Nonlinear Differ. Equ. Appl. (2018) 25:4

1021-9722/18/010001-20
published online January 24, 2018
https://doi.org/10.1007/s00030-018-0496-8

Nonlinear Differential Equations and Applications NoDEA

Carleman estimate for a linearized bidomain model in electrocardiology and its applications

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Abstract. This paper concerns Carleman estimate and its applications for a linearized bidomain model in electrocardiology, which describes the electrical activity in the cardiac tissue. We first establish a new Carleman estimate for this reaction–diffusion system. By means of this Carleman estimate, we study two problems for the linearized bidomian model, a Cauchy problem and an inverse conductivities problem. We prove a conditional stability result for the Cauchy problem and a Hölder stability result for the inverse conductivities problem.

Mathematics Subject Classification. 35L05, 35L10, 35R09, 35R30.

Keywords. Electrocardiology, Cauchy problem, Conditional stability.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bound domain with smooth boundary $\partial\Omega$. We further set $Q_T := \Omega \times (0, T)$, $\Sigma_T := \partial\Omega \times (0, T)$. Then the bidomain model describing the electrical activity in the cardiac tissue can be written as follows [13]:

$$\begin{cases} c_m\partial_t v - \nabla \cdot (M_i(x)\nabla u_i) + I_{ion}(v, w) = I_i, & (x, t) \in Q_T, \\ c_m\partial_t v + \nabla \cdot (M_e(x)\nabla u_e) + I_{ion}(v, w) = I_e, & (x, t) \in Q_T, \\ \partial_t w - H(v, w) = h, & (x, t) \in Q_T, \end{cases} \tag{1.1}$$

where $u_i = u_i(x, t)$, $u_e = u_e(x, t)$ represent the intracellular and extracellular electric potentials respectively, and their difference, $v = u_i - u_e$ is called the transmembrane potential. $w = w(x, t)$ is the gating or recovery variable, which represents the ionic current variables. The positive constant c_m is the surface capacitance of the membrane. The anisotropic properties of the media are modeled by intracellular and extracellular conductivity tensors $M_i(x)$ and $M_e(x)$, details see [13].

We denote by I_i and I_e the internal and the external current stimulus respectively. Moreover, $H(v, w)$ and $I_{ion}(v, w)$ are functions which correspond to the widely known FitzHugh–Nagumo model for the membrane and ionic currents, i.e.

$$H(v, w) = av - bw, \tag{1.2}$$

$$I_{ion}(v, w) = -\sigma(w - v(1 - v)(v - \theta)), \tag{1.3}$$

where a, b, σ, θ are given positive constants, see [4].

As for the direct problem about the bidomain model, [7,25] and the references therein, proved the existence of the weak or the strong solution in the framework of the suitable Banach spaces. Additionally, different numerical methods have been used for solving this type of bidomain model in [10,11,26].

When dropping the effect of w in (1.1) and letting

$$M_i(x) = \mu M_e(x), \quad x \in \Omega \tag{1.4}$$

with some constant $\mu \in \mathbb{R}$, the bidomain model (1.1) can be rewritten as the following monodomain model [18]

$$\begin{cases} c_m \partial_t v - \frac{\mu}{\mu+1} \nabla \cdot (M_e(x) \nabla v) + h(v) = f, \ (x, t) \in Q_T, \\ -\nabla \cdot (M(x) \nabla u_e) = \nabla \cdot (M_i(x) \nabla v), \qquad (x, t) \in Q_T, \end{cases} \tag{1.5}$$

with $M = M_i + M_e$ and suitable f, $h(v)$. Bendahmane and Chaves-Silva [5] proved null controllability of the approximating system of (1.5) by means of a single control on ω:

$$\begin{cases} c_m \partial_t v - \frac{\mu}{\mu+1} \nabla \cdot (M_e(x) \nabla v) + h(v) = f 1_\omega, \ (x, t) \in Q_T, \\ \varepsilon \partial_t u_e - \nabla \cdot (M(x) \nabla u_e) = \nabla \cdot (M_i(x) \nabla v), \ (x, t) \in Q_T, \end{cases} \tag{1.6}$$

with any sufficiently small $\varepsilon > 0$ by Carleman estimate. Further the null controllability for a monodomain model (1.5) was shown. Ainseba, Bendahmane and He [1] established a Carleman estimate for a linearized version of (1.6). Then by using this Carleman estimate, a stability result for recovering the conductivities was obtained. In these two papers, the weight function of Carleman estimate is singular like $\frac{e^{\lambda\psi(x)} - e^{2\lambda\|\psi\|_{C(\overline{\Omega})}}}{t(T-t)}$ with suitable ψ, which is very difficult applied to study Cauchy problem for the bidomain model. Additionally, Lassoued, Mahjoub and Zemzemi [19] studied a parameter identification inverse problem in cardiac electro-physiology. Boulakia and Schenone [6] proved a Carleman estimate for a reaction–diffusion equation coupled with an ODE. However, to the best of our knowledge there is no publications about Carleman estimates for the bidomain model.

Carleman estimate is a class of weighted energy estimates connected with the differential operator, which can be applied to many aspects, such as stabilization and control theory [14,16], coefficient inverse problems [8,22,27] and unique continuation [12,21,23,28] and so on.

The main objective of this paper is to obtain Carleman estimate for the linearized bidomain model. As its applications, we consider the following two problems, a Cauchy problem and an inverse conductivities problem.

Cauchy problem. Letting Γ be an arbitrary non-empty sub-boundary of $\partial\Omega$ and $\Gamma_T := \Gamma \times (0,T)$, we determine (u_i, u_e) in

$$\begin{cases} c_m\partial_t v - \nabla\cdot(M_i(x)\nabla u_i) + av = f, \ (x,t)\in Q_T, \\ c_m\partial_t v + \nabla\cdot(M_e(x)\nabla u_e) + bv = g, \ (x,t)\in Q_T, \end{cases} \tag{1.7}$$

by a lateral Cauchy data

$$(u_i, u_e)\,|_{\Gamma_T} = (p,q). \tag{1.8}$$

Inverse conductivities problem. Letting ω be a given sub-domain such that $\partial\omega \supset \partial\Omega$ and t_0 be a given time, we determine M_i, M_e in

$$\begin{cases} c_m\partial_t v - \nabla\cdot(M_i(x)\nabla u_i) + a_{11}v + a_{12}w = f, & (x,t)\in Q_T, \\ c_m\partial_t v + \nabla\cdot(M_e(x)\nabla u_e) + a_{21}v + a_{22}w = g, & (x,t)\in Q_T, \\ \partial_t w + a_{31}v + a_{32}w = h, & (x,t)\in Q_T, \\ u_i(x,t) = u_e(x,t) = 0, & (x,t)\in \Sigma_T, \\ v(x,0) = u_{i,0}(x) - u_{e,0}(x) := v_0(x), w(x,0) = w_0(x), & (x,t)\in\Omega, \end{cases} \tag{1.9}$$

by observation data

$$(u_i,u_e)|_{\omega\times(0,T)} \text{ and } \left(u_i(x,t_0), u_e(x,t_0), w(x,t_0)\right), \quad x\in\Omega. \tag{1.10}$$

Remark 1.1. Generally, the conductivities $\mathbf{M}_i$ and $\mathbf{M}_e$ are two matrices given by

$$\mathbf{M}_j(x) = \sigma_j^t(x)\mathbf{I} + (\sigma_j^l(x) - \sigma_j^t(x))\mathbf{a}_l(x)\mathbf{a}_l^T(x),$$

where σ_j^l and σ_j^t, $j\in\{i,e\}$ are the intra- and extracellular conductivities along and transversal to the direction of the fiber (parallel to $\mathbf{a}_l(x)$), respectively. In the case of equal anisotropy [20], i.e. the so-called anisotropy ratios $\sigma_i^l/\sigma_i^t = 1$ and $\sigma_e^l/\sigma_e^t = 1$, the $\mathbf{M}_i$ and $\mathbf{M}_e$ are simplified as $\mathbf{M}_i(x) = M_i(x)\mathbf{I}$ and $\mathbf{M}_e(x) = M_e(x)\mathbf{I}$ with $M_i(x) = \sigma_i^t(x)$, $M_e(x) = \sigma_e^t(x)$, which is the case we discussed. If $\mathbf{M}_i$ and $\mathbf{M}_e$ in bidomain model are two metrics, such an inverse problem is still open and more complicated. For example, Yuan and Yamamoto [29] studied an inverse problem for recovering the matrix $\mathbf{A}$ in the following parabolic equation:

$$y_t - \nabla\cdot(\mathbf{A}(x)\nabla y) = h(x,t), \quad (x,t)\in Q_T,$$

where $\mathbf{A}(x) = (a_{ij}(x))_{1\le i,j\le n}$ with $a_{ij} = a_{ji}$. In order to prove a stability result, they need $n(n+3)/2$ times measurement data to determine $n(n+1)/2$ unknown a_{ij}. Additionally, some technique conditions are included in the stability results, details see [29].

We make the following assumptions for Cauchy problem.

(A1) $M_i, M_e \in C^1(\overline{\Omega})$ such that

$$M_i > \epsilon_0, \quad M_e > \epsilon_0 \quad \text{in } \overline{\Omega}$$

with a positive constant ϵ_0;

(A2) $f, g\in L^2(Q_T)$, $a, b\in C(\overline{Q}_T)$;

(A3) $p, q\in H^1(\Gamma_T)$.

Theorem 1.1. *Let $\Gamma \subset \partial\Omega$ be an arbitrary non-empty sub-boundary and (A1)-(A3) be held, and let $(u_i, u_e) \in \left(L^2(0,T;H^1(\Omega))\right)^2$ be a solution of (1.7) and (1.8). For any $\varepsilon > 0$ and an arbitrary bounded domain Ω_0 such that $\overline{\Omega}_0 \subset \Omega \cup \Gamma$, $\partial\Omega_0 \cap \partial\Omega$ is a non-empty open subset of $\partial\Omega$ and $\partial\Omega_0 \cap \partial\Omega \subsetneqq \Gamma$, there exist positive constants C and $\kappa \in (0,1)$ such that*

$$||u_i||_{L^2(\varepsilon,T-\varepsilon;H^1(\Omega_0))} + ||u_e||_{L^2(\varepsilon,T-\varepsilon;H^1(\Omega_0))} \leq C I^{1-\kappa} J^{\kappa}, \tag{1.11}$$

where

$$\begin{aligned} I &= \|u_i\|_{L^2(0,T;H^1(\Omega))} + ||u_e||_{L^2(0,T;H^1(\Omega))}, \\ J &= \|f\|_{L^2(Q_T)} + \|g\|_{L^2(Q_T)} + ||p||_{H^1(\Gamma_T)} + ||q||_{H^1(\Gamma_T)}. \end{aligned}$$

Since $\varepsilon > 0$ and $\Omega_0 \subset \Omega$ are chosen arbitrary provided that the constraints on Ω_0 in Theorem 1.1 are fulfilled, we have the following unique continuation result.

Corollary 1.2. *Under the same assumptions as in Theorem 1.1, if*

$$f(x,t) = g(x,t) = 0, \ (x,t) \in Q_T, \quad p(x,t) = q(x,t) = 0, \quad (x,t) \in \Gamma_T,$$

then $u_i = u_e = 0$ a.e. in Q_T.

To state our second main result, i.e. Hölder stability for our inverse conductivities problem, we first introduce the set

$$\begin{aligned} \mathcal{W} = \Big\{ &(M_i, M_e) \in \left(C^1(\overline{\Omega})\right)^2; \quad M_i > \epsilon_0, \ M_e > \epsilon_0 \text{ in } \overline{\Omega}, \\ &(M_i, M_e)|_{\partial\Omega} = (a_i, a_e), \quad (\nabla M_i, \nabla M_e)|_{\partial\Omega} = (\mathbf{b}_i, \mathbf{b}_e), \\ &|\nabla u_i[M_i, M_e](x,t_0) \cdot (x - x_0)| \geq \epsilon_0, \ |\nabla u_e[M_i, M_e](x,t_0) \cdot (x - x_0)| \geq \epsilon_0 \text{ in } \Omega \Big\} \end{aligned}$$

for fixed sufficiently smooth functions $a_i, a_e, \mathbf{b}_i, \mathbf{b}_e$ on Γ and a positive constant ϵ_0, where $x_0 \in \mathbb{R}^N \setminus \overline{\Omega}$ is a fixed point. Here $(u_i[M_i, M_e], u_e[M_i, M_e])$ denotes the solution of the problem (1.9) corresponding to (M_i, M_e).

Assumptions.(A1') $\omega \subset \Omega$ is a given sub-domain such that $\partial\omega \supset \partial\Omega$, i.e. ω is a small neighborhood near $\partial\Omega$ inside Ω;

(A2') $f, g, h \in H^2(Q_T)$, $a_{ij} \in C^2(\overline{Q}_T)(1 \leq i \leq 3, 1 \leq j \leq 2)$, $u_{i,0}, u_{e,0} \in H^3(\Omega)$, $w_0 \in H^1(\Omega)$;

(A3') $u_i, u_e \in C^2([0,T]; W^{2,\infty}(\Omega)) \cap C([0,T]; H^3(\Omega))$, $w \in W^{2,\infty}(Q_T) \cap C([0,T];\ H^1(\Omega))$.

Remark 1.2. As [1] or [2], we need the technical condition $|\nabla u_i \cdot (x - x_0)| \geq \epsilon_0$ and $|\nabla u_e \cdot (x - x_0)| \geq \epsilon_0$ to apply Lemma 4.2 below. However, it is very hard to find how a suitable condition should be imposed to (M_i, M_e) to guarantee the existence of ϵ_0.

Remark 1.3. Similar to [3], the condition $\partial\omega \supset \partial\Omega$ is used to apply Carleman estimate to solutions with compact supports.

Remark 1.4. Condition (A3') is a regularity requirement on u_i, u_e and w for our stability results. In fact, we could deduce such regularity as (A3') from the method proposed by Colli-Franzone and Savaré in [9], in which the global existence in time and uniqueness for the solution of the bidomain model is proved. Since our paper focuses on the stability for our inverse conductivities problem, we do not stick to the exact condition to yield the regularity condition (A3'). For this reason, without loss of generality we assume that a_{ij} are constants and the functions f, g, h, v_0, w_0 are sufficiently smooth. By a simple calculation, we obtain for $j = 1, 2, 3, 4$ that

$$\begin{cases} c_m \partial_t^{j+1} v - \nabla \cdot (M_i(x) \nabla \partial_t^j u_i) + a_{11} \partial_t^j v + a_{12} \partial_t^j w = \partial_t^j f, & (x,t) \in Q_T, \\ c_m \partial_t^{j+1} v + \nabla \cdot (M_e(x) \nabla \partial_t^j u_e) + a_{21} \partial_t^j v + a_{22} \partial_t^j w = \partial_t^j g, & (x,t) \in Q_T, \\ \partial_t^{j+1} w + a_{31} \partial_t^j v + a_{32} \partial_t^j w = \partial_t^j h, & (x,t) \in Q_T, \\ \partial_t^j u_i(x,t) = \partial_t^j u_e(x,t) = 0, & (x,t) \in \Sigma_T, \\ \partial_t^j v(x,0) = v_j(x), \ \partial_t^j w(x,0) = w_j(x), & (x,t) \in \Omega, \end{cases} \tag{1.12}$$

with

$$\begin{aligned} & v_j(x) \\ & \quad = \frac{1}{c_m} \Big[\nabla \cdot (M_i \nabla v_{j-1}(x)) + \nabla \cdot (M_i \nabla u_{e,j-1}(x)) - a_{11} v_{j-1}(x) \\ & \qquad - a_{12} w_{j-1}(x) + \partial_t^{j-1} f(x,0) \Big], \\ & w_j(x) = -a_{31} v_{j-1}(x) - a_{32} w_{j-1}(x) + \partial_t^{j-1} h(x,0), \end{aligned}$$

where $u_{e,j-1}$ is the solution of the following elliptic problem

$$\begin{cases} -\nabla \cdot ((M_i + M_e)(x) \nabla u_{e,j-1}) = \nabla \cdot (M_i(x) \nabla v_{j-1}) \\ \quad -(a_{11} - a_{21}) v_{j-1} - (a_{12} - a_{22}) w_{j-1} + \partial_t^{j-1} f(x,0) - \partial_t^{j-1} g(x,0), & x \in \Omega, \\ u_{e,j-1}(x) = 0, & x \in \partial\Omega. \end{cases} \tag{1.13}$$

From Theorem 2 and Remark 3.2 in [9], we deduce that $\partial_t^4 u_i, \partial_t^4 u_e \in L^2(0,T; H^2(\Omega))$, i.e. $u_i, u_e \in H^4(0,T; H^2(\Omega))$ for sufficiently smooth f, g, h and v_0, w_0. Then by the standard theory for the ordinary differential equation, we obtain $\partial_t^3 w \in C([0,T]; H^2(\Omega))$ because of

$$\begin{cases} \partial_t \left(\partial_t^3 w \right) + a_{32} \partial_t^3 w = \partial_t^3 h - a_{31} \partial_t^3 v \in L^2(0,T; H^2(\Omega)), \\ \partial_t^3 w(x,0) = w_3(x) \in H^2(\Omega). \end{cases} \tag{1.14}$$

Obviously,

$$\begin{cases} -\nabla \cdot (M_i(x) \nabla \partial_t^3 u_i) = \partial_t^3 f - c_m \partial_t^4 v - a_{11} \partial_t^3 v - a_{12} \partial_t^3 w \in L^2(0,T; H^2(\Omega)), \\ \partial_t^3 u_i(x,t) = 0, \qquad (x,t) \in \Sigma_T. \end{cases} \tag{1.15}$$

From the regularity of the elliptic equation, it follows that $\partial_t^3 u_i \in L^2(0,T; H^4(\Omega))$, i.e. $u_i \in H^3(0,T; H^4(\Omega)) \hookrightarrow C^2([0,T]; W^{2,\infty}(\Omega)) \cap C([0,T]; H^3(\Omega))$. Similarly, the same regularity also holds for u_e. The regularity of w is easy to obtain by the equation of w and $v \in H^3(0,T; H^4(\Omega))$.

Now we state our second main result in this paper.

Theorem 1.3. *Let* $(M_i, M_e), (\tilde{M}_i, \tilde{M}_e) \in \mathcal{W}$ *and (A1')-(A3') be held, and let* (u_i, v_i, w) *and* $(\tilde{u}_i, \tilde{u}_e, \tilde{w})$ *be two solutions of* (1.9) *corresponding to* (M_i, M_e) *and* $(\tilde{M}_i, \tilde{M}_e)$. *Then there exist positive constants* C *and* $\mu \in (0,1)$ *such that*

$$\|M_i - \tilde{M}_i\|_{H^1(\Omega)} + \|M_e - \tilde{M}_e\|_{H^1(\Omega)} \le CK^\mu L^{(1-\mu)} \tag{1.16}$$

where

$$\begin{aligned} K =& ||u_i - \tilde{u}_i||_{H^2(0,T;L^2(\Omega))} + ||u_e - \tilde{u}_e||_{H^2(0,T;L^2(\Omega))} + ||w - \tilde{w}_e||_{H^2(0,T;L^2(\Omega))}, \\ L =& \|u_i - \tilde{u}_i\|_{H^2(0,T;H^1(\omega))} + \|u_e - \tilde{u}_e\|_{H^2(0,T;H^1(\omega))} \\ &+ \|(u_i - \tilde{u}_i)(\cdot, t_0)\|_{H^3(\Omega)} + \|(u_e - \tilde{u}_e)(\cdot, t_0)\|_{H^3(\Omega)} + \|(w - \tilde{w}_e)(\cdot, t_0)\|_{H^1(\Omega)}. \end{aligned}$$

Remark 1.5. Similarly to [1], we can expect the Lipschitz stability in place of (1.16). For it, we have to estimate K by $M_i - \tilde{M}_i$ and $M_e - \tilde{M}_e$.

The following uniqueness is a direct result from Theorem 1.3.

Corollary 1.4. *Under the same assumptions as in Theorem* 1.3, *if*

$$\begin{aligned} &u_i(x,t) = \tilde{u}_i(x,t),\ u_e(x,t) = \tilde{u}_e(x,t), \quad (x,t) \in \omega \times (0,T), \\ &u_i(x,t_0) = \tilde{u}_i(x,t_0),\ u_e(x,t_0) = \tilde{u}_e(x,t_0),\ w(x,t_0) = \tilde{w}(x,t_0), \quad x \in \Omega, \end{aligned}$$

then $M_i = \tilde{M}_i,\ M_e = \tilde{M}_e$ *a.e. in* Ω.

The remainder of the paper is organized as follows. In the next section, we prove a Carleman estimate for the linearized bidomain model. In Sect. 3, we give the proof of the stability result of Cauchy problem, i.e. Theorem 1.1. In last section, we prove Hölder stability for our inverse conductivities problem, i.e. Theorem 1.3.

2. Carleman estimate

This section is devoted to prove a Carleman estimate for the linearized bidomain model. In order to formulate our Carleman estimate, we introduce a function ψ

$$\psi(x,t) = d(x) - \beta(t-t_0)^2, \quad \varphi(x,t) = e^{\lambda\psi(x,t)}, \quad (x,t) \in Q_T, \tag{2.1}$$

with a parameter $\beta > 0$ and a large parameter $\lambda > 0$, where $d \in C^2(\overline{\Omega})$ satisfies $|\nabla d| \neq 0$ on $\overline{\Omega}$.

Now we state the main result in this section.

Theorem 2.1. *Let* $D \subset \overline{Q}_T$ *with smooth boundary,* $F, G \in L^2(Q_T)$ *and (A1), (A2) be held. Then there exist positive constants* $\lambda_0 = \lambda_0(\Omega, T, \beta)$, $s_0 = s_0(\Omega, T, \beta, \lambda_0)$ *and* $C = C(\Omega, T, \beta, \lambda)$ *such that*

$$\begin{aligned} &\int_D \left[|\partial_t v|^2 + s(|\nabla u_i|^2 + |\nabla u_e|^2) + s^3(|u_i|^2 + |u_e|^2)\right] e^{2s\varphi} \mathrm{d}x\mathrm{d}t \\ &\quad \le C\int_D s\left(|F|^2 + |G|^2\right) e^{2s\varphi}\mathrm{d}x\mathrm{d}t + Cse^{Cs}\left(\|u_i\|^2_{H^1(\partial D)} + \|u_e\|^2_{H^1(\partial D)}\right) \end{aligned} \tag{2.2}$$

for all $\lambda > \lambda_0$ *and* $s \geq s_0$*, provided that* $u_i, u_e \in L^2(0,T;H^2(\Omega))$*,* $v \in H^1(0,T;L^2(\Omega))$ *satisfies*

$$\begin{cases} c_m \partial_t v - \nabla \cdot (M_i(x)\nabla u_i) = F, \ (x,t) \in Q_T, \\ c_m \partial_t v + \nabla \cdot (M_e(x)\nabla u_e) = G, \ (x,t) \in Q_T. \end{cases} \tag{2.3}$$

Proof. By using $u_i = v + u_e$, we have

$$\begin{cases} c_m \partial_t v - \nabla \cdot (M_i(x)\nabla v) - \nabla \cdot (M_i(x)\nabla u_e) = F, & (x,t) \in Q_T, \\ c_m \partial_t v + \nabla \cdot (M_e(x)\nabla u_e) = G, & (x,t) \in Q_T. \end{cases} \tag{2.4}$$

Further, by the second equation in (2.4), we can obtain

$$-\nabla \cdot (M_i(x)\nabla u_e) = \frac{c_m M_i(x)\partial_t v - M_i(x) G}{M_e(x)} + \mathbf{A}(x) \cdot \nabla u_e \tag{2.5}$$

with

$$\mathbf{A}(x) := \frac{M_i(x)\nabla M_e(x) - M_e(x)\nabla M_i(x)}{M_e(x)}.$$

Substituting (2.5) into (2.4) yields

$$\begin{cases} c_m \left(1 + \frac{M_i(x)}{M_e(x)}\right) \partial_t v - \nabla \cdot (M_i(x)\nabla v) = F + \frac{M_i(x)}{M_e(x)} G - \mathbf{A}(x) \cdot \nabla u_e, & (x,t) \in Q_T, \\ c_m \partial_t v + \nabla \cdot (M_e(x)\nabla u_e) = G, & (x,t) \in Q_T. \end{cases} \tag{2.6}$$

Further, letting $\hat{u}_e = \varphi^{-\frac{1}{2}} u_e$ and then we have

$$\begin{cases} c_m \left(1 + \frac{M_i(x)}{M_e(x)}\right) \partial_t v - \nabla \cdot (M_i(x)\nabla v) = F + \frac{M_i(x)}{M_e(x)} G - \mathbf{A}(x) \cdot \nabla u_e, & (x,t) \in Q_T, \\ -\nabla \cdot (M_e(x)\nabla \hat{u}_e) = \varphi^{-\frac{1}{2}} \left(c_m \partial_t v - G\right) + H(\hat{u}_e, \nabla \hat{u}_e), & (x,t) \in Q_T, \end{cases} \tag{2.7}$$

where

$$H(\hat{u}_e, \nabla \hat{u}_e) = \lambda M_e(x) \nabla \psi \cdot \nabla \hat{u}_e + \left[\frac{1}{2}\lambda \nabla \psi \cdot \nabla M_e + \left(\frac{1}{4}\lambda^2 |\nabla \psi|^2 + \frac{1}{2}\lambda \Delta \psi\right) M_e\right] \hat{u}_e.$$

Applying the Carleman estimate for the parabolic equation (Theorem 3.2, [28]) to the equation of v in (2.7), we obtain that there exist positive constants λ_1, $s_1(\lambda)$ and C such that

$$\begin{aligned} &\int_D \left[s^{-1}\varphi^{-1}|\partial_t v|^2 + s\lambda^2\varphi|\nabla v|^2 + s^3\lambda^4\varphi^3|v|^2\right] e^{2s\varphi} \mathrm{d}x\mathrm{d}t \\ &\quad \leq C \int_D \left(|F|^2 + |G|^2 + |\nabla u_e|^2\right) e^{2s\varphi} \mathrm{d}x\mathrm{d}t + C e^{C(\lambda)s} \int_{\partial D} \left(|\nabla_{x,t} v|^2 + |v|^2\right) \mathrm{d}x\mathrm{d}t \end{aligned} \tag{2.8}$$

for all $\lambda \geq \lambda_1$, $s \geq s_1(\lambda)$, which yields

$$\begin{aligned} &\int_D \left(\varphi^{-1}|\partial_t v|^2 + s^2\lambda^2\varphi|\nabla v|^2 + s^4\lambda^4\varphi^3|v|^2\right) e^{2s\varphi} \mathrm{d}x\mathrm{d}t \\ &\quad \leq C \int_D s \left(|F|^2 + |G|^2 + |\nabla u_e|^2\right) \mathrm{d}x\mathrm{d}t + C s e^{C(\lambda)s} \int_{\partial D} \left(|\nabla_{x,t} v|^2 + |v|^2\right) \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{2.9}$$

Note that Theorem 3.2 in [28] holds also for elliptic operator, since all terms on the left-hand side of Carleman estimate are derived from the decomposition of elliptic operators $e^{s\varphi}\nabla\cdot(M_e(x)\nabla\hat{u}_e)$. Then similar to (2.8), we obtain the following Carleman estimate for $\hat{u}_e$:

$$\begin{aligned}&\int_D (s\lambda^2\varphi|\nabla\hat{u}_e|^2+s^3\lambda^4\varphi^3|\hat{u}_e|^2)e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\ &\quad\le C\int_D\Big[\varphi^{-1}\left(|G|^2+|\partial_t v|^2\right)+\left(\lambda^4|\hat{u}_e|^2+\lambda^2|\nabla\hat{u}_e|^2\right)\Big]e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\ &\quad+Ce^{C(\lambda)s}\int_{\partial D}\left(|\nabla_{x,t}\hat{u}_e|^2+|\hat{u}_e|^2\right)\mathrm{d}x\mathrm{d}t. \end{aligned}\tag{2.10}$$

Obviously,

$$C\varphi^{-\frac12}|\nabla u_e|-C\lambda\varphi^{-\frac12}|u_e|\le|\nabla\hat{u}_e|\le C\varphi^{-\frac12}|\nabla u_e|+C\lambda\varphi^{-\frac12}|u_e|.\tag{2.11}$$

Noticing that φ has positive lower and upper bound depending λ and using (2.10) and (2.11), we have

$$\begin{aligned}&\int_D\Big[s\lambda^2|\nabla u_e|^2+\left(s^3\lambda^4\varphi^2-s\lambda^4\right)|u_e|^2\Big]e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\ &\quad\le C\int_D\varphi^{-1}\left(|G|^2+|\partial_t v|^2\right)e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\ &\qquad+C(\lambda)e^{C(\lambda)s}\int_{\partial D}\left(|\nabla_{x,t}u_e|^2+|u_e|^2\right)\mathrm{d}x\mathrm{d}t,\end{aligned}\tag{2.12}$$

if we choose s such that $s\ge 2C\varphi^{-1}$. Multiplying (2.9) by $(C+1)$ and adding up (2.12) to absorb the term of v_t on the right-hand side of (2.12), and choosing λ such that $\lambda\ge C(C+1)$ to absorb the term ∇u_e on the right-hand side of (2.9), we can obtain

$$\begin{aligned}&\int_D\left(\varphi^{-1}|\partial_t v|^2+s^2\lambda^2\varphi|\nabla v|^2+s^4\lambda^4\varphi^3|v|^2\right)e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\ &\quad+\int_D\left(s\lambda^2|\nabla u_e|^2+s^3\lambda^4\varphi^2|u_e|^2\right)e^{2s\varphi}\mathrm{d}x\mathrm{d}t\\ &\le C(\lambda)\int_D s\left(|F|^2+|G|^2\right)e^{2s\varphi}\mathrm{d}x\mathrm{d}t+C(\lambda)se^{C(\lambda)s}\left(\|v\|^2_{H^1(\partial D)}+\|u_e\|^2_{H^1(\partial D)}\right)\end{aligned}$$

for sufficiently large s. Finally, noting that $v=u_i-u_e$, we can obtain the desired estimate (2.2). This completes the proof of Theorem 2.1. □

3. Proof of the Theorem 1.1

Now we prove the conditional stability of our inverse problem, i.e. Theorem 1.1. The proof is based on the idea used in [24].

Proof of Theorem 1.1. We first choose a bounded Ω_1 with smooth boundary such that

$$\Omega\subsetneqq\Omega_1,\quad\overline{\Gamma}=\overline{\partial\Omega\cap\Omega_1},\quad\partial\Omega\backslash\Gamma\subset\partial\Omega_1\tag{3.1}$$

where Ω_1 is constructed by taking a union of Ω and a domain $\widetilde{\Omega}$ such that $\partial\widetilde{\Omega} \cap \overline{\Omega} = \Gamma$. Furthermore, we introduce $d \in C^2(\overline{\Omega}_1)$ such that

$$d(x) > 0 \quad x \in \Omega_1, \quad d|_{\partial\Omega_1} = 0, \quad |\nabla d(x)| > 0, \quad x \in \overline{\Omega}. \tag{3.2}$$

Then, since $\overline{\Omega}_0 \subset \Omega_1$, we can choose a sufficiently large $N > 1$ such that

$$\left\{ x \in \Omega_1; d(x) > \frac{4}{N}||d||_{C(\overline{\Omega}_1)} \right\} \cap \overline{\Omega} \supset \Omega_0 \tag{3.3}$$

Moreover for any given $0 < \varepsilon < 1$, we choose $\beta > 0$ such that

$$\beta\varepsilon^2 > ||d||_{C(\overline{\Omega}_1)} > \frac{1}{2}\beta\varepsilon^2 \tag{3.4}$$

For $0 < \varepsilon < 1$ and given N, there exist finite $t_j, j = 1, 2, \ldots, n_0$ such that $t_j \in [\varepsilon, T-\varepsilon]$ and $(\varepsilon, T-\varepsilon) \subset \cup_{j=1}^{n_0}(t_j - \frac{\varepsilon}{\sqrt{2N}}, t_j + \frac{\varepsilon}{\sqrt{2N}})$. Further we set $\varphi_j(x,t) = e^{\lambda\psi_j(x,t)}$, $\psi_j(x,t) = d(x) - \beta(t-t_j)^2$ and $D_j = \{(x,t); x \in \overline{\Omega}, \varphi_j(x,t) > \mu_1\}$ for fixed j, where $\mu_k = \exp(\lambda(\frac{k}{N}||d||_{C(\overline{\Omega}_1)} - \frac{\beta\varepsilon^2}{2N})), k = 1, 2, 3, 4$. Similar to [24], we can verify that

$$\Omega_0 \times \left(t_j - \frac{\varepsilon}{\sqrt{2N}}, t_j + \frac{\varepsilon}{\sqrt{2N}} \right) \subset D_j \subset \overline{\Omega} \times (t_j - \varepsilon, t_j + \varepsilon) \tag{3.5}$$

and $\partial D_j \subset \Gamma_T \cup \Sigma_1$ with $\Sigma_1 = \{(x,t); x \in \Omega, \varphi_j(x,t) = \mu_1\}$.

Let $\chi_1 \in C^\infty(\mathbb{R}^{n+1})$ such that $0 \le \chi_1 \le 1$ and

$$\chi_1(x,t) = \begin{cases} 1, & \varphi_j(x,t) > \mu_3, \\ 0, & \varphi_j(x,t) < \mu_2. \end{cases}$$

We set $\hat{u}_i = \chi_1 u_i, \hat{u}_e = \chi_1 u_e, \hat{f} = \chi_1 f, \hat{g} = \chi_1 g, \hat{v} = \chi_1 v$, and see that $(\hat{u}_i, \hat{u}_e)$ satisfies

$$\begin{cases} c_m\partial_t\hat{v} - \nabla\cdot(M_i(x)\nabla\hat{u}_i) \\ \qquad = \hat{f} + c_m\partial_t\chi_1 v - a\hat{v} - \nabla\cdot(M_i(x)u_i\nabla\chi_1) - M_i(x)\nabla\chi_1\cdot\nabla u_i, \ (x,t) \in D_j, \\ c_m\partial_t\hat{v} + \nabla\cdot(M_e(x)\nabla\hat{u}_e) \\ \qquad = \hat{g} + c_m\partial_t\chi_1 v - b\hat{v} + \nabla\cdot(M_e(x)u_e\nabla\chi_1) + M_e(x)\nabla\chi_1\cdot\nabla u_e, \ (x,t) \in D_j. \end{cases}$$

Additionally, by $\mu_1 < \mu_2$ and the definition of χ_1, we see that $\hat{u}_i = \hat{u}_e = |\nabla\hat{u}_i| = |\nabla\hat{u}_e| = 0$ on Σ_1. Hence by Theorem 2.1 we find that

$$\begin{aligned} &\int_{D_j} \left[s\left(|\nabla\hat{u}_i|^2 + |\nabla\hat{u}_e|^2\right) + s^3\left(|\hat{u}_i|^2 + |\hat{u}_e|^2\right)\right] e^{2s\varphi_j} \mathrm{d}x\mathrm{d}t \\ &\quad \le C\int_{D_j} s\left(|\hat{f}|^2 + |\hat{g}|^2 + |\hat{v}|^2\right) e^{2s\varphi_j}\mathrm{d}x\mathrm{d}t + Cse^{Cs}\left(||u_i||^2_{H^1(\Gamma_T)} + ||u_e||^2_{H^1(\Gamma_T)}\right) \\ &\qquad + C\int_{D_j} s\left(|\partial_t\chi_1|^2 + |\nabla\chi_1|^2 + |\Delta\chi_1|^2\right) \times (|\nabla u_i|^2 + |\nabla u_e|^2 + |u_i|^2 + |u_e|^2)e^{2s\varphi_j}\mathrm{d}x\mathrm{d}t \end{aligned} \tag{3.6}$$

for all $s \geq s_0$. Since $|\partial_t \chi_1|^2 + |\nabla \chi_1|^2 + |\Delta \chi_1|^2 \neq 0$ on $\{(x,t) \in Q_T \mid \mu_2 \leq \varphi_j(x,t) \leq \mu_3\}$ and $|\hat{v}| \leq C(|\hat{u}_i| + |\hat{u}_e|)$, we further have

$$\begin{aligned}\int_{D_j} &\left[s\left(|\nabla \hat{u}_i|^2 + |\nabla \hat{u}_e|^2\right) + s^3\left(|\hat{u}_i|^2 + |\hat{u}_e|^2\right)\right] e^{2s\varphi_j} \mathrm{d}x\mathrm{d}t \\ &\leq Cse^{2\mu_3 s} I^2 + Cse^{Cs} J^2. \end{aligned} \tag{3.7}$$

On the other hand, by $\varphi_j(x,t) \geq \mu_4$ for $(x,t) \in \Omega_0 \times (t_j - \frac{\varepsilon}{\sqrt{2N}}, t_j + \frac{\varepsilon}{\sqrt{2N}}) \subset D_j$ and $\hat{u}_i = u_i, \hat{u}_e = u_e$ when $\varphi_j(x,t) \geq \mu_4$, we find that

$$\begin{aligned}&\int_{D_j} \left[s\left(|\nabla \hat{u}_i|^2 + |\nabla \hat{u}_e|^2\right) + s^3\left(|\hat{u}_i|^2 + |\hat{u}_e|^2\right)\right] e^{2s\varphi_j} \mathrm{d}x\mathrm{d}t \\ &\quad\geq e^{2s\mu_4} \int_{t_j - \frac{\varepsilon}{\sqrt{2N}}}^{t_j + \frac{\varepsilon}{\sqrt{2N}}} \int_{\Omega_0} \left[s\left(|\nabla u_i|^2 + |\nabla u_e|^2\right) + s^3\left(|u_i|^2 + |u_e|^2\right)\right] \mathrm{d}x\mathrm{d}t \end{aligned} \tag{3.8}$$

Hence, by (3.7) and (3.8) we have

$$\begin{aligned}&s\left(||u_i||^2_{L^2\left(t_j - \frac{\varepsilon}{\sqrt{2N}}, t_j + \frac{\varepsilon}{\sqrt{2N}}; H^1(\Omega_0)\right)} + ||u_e||^2_{L^2\left(t_j - \frac{\varepsilon}{\sqrt{2N}}, t_j + \frac{\varepsilon}{\sqrt{2N}}; H^1(\Omega_0)\right)}\right) \\ &\quad\leq Cse^{2s(\mu_3 - \mu_4)} I^2 + Cse^{Cs} J^2 \end{aligned} \tag{3.9}$$

for all $s \geq s_0$ and $j = 1, 2, \ldots, n_0$. Summing up over j, we obtain

$$\begin{aligned}&||u_i||^2_{L^2(\varepsilon, T-\varepsilon; H^1(\Omega_0))} + ||u_e||^2_{L^2(\varepsilon, T-\varepsilon; H^1(\Omega_0))} \\ &\quad\leq Ce^{2s(\mu_3 - \mu_4)} I^2 + Ce^{Cs} J^2 \end{aligned} \tag{3.10}$$

for all $s \geq s_0$. Setting $s := s + s_0$ and replacing C by Ce^{Cs_0}, we obtain (3.10) for all $s \geq 0$. Finally, minimizing the right-hand side of (3.10) with respect to s, we obtain

$$||u_i||^2_{L^2(\varepsilon, T-\varepsilon; H^1(\Omega_0))} + ||u_e||^2_{L^2(\varepsilon, T-\varepsilon; H^1(\Omega_0))} \leq 2CI^{2(1-\kappa)} J^{2\kappa} \tag{3.11}$$

with $\kappa = \frac{2(\mu_4 - \mu_3)}{C + 2(\mu_4 - \mu_3)}$. The proof of Theorem 1.1 is completed. □

4. Proof of the Theorem 1.3

This section devotes to proving Hölder stability for our inverse conductivities problem, i.e. Theorem 1.3, by means of Carleman estimate (2.2). In this section, we choose

$$d(x) = (x - x_0)^2 + M, \quad x \in \Omega$$

with a fixed point $x_0 \in \mathbb{R}^N \backslash \overline{\Omega}$ and a constant M such that

$$\psi(x,t) = (x - x_0)^2 - \beta(t - t_0)^2 + M > 0, \quad (x,t) \in Q_T.$$

Obviously, we have $|\nabla d| \neq 0$ on $\overline{\Omega}$. Fix $\delta > 0$ and $\beta > 0$ such that

$$\beta \min\left\{t_0^2, (T - t_0)^2\right\} > \max_{x \in \overline{\Omega}} (x - x_0)^2 + 2\delta. \tag{4.1}$$

Then the function ψ satisfies the following properties

$$\max_{x\in\overline{\Omega}}\psi(x,0)\le M-2\delta,\quad \max_{x\in\overline{\Omega}}\psi(x,T)\le M-2\delta \tag{4.2}$$

and

$$\min_{x\in\overline{\Omega}}\psi(x,t_0)\ge M. \tag{4.3}$$

By (4.2), we can choose ε sufficiently small to satisfy

$$\max_{x\in\overline{\Omega}}\psi(x,t)\le M-\delta,\quad t\in[0,2\varepsilon]\cup[T-2\varepsilon,T]. \tag{4.4}$$

In the following we fix $\lambda=\lambda_0$ and use C to denote a generic positive constant depending on $x_0,\Omega,T,\beta,\epsilon_0$ and λ_0, but not independent of s.

Let (u_i,u_e,w) and $(\tilde u_i,\tilde u_e,\tilde w)$ be two solutions of (1.9) corresponding to (M_i,M_e) and $(\tilde M_i,\tilde M_e)$ respectively, and let $(\overline{u}_i,\overline{u}_e,\overline{w})=(u_i-\tilde u_i,u_e-\tilde u_e,$ $w-\tilde w)$, $\overline{v}=\overline{u}_i-\overline{u}_e$, $(\overline{M}_i,\overline{M}_e)=\left(M_i-\tilde M_i,M_e-\tilde M_e\right)$. Then for $j=1,2,3$ we have

$$\begin{cases} c_m\partial_t^j\overline{v}-\nabla\cdot(M_i(x)\nabla\partial_t^{j-1}\overline{u}_i)+a_{11}\partial_t^{j-1}\overline{v}+a_{12}\partial_t^{j-1}\overline{w}=F_j, & (x,t)\in Q_T,\\ c_m\partial_t^j\overline{v}+\nabla\cdot(M_e(x)\nabla\partial_t^{j-1}\overline{u}_e)+a_{21}\partial_t^{j-1}\overline{v}+a_{22}\partial_t^{j-1}\overline{w}=G_j, & (x,t)\in Q_T,\\ \partial_t^j\overline{w}+a_{31}\partial_t^{j-1}\overline{v}+a_{32}\partial_t^{j-1}\overline{w}=H_j, & (x,t)\in Q_T,\\ \partial_t^{j-1}\overline{u}_i(x,t)=\partial_t^{j-1}\overline{u}_e(x,t)=0, & (x,t)\in\Sigma_T, \end{cases} \tag{4.5}$$

where

$$F_j=\begin{cases}\nabla\cdot\left(\overline{M}_i(x)\nabla\partial_t^{j-1}\tilde u_i\right), & j=1,\\ \nabla\cdot\left(\overline{M}_i(x)\nabla\partial_t^{j-1}\tilde u_i\right)-\partial_t^{j-1}a_{11}\overline{v}-\partial_t^{j-1}a_{12}\overline{w}, & j=2,\\ \nabla\cdot\left(\overline{M}_i(x)\nabla\partial_t^{j-1}\tilde u_i\right)-\sum_{k=1}^2 k\partial_t^{j-k}a_{11}\partial_t^{k-1}\overline{v}-\sum_{k=1}^2 k\partial_t^{j-k}a_{12}\partial_t^{k-1}\overline{w}, & j=3,\end{cases}$$

$$G_j=\begin{cases}-\nabla\cdot\left(\overline{M}_e(x)\nabla\partial_t^{j-1}\tilde u_e\right), & j=1,\\ -\nabla\cdot\left(\overline{M}_e(x)\nabla\partial_t^{j-1}\tilde u_e\right)-\partial_t^{j-1}a_{21}\overline{v}-\partial_t^{j-1}a_{22}\overline{w}, & j=2,\\ -\nabla\cdot\left(\overline{M}_e(x)\nabla\partial_t^{j-1}\tilde u_e\right)-\sum_{k=1}^2 k\partial_t^{j-k}a_{21}\partial_t^{k-1}\overline{v}-\sum_{k=1}^2 k\partial_t^{j-k}a_{22}\partial_t^{k-1}\overline{w}, & j=3,\end{cases}$$

$$H_j=\begin{cases}0, & j=1,\\ -\partial_t^{j-1}a_{31}\overline{v}-\partial_t^{j-1}a_{32}\overline{w}, & j=2,\\ -\sum_{k=1}^2 k\partial_t^{j-k}a_{31}\partial_t^{k-1}\overline{v}-\sum_{k=1}^2 k\partial_t^{j-k}a_{32}\partial_t^{k-1}\overline{w}, & j=3.\end{cases}$$

In order to apply Carleman estimate (2.2) to (4.5), we need to introduce a cut function $\chi_2(x,t)=\xi(x)\eta(t)\in C_0^\infty(Q_T)$ such that $0\le\xi\le 1$, $0\le\eta\le 1$ and

$$\begin{cases}\xi(x)=1, & x\in\Omega\setminus\omega,\\ \xi(x)=0, & x\in\mathbb{R}^N\setminus\Omega,\end{cases}\qquad \begin{cases}\eta(t)=1, & t\in[2\varepsilon,T-2\varepsilon],\\ \eta(t)=0, & t\in[0,\varepsilon]\cup[T-\varepsilon,T].\end{cases} \tag{4.6}$$

Letting $(\hat{u}_i, \hat{u}_e, \hat{w}) = (\chi_2 \partial_t^{j-1}\overline{u}_i, \chi_2 \partial_t^{j-1}\overline{u}_e, \chi_2 \partial_t^{j-1}\overline{w})$, $\hat{v} = \hat{u}_i - \hat{u}_e$ and $\hat{F} = \chi_2 F_j$, $\hat{G} = \chi_2 G_j$, $\hat{H} = \chi_2 H_j$, by a direct calculation we obtain

$$\begin{cases} c_m \partial_t \hat{v} - \nabla\cdot(M_i(x)\nabla\hat{u}_i) = \hat{F} + c_m\partial_t\chi_2\left(\partial_t^{j-1}\overline{u}_i - \partial_t^{j-1}\overline{u}_e\right) \\ \quad -a_{11}\hat{v} - a_{12}\hat{w} - \nabla\cdot(M_i(x)\partial_t^{j-1}\overline{u}_i\nabla\chi_2) - M_i(x)\nabla\chi_2\cdot\nabla\partial_t^{j-1}\overline{u}_i, \ \ (x,t)\in Q_T, \\ c_m \partial_t \hat{v} + \nabla\cdot(M_e(x)\nabla\hat{u}_e) = \hat{G} + c_m\partial_t\chi_2\left(\partial_t^{j-1}\overline{u}_i - \partial_t^{j-1}\overline{u}_e\right) \\ \quad -a_{21}\hat{v} - a_{22}\hat{w} + \nabla\cdot(M_e(x)\partial_t^{j-1}\overline{u}_e\nabla\chi_2) + M_e(x)\nabla\chi_2\cdot\nabla\partial_t^{j-1}\overline{u}_e, \ (x,t)\in Q_T, \\ \partial_t\hat{w} + a_{31}\hat{v} + a_{32}\hat{w} = \hat{H} + \partial_t\chi_2\partial_t^{j-1}\overline{w}, \qquad (x,t)\in Q_T. \end{cases}$$

Then we have

Lemma 4.1. *Let (A1')-(A3') be held. Then there exist positive constants* s_1 *and* C *such that*

$$\begin{aligned} &\sum_{j=1}^{3}\int_{2\varepsilon}^{T-2\varepsilon}\int_\Omega \Big[s\left(|\partial_t^{j-1}\nabla\overline{u}_i|^2 + |\partial_t^{j-1}\nabla\overline{u}_e|^2\right) \\ &\quad + s^3\left(|\partial_t^{j-1}\overline{u}_i|^2 + |\partial_t^{j-1}\overline{u}_e|^2\right)\Big] e^{2s\varphi}\mathrm{d}x\mathrm{d}t \\ &\le C\int_{Q_T} s^2\left(|\overline{M}_i|^2 + |\nabla\overline{M}_i|^2 + |\overline{M}_e|^2 + |\nabla\overline{M}_e|^2\right)e^{2s\varphi}\mathrm{d}x\mathrm{d}t \\ &\quad + Cs\exp(2se^{\lambda_0(M-\delta)})K^2 + Cs^3e^{Cs}L^2 \end{aligned} \tag{4.7}$$

for $s \ge s_1$.

Proof. By applying Theorem 2.1 to $(\hat{u}_i, \hat{u}_e)$, we have

$$\begin{aligned} &\int_{Q_T}\left[s\left(|\nabla\hat{u}_i|^2 + |\nabla\hat{u}_e|^2\right) + s^3\left(|\hat{u}_i|^2 + |\hat{u}_e|^2\right)\right]e^{2s\varphi}\mathrm{d}x\mathrm{d}t \\ &\le C\int_{Q_T} s\left(|\hat{F}|^2 + |\hat{G}|^2 + |\hat{w}|^2\right)e^{2s\varphi}\mathrm{d}x\mathrm{d}t \\ &+ C\int_{Q_T} s|\partial_t\chi_2|^2\left(\left|\partial_t^{j-1}\overline{u}_i\right|^2 + \left|\partial_t^{j-1}\overline{u}_e\right|^2\right)e^{2s\varphi}\mathrm{d}x\mathrm{d}t \\ &+ C\int_{Q_T} s\left(|\nabla\chi_2|^2 + |\Delta\chi_2|^2\right) \\ &\times\left(|\nabla\partial_t^{j-1}\overline{u}_i|^2 + |\nabla\partial_t^{j-1}\overline{u}_e|^2 + |\partial_t^{j-1}\overline{u}_i|^2 + |\partial_t^{j-1}\overline{u}_e|^2\right)e^{2s\varphi}\mathrm{d}x\mathrm{d}t \end{aligned} \tag{4.8}$$

for all $s \ge s_0$. On the other hand, we can write $\hat{w}$ as

$$\begin{aligned} \hat{w}(x,t) =& e^{-\int_{t_0}^t a_{32}(x,\tau)\mathrm{d}\tau}\Big[\hat{w}(x,t_0) \\ &+ \int_{t_0}^t e^{\int_{t_0}^\tau a_{32}(x,s)\mathrm{d}s}\left(\hat{H} + \partial_t\chi_2\partial_t^{j-1}\overline{w} - a_{31}\hat{v}\right)(x,\tau)\mathrm{d}\tau\Big], \end{aligned}$$

which leads to

$$\begin{aligned}
&\int_{Q_T} |\hat{w}|^2 e^{2s\varphi} \mathrm{d}x\mathrm{d}t \leq C \int_{Q_T} |\partial_t^{j-1} \overline{w}(x,t_0)|^2 e^{2s\varphi} \mathrm{d}x\mathrm{d}t \\
&\quad + C \int_{Q_T} \left| \int_{t_0}^{t} e^{\int_{t_0}^{\tau} a_{32}(x,s)\mathrm{d}s} \left(\hat{H} + \partial_t \chi_2 \partial_t^{j-1} \overline{w} - a_{31} \hat{v} \right)(x,\tau) \mathrm{d}\tau \right|^2 e^{2s\varphi} \mathrm{d}x\mathrm{d}t \\
&\leq C \int_{Q_T} |\partial_t^{j-1} \overline{w}(x,t_0)|^2 e^{2s\varphi} \mathrm{d}x\mathrm{d}t \\
&\quad + C \int_{Q_T} (t-t_0) \int_{t_0}^{t} \left| \left(\hat{H} + \partial_t \chi_2 \partial_t^{j-1} \overline{w} - a_{31} \hat{v} \right)(x,\tau) \right|^2 \mathrm{d}\tau e^{2s\varphi} \mathrm{d}x\mathrm{d}t \\
&\leq C \int_{Q_T} |\partial_t^{j-1} \overline{w}(x,t_0)|^2 e^{2s\varphi} \mathrm{d}x\mathrm{d}t \\
&\quad - C s^{-1} \int_{Q_T} \int_{t_0}^{t} \left| \left(\hat{H} + \partial_t \chi_2 \partial_t^{j-1} \overline{w} - a_{31} \hat{v} \right)(x,\tau) \right|^2 \mathrm{d}\tau \left(e^{2s\varphi} \right)_t \mathrm{d}x\mathrm{d}t \\
&\leq C \int_{Q_T} |\partial_t^{j-1} \overline{w}(x,t_0)|^2 e^{2s\varphi} \mathrm{d}x\mathrm{d}t \\
&\quad + C s^{-1} \int_{Q_T} \left(|\hat{H}|^2 + |\partial_t \chi_2 \partial_t^{j-1} \overline{w}|^2 + |\hat{v}|^2 \right) e^{2s\varphi} \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Namely,

$$\begin{aligned}
\int_{Q_T} s^2 |\hat{w}|^2 e^{2s\varphi} \mathrm{d}x\mathrm{d}t &\leq C \int_{Q_T} s^2 |\partial_t^{j-1} \overline{w}(x,t_0)|^2 e^{2s\varphi} \mathrm{d}x\mathrm{d}t \\
&\quad + C \int_{Q_T} s \left(|\hat{H}|^2 + |\partial_t \chi_2 \partial_t^{j-1} \overline{w}|^2 + |\hat{v}|^2 \right) e^{2s\varphi} \mathrm{d}x\mathrm{d}t. \qquad (4.9)
\end{aligned}$$

Therefore, by adding up (4.8) and (4.9) we find that for $j = 1, 2, 3$,

$$\begin{aligned}
&\int_{Q_T} s \left(\left| \nabla \left(\chi_2 \partial_t^{j-1} \overline{u}_i \right) \right|^2 + \left| \nabla \left(\chi_2 \partial_t^{j-1} \overline{u}_e \right) \right|^2 \right) e^{2s\varphi} \mathrm{d}x\mathrm{d}t \\
&\quad + \int_{Q_T} \left[s^3 \left(|\chi_2 \partial_t^{j-1} \overline{u}_i|^2 + |\chi_2 \partial_t^{j-1} \overline{u}_e|^2 \right) + s^2 |\chi_2 \partial_t^{j-1} \overline{w}|^2 \right] e^{2s\varphi} \mathrm{d}x\mathrm{d}t \\
&\quad \leq C \int_{Q_T} s |\chi_2|^2 \left(|F_j|^2 + |G_j|^2 + |H_j|^2 \right) e^{2s\varphi} \mathrm{d}x\mathrm{d}t \\
&\quad + C \int_{Q_T} s^2 |\partial_t^{j-1} \overline{w}(x,t_0)|^2 e^{2s\varphi} \mathrm{d}x\mathrm{d}t \\
&\quad + C \int_{Q_T} s \left(|\nabla \chi_2|^2 + |\Delta \chi_2|^2 \right) (|\nabla \partial_t^{j-1} \overline{u}_i|^2 + |\nabla \partial_t^{j-1} \overline{u}_e|^2 + |\partial_t^{j-1} \overline{u}_i|^2 \\
&\quad + |\partial_t^{j-1} \overline{u}_e|^2) e^{2s\varphi} \mathrm{d}x\mathrm{d}t \\
&\quad + C \int_{Q_T} s |\partial_t \chi_2|^2 \left(\left| \partial_t^{j-1} \overline{u}_i \right|^2 + \left| \partial_t^{j-1} \overline{u}_e \right|^2 + \left| \partial_t^{j-1} \overline{w} \right|^2 \right) e^{2s\varphi} \mathrm{d}x\mathrm{d}t.
\end{aligned}$$
(4.10)

Obviously,

$$\begin{aligned}
&\sum_{j=1}^{3} |\chi_2| \left(|F_j| + |G_j| + |H_j|\right) \\
&\quad \le C \left(|\overline{M}_i| + |\nabla \overline{M}_i| + |\overline{M}_e| + |\nabla \overline{M}_e|\right) + C \sum_{j=1}^{2} |\chi_2| \left(|\partial_t^{j-1} \overline{v}| + |\partial_t^{j-1} \overline{w}|\right) \\
&\quad \le C \left(|\overline{M}_i| + |\nabla \overline{M}_i| + |\overline{M}_e| + |\nabla \overline{M}_e|\right) \\
&\qquad + C \sum_{j=1}^{2} \left(|\chi_2 \partial_t^{j-1} \overline{u}_i| + |\chi_2 \partial_t^{j-1} \overline{u}_e| + |\chi_2 \partial_t^{j-1} \overline{w}|\right). \qquad (4.11)
\end{aligned}$$

By (4.5), we obtain

$$\begin{aligned}
&\sum_{j=1}^{3} |\partial_t^{j-1} \overline{w}(\cdot, t_0)| \\
&\quad \le C \left(|\overline{w}(\cdot, t_0)| + |\overline{v}(\cdot, t_0)| + |\partial_t \overline{v}(\cdot, t_0)|\right) \\
&\quad \le C \left(|\overline{w}(\cdot, t_0)| + |\overline{u}_i(\cdot, t_0)| + |\overline{u}_e(\cdot, t_0)| + |\Delta \overline{u}_i(\cdot, t_0)| + |\overline{M}_i| + |\nabla \overline{M}_i|\right). \qquad (4.12)
\end{aligned}$$

Then summing up over j in (4.10), substituting (4.11) and (4.12) into (4.10) and choosing s sufficiently large to absorb the terms $\partial_t^{j-1} \overline{u}_i, \partial_t^{j-1} \overline{u}_e$ and $\partial_t^{j-1} \overline{w}$ on the right-hand side of (4.11), we find that

$$\begin{aligned}
&\sum_{j=1}^{3} \int_{2\varepsilon}^{T-2\varepsilon} \int_{\Omega \setminus \omega} \left[s \left(|\partial_t^{j-1} \nabla \overline{u}_i|^2 + |\partial_t^{j-1} \nabla \overline{u}_e|^2\right) + s^3 \left(|\partial_t^{j-1} \overline{u}_i|^2 + |\partial_t^{j-1} \overline{u}_e|^2\right)\right] e^{2s\varphi} \mathrm{d}x\mathrm{d}t \\
&\quad \le C \int_{Q_T} s^2 \left(|\overline{M}_i|^2 + |\nabla \overline{M}_i|^2 + |\overline{M}_e|^2 + |\nabla \overline{M}_e|^2\right) e^{2s\varphi} \mathrm{d}x\mathrm{d}t \\
&\qquad + Cs \exp(2se^{\lambda_0 (M-\delta)}) \left(||\overline{u}_i||^2_{H^2(0,T;L^2(\Omega))} + ||\overline{u}_e||^2_{H^2(0,T;L^2(\Omega))} + ||\overline{w}||^2_{H^2(0,T;L^2(\Omega))}\right) \\
&\qquad + Cs^2 e^{Cs} \left(\|\overline{u}_i(\cdot, t_0)\|^2_{H^2(\Omega)} + \|\overline{u}_e(\cdot, t_0)\|^2_{L^2(\Omega)} + \|\overline{w}(\cdot, t_0)\|^2_{L^2(\Omega)}\right) \\
&\qquad + Cse^{Cs} \left(\|\overline{u}_i\|^2_{H^2(0,T;H^1(\omega))} + \|\overline{u}_e\|^2_{H^2(0,T;H^1(\omega))}\right), \qquad (4.13)
\end{aligned}$$

where we have used $\mathrm{Supp}(\nabla \chi_2), \mathrm{Supp}(\Delta \chi_2) \subset \omega$, $\mathrm{Supp}(\partial_t \chi_2) \subset [\varepsilon, 2\varepsilon] \cup [T - 2\varepsilon, T - \varepsilon]$, (4.4) and (4.6). This yields the desired estimate (4.7). This completes the proof of Lemma 4.1. □

To prove our stability result, we also need a Carleman estimate for the following first-order partial differential equation:

$$\mathbf{B}(x) \cdot \nabla \vartheta(x) + B_0(x) \vartheta(x) = R(x), \quad x \in \Omega, \qquad (4.14)$$

where $\mathbf{B} \in \left(W^{1,\infty}(\Omega)\right)^3$ and $B_0 \in W^{1,\infty}(\Omega)$. The follow Lemma can be found in [15] or [2].

Lemma 4.2. *We assume*

$$|\mathbf{B}(x) \cdot \nabla d| > 0, \quad x \in \Omega.$$

Then there exists positive constants s_2 and C such that

$$s^2 \int_\Omega |\vartheta|^2 e^{s\varphi(x,t_0)} \mathrm{d}x \leq C \int_\Omega |R|^2 e^{s\varphi(x,t_0)} \mathrm{d}x \tag{4.15}$$

and

$$s^2 \int_\Omega |\nabla\vartheta|^2 e^{s\varphi(x,t_0)} \mathrm{d}x \leq C \int_\Omega \left(|R|^2 + |\nabla R|^2\right) e^{s\varphi(x,t_0)} \mathrm{d}x \tag{4.16}$$

for all $s \geq s_2$ and $\vartheta \in H^2(\Omega)$ satisfying $\vartheta(x) = 0,\ \nabla\vartheta(x) = 0, \quad x \in \partial\Omega$.

Now we prove Theorem 1.3.

Proof of Theorem 1.3. By (4.5), we have

$$\begin{aligned} &\nabla \overline{M}_i(x) \cdot \nabla \tilde{u}_i(x,t_0) + \overline{M}_i(x) \Delta \tilde{u}_i(x,t_0) \\ &\quad = c_m \partial_t \overline{v}(x,t_0) - \nabla \cdot (M_i(x) \nabla \overline{u}_i(x,t_0)) \\ &\qquad + a_{11}(x,t_0)\overline{v}(x,t_0) + a_{12}(x,t_0)\overline{w}(x,t_0) \end{aligned} \tag{4.17}$$

Applying Lemma 4.2 to (4.17), we find that

$$\begin{aligned} &\int_\Omega s^2 \left(|\overline{M}_i|^2 + |\nabla \overline{M}_i|^2\right) e^{2s\varphi(x,t_0)} \mathrm{d}x \\ &\quad \leq C \int_\Omega \left(|\partial_t \overline{v}(x,t_0)|^2 + |\partial_t \nabla \overline{v}(x,t_0)|^2\right) e^{2s\varphi(x,t_0)} \mathrm{d}x \\ &\qquad + Ce^{Cs} \left(\|\overline{u}_i(\cdot,t_0)\|^2_{H^3(\Omega)} + \|\overline{u}_e(\cdot,t_0)\|^2_{H^3(\Omega)} + \|\overline{w}(\cdot,t_0)\|^2_{H^1(\Omega)}\right) \end{aligned} \tag{4.18}$$

for all $s \geq s_2$. A similar estimate holds for $\overline{M}_e$. Therefore we obtain

$$\begin{aligned} &\int_\Omega s^2 \left(|\overline{M}_i|^2 + |\overline{M}_e|^2 + |\nabla \overline{M}_i|^2 + |\nabla \overline{M}_e|^2\right) e^{2s\varphi(x,t_0)} \mathrm{d}x \\ &\quad \leq C \int_\Omega \left(|\partial_t \overline{v}(x,t_0)|^2 + |\partial_t \nabla \overline{v}(x,t_0)|^2\right) e^{2s\varphi(x,t_0)} \mathrm{d}x + Ce^{Cs} L^2. \end{aligned} \tag{4.19}$$

Let $\tilde{\eta} \in C_0^\infty[0,T]$ such that $\tilde{\eta}(t_0) = 1$ and $\mathrm{Supp}(\tilde{\eta}) \subset [2\varepsilon, T-2\varepsilon]$. Then we have

$$\begin{aligned} &\int_\Omega \left(|\partial_t \overline{v}(x,t_0)|^2 + |\partial_t \nabla \overline{v}(x,t_0)|^2\right) e^{2s\varphi(x,t_0)} \mathrm{d}x \\ &\quad = \int_0^{t_0} \int_\Omega \left[\tilde{\eta}(t) \left(|\partial_t \overline{v}(x,t)|^2 + |\partial_t \nabla \overline{v}(x,t)|^2\right) e^{2s\varphi(x,t)}\right]_t \mathrm{d}x\mathrm{d}t \\ &\quad \leq \int_0^{t_0} \int_\Omega (\partial_t \tilde{\eta} + 2s\tilde{\eta}\partial_t \varphi) \left(|\partial_t \overline{v}|^2 + |\partial_t \nabla \overline{v}|^2\right) e^{2s\varphi} \mathrm{d}x\mathrm{d}t \\ &\qquad + \int_0^{t_0} \int_\Omega 2\tilde{\eta} \left(\partial_t \overline{v} \partial_t^2 \overline{v} + \partial_t \nabla \overline{v} \cdot \partial_t^2 \nabla \overline{v}\right) e^{2s\varphi} \mathrm{d}x\mathrm{d}t \\ &\quad \leq C \sum_{j=2}^3 \int_{2\varepsilon}^{T-2\varepsilon} \int_\Omega s \left(|\partial_t^{j-1} \overline{v}|^2 + |\partial_t^{j-1} \nabla \overline{v}|^2\right) e^{2s\varphi} \mathrm{d}x\mathrm{d}t \\ &\quad \leq C \sum_{j=2}^3 \int_{2\varepsilon}^{T-2\varepsilon} \int_\Omega s \left(|\partial_t^{j-1} \overline{u}_i|^2 + |\partial_t^{j-1} \overline{u}_e|^2 + |\partial_t^{j-1} \nabla \overline{u}_i|^2 + |\partial_t^{j-1} \nabla \overline{u}_e|^2\right) e^{2s\varphi} \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{4.20}$$

From (4.19) and (4.20), it follows that

$$\int_\Omega s^2\left(|\overline{M}_i|^2+|\overline{M}_e|^2+|\nabla\overline{M}_i|^2+|\nabla\overline{M}_e|^2\right)e^{2s\varphi(x,t_0)}\mathrm{d}x$$
$$\le C\sum_{j=2}^{3}\int_{2\varepsilon}^{T-2\varepsilon}\int_\Omega s\left(|\partial_t^{j-1}\overline{u}_i|^2+|\partial_t^{j-1}\overline{u}_e|^2+|\partial_t^{j-1}\nabla\overline{u}_i|^2+|\partial_t^{j-1}\nabla\overline{u}_e|^2\right)e^{2s\varphi}\mathrm{d}x\mathrm{d}t$$
$$+Ce^{Cs}L^2. \tag{4.21}$$

Substituting (4.21) into (4.7) leads to

$$\int_\Omega s^2\left(|\overline{M}_i|^2+|\overline{M}_e|^2+|\nabla\overline{M}_i|^2+|\nabla\overline{M}_e|^2\right)e^{2s\varphi(x,t_0)}\mathrm{d}x$$
$$\le C\int_{Q_T} s^2\left(|\overline{M}_i|^2+|\nabla\overline{M}_i|^2+|\overline{M}_e|^2+|\nabla\overline{M}_e|^2\right)e^{2s\varphi}\mathrm{d}x\mathrm{d}t$$
$$+Cs\exp(2se^{\lambda_0(M-\delta)})K^2+Cs^3e^{Cs}L^2 \tag{4.22}$$

for all $s\ge s^*=\max\{s_1,s_2\}$. Since $\varphi(x,t_0)>\varphi(x,t)$ for $t\ne t_0$, Lebesgue's dominated convergence theorem yields

$$\int_{Q_T}\left(|\overline{M}_i|^2+|\nabla\overline{M}_i|^2+|\overline{M}_e|^2+|\nabla\overline{M}_e|^2\right)e^{2s\varphi}\mathrm{d}x\mathrm{d}t$$
$$=\int_\Omega\left(|\overline{M}_i|^2+|\overline{M}_e|^2+|\nabla\overline{M}_i|^2+|\nabla\overline{M}_e|^2\right)e^{2s\varphi(x,t_0)}$$
$$\times\left(\int_0^T e^{-2s(\varphi(x,t_0)-\varphi(x,t))}\right)\mathrm{d}t\mathrm{d}x$$
$$\le\epsilon\int_\Omega\left(|\overline{M}_i|^2+|\overline{M}_e|^2+|\nabla\overline{M}_i|^2+|\nabla\overline{M}_e|^2\right)e^{2s\varphi(x,t_0)}\mathrm{d}x \tag{4.23}$$

for sufficiently small $\epsilon>0$, as $s\to+\infty$. We can choose ϵ sufficiently small to absorb the first term on the right-hand side into the left-hand side to obtain

$$s^2\int_\Omega\left(|\overline{M}_i|^2+|\overline{M}_e|^2+|\nabla\overline{M}_i|^2+|\nabla\overline{M}_e|^2\right)e^{2s\varphi(x,t_0)}\mathrm{d}x$$
$$\le Cs\exp(2se^{\lambda_0(M-\delta)})K^2+Cs^3e^{Cs}L^2. \tag{4.24}$$

Additionally, since $\varphi(x,t_0)\ge\exp(\lambda_0 M)$ in Ω, we have

$$s^2\int_\Omega\left(|\overline{M}_i|^2+|\overline{M}_e|^2+|\nabla\overline{M}_i|^2+|\nabla\overline{M}_e|^2\right)e^{2s\varphi(x,t_0)}\mathrm{d}x$$
$$\ge Cs^2\exp(2se^{\lambda_0 M})\left(\|\overline{M}_i\|^2_{H^1(\Omega)}+\|\overline{M}_e\|^2_{H^1(\Omega)}\right). \tag{4.25}$$

Therefore, it follows from (4.24) and (4.25) that

$$\|\overline{M}_i\|^2_{H^1(\Omega)}+\|\overline{M}_e\|^2_{H^1(\Omega)}\le Ce^{-2\sigma s}K^2+Cs\exp(-2se^{\lambda_0 M})e^{Cs}L^2$$
$$\le Ce^{-2\sigma s}K^2+Ce^{Cs}L^2 \tag{4.26}$$

for $s\ge s^*$ with $s^*\exp(-2s^*e^{\lambda_0 M})<1$, where $\sigma=e^{\lambda_0 M}-e^{\lambda_0(M-\delta)}$. Setting $s:=s+s^*$ and replacing C by Ce^{Cs^*}, we obtain (4.26) for all $s\ge 0$. Finally, minimizing the right-hand side of (4.26) with respect to s, we obtain

$$\|\overline{M}_i\|^2_{H^1(\Omega)} + \|\overline{M}_e\|^2_{H^1(\Omega)} \leq CK^{2\mu}L^{2(1-\mu)} \tag{4.27}$$

with $\mu = \frac{C}{C+2\sigma}$. This completes the proof of Theorem 1.3. □

5. Conclusion

In this paper, we prove two stability results for a linearized bidomain model in electrocardiology. One is a conditional stability for Cauchy problem (Theorem 1.1), which shows that we could determine (u_i, u_e) in an arbitrary sub-domain Ω_0 by the lateral data of (u_i, u_e) on arbitrary non-empty sub-boundary of $\partial\Omega$. In other words, in order to obtain the data in interior domain, we only need to measure the lateral data on some sub-boundary rather than the whole boundary in engineering environment. This can greatly reduce the measurement data. In equal anisotropy case, our another stability (Theorem 1.3) gives a Hölder stability for recovering two conductivity functions by the data in a suitable small interior domain and the data at a fixed time. Such kinds of inverse problems of determining the physical parameters in applied model have not only a great theoretical value but also a certain realistic value, which would provide theoretical support for researchers to develop stable and efficient numerical methods. Widely open is the case of strong anisotropy. Since in this case the conductivities in the longitudinal direction are higher than those across the fiber, one has to identify the unknown $\sigma_i^l, \sigma_i^t, \sigma_e^l, \sigma_e^t$ and a direction $\mathbf{a}_l$ in matrices $\mathbf{M}_i$ and $\mathbf{M}_e$, which needs more measurement data and more elegant mathematical analysis.

Our stability solves what data could determine the conductivities in mathematics. However, how to measure the data without injury, especially in the interior of the heart, is still a hard problem. A method in medicine is using catheter interventions. Measuring the data on the surface of the torso is noninvasive and much easier to manipulate. Therefore, similar inverse problems for a more complicated model consisting of a geometric torso model and a model of the electric activation in the heart [20] have more realistic meaning. Our future research will focus on this subject.

Acknowledgements

This work is supported by NSFC (Nos. 11661004, 11601240)

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Received: 29 December 2016.
Accepted: 8 January 2018.