_0^{q_0-2}\varphi_1$$
$$+\int_\Omega q_0(\nabla u_0\nabla\varphi_1)u_0^{q_0-1}.$$

Then, the divergence theorem provides

$$\int_\Omega \mathrm{div}\left(\nabla u_0 q_0 u_0^{q_0-1}\varphi_1\right)=\int_{\partial\Omega}\frac{\partial u_0}{\partial\nu}q_0u_0^{q_0-1}\varphi_1=0.$$

In a similar manner, we deduce by a direct computation that

$$\int_\Omega \mathrm{div}\,(\nabla\varphi_1 u_0^{q_0})=\int_\Omega(\Delta\varphi_1)u_0^{q_0}+\int_\Omega(\nabla\varphi_1\nabla u_0)q_0u_0^{q_0-1},$$

and by use of the divergence theorem that

$$\int_\Omega \mathrm{div}\,(\nabla\varphi_1 u_0^{q_0})=\int_{\partial\Omega}\frac{\partial\varphi_1}{\partial\nu}u_0^{q_0}=0.$$

Combining these assertions, we obtain (4.2). Now, it follows from (4.2) that

$$\begin{aligned}&\int_\Omega|\nabla u_0|^2q_0(q_0-1)u_0^{q_0-2}\varphi_1\\&\quad=\int_\Omega(-\Delta u_0)q_0u_0^{q_0-1}\varphi_1+u_0^{q_0}(\Delta\varphi_1)\\&\quad=\int_\Omega(a(x)u_0^{q_0})q_0u_0^{q_0-1}\varphi_1+u_0^{q_0}(-q_0a(x)u_0^{q_0-1}\varphi_1-\gamma_1\varphi_1)\\&\quad=-\gamma_1\int_\Omega u_0^{q_0}\varphi_1,\end{aligned}$$

and thus that

$$\gamma_1=\frac{q_0(1-q_0)\int_\Omega|\nabla u_0|^2u_0^{q_0-2}\varphi_1}{\int_\Omega u_0^{q_0}\varphi_1}>0,$$

as desired.

The assertion $\gamma_1>0$ tells us that $\mathcal{F}_u(q_0,u_0)$ is bijective. Since $\mathcal{F}_u(q_0,u_0)$ is continuous, the bounded inverse theorem yields (4.1).

We are now ready to apply the implicit function theorem to $\mathcal{F}$ at (q_0,u_0), which provides us with some $\delta_0>0$ such that $\mathcal{F}(q,u(q))=0$ for $q\in(q_0-\delta_0,q_0+\delta_0)$, $q\mapsto u(q)\in W_N^{2,r}(\Omega)$ is continuous, and $u(q_0)=u_0$. In particular, $q\mapsto u(q)\in\mathcal{C}^1(\overline{\Omega})$ is continuous, so that $u(q)\in\mathcal{P}^\circ$ for every $q\in(q_0-\delta_0,q_0+\delta_0)$, since $u(q_0)\in\mathcal{P}^\circ$. Therefore, $(q_0-\delta_0,q_0+\delta_0)\subset\mathcal{I}$, i.e. $\mathcal{I}$ is open. □

4.2. Proof of Theorem 1.4 (i)

First we note, as a consequence of Theorem 1.2, that $\mathcal{I}\neq\emptyset$ since (H_0) holds.

Let $q_0\in\mathcal{I}$ and $u_0\in\mathcal{P}^\circ$ be a corresponding solution of (P_{a,q_0}). Given $q\in(q_0,1)$, define

$$\gamma:=\frac{1-q_0}{1-q}>1,\quad\text{and}\quad w:=\gamma^{\frac{-1}{1-q}}u_0^\gamma.$$

Then, a brief computation yields that

$$\begin{aligned}
-\Delta w &= -\gamma^{\frac{-1}{1-q}}\gamma\left((\gamma-1)\,u_0^{\gamma-2}\,|\nabla u_0|^2 + u_0^{\gamma-1}\Delta u_0\right)\\
&\leq \gamma^{\frac{-1}{1-q}}\gamma u_0^{\gamma-1} a(x)\, u_0^{q_0}\\
&= a(x)\, w^q \quad a.e. \text{ in } \Omega
\end{aligned}$$

and $\frac{\partial w}{\partial \nu} = 0$ on $\partial\Omega$. In other words, w is a subsolution of $(P_{a,q})$ belonging to $\mathcal{P}^\circ$. So, recalling Remark 3.1 (ii), we obtain a solution $u \in \mathcal{P}^\circ$ of $(P_{a,q})$, and thus $q \in \mathcal{I}$. Therefore, defining $q_i := \inf \mathcal{I}$ and noting Proposition 4.1, the former assertion follows.

Since (H_0) holds, one can see by a variational approach that $(P_{a,q})$ has a nontrivial nonnegative solution for any $0 < q < 1$ (see e.g. the proof of [16, Corollary 1.8]), and thus $\mathcal{A} \subseteq \mathcal{I}$. Assume now (H_1') and (H_+). Let $q \in (0,1)$, and suppose by contradiction that there exist u and v nontrivial nonnegative solutions of $(P_{a,q})$ with $u \not\equiv v$. We claim that $u \not\equiv 0$ in Ω_+. Indeed, if not, then $\Delta u \geq 0$ in Ω and $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, and therefore the maximum principle says that $u \equiv 0$ in Ω, which is not possible. Now, taking into account that Ω_+ is connected (by (H_1')), arguing as in Lemma 2.2 in [4] we infer that $u > 0$ in Ω_+. But, since the same reasoning applies to v, this contradicts the uniqueness result of [4, Theorem 3.1] (see Remark 1.1 (ii)).

Finally, recalling that $\mathcal{A} \subseteq \mathcal{I}$, we deduce that $\mathcal{A} = \mathcal{I}$, and thus, the latter assertion follows. □

4.3. Proof of Theorem 1.4 (ii)

After a dilation and a translation, we can assume that $\Omega := (-2, 2)$. For any $q \in (0,1)$, we shall construct $a \in \mathcal{C}(\overline{\Omega})$ such that $(P_{a,q})$ has *one* solution in $\mathcal{P}^\circ$ and *two* nontrivial nonnegative solutions having nonempty dead cores. This result will be proved in two parts, in accordance with the value of q.

(i) First we consider $q \in \left[\frac{1}{3}, 1\right)$. We define

$$r := \frac{2}{1-q} \in [3, \infty) \quad \text{and} \quad f(x) := \frac{(x+1)^r}{r}.$$

Note that $rq = r - 2$. Let p be the polynomial given by

$$p(x) := \alpha x^3 + \beta x^2 + \gamma x + \delta,$$

where

$$\alpha := -\frac{2^{r-2}(r+1)}{3}, \quad \beta := 2^{r-3}(3r+1),$$
$$\gamma := -2^{r-1}(r-1), \quad \delta := \frac{2^{r-3}}{3}\left(\frac{24}{r} + 5r - 13\right).$$

One can verify that

$$p(1) = f(1), \quad p'(1) = f'(1), \quad p''(1) = f''(1), \quad p'(2) = 0. \tag{4.3}$$

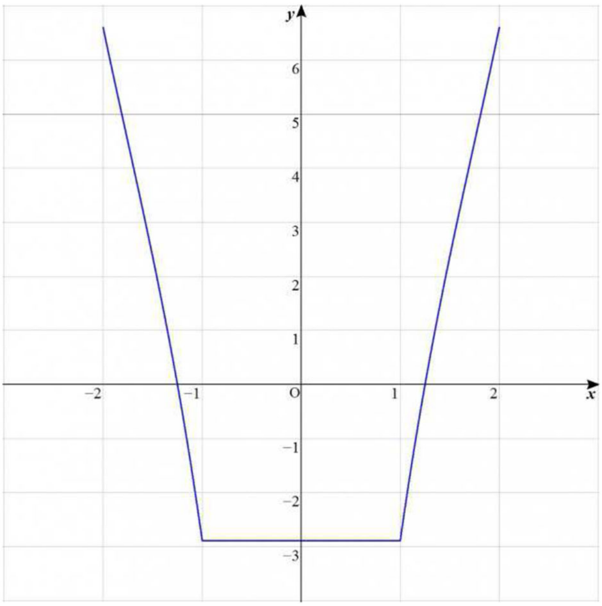

FIGURE 3. The indefinite weight a in the case $q = \frac{1}{3}$

Moreover, it also holds that p is increasing in $(1, 2)$, and in particular it follows that $p > 0$ in $[1, 2]$. Set

$$a(x) := \begin{cases} -(r-1)\,r^q & \text{if } x \in [0,1], \\ -\frac{p''(x)}{[p(x)]^q} & \text{if } x \in [1,2], \\ a(-x) & \text{if } x \in [-2,0], \end{cases}$$

(see Fig. 3) and observe that $a \in \mathcal{C}(\overline{\Omega})$ since (recall that $rq = r - 2$)

$$-\frac{p''(1)}{[p(1)]^q} = -\frac{f''(1)}{[f(1)]^q} = -(r-1)\,r^q.$$

Also, since $p > 0$ in $[1, 2]$, it follows from the definition that a changes sign in $(1, 2)$. Furthermore,

$$\int_1^2 a = -\int_1^2 \frac{p''(x)}{[p(x)]^q} = -\left[\frac{p'(x)}{[p(x)]^q}\bigg|_1^2 + q\int_1^2 \frac{[p'(x)]^2}{[p(x)]^{q+1}} \right]$$
$$< \frac{p'(x)}{([p(x)])^q}\bigg|_2^1 = \frac{p'(1)}{[p(1)]^q} = \frac{f'(1)}{[f(1)]^q} = 2r^q$$

and hence

$$\int_0^2 a < 2r^q - (r-1)\,r^q \le 0$$

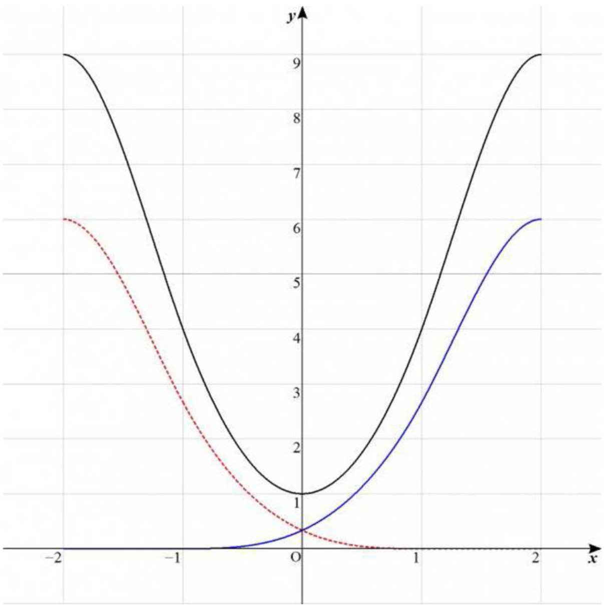

Figure 4. The nontrivial nonnegative solutions u_1, u_2 with dead cores and a solution u in $\mathcal{P}^\circ$ (which is even, by uniqueness) in the case $q = \frac{1}{3}$

since $r \geq 3$. Therefore, $\int_\Omega a < 0$. Define now

$$u_1(x) := \begin{cases} 0 & \text{if } x \in [-2,-1], \\ f(x) & \text{if } x \in [-1,1], \\ p(x) & \text{if } x \in [1,2], \end{cases}$$

and $u_2(x) := u_1(-x)$. Taking into account (4.3), we see that $u_1, u_2 \in \mathcal{C}^2(\overline{\Omega})$. Moreover, one can see that u_1 and u_2 are two distinct nonnegative nontrivial solutions of the problem

$$\begin{cases} -u'' = a(x)u^q & \text{in } \Omega, \\ u' = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

Now, a simple integration by parts shows that

$$\max(u_1(x), u_2(x)) = \begin{cases} u_2(x) & \text{in } [-2,0], \\ u_1(x) & \text{in } [0,2], \end{cases}$$

is a strictly positive weak subsolution of (4.4). Thus, since $\int_\Omega a < 0$, by Remark 3.1 (ii) there exist arbitrary large supersolutions of (4.4) and we then obtain a solution $u \in \mathcal{P}^\circ$ of (4.4), see Fig. 4. It follows that $q \in \mathcal{I}$, but $q \notin \mathcal{A}$, since u_1 and u_2 are nontrivial nonnegative solutions having nonempty dead cores.

(ii) Now we consider $q \in \left(0, \frac{1}{3}\right)$. We proceed as above, the only difference being the definition of p. For $K > 0$, let

$$p(x) = p_K(x) := \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \mu,$$

where

$$\alpha := 3\left(\frac{2^r}{r} - K\right) + 2^{r-3}(r+7), \quad \beta := -16\left(\frac{2^r}{r} - K\right) + 2^{r-3}(-6r-38),$$
$$\gamma := 30\left(\frac{2^r}{r} - K\right) + 2^{r-3}(13r+71), \quad \delta := -24\left(\frac{2^r}{r} - K\right) - 2^{r-3}(12r+52),$$
$$\mu := 8\frac{2^r}{r} - 7K + 2^{r-3}(4r+12).$$

One can check that

$$p(1) = f(1), \quad p'(1) = f'(1), \quad p''(1) = f''(1), \qquad p'(2) = 0, \quad p(2) = K. \tag{4.5}$$

We observe that $p > 0$ in $[1,2]$. Indeed, since

$$p(1), p'(1), p''(1), p(2) > 0 = p'(2),$$

if $p \leq 0$ somewhere, then p'' would vanish at least at three points in $(1,2)$, which is impossible since p'' has degree 2. It follows that

$$a_K(x) := -\frac{p_K''(x)}{[p_K(x)]^q} \in \mathcal{C}([1,2]).$$

We claim now that for all $K > 0$ large enough, a_k changes sign in $(1,2)$ and $\int_1^2 a_K < 0$. Indeed, $a_K(1) < 0$, and for K sufficiently large we have $p_K''(2) < 0$, so that $a_K(2) > 0$. Hence, the first assertion follows. To show the second one, we first note that

$$\int_1^2 a_K = -K^{1-q} \int_1^2 \frac{p_K''(x)}{K}\left(\frac{K}{p_K(x)}\right)^q, \tag{4.6}$$

$$\lim_{K\to\infty} \frac{p_K(x)}{K} = -3x^4 + 16x^3 - 30x^2 + 24x - 7 = (7-3x)(x-1)^3 := g(x),$$

and

$$\lim_{K\to\infty} \frac{p_K''(x)}{K} = -36x^2 + 96x - 60 = 12(5-3x)(x-1) := 12h(x).$$

Define

$$H(x) := -x^3 + 4x^2 - 5x + 2 = (2-x)(x-1)^2 > 0 \quad \text{in } (1,2).$$

Then, $H' = h$. Also, since $2 - 3q > 0$ we see that $\lim_{x\to 1^+} H(x)g(x)^{-q} = 0$. Therefore, an integration by parts yields that

$$\int_1^2 \frac{h(x)}{g^q(x)} = H(x)g(x)^{-q}\Big|_1^2 - \int_1^2 H(x)\left(g(x)^{-q}\right)' \tag{4.7}$$
$$= q\int_1^2 H(x)g(x)^{-(q+1)}g'(x) > 0$$

because $g' > 0$ in $(1,2)$. It follows from (4.6) and (4.7) that

$$\lim_{K\to\infty} \int_1^2 a_K = -\infty,$$

and therefore the claim is proved. We can then fix some $K > 0$ such that a_K changes sign in Ω and $\int_\Omega a_K < 0$, and thus the proof can be completed as in the previous case.

□

Remark 4.2. Let us point out that, by the uniqueness results in [4], for every $q \in (0,1)$, the problem $(P_{a,q})$ with the weight a_q constructed in the above proof has *exactly* three (nontrivial) nonnegative solutions. Indeed, one can verify that $\Omega_+(a_q)$ has exactly two connected components (taking K large if $q < 1/3$), say $\mathcal{O}_1$ and $\mathcal{O}_2$. Now, by [4, Theorem 3.1] there exists at most one nonnegative solution which is positive in $\mathcal{O}_1$ and zero in $\mathcal{O}_2$, and vice-versa. Also, by the aforementioned theorem, there exists at most one nonnegative solution which is positive in both $\mathcal{O}_1$ and $\mathcal{O}_2$. Since the nontrivial nonnegative solutions u satisfy that either $u > 0$ in $\mathcal{O}_i$ or $u \equiv 0$ in $\mathcal{O}_i$ (see [4, Lemma 2.2]), our assertion follows because from the maximum principle we deduce that there is no nontrivial nonnegative solution vanishing in both $\mathcal{O}_1$ and $\mathcal{O}_2$. Let us also remark that the solution in $\mathcal{P}^\circ$ is even: indeed, if not, we would have four nontrivial nonnegative solutions. Summing up, for this family of even weights a_q, there exist two (nontrivial) noneven nonnegative solutions with nonempty dead cores, and one even solution in $\mathcal{P}^\circ$.

Remark 4.3. Let a_q be as in the first case of the proof of Theorem 1.4 (ii), but now with $q \in [0,1)$. A quick look at the aforementioned proof shows that $\int_\Omega a_q > 0$ for $q > 0$ close enough to 0. Indeed, this follows easily from the fact that $\int_\Omega a_0 = 2$. Furthermore, for such q's, reasoning as therein we obtain two (nontrivial) nonnegative solutions of $(P_{a_q,q})$. In other words, this result shows that, unlike for the existence of positive solutions, the condition $\int_\Omega a < 0$ is *not necessary* in order to have existence of (nontrivial) nonnegative solutions of $(P_{a,q})$. Let us add that this matter has already been noted in [4, Section 2.3].

As an immediate consequence of Theorems 1.8 and 1.4 (i), we have the following result:

Corollary 4.4. *Let $\Omega := B_R$ and $a \in L^\infty(\Omega)$ be a radial function such that $\int_\Omega a < 0$ and $a \geq 0$ in $A_{R_0,R}$ for some $R_0 \in (0,R)$. Then,*

$$\left(\frac{1-KN}{1-KN+2K}, 1\right) \subseteq \mathcal{I}, \quad \text{where} \quad K = K(a) := \frac{\int_{A_{R_0,R}} a}{\omega_{N-1} R_0^N \left\|a^-\right\|_{L^\infty(B_{R_0})}}.$$

Moreover,

$$\left(\frac{1-KN}{1-KN+2K}, 1\right) \subseteq \mathcal{A}$$

if $a \leq 0$ in B_{R_0}.

Proof. Since $\int_\Omega a < 0$, a direct computation gives that $KN < 1$. Let $q \in \left(\frac{1-KN}{1-KN+2K}, 1\right)$. Then one can check that (1.4) is satisfied and thus there exists $u \in \mathcal{P}^\circ$ solution of $(P_{a,q})$, so that $q \in \mathcal{I}$. The last assertion of the corollary is now immediate from Theorem 1.4 (i2). □

Remark 4.5. Let us point out that $\mathcal{I}(a)$ may approach the whole interval $(0,1)$ as the coefficient a varies. To show this, we may use either the sub and supersolutions method (Corollary 4.4), or a bifurcation analysis (Proposition 2.3). Let us also add that, however, we believe that there is no a such that $\mathcal{I}(a) = (0,1)$, but we are not able to prove it.

(i) Given any fixed $q_0 \in (0,1)$, Corollary 4.4 provides some cases in which $(q_0, 1) \subseteq \mathcal{I} = \mathcal{A}$. Indeed, in order to see this it suffices to find a such that $K(a)$ satisfies $1 > K(a)N \approx 1$. One may take for instance $\Omega := B_1$ and

$$a(x) := \sigma \chi_{A_{\frac{1}{2},1}}(x) - \chi_{B_{\frac{1}{2}}}(x), \qquad \text{for } x \in \Omega,$$

where $\sigma > 0$. Since $K(a)N = \sigma\left(2^N - 1\right)$, it is easy to choose σ adequately.

(ii) Let $a \in L^\infty(\Omega)$ be given by $a(x) := \sigma\chi_{\Omega_1}(x) - \chi_{\Omega_2}(x)a_2(x)$, where $\sigma > 0$, $a_2 \geq 0$, and Ω_1, Ω_2 are disjoint subsets of Ω such that $\int_\Omega a = 0$. Then, for any $\varepsilon > 0$ small, we see that $a - \varepsilon$ changes sign, $\int_\Omega (a-\varepsilon) < 0$, and $\Omega_+(a-\varepsilon) = \Omega_+(a)$. By combining Theorem 1.4 (i) and Proposition 2.3, we see that $\mathcal{I}_{a-\varepsilon}$ approaches $(0,1)$ as $\varepsilon \to 0^+$. Additionally, if $\Omega_+(a)$ satisfies (H_1') and (H_+), then $\mathcal{A}_{a-\varepsilon} = \mathcal{I}_{a-\varepsilon}$ approaches $(0,1)$ as $\varepsilon \to 0^+$.

5. Proof of Theorem 1.10

Proof of Theorem 1.10 (i): Let Ω_0 be a tubular neighborhood of $\partial\Omega$ such that $a > 0$ *a.e.* in Ω_0, with smooth boundary $\partial\Omega_0 = \partial\Omega \cup \Gamma_0$, where $\Gamma_0 = \partial\Omega_0 \cap \Omega$. We consider the following concave mixed problem

$$\begin{cases} -\Delta v = a^+(x)v^q & \text{in } \Omega_0, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ v = 0 & \text{on } \Gamma_0. \end{cases} \tag{5.1}$$

Proceeding as in [2, Lemma 3.3], we see that the comparison principle holds for (5.1), i.e. $\underline{v} \leq \overline{v}$ on $\overline{\Omega_0}$ for any nonnegative supersolution $\overline{v}$ and subsolution $\underline{v}$ of (5.1) such that $\overline{v}, \underline{v} > 0$ in Ω_0.

Let u be a nontrivial nonnegative solution of $(P_{a,q})$. Then u is a supersolution of (5.1). In addition, $u > 0$ in Ω_0. Indeed, recalling (H_1'), we observe that

$$0 < \int_\Omega |\nabla u|^2 = \int_\Omega a(x)u^{q+1} \leq \int_{\text{supp } a^+} a(x)u^{q+1} = \int_{\Omega_+} a(x)u^{q+1}.$$

It follows that $u \not\equiv 0$ in Ω_+. Also, since Ω_+ is connected (by (H_1')), the strong maximum principle yields that $u > 0$ in Ω_+, and consequently $u > 0$ in Ω_0 as claimed.

On the other hand, in order to construct a subsolution, we consider the following mixed eigenvalue problem:

$$\begin{cases} -\Delta\psi = \sigma a^+(x)\psi & \text{in } \Omega_0, \\ \frac{\partial\psi}{\partial\nu} = 0 & \text{on } \partial\Omega, \\ \psi = 0 & \text{on } \Gamma_0. \end{cases} \tag{5.2}$$

By $\sigma_1 > 0$, we denote the smallest eigenvalue of (5.2), and by ψ_1 an eigenfunction associated to σ_1 satisfying $\psi_1 > 0$ on $\overline{\Omega_0} \setminus \Gamma_0$. Then, we see that $\varepsilon_q\psi_1$ is a subsolution of (5.2) for some $\varepsilon_q > 0$ small. By the comparison principle, we deduce that $\varepsilon_q\psi_1 \le u$ on $\overline{\Omega_0}$, from which the desired conclusion follows. $\square$

The following result will be used in the proof of Theorem 1.10 (ii):

Lemma 5.1. *Under the conditions of Theorem 1.10 (ii), let $\overline{q} \in (0,1)$ and $\delta_2 > 0$ such that $\int_\Omega a_{\delta_2} < 0$. Then, there exists $C > 1$ such that $\|u\|_{L^\infty(\Omega)} < C$ for all nonnegative solutions u of $(P_{a_\delta,q})$ with $q \in (0,\overline{q}]$ and $\delta \ge \delta_2$.*

Proof. Let u be a nonnegative solution of $(P_{a_\delta,q})$ with $q \in (0,\overline{q}]$ and $\delta \ge \delta_2$. Then u is a subsolution of $(P_{a_\delta,q})$ for $\delta = \delta_2$. In view of Remark 3.1 (ii), we can construct a supersolution w of $(P_{a_{\delta_2},q})$ such that $u \le w$. Hence, the sub and supersolutions method ensures the existence of a nonnegative solution v of $(P_{a_{\delta_2},q})$ such that $u \le v \le w$. By Proposition 2.1 (ii), $(P_{a_{\delta_2},q})$ has an *a priori* bound for nonnegative solutions in $L^\infty(\Omega)$, which is uniform in $q \in (0,\overline{q}]$. The lemma now follows. $\square$

Proof of Theorem 1.10 (ii): We proceed as in the proofs of [9, Theorem 1(iv)] and [8, Theorem 3.1]. Let $\overline{q} \in (0,1)$ and $\delta_2 > 0$ be the constant given by Lemma 5.1, and u be a nontrivial nonnegative solution of $(P_{a_\delta,q})$ with $q \in (0,\overline{q}]$ and $\delta \ge \delta_2$. Given $\sigma > 0$, we pick $a_0 > 0$ such that

$$b_2(z) \ge a_0 \quad \text{for all } z \in G_{\sigma/2}, \tag{5.3}$$

where we have used the continuity of b_2. Let us fix $x \in G_{\sigma/2}$, and consider $d(x) := \operatorname{dist}(x, \partial G_{\sigma/2})$, where $d(x) > 0$ since $G_{\sigma/2}$ is open. We then define

$$v_1(y) := d(x)^{-\alpha}u(x + d(x)y), \quad \text{for } |y| \le 1. \tag{5.4}$$

Let $\alpha := 2/(1-q)$, so that $2 - \alpha + \alpha q = 0$. If $|y| < 1$ then, using (5.3), a brief computation yields

$$\begin{aligned} -\Delta v_1(y) &= d(x)^{2-\alpha}a_\delta(x + d(x)y)\, u(x + d(x)y)^q \\ &= -\delta b_2(x + d(x)y)\, v_1(y)^q \\ &\le -\delta a_0\, v_1(y)^q. \end{aligned}$$

Here, we have used the fact that $x + d(x)y \in G_{\sigma/2}$. If $|y| = 1$, then we have

$$v_1(y) \le d(x)^{-\alpha}C, \tag{5.5}$$

where $C > 1$ is provided by Lemma 5.1.

Given $\varepsilon > 0$, we now consider the problem

$$(Q_{\delta,\varepsilon}) \qquad \begin{cases} -\Delta v = -\delta a_0 v^q & \text{in } B_1, \\ v = \varepsilon & \text{on } \partial B_1. \end{cases}$$

We observe from (5.5) that v_1 is a subsolution of $(Q_{\delta,\varepsilon})$ if

$$d(x)^{-\alpha} C \leq \varepsilon. \tag{5.6}$$

Next, we construct a supersolution of $(Q_{\delta,\varepsilon})$. For $r = |y|$, we define

$$z_1(r) := \begin{cases} 0, & 0 \leq r \leq \frac{1}{2}, \\ A\left(r - \frac{1}{2}\right)^\alpha, & \frac{1}{2} < r \leq 1, \end{cases} \tag{5.7}$$

where A is a positive constant to be determined. Since $\alpha > 2$, we have $z_1 \in \mathcal{C}^2(\overline{B_1})$, and in addition,

$$\begin{aligned} \Delta z_1 = z_1'' + \frac{N-1}{r} z_1' &= A\alpha(\alpha - 1)\left(r - \frac{1}{2}\right)^{\alpha-2} + \frac{N-1}{r} A\alpha \left(r - \frac{1}{2}\right)^{\alpha-1} \\ &\leq A\alpha(\alpha-1)\left(r - \frac{1}{2}\right)^{\alpha-2} + (N-1)A\alpha\left(r - \frac{1}{2}\right)^{\alpha-2} \\ &\leq \delta a_0 \left(A\left(r - \frac{1}{2}\right)^\alpha\right)^q = \delta a_0 z_1^q \quad \text{for } \frac{1}{2} < r < 1, \end{aligned}$$

if $A\alpha(\alpha-1) + (N-1)A\alpha \leq \delta a_0 A^q$, i.e.

$$A \leq \left(\frac{\delta a_0}{\alpha(\alpha-1) + (N-1)\alpha}\right)^{\frac{1}{1-q}}. \tag{5.8}$$

Moreover, we note that

$$z_1(1) = A\left(\frac{1}{2}\right)^\alpha \geq \varepsilon,$$

provided that

$$A \geq 2^\alpha \varepsilon. \tag{5.9}$$

Hence, from (5.6), (5.8) and (5.9), it follows that if

$$2^\alpha d(x)^{-\alpha} C \leq 2^\alpha \varepsilon \leq A \leq \left(\frac{\delta a_0}{\alpha(\alpha-1) + (N-1)\alpha}\right)^{\frac{1}{1-q}},$$

i.e.

$$d(x) \geq 2C^{\frac{1}{\alpha}} \left(\frac{\alpha(\alpha-1) + (N-1)\alpha}{\delta a_0}\right)^{\frac{1}{2}},$$

then v_1 is a subsolution of $(Q_{\delta,\varepsilon})$, and in addition, z_1 is a supersolution of $(Q_{\delta,\varepsilon})$. Since $2 < \alpha \leq 2/(1-\overline{q}) =: \overline{\alpha}$, this occurs for some $\varepsilon = \varepsilon_{\delta,x}$ if

$$d(x) \geq 2\left(\frac{\overline{\alpha}(\overline{\alpha}-1) + (N-1)\overline{\alpha}}{C^{-1}\delta a_0}\right)^{\frac{1}{2}} =: d_\delta. \tag{5.10}$$

Now, using the comparison principle for $(Q_{\delta,\varepsilon})$ (which is deduced from the weak maximum principle) we derive that $v_1 \leq z_1$, so that

$$d(x)^{-\alpha} u(x) = v_1(0) \leq z_1(0) = 0,$$

and consequently, $u(x) = 0$. Therefore, we have proved that if $x \in G_{\sigma/2}$ satisfies (5.10), then $u(x) = 0$ for any nontrivial nonnegative solution u of $(P_{a_\delta,q})$ with $q \in (0, \overline{q}]$. Since d_δ in (5.10) does not depend on u or q, and converges to 0 as $\delta \to \infty$, we have the desired conclusion. □

Remark 5.2.

(i) The conclusion of Theorem 1.10 (ii) still holds if $a_\delta := b - \delta\chi_G$, with b, G satisfying

$$(H_2') \begin{cases} b \in L^\infty(\Omega), \ 0 \not\equiv b \geq 0, \text{ and} \\ \emptyset \neq G \subset \Omega \text{ is an open subset such that } \operatorname{supp} b \cap G = \emptyset. \end{cases}$$

Here $\delta > 0$ and χ_G is the characteristic function of G.

(ii) Let $a_\delta := b_1 - \delta b_2$ with $b_1, b_2 \not\equiv 0$ satisfying (H_2), and $\delta > 0$.

(ii1) In addition to (H_1'), let us assume that

$$\operatorname{supp} b_1 \cup \{x \in \Omega : b_2(x) > 0\} = \Omega.$$

Let $q \in (0, 1)$. Theorem 1.10 (ii) then shows that the support of any nontrivial nonnegative solution of $(P_{a_\delta,q})$ approaches Ω_+ (in some sense) as $\delta \to \infty$.

(ii2) Combining Theorem 1.2 and Theorem 1.10 (ii), we find $\delta_1 > 0$ and $0 < q_1 \leq q_0 < 1$ such that any nontrivial nonnegative solution of $(P_{a_\delta,q})$ with $\delta = \delta_1$ has a nonempty dead core for $q \in (0, q_1]$, whereas this problem has a unique solution in $\mathcal{P}^\circ$ and no other nontrivial nonnegative solutions for $q \in (q_0, 1)$. Furthermore, according to Theorem 1.4 (i) and Theorem 1.10 (i), we see that if (H_1') and (H_+) hold and Ω_+ contains a tubular neighborhood of $\partial\Omega$, then $q_1 = q_0$, and the nontrivial nonnegative solution for $q \in (0, q_0]$ is also unique (see Fig. 2).

(ii3) As we shall see from its proof, Theorem 1.10 (ii) holds also for the Dirichlet counterpart of $(P_{a_\delta,q})$. In particular, it complements [16, Theorem 1.1] as follows: given $q \in (0, 1)$ there exist $0 < \delta_1 < \delta_0$ such that every nontrivial nonnegative solution u of

$$\begin{cases} -\Delta u = a_\delta(x) u^q & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

satisfies $u > 0$ in Ω and $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$ for $\delta < \delta_1$, whereas u has a nonempty dead core for $\delta > \delta_0$.

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References

[1] Alama, S.: Semilinear elliptic equations with sublinear indefinite nonlinearities. Adv. Differ. Equ. **4**, 813–842 (1999)

[2] Ambrosetti, A., Brezis, H., Cerami, G.: Combined effects of concave and convex nonlinearities in some elliptic problems. J. Funct. Anal. **122**, 519–543 (1994)

[3] Bandle, C., Pozio, M., Tesei, A.: The asymptotic behavior of the solutions of degenerate parabolic equations. Trans. Am. Math. Soc. **303**, 487–501 (1987)

[4] Bandle, C., Pozio, A.M., Tesei, A.: Existence and uniqueness of solutions of nonlinear Neumann problems. Math. Z. **199**, 257–278 (1988)

[5] Brown, K.J., Hess, P.: Stability and uniqueness of positive solutions for a semi-linear elliptic boundary value problem. Differ. Integral Equ. **3**(2), 201–207 (1990)

[6] Cuesta, M., Takáč, P.: A strong comparison principle for positive solutions of degenerate elliptic equations. Differ. Integral Equ. **13**, 721–746 (2000)

[7] Delgado, M., Suárez, A.: On the uniqueness of positive solution of an elliptic equation. Appl. Math. Lett. **18**, 1089–1093 (2005)

[8] Friedman, A., Phillips, D.: The free boundary of a semilinear elliptic equation. Trans. Am. Math. Soc. **282**, 153–182 (1984)

[9] García-Melián, J., Rossi, J.D., Sabina de Lis, J.C.: A bifurcation problem governed by the boundary condition II. Proc. Lond. Math. Soc. **94**, 1–25 (2007)

[10] Godoy, T., Kaufmann, U.: On strictly positive solutions for some semilinear elliptic problems. NoDEA Nonlinear Differ. Equ. Appl. **20**, 779–795 (2013)

[11] Godoy, T., Kaufmann, U.: Existence of strictly positive solutions for sublinear elliptic problems in bounded domains. Adv. Nonlinear Stud. **14**, 353–359 (2014)

[12] Gurtin, M.E., MacCamy, R.C.: On the diffusion of biological populations. Math. Biosci. **33**, 35–49 (1977)

[13] Hernández, J., Mancebo, F., Vega, J.: On the linearization of some singular, nonlinear elliptic problems and applications. Ann. Inst. H. Poincaré Anal. Non Linéaire **19**, 777–813 (2002)

[14] Jeanjean, L.: Some continuation properties via minimax arguments. Electron. J. Differ. Equ. **2011**(48), 1–10 (2011)

[15] Kajikiya, R.: Positive solutions of semilinear elliptic equations with small perturbations. Proc. Am. Math. Soc. **141**, 1335–1342 (2013)

[16] Kaufmann, U., Ramos Quoirin, H., Umezu, K.: Positivity results for indefinite sublinear elliptic problems via a continuity argument. J. Differ. Equ. **263**, 4481–4502 (2017)

[17] Namba, T.: Density-dependent dispersal and spatial distribution of a population. J. Theor. Biol. **86**, 351–363 (1980)

[18] Pozio, M.A., Tesei, A.: Support properties of solution for a class of degenerate parabolic problems. Commun. Partial Differ. Equ. **12**, 47–75 (1987)

[19] Zeidler, E.: Nonlinear Functional Analysis and Its Applications I: Fixed-Point Theorems. Springer, New York (1986)

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